

$K(E_9)$ from $K(E_{10})$

Axel Kleinschmidt, Hermann Nicolai and Jakob Palmkvist

*Max Planck Institute for Gravitational Physics, Albert Einstein Institute,
Am Mühlenberg 1, 14476 Potsdam, Germany*

*E-mail: axel.kleinschmidt@aei.mpg.de, hermann.nicolai@aei.mpg.de,
jakob.palmkvist@aei.mpg.de*

ABSTRACT: We analyse the M-theoretic generalisation of the tangent space structure group after reduction of the $D = 11$ supergravity theory to two space-time dimensions in the context of hidden Kac-Moody symmetries. The action of the resulting infinite-dimensional ‘R symmetry’ group $K(E_9)$ on certain unfaithful, finite-dimensional spinor representations inherited from $K(E_{10})$ is studied. We explain in detail how these representations are related to certain finite codimension ideals within $K(E_9)$, which we exhibit explicitly, and how the known, as well as new finite-dimensional ‘generalised holonomy groups’ arise as quotients of $K(E_9)$ by these ideals. In terms of the loop algebra realisations of E_9 and $K(E_9)$ on the fields of maximal supergravity in two space-time dimensions, these quotients are shown to correspond to (generalised) evaluation maps, in agreement with previous results of [1]. The outstanding question is now whether the related unfaithful representations of $K(E_{10})$ can be understood in a similar way.

KEYWORDS: Global Symmetries, Gauge Symmetry, Extended Supersymmetry, Supergravity Models.

Contents

1.	Introduction	1
2.	E_8, E_9 and E_{10}	4
2.1	E_8 via A_7	5
2.2	E_9 as extended current algebra	7
2.3	Embedding of E_9 in E_{10}	7
3.	$K(E_9)$ spinor representations from $K(E_{10})$	12
3.1	Dirac-spinor ideal	15
3.2	Vector-spinor ideal	18
4.	Relation to current algebra realisation	22
4.1	$\mathfrak{so}(16) \subset E_{8(8)}$	23
4.2	Current algebra fermion transformations	25
A.	Gamma matrix conventions	27
B.	Relation between the two E_8 bases	28

1. Introduction

The study of a one-dimensional bosonic geodesic σ -model based on the the Kac-Moody coset $E_{10}/K(E_{10})$ has revealed a tantalizing dynamical link to the bosonic dynamics of maximal $D = 11$ supergravity in the vicinity of a space-like singularity [2] (see also [3]).¹ Though striking, this link is limited to truncations on both the Kac-Moody side and the supergravity side. Further progress is partly inhibited by a lack of understanding of the structure of E_{10} and of its maximal compact subgroup $K(E_{10})$ which is not even of Kac-Moody type [7]. The extension of the partial results in the bosonic sector to fermionic fields requires the representation theory of the infinite-dimensional $K(E_{10})$. As an important first step it was shown in [8–11] that $K(E_{10})$ admits (unfaithful) spinor representations of dimensions 320 and 32 with the correct properties to parallel the promising features of the bosonic model. In particular, it was shown there that the fermionic field equations of maximal supergravity (with appropriate truncations) take the form of a $K(E_{10})$ covariant ‘Dirac equation’. Furthermore, the decomposition of these spinor representations under those subgroups of $K(E_{10})$ which are known to lead to the massive type IIA and type

¹An alternative covariant approach to the bosonic dynamics of $D = 11$ supergravity based on E_{11} and the conformal group was initiated in [4, 5]. See also [6] for a proposal combining some features of [5] and [2].

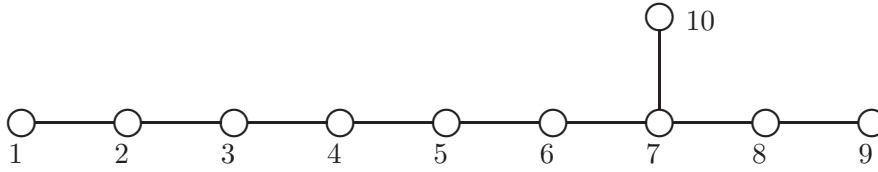


Figure 1: Dynkin diagram of E_{10} with numbering of nodes.

IIB theories were shown to result in precisely the right (respectively, vector-like and chiral) fermionic field representations of type IIA and type IIB supergravity [12] (the corresponding embeddings of the bosonic sectors had already been studied previously in [13, 14] for E_{10} , and [15–17] for E_{11}). In this way the E_{10} and $K(E_{10})$ structures incorporate kinematically and dynamically the known duality relations between the maximal supergravity theories for bosons and fermions alike.²

In this paper we extend the analysis of the unfaithful $K(E_{10})$ representations to a decomposition under its $K(E_9)$ subgroup. The latter is the maximal compact subgroup of the affine E_9 which is known to be a symmetry of the field equations of maximal $N = 16$ supergravity in $D = 2$ [20–23].³ While the finite-dimensional exceptional ‘hidden symmetries’ E_n of maximal supergravity in $D = 11 - n$ for $n \leq 8$ can be directly realised on the supergravity fields [26, 27], the infinite-dimensional affine symmetries of the $D = 2$ theory are realised via a linear system whose integrability condition yields the equations of motion. The fermionic fields (as well as the supercharges) form linear representations of the maximal compact subgroup $K(E_n)$ for $n \leq 9$. Here we will show how, using $K(E_{10})$ and its representations, the $K(E_9)$ transformation rules for the fermions in two space-time dimensions can be derived purely algebraically from the reduction. This constitutes the first direct proof of the $K(E_9)$ properties of $D = 2$ supergravity that does not resort to the linear system. Moreover, we will show that our algebraic action is equivalent to the analytic description of the $K(E_9)$ action in terms of a spectral parameter via a ‘generalised evaluation map’ [1]. The equivalence of the latter with the algebraic construction of the present work suggests that $K(E_{10})$ may admit a similar realisation – a tantalizing opportunity for future research, since it may also lead to a new realisation of the hyperbolic Kac-Moody algebra E_{10} itself!

A major tool in our investigation is the so-called level decomposition of the global hidden symmetries E_n . In figure 1, we display the Dynkin diagram of E_{10} with our labelling conventions; the lower rank exceptional algebras are obtained by removing nodes from the left. The level decomposition with regard to the $A_{n-1} \equiv \mathfrak{sl}(n)$ subalgebras of E_n allows one to identify the physical fields from the adjoint representation of E_n in terms of $\text{SL}(n)$ tensors. More specifically, these level decompositions follow the scheme presented in table 1 for $n = 5, \dots, 9$, where we label the relevant $\text{SL}(n)$ representations by bold face letters in

²The correct structure for the non-maximal type I supergravity theory in $D = 10$ is $DE_{10} \subset E_{10}$ [18]. The **32** and **320** representations of $K(E_{10})$ decompose into the correct spinors of $K(DE_{10})$. The bosonic sector of this theory was previously studied in relation to DE_{11} in [19].

³See also [24, 25] for similar infinite-dimensional symmetries in pure Einstein gravity.

$E_n \setminus \ell$	-4	-3	-2	-1	0	1	2	3	4
E_5				10	\oplus (24 \oplus 1)	\oplus 10			
E_6			1	\oplus 20	\oplus (35 \oplus 1)	\oplus 20	\oplus 1		
E_7			7	\oplus 35	\oplus (48 \oplus 1)	\oplus 35	\oplus 7		
E_8		$\overline{\mathbf{8}}$	\oplus 28	\oplus $\overline{\mathbf{56}}$	\oplus (63 \oplus 1)	\oplus 56	\oplus $\overline{\mathbf{28}}$	\oplus 8	
E_9	\dots	\oplus 80	\oplus 84	\oplus $\overline{\mathbf{84}}$	\oplus (80 \oplus 1 \oplus 1)	\oplus 84	\oplus $\overline{\mathbf{84}}$	\oplus 80	\oplus \dots

Table 1: The level decompositions of the global E_n hidden symmetries in $D = 11 - n$ dimensions under the gravity $SL(n)$ subgroup. The column headings ℓ refer to the level in this level decomposition. For $\ell = 0$, the adjoint of $SL(n)$ always combines with the singlet into the adjoint of $GL(n)$, in the affine case also extended by the derivation d and the central element c of E_9 . The derivation is part of $\mathfrak{gl}(9)$.

the usual way, noting that the entries of the columns $\ell = 1, 2$ always correspond to the three- and six-form representations of $SL(n)$, respectively (and the columns $\ell = -1, -2$ to their contragredient representations). Naturally, E_6 in five dimensions is the first time the six-forms appear in the scalar coset.

For the finite-dimensional algebras in this series (that is, for $n \leq 8$) these results have been known for a long time (for a systematic analysis, see [27]). For $n = 9$, the triple of representations $\overline{\mathbf{84}} \oplus \mathbf{80} \oplus \mathbf{84}$ is repeated an infinite number of times, giving rise to the affine extension of E_8 in the standard way (the two singlets appearing in the middle column for E_9 are the central charge c and the derivation d). For $n = 10$ and $n = 11$, we can no longer display the representations in such a simple fashion, as the number of representations ‘explodes’; but see [28] for the tables up to levels $\ell = 18$ and $\ell = 10$, respectively, which were obtained by computer algebra,⁴ and also [2] and [29] for earlier results on very low levels of E_{10} and E_{11} , respectively.

We conclude this introduction with some comments on the link between the mathematical structures (ideals, and unfaithful representations of infinite-dimensional compact subgroups of hidden symmetries) exhibited in the main part of this paper, and the so-called ‘generalised holonomies’ discussed in the recent literature. *Quite generally, the latter should be identified with quotients of the infinite-dimensional algebras $K(E_9)$ and $K(E_{10})$ by certain finite codimension ideals.* Given any Lie algebra \mathfrak{k} and a linear representation space V , the subspace

$$i_V := \{x \in \mathfrak{k} \mid x \cdot v = 0 \quad \forall v \in V\} \subset \mathfrak{k} \tag{1.1}$$

defines an *ideal* in \mathfrak{k} . The representation is unfaithful if $i_V \neq \{0\}$. The existence of non-trivial ideals implies in particular that the Lie algebra \mathfrak{k} is not simple. For any i_V , we can define the quotient algebra

$$\mathfrak{q}_V := \mathfrak{k}/i_V \subset \mathfrak{gl}(V). \tag{1.2}$$

⁴In fact, for E_{10} , the tables are available up to A_9 level $\ell = 28$ with a total of 4 400 752 653 representations [28].

The unfaithful *finite-dimensional* spinorial representations of $K(E_9)$ and $K(E_{10})$ discovered in [1, 8–11] are directly related to the Dirac- and vector (gravitino) spinors appearing in maximal supergravities. For instance, the relevant representations for $K(E_{10})$ are the **32** and the **320** [9, 10]. These representations are inherited by $K(E_9) \subset K(E_{10})$, such that the **32** decomposes into two inequivalent 16-dimensional Dirac-type representations of $K(E_9)$. *As one of our main results we are able to present the associated ideals in $K(E_9)$ in complete detail*, cf. section 3. Because a single ideal may be associated to more than one (and sometimes infinitely many) representations, the description of these structures in terms of ideals appears to be the most economical way to study them.

It is perhaps worth stressing that the quotient group $SO(16)_+ \times SO(16)_-$ associated to the $\mathbf{16}_+ \oplus \mathbf{16}_-$ representation of $K(E_9)$ is *not* a subgroup of $K(E_9)$, because the would-be $SO(16)_+ \times SO(16)_-$ generators are *distributional objects*, as we will explain (see also [11]). The latter group has been proposed as a ‘generalised holonomy group’ of M-theory [30, 31], generalising the $SO(9)$ Lorentz structure group of the tangent space of the nine torus on which the $D = 11$ theory was reduced. By studying its subgroups and the branching of the **32** representation under these, supersymmetric solutions can be studied and classified [32, 30, 31, 33]. On the other hand, it is known that neither this generalised holonomy group, nor its extensions $SO(32)$ and $SL(32)$, can extend to symmetries of the full equations because of global obstructions [34]. In addition, the generalised holonomies proposed so far do not admit acceptable vector-spinor representations, and as such are restricted to the Killing spinor equation instead of the full supergravity system (in particular, the Rarita Schwinger equation). Our results strengthen the case for $K(E_9)$ and for $K(E_{10})$ as the correct generalised holonomy (and R symmetry) groups since both groups do allow for all the required spinor representations. Moreover, $K(E_9)$ is a genuine local symmetry of the reduced theory.

This article is organised as follows. Section 2 summarizes some (largely known) results on the embedding of E_8 and E_9 in a notation adapted to the level decomposition, and goes on to derive their embedding into E_{10} . Informed readers may skip the bulk of this section and proceed directly to section 3, where we derive the branching of the unfaithful $K(E_{10})$ spinors under the $K(E_9)$ subalgebra. The resulting $K(E_9)$ transformation rules are compared to those of the linear system in section 4. Using relations provided in two appendices, we establish complete agreement with previous results of [1].

2. E_8 , E_9 and E_{10}

We here study the chain of embeddings $E_8 \subset E_9 \subset E_{10}$ in $A_7 \subset A_8 \subset A_9$ level decompositions and fix necessary notation for our analysis of the spinor representations in the next section. Throughout this paper, except for the appendices, we adopt the following indexing conventions for the $SL(n)$ tensors arising in the decomposition of the algebras E_8 ,

A_7 level ℓ in E_8	Generator	SL(8) representation
-3	Z_i	$\bar{8}$
-2	$Z_{i_1 \dots i_6}$	28
-1	$Z_{i_1 i_2 i_3}$	$\bar{56}$
0	G^i_j	63 \oplus 1
1	$Z^{i_1 i_2 i_3}$	56
2	$Z^{i_1 \dots i_6}$	$\bar{28}$
3	Z^i	8

Table 2: A_7 decomposition of E_8 .

E_9 and E_{10} :

$$\begin{aligned}
 E_{10} &\leftrightarrow a, b, \dots \in \{1, \dots, 10\} \\
 E_9 &\leftrightarrow \alpha, \beta, \dots \in \{2, \dots, 10\} \\
 E_8 &\leftrightarrow i, j, \dots \in \{3, \dots, 10\}
 \end{aligned}
 \tag{2.1}$$

2.1 E_8 via A_7

The E_8 subalgebra of E_{10} is generated by nodes 3 through to 10 of figure 1 and can be written in terms of irreducible tensors of its $A_7 \cong \mathfrak{sl}(8)$ subalgebra (corresponding to nodes 3 through to 9). By adjoining the eighth Cartan generator, this $\mathfrak{sl}(8)$ subalgebra can be extended to a $\mathfrak{gl}(8)$ subalgebra generated by

$$G^i_j, \quad \text{with} \quad [G^i_j, G^k_l] = \delta_j^k G^i_l - \delta_l^i G^k_j,
 \tag{2.2}$$

where the indices take values $i, j = 3, \dots, 10$. The A_7 decomposition of E_8 gives the $\mathfrak{sl}(8)$ tensors displayed in table 2 [27].

In the left column we have indicated the $\mathfrak{sl}(8)$ level, that is the number of times the exceptional simple root α_{10} occurs in the associated roots. All indices i, j, \dots run from $3, \dots, 10$ and all tensors, except for G^i_j , are totally anti-symmetric in their SL(8) (co-)vector indices. The Chevalley transposition $(\cdot)^T$ acts by $(G^i_j)^T = G^j_i$ and $(Z_{i_1 i_2 i_3})^T = Z^{i_1 i_2 i_3}$, etc. The $\mathfrak{gl}(8)$ tensors in the table with upper (lower) indices correspond to positive (negative) roots. In E_{10} language, the former correspond to the ‘E-type’ generators, while the latter transform in the contragredient representations and correspond to the ‘F-type’ generators in the notation of [3].

The commutation relations between G^i_j and the positive and negative $\mathfrak{gl}(8)$ level ‘step

operators' are

$$\begin{aligned}
 [G_j^i, Z^{k_1 k_2 k_3}] &= 3\delta_j^{[k_1} Z^{k_2 k_3]i}, \\
 [G_j^i, Z^{k_1 \dots k_6}] &= -6\delta_j^{[k_1} Z^{k_2 \dots k_6]i}, \\
 [G_j^i, Z^k] &= \delta_j^k Z^i + \delta_j^i Z^k, \\
 [G_j^i, Z_{k_1 k_2 k_3}] &= -3\delta_{[k_1}^i Z_{k_2 k_3]j}, \\
 [G_j^i, Z_{k_1 \dots k_6}] &= 6\delta_{[k_1}^i Z_{k_2 \dots k_6]j}, \\
 [G_j^i, Z_k] &= -\delta_k^i Z_j - \delta_j^i Z_k.
 \end{aligned} \tag{2.3}$$

Note the trace terms in the commutation relations involving the $\mathfrak{gl}(8)$ vectors Z^i and Z_i which are needed for the correct transformation under the trace of $\mathfrak{gl}(8)$, and for the consistency of the first two relations with the second relation in (2.4) below. Furthermore,

$$\begin{aligned}
 [Z^{i_1 i_2 i_3}, Z^{i_4 i_5 i_6}] &= Z^{i_1 \dots i_6}, \\
 [Z^{i_1 i_2 i_3}, Z^{i_4 \dots i_9}] &= 3Z^{[i_1} \epsilon^{i_2 i_3] i_4 \dots i_9},
 \end{aligned} \tag{2.4}$$

where $\epsilon^{i_1 \dots i_8}$ is the $\text{SL}(8)$ totally anti-symmetric tensor. Similar expressions are obtained for the negative level generators by applying the Chevalley transposition.

The mixed commutation relations are

$$\begin{aligned}
 [Z^{i_1 i_2 i_3}, Z_{j_1 j_2 j_3}] &= -2\delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} G + 18\delta_{[j_1 j_2}^{[i_1 i_2} G_{j_3]}^{i_3]}, \\
 [Z^{i_1 i_2 i_3}, Z_{j_1 \dots j_6}] &= -5! \delta_{[j_1 j_2 j_3}^{i_1 i_2 i_3} Z_{j_4 j_5 j_6]}, \\
 [Z^{i_1 \dots i_6}, Z_{j_1 \dots j_6}] &= -\frac{2}{3} \cdot 6! \delta_{j_1 \dots j_6}^{i_1 \dots i_6} G + 6 \cdot 6! \delta_{[j_1 \dots j_5}^{[i_1 \dots i_5} G_{j_6]}^{i_6]}, \\
 [Z^{i_1 i_2 i_3}, Z_j] &= \frac{1}{5!} \epsilon^{i_1 i_2 i_3 k_1 \dots k_5} Z_{k_1 \dots k_5 j}, \\
 [Z^{i_1 \dots i_6}, Z_j] &= \frac{1}{2} \epsilon^{i_1 \dots i_6 k_1 k_2} Z_{k_1 k_2 j}, \\
 [Z^i, Z_j] &= G_j^i.
 \end{aligned} \tag{2.5}$$

Here, $G \equiv \sum_{k=3}^{10} G^k$. Equations (2.3), (2.4) and (2.5), together with their Chevalley transposes, constitute a complete set of E_8 commutation relations. The normalisations of the generators are

$$\begin{aligned}
 \langle G_j^i | G_l^k \rangle &= \delta_l^i \delta_j^k + \delta_j^i \delta_l^k, \\
 \langle Z^{i_1 i_2 i_3} | Z_{j_1 j_2 j_3} \rangle &= 3! \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3}, \\
 \langle Z^{i_1 \dots i_6} | Z_{j_1 \dots j_6} \rangle &= 6! \delta_{j_1 \dots j_6}^{i_1 \dots i_6}, \\
 \langle Z^i | Z_j \rangle &= \delta_j^i.
 \end{aligned} \tag{2.6}$$

Modulo normalisation factors, the same relations have been given for example in [27, 35]. In comparison with the notation of [35] the tensors on levels $\ell = \pm 2$ have been dualised using the ϵ -tensor of $\text{SL}(8)$ and some of the normalisations have changed.

2.2 E_9 as extended current algebra

As is well known (see e.g. [36]), the affine Lie algebra $E_9 \equiv E_8^{(1)}$ is obtained from E_8 by ‘affinization’, that is by embedding E_8 in its current algebra (parametrized by the spectral parameter t), and by adjoining two more Lie algebra elements, the central charge c and the derivation d : $E_9 = E_8[[t, t^{-1}]] \oplus \mathbb{R}c \oplus \mathbb{R}d$ (as always, we restrict attention to the split real forms of these Lie algebras). By writing $X^{(m)} \equiv X \otimes t^m$ (for $m \in \mathbb{Z}$) the E_9 commutation relations are

$$\begin{aligned} [X^{(m)}, Y^{(n)}] &= [X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{m+n} + m\delta_{m+n,0} \langle X|Y \rangle c, \\ [d, X^{(m)}] &= mX^{(m)}, \\ [c, X^{(m)}] &= 0, \quad [c, d] = 0. \end{aligned} \tag{2.7}$$

They can thus be read off directly from the E_8 commutation relations above in the standard fashion. The inner product between c and d is $\langle c|d \rangle = 1$. The ‘horizontal’ E_8 at affine level 0 is isomorphic to E_8 and we will often write $X \equiv X^{(0)}$ for any E_8 generator X , for example

$$G^i_j \equiv G^{(0)i}_j, \quad Z_i \equiv Z_i^{(0)}, \quad \text{etc.} \tag{2.8}$$

Next, we will study how the current algebra generators emerge from E_{10} , that is how they are obtained from the latter algebra by truncation and by ‘dimensional reduction’.

2.3 Embedding of E_9 in E_{10}

With regard to the E_{10} Dynkin diagram, the E_9 subalgebra of E_{10} is obtained by deleting node 1 from the diagram 1, or equivalently by restricting to level zero in an E_9 level decomposition⁵ which counts the number of occurrences of the simple root α_1 . However, one does keep the Cartan generator h_1 which is needed to ‘desingularize’ the metric on the root lattice (so the Cartan subalgebra of E_9 can be identified with the one of E_{10} , h_1 appears only in d). Using the $\mathfrak{gl}(10)$ basis of E_{10} , where small Latin indices take values $a = 1, \dots, 10$,

$$K^a_b, \quad \text{with} \quad [K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b, \tag{2.9}$$

the E_{10} Cartan generators are

$$\begin{aligned} h_a &= K^a_a - K^{a+1}_{a+1} \quad (a = 1, \dots, 9), \\ h_{10} &= -\frac{1}{3} \sum_{a=1}^{10} K^a_a + K^8_8 + K^9_9 + K^{10}_{10}. \end{aligned} \tag{2.10}$$

The invariant inner product of these generators is given by

$$\langle K^a_b | K^c_d \rangle = \delta_d^a \delta_b^c - \delta_b^a \delta_d^c. \tag{2.11}$$

⁵In comparison to the A_9 level decomposition of E_{10} which can be thought of as a space-like foliation of the Lorentzian root lattice, the E_9 decomposition foliates the root lattice by *null* planes.

The coefficient of the second term is not fixed by invariance but by requiring that $\langle h_{10}|h_{10}\rangle = 2$, where h_{10} in (2.10) was fixed by requiring the right $\mathfrak{gl}(10)$ commutation relation with the A_9 level $\ell = 1$ generator E^{abc} .⁶ We follow the conventions of [3] except for two differences: Firstly, we take e_{10} to be E^{8910} since the exceptional node is attached at the other end. Secondly, we rescale the A_9 level $\ell = \pm 3$ generators by a factor $1/3$.

In terms of the A_9 level decomposition of E_{10} the E_9 elements are precisely those contained in the ‘gradient representations’ of [2] where indices are restricted to take values $2, \dots, 10$. As shown there (see also [28]), every n th order gradient generator contains n sets of nine anti-symmetric indices, and thus has A_9 Dynkin labels $[n * * * * * * * * *]$. For instance, at A_9 level $\ell = 3n + 1$, we have the following gradient generators

$$E^{a_1^{(1)} \dots a_9^{(1)} | \dots | a_1^{(n)} \dots a_9^{(n)} | bcd} \quad \text{with } a_i^{(j)}, b, c, d \in \{1, \dots, 10\},$$

which are antisymmetric in all 9 -tuples $(a_1^{(j)} \dots a_9^{(j)})$. Restricting all indices on the above element to the values $2, \dots, 10$, we see that, up to permutations, there is only one choice of filling indices into these 9 -tuples, and we thus only need to remember that there were n such sets. In fact, this restriction is physically motivated since E_9 arises in the reduction to two dimensions with one left-over non-compact spatial direction 1 (obviously, there are nine alternative choices for this residual spatial dimension, corresponding to ten distinguished E_9 subgroups in E_{10}). Accordingly, we introduce the following shorthand notation for the gradient generators

$$\begin{aligned} E^{\overbrace{2 \dots 10 | \dots | 2 \dots 10}^{n \text{ times}} | \alpha_1 \alpha_2 \alpha_3} &\equiv E^{(n)}_{\alpha_1 \alpha_2 \alpha_3} \\ E^{\overbrace{2 \dots 10 | \dots | 2 \dots 10}^{n \text{ times}} | \alpha_1 \dots \alpha_6} &\equiv E^{(n)}_{\alpha_1 \dots \alpha_6} \\ E^{\overbrace{2 \dots 10 | \dots | 2 \dots 10}^{n \text{ times}} | \alpha_0 | \alpha_1 \dots \alpha_8} &\equiv E^{(n)}_{\alpha_0 | \alpha_1 \dots \alpha_8} \end{aligned} \tag{2.12}$$

where $\alpha_0, \alpha_1, \alpha_2, \dots = 2, \dots, 10$. The ‘F-type’ gradient generators are defined analogously. Our notation is summarized in table 3: the indices here take values $\alpha = 2, \dots, 10$, and together with K^α_β and K^1_1 from A_9 level $\ell = 0$ they constitute *all* E_9 generators expressed in E_{10} variables. As will be seen below, the central charge c of E_9 in terms of E_{10} generators is proportional to K^1_1 and commutes with all elements of E_9 whence the restriction of indices to $\alpha = 2, \dots, 10$ is the correct restriction to E_9 . That the suppression of the blocks of nine indices is justified will be shown below. Now we want to relate these generators to the E_9 generators of section 2.

The generators of E_8 are embedded regularly in E_{10} and, away from the Cartan subalgebra, are identical to those of E_{10} for levels $|\ell| \leq 3$ if the indices are restricted to the

⁶This is the reason for the minus sign in the bilinear form (2.11), resulting in the indefiniteness of the inner product (2.11). By contrast, (2.6) has a plus sign in the corresponding formula, whence the inner product is positive definite for E_8 .

A_9 level in E_{10}	(Restricted) gradient generator	$\mathfrak{sl}(9)$ irrep
$\ell = 3n + 3$	$E^{(n)}_{\alpha_0 \alpha_1\dots\alpha_8}$	80
$\ell = 3n + 2$	$E^{(n)}_{\alpha_1\dots\alpha_6}$	$\overline{84}$
$\ell = 3n + 1$	$E^{(n)}_{\alpha_1\alpha_2\alpha_3}$	84
$\ell = -3n - 1$	$F^{(n)}_{\alpha_1\alpha_2\alpha_3}$	$\overline{84}$
$\ell = -3n - 2$	$F^{(n)}_{\alpha_1\dots\alpha_6}$	84
$\ell = -3n - 3$	$F^{(n)}_{\alpha_0 \alpha_1\dots\alpha_8}$	80

Table 3: Identification of the E_9 generators in terms of E_{10} gradient generators.

range $\{3, \dots, 10\}$. Therefore we find immediately that

$$\begin{aligned}
 Z^{(0)i_1i_2i_3} &= E^{(0)i_1i_2i_3}, & Z^{(0)}_{i_1i_2i_3} &= F^{(0)}_{i_1i_2i_3}, \\
 Z^{(0)i_1\dots i_6} &= E^{(0)i_1\dots i_6}, & Z^{(0)}_{i_1\dots i_6} &= F^{(0)}_{i_1\dots i_6}, \\
 \epsilon^{k_1\dots k_8} Z^{(0)i} &= E^{(0)i|k_1\dots k_8}, & \epsilon_{k_1\dots k_8} Z^{(0)}_i &= F^{(0)}_{i|k_1\dots k_8},
 \end{aligned}
 \tag{2.13}$$

where the superscript on the l.h.s. denotes the affine level, whereas the superscript on the r.h.s. denotes the ‘gradient’ level as explained in (2.12). As a mnemonic and notational device to distinguish between these two kinds of levels we place the superscripts slightly differently, as evident from the preceding equation. The objects on the r.h.s. are $GL(8)$ tensors, and we recall that, for the comparison between E_8 and E_{10} we must restrict the indices on the $SL(10)$ tensors appearing in the A_9 decomposition of E_{10} to the values $i = 3, \dots, 10$. To identify the $GL(8)$ generators in terms of the Cartan generators we note that the only difference can be in the diagonal part of $G^{(0)i}_j$ since the off-diagonal elements correspond to step operators. A simple calculation shows that the correct identification between $G^i_j \in E_8$ and $K^i_j \in E_{10}$ is⁷

$$G^i_j \equiv G^{(0)i}_j = K^i_j + \delta^i_j(c - d), \tag{2.14}$$

where the central element c and derivation d of E_9 in terms of the $\mathfrak{gl}(10)$ generators are given by

$$d = K^2_2, \quad c = -K^1_1. \tag{2.15}$$

It is easy to see that c indeed commutes with all elements of E_9 and has inner product +1 with d . Furthermore, d commutes with E_8 of (2.13) as it should. Evidently, the affine level operator d counts the number of tensor indices taking the value 2 (with (+1) for upper

⁷One way to see the necessity of this redefinition is to compute $[Z^{(0)8910}, Z^{(0)}_{8910}]$ both in E_8 and E_{10} , and to demand that the central charge c and the derivation d drop out from this commutator for E_8 .

and (-1) for lower indices). The extra terms in (2.14) also induce the relative change in sign between (2.6) and (2.11).

Using the relation of the general linear subalgebras (2.14) we can show that the blocks of nine anti-symmetric indices suppressed in the gradient generators are not ‘seen’ by the $\mathfrak{gl}(8)$ generators, as we already claimed above. Consider a generator $X^{2k_1\dots k_8}$ which is totally anti-symmetric in its nine indices and $k \in \{3, \dots, 10\}$. Then

$$\left[G^{(0)i}{}_j, X^{2k_1\dots k_8} \right] = 8\delta_j^{[k_1} X^{k_2\dots k_8]2i} - \delta_j^i X^{2k_1\dots k_8} = -9\delta_j^i X^{k_1\dots k_8]2} = 0 \quad (2.16)$$

by Schouten’s identity; the last term in the middle expression is due to the correction term with d in (2.14), which is thus crucial for the vanishing of the above commutator. This confirms that we can indeed replace each 9 -tuple of indices by a label indicating the number of such 9 -tuples and assume that the 9 -tuples are filled in some fixed way by $2, \dots, 10$.

From the form of d in (2.15) we see that the number of upper indices equal to 2 on a positive step generators is the affine level and similarly for negative step operators. It is not hard to identify the following affine level $+1$ generators among the E_{10} generators,

$$\begin{aligned} Z_j^{(1)} &= K^2{}_j, \\ Z_{j_1\dots j_6}^{(1)} &= \frac{1}{2}\epsilon_{j_1\dots j_6 k_1 k_2} E^{(0)}_{k_1 k_2 2}, \\ Z_{j_1 j_2 j_3}^{(1)} &= \frac{1}{5!}\epsilon_{j_1 j_2 j_3 k_1\dots k_5} E^{(0)}_{k_1\dots k_5 2}, \\ G^{(1)i}{}_j &= -\frac{1}{7!}\epsilon_{j k_1\dots k_7} E^{(0)}_{i|2k_1\dots k_7} - \frac{1}{8!}\delta_j^i \epsilon_{k_1\dots k_8} E^{(0)}_{2|k_1\dots k_8}. \end{aligned} \quad (2.17)$$

This involves only generators with A_9 level $\ell = 0, \dots, 3$ in the E_{10} decomposition. Similarly, at affine level -1 we have

$$\begin{aligned} Z^{(-1)i} &= K^i{}_2, \\ Z^{(-1)i_1\dots i_6} &= \frac{1}{2}\epsilon^{i_1\dots i_6 k_1 k_2} F^{(0)}_{k_1 k_2 2}, \\ Z^{(-1)i_1 i_2 i_3} &= \frac{1}{5!}\epsilon^{i_1 i_2 i_3 k_1\dots k_5} F^{(0)}_{k_1\dots k_5 2}, \\ G^{(-1)i}{}_j &= -\frac{1}{7!}\epsilon^{i k_1\dots k_7} F^{(0)}_{j|2k_1\dots k_7} - \frac{1}{8!}\delta_j^i \epsilon^{k_1\dots k_8} F^{(0)}_{2|k_1\dots k_8}. \end{aligned} \quad (2.18)$$

Again, the redefinition (2.14) is crucial for the correct E_9 transformation rules, e.g.

$$\begin{aligned} \left[G^{(0)i}{}_j, Z_{k_1\dots k_6}^{(1)} \right] &= \frac{1}{2}\epsilon_{k_1\dots k_6 l_1 l_2} \left[K^i{}_j - \delta_j^i d, E^{(0)}_{l_1 l_2 2} \right] \\ &= \frac{1}{2}\epsilon_{k_1\dots k_6 l_1 l_2} \left(2\delta_j^{l_1} E^{(0)}_{i l_2 2} - \delta_j^i E^{(0)}_{l_1 l_2 2} \right) \\ &= \frac{1}{2 \cdot 6!}\epsilon_{k_1\dots k_6 l_1 l_2} \left(2\delta_j^{l_1} \epsilon^{i l_2 m_1\dots m_6} - \delta_j^i \epsilon^{l_1 l_2 m_1\dots m_6} \right) Z_{m_1\dots m_6}^{(1)} \\ &= 6\delta_{[k_1}^i Z_{k_2\dots k_6]j}^{(1)}, \end{aligned} \quad (2.19)$$

in agreement with (2.3) for affine level +1. We identify also the following elements at affine level ± 2 ,

$$\begin{aligned} Z_j^{(2)} &= -\frac{1}{7!} \epsilon_{jk_1 \dots k_7} E^{(0)}_{2|2k_1 \dots k_7}, \\ Z^{(-2)i} &= -\frac{1}{7!} \epsilon^{ik_1 \dots k_7} F^{(0)}_{2|2k_1 \dots k_7}. \end{aligned} \quad (2.20)$$

Indeed, one can check from these relations that

$$\left[Z^{(-2)i}, Z_j^{(2)} \right] = G^{(0)i}_j - 2\delta_j^i c \quad (2.21)$$

as it should be for this affine commutator. Again we see, that the affine level is equal to the difference between the number of upper and lower indices equalling 2. With relations (2.13), (2.14), (2.15), (2.17), (2.18) and (2.20) we have identified all E_9 generators appearing on A_9 levels $-3 \leq \ell \leq 3$ in E_{10} . It should now be clear how to obtain the higher affine levels: the scheme repeats itself after shifting $\ell \rightarrow \ell + 3$, as illustrated in figure 2. As evident from these formulæ, the affine level and the A_9 level are ‘oblique’ w.r.t. each other: The elements of affine level n are spread over the A_9 levels $3n - 3 \leq \ell \leq 3n + 3$. This is also shown in figure 2.

For completeness, we write the general formulæ for $n > 1$,

$$\begin{aligned} Z_i^{(n)} &= -\frac{1}{7!} \epsilon_{ik_1 \dots k_7} E^{(n-2)}_{2|2k_1 \dots k_7}, \\ Z_{i_1 \dots i_6}^{(n)} &= \frac{1}{2} \epsilon_{i_1 \dots i_6 k_1 k_2} E^{(n-1)}_{k_1 k_2 2}, \\ Z_{i_1 i_2 i_3}^{(n)} &= \frac{1}{5!} \epsilon_{i_1 i_2 i_3 k_1 \dots k_5} E^{(n-1)}_{k_1 \dots k_5 2}, \\ G^{(n)i}_j &= -\frac{1}{7!} \epsilon_{jk_1 \dots k_7} E^{(n-1)}_{i|2k_1 \dots k_7} - \frac{1}{8!} \delta_j^i \epsilon_{k_1 \dots k_8} E^{(n-1)}_{2|k_1 \dots k_8}, \\ Z^{(n)i_1 i_2 i_3} &= E^{(n)}_{i_1 i_2 i_3}, \\ Z^{(n)i_1 \dots i_6} &= E^{(n)}_{i_1 \dots i_6}, \\ Z^{(n)i} &= \frac{1}{8!} \epsilon_{k_1 \dots k_8} E^{(n)}_{i|k_1 \dots k_8} \end{aligned} \quad (2.22)$$

for the positive current modes and

$$\begin{aligned} Z_i^{(-n)} &= \frac{1}{8!} \epsilon^{k_1 \dots k_8} F_{i|k_1 \dots k_8}^{(n)}, \\ Z_{i_1 \dots i_6}^{(-n)} &= F_{i_1 \dots i_6}^{(n)}, \\ Z_{i_1 i_2 i_3}^{(-n)} &= F_{i_1 i_2 i_3}^{(n)}, \\ G^{(-n)i}_j &= -\frac{1}{7!} \epsilon^{ik_1 \dots k_7} F_{j|2k_1 \dots k_7}^{(n-1)} - \frac{1}{8!} \delta_j^i \epsilon_{k_1 \dots k_8} F_{2|k_1 \dots k_8}^{(n-1)}, \\ Z^{(-n)i_1 i_2 i_3} &= \frac{1}{5!} \epsilon^{i_1 i_2 i_3 k_1 \dots k_5} F_{k_1 \dots k_5 2}^{(n-1)}, \\ Z^{(-n)i_1 \dots i_6} &= \frac{1}{2} \epsilon^{i_1 \dots i_6 k_1 k_2} F_{k_1 k_2 2}^{(n-1)}, \\ Z^{(-n)i} &= -\frac{1}{7!} \epsilon^{ik_1 \dots k_7} F_{2|2k_1 \dots k_7}^{(n-2)} \end{aligned} \quad (2.23)$$

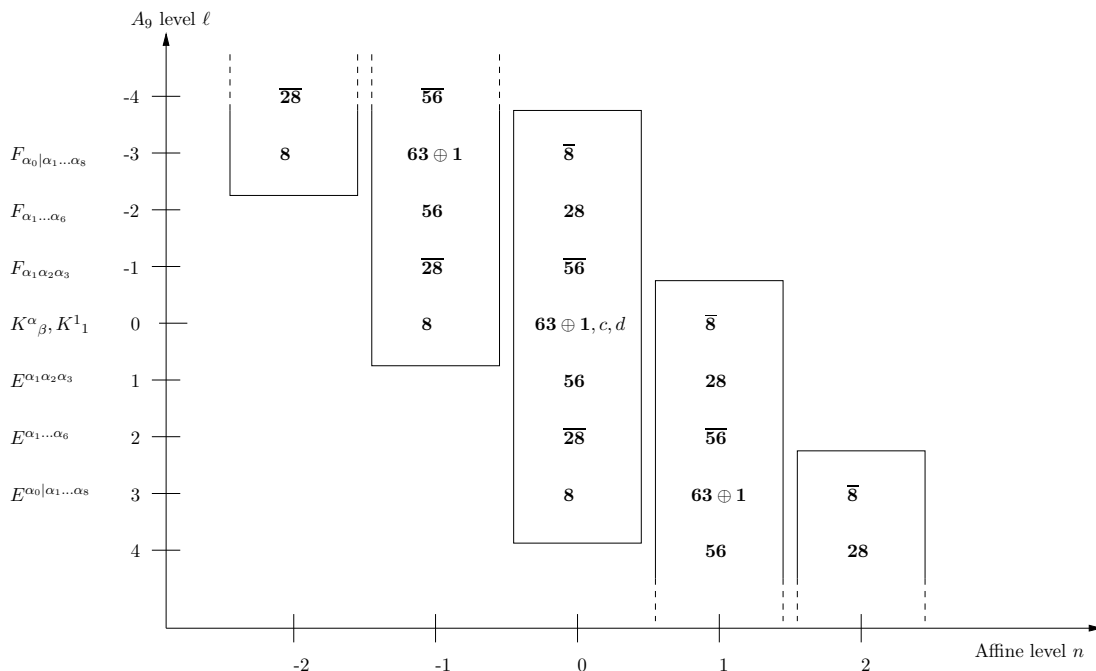


Figure 2: Diagram illustrating the distribution of A_9 levels and affine levels in E_{10} . The affine level n is given by the number of upper 2s minus the number of lower 2s on an E_{10} generator. The indices on the A_9 level $\ell \neq 0$ generators range over $\alpha = 2, \dots, 10$. The boxed set of generators correspond to copies of E_8 ; at affine level 0, the central charge and derivation are also included.

for the negative current modes with $n > 1$. Observe that the $\mathfrak{sl}(8)$ representations appearing in the vertical lines in figure 2 combine ‘sideways’ into the required $\mathfrak{sl}(9)$ representations in accordance with the decompositions

$$\begin{aligned}
 80 &\rightarrow 8 \oplus (63 \oplus 1) \oplus \bar{8}, \\
 84 &\rightarrow 56 \oplus 28, \\
 \bar{84} &\rightarrow \bar{56} \oplus \bar{28}.
 \end{aligned}
 \tag{2.24}$$

3. $K(E_9)$ spinor representations from $K(E_{10})$

The generators of $K(E_{10})$ are the anti-symmetric elements under the Chevalley transposition (see e.g. [11]). Therefore, we can construct a $K(E_{10})$ generators for any positive root step operator E by taking $J = E - E^T \equiv E - F$. The restriction to $K(E_9)$ is then obtained by considering only those positive step operators of table 3. As mentioned in the introduction $K(E_9)$ is not of Kac-Moody type (nor is $K(E_{10})$). The reason for this is that the invariant inner product

$$(x|y) := -\langle x|y \rangle \quad \text{for all } x, y \in K(E_9) \text{ (or } K(E_{10})), \tag{3.1}$$

inherited from the invariant bilinear form on E_9 (E_{10}), is positive definite on the compact subalgebras [7].

Despite this complication, finite-dimensional, hence *unfaithful*, representations corresponding to Dirac-spinor and vector-spinor (gravitino) representations of $K(E_{10})$ have been constructed in [8–11]. We now study the branching of these representations to $K(E_9) \subset K(E_{10})$. Before doing so we derive the complete $K(E_9)$ commutation relations in a form convenient for this computation.

The $K(E_{10})$ generators at ‘ A_9 levels’ $\ell = 0, \dots, 3$ are defined by

$$\begin{aligned}
 J_{(0)}^{ab} &= K^a{}_b - K^b{}_a, \\
 J_{(1)}^{a_1 a_2 a_3} &= E^{(0)}_{a_1 a_2 a_3} - F^{(0)}_{a_1 a_2 a_3}, \\
 J_{(2)}^{a_1 \dots a_6} &= E^{(0)}_{a_1 \dots a_6} - F^{(0)}_{a_1 \dots a_6}, \\
 J_{(3)}^{a_0 | a_1 \dots a_8} &= E^{(0)}_{a_0 | a_1 \dots a_8} - F^{(0)}_{a_0 | a_1 \dots a_8},
 \end{aligned} \tag{3.2}$$

for $a, b = 1, \dots, 10$. Observe that on the l.h.s. the position of indices no longer matters, as these tensors transform only under the $SO(10)$ subgroup of $K(E_{10})$ and indices can be raised and lowered with the invariant δ^{ab} . The lower indices in parentheses on the l.h.s. indicate the A_9 level in E_{10} (or A_8 level in E_9), where as the indices placed above the generators on the r.h.s. indicate the gradient level of (2.12). As before, the $K(E_9)$ generators are obtained from these by ‘dimensional reduction’, that is by restricting the indices to $\alpha, \beta = 2, \dots, 10$, corresponding to the A_8 level decomposition of E_9 . The relation between the A_8 decomposition and the current algebra decomposition of E_9 was explained in the preceding section.

In the remainder we will make use of the following notation for the $K(E_9)$ generators in $K(E_{10})$ for $k \geq 0$,

$$\begin{aligned}
 J_{(0)}^{\alpha\beta} &= K^\alpha{}_\beta - K^\beta{}_\alpha, \\
 J_{(3k+1)}^{\alpha_1 \alpha_2 \alpha_3} &= E^{(k)}_{\alpha_1 \alpha_2 \alpha_3} - F^{(k)}_{\alpha_1 \alpha_2 \alpha_3}, \\
 J_{(3k+2)}^{\alpha_1 \dots \alpha_6} &= E^{(k)}_{\alpha_1 \dots \alpha_6} - F^{(k)}_{\alpha_1 \dots \alpha_6}, \\
 J_{(3k+3)}^{\alpha_0 | \alpha_1 \dots \alpha_8} &= E^{(k)}_{\alpha_0 | \alpha_1 \dots \alpha_8} - F^{(k)}_{\alpha_0 | \alpha_1 \dots \alpha_8},
 \end{aligned} \tag{3.3}$$

using the notation of (2.12) and table 3. The generator at A_8 level $(3k+3)$ is $\mathfrak{so}(9)$ reducible and decomposes after dualisation into

$$J_{(3k+3)}^{\beta | \alpha_1 \dots \alpha_8} = \left(J_{(3k+3)}^{\beta\gamma} + S_{(3k+3)}^{\beta\gamma} \right) \epsilon^{\gamma \alpha_1 \dots \alpha_8} \quad \text{for } k \geq 0. \tag{3.4}$$

Here, the anti-symmetric tensor $J_{(3k+3)}^{\alpha\beta} = -J_{(3k+3)}^{\beta\alpha}$ is the trace part of the original tensor $J_{(3k+3)}^{\alpha_0 | \alpha_1 \dots \alpha_8}$, and the symmetric $S_{(3k+3)}^{\alpha\beta} = +S_{(3k+3)}^{\beta\alpha}$ is traceless, $S_{(3k+3)}^{\gamma\gamma} = 0$, according to the original Young symmetry. The anti-symmetric part has the same representation structure as $J_{(0)}^{\alpha\beta}$; by contrast, the symmetric generators $S_{(3n)}^{\alpha\beta}$ have no zero mode part, and exist only for $n \geq 1$.

From (2.22) and (2.23) we deduce the following $K(E_9)$ relations (for $m \geq n$),

$$\begin{aligned}
\left[J_{(3m)}^{\alpha\beta}, J_{(3n)}^{\gamma\delta} \right] &= 2\delta^{\beta\gamma} J_{(3(m+n))}^{\alpha\delta} + 2\delta^{\beta\gamma} J_{(3(m-n))}^{\alpha\delta}, \\
\left[J_{(3m)}^{\alpha\beta}, S_{(3n)}^{\gamma\delta} \right] &= 2\delta^{\beta\gamma} S_{(3(m+n))}^{\alpha\delta} - 2\delta^{\beta\gamma} S_{(3(m-n))}^{\alpha\delta}, \\
\left[S_{(3m)}^{\alpha\beta}, J_{(3n)}^{\gamma\delta} \right] &= 2\delta^{\beta\gamma} S_{(3(m+n))}^{\alpha\delta} + 2\delta^{\beta\gamma} S_{(3(m-n))}^{\alpha\delta}, \\
\left[S_{(3m)}^{\alpha\beta}, S_{(3n)}^{\gamma\delta} \right] &= 2\delta^{\beta\gamma} J_{(3(m+n))}^{\alpha\delta} - 2\delta^{\beta\gamma} J_{(3(m-n))}^{\alpha\delta}, \\
\left[J_{(3m)}^{\alpha\beta}, J_{(3n+1)}^{\gamma_1\gamma_2\gamma_3} \right] &= 3\delta^{\beta\gamma_1} J_{(3(m+n)+1)}^{\alpha\gamma_2\gamma_3} - \frac{3}{6!} \delta^{\beta\gamma_1} \epsilon^{\alpha\gamma_2\gamma_3\delta_1\dots\delta_6} J_{(3(m-n)-1)}^{\delta_1\dots\delta_6}, \\
\left[J_{(3n)}^{\alpha\beta}, J_{(3m+1)}^{\gamma_1\gamma_2\gamma_3} \right] &= 3\delta^{\beta\gamma_1} J_{(3(m+n)+1)}^{\alpha\gamma_2\gamma_3} + 3\delta^{\beta\gamma_1} J_{(3(m-n)+1)}^{\alpha\gamma_2\gamma_3}, \\
\left[S_{(3m)}^{\alpha\beta}, J_{(3n+1)}^{\gamma_1\gamma_2\gamma_3} \right] &= 3\delta^{\beta\gamma_1} J_{(3(m+n)+1)}^{\alpha\gamma_2\gamma_3} + \frac{3}{6!} \delta^{\beta\gamma_1} \epsilon^{\alpha\gamma_2\gamma_3\delta_1\dots\delta_6} J_{(3(m-n)-1)}^{\delta_1\dots\delta_6} \\
&\quad - \frac{1}{3} \delta^{\alpha\beta} J_{(3(m+n)+1)}^{\gamma_1\gamma_2\gamma_3} - \frac{1}{3 \cdot 6!} \delta^{\alpha\beta} \epsilon^{\gamma_1\gamma_2\gamma_3\delta_1\dots\delta_6} J_{(3(m-n)-1)}^{\delta_1\dots\delta_6}, \\
\left[S_{(3n)}^{\alpha\beta}, J_{(3m+1)}^{\gamma_1\gamma_2\gamma_3} \right] &= 3\delta^{\beta\gamma_1} J_{(3(m+n)+1)}^{\alpha\gamma_2\gamma_3} - 3\delta^{\beta\gamma_1} J_{(3(m-n)-1)}^{\alpha\gamma_2\gamma_3} \\
&\quad - \frac{1}{3} \delta^{\alpha\beta} J_{(3(m+n)+1)}^{\gamma_1\gamma_2\gamma_3} + \frac{1}{3} \delta^{\alpha\beta} J_{(3(m-n)-1)}^{\gamma_1\gamma_2\gamma_3}, \\
\left[J_{(3m)}^{\alpha\beta}, J_{(3n+2)}^{\gamma_1\dots\gamma_6} \right] &= 6\delta^{\beta\gamma_1} J_{(3(m+n)+2)}^{\alpha\gamma_2\dots\gamma_6} - \delta^{\beta\gamma_1} \epsilon^{\alpha\gamma_2\dots\gamma_6\delta_1\delta_2\delta_3} J_{(3(m-n)-2)}^{\delta_1\delta_2\delta_3}, \\
\left[J_{(3n)}^{\alpha\beta}, J_{(3m+2)}^{\gamma_1\dots\gamma_6} \right] &= 6\delta^{\beta\gamma_1} J_{(3(m+n)+2)}^{\alpha\gamma_2\dots\gamma_6} + 6\delta^{\beta\gamma_1} J_{(3(m-n)+2)}^{\alpha\gamma_2\dots\gamma_6}, \\
\left[S_{(3m)}^{\alpha\beta}, J_{(3n+2)}^{\gamma_1\dots\gamma_6} \right] &= 6\delta^{\beta\gamma_1} J_{(3(m+n)+2)}^{\alpha\gamma_2\dots\gamma_6} + \delta^{\beta\gamma_1} \epsilon^{\alpha\gamma_2\dots\gamma_6\delta_1\delta_2\delta_3} J_{(3(m-n)-2)}^{\delta_1\delta_2\delta_3} \\
&\quad - \frac{2}{3} \delta^{\alpha\beta} J_{(3(m+n)+2)}^{\gamma_1\dots\gamma_6} - \frac{1}{9} \delta^{\alpha\beta} \epsilon^{\gamma_1\dots\gamma_6\delta_1\delta_2\delta_3} J_{(3(m-n)-2)}^{\delta_1\delta_2\delta_3}, \\
\left[S_{(3n)}^{\alpha\beta}, J_{(3m+2)}^{\gamma_1\dots\gamma_6} \right] &= 6\delta^{\beta\gamma_1} J_{(3(m+n)+2)}^{\alpha\gamma_2\dots\gamma_6} - 6\delta^{\beta\gamma_1} J_{(3(m-n)+2)}^{\alpha\gamma_2\dots\gamma_6} \\
&\quad - \frac{2}{3} \delta^{\alpha\beta} J_{(3(m+n)+2)}^{\gamma_1\dots\gamma_6} + \frac{2}{3} \delta^{\alpha\beta} J_{(3(m-n)+2)}^{\gamma_1\dots\gamma_6}, \\
\left[J_{(3m+1)}^{\alpha_1\alpha_2\alpha_3}, J_{(3n+1)}^{\beta_1\beta_2\beta_3} \right] &= J_{(3(m+n)+2)}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} - 18\delta^{\alpha_1\beta_1} \delta^{\alpha_2\beta_2} \left(J_{(3(m-n))}^{\alpha_3\beta_3} + S_{(3(m-n))}^{\alpha_3\beta_3} \right), \\
\left[J_{(3m+1)}^{\alpha_1\alpha_2\alpha_3}, J_{(3n+2)}^{\beta_1\dots\beta_6} \right] &= 3\epsilon^{\gamma\beta_1\dots\beta_6\alpha_1\alpha_2} \left(J_{(3(m+n)+3)}^{\alpha_3\gamma} + S_{(3(m+n)+3)}^{\alpha_3\gamma} \right) \\
&\quad + \frac{1}{6} \delta_{\alpha_1\alpha_2\alpha_3}^{\beta_1\beta_2\beta_3} \epsilon^{\beta_4\beta_5\beta_6\gamma_1\dots\gamma_6} J_{(3(m-n)-1)}^{\gamma_1\dots\gamma_6}, \\
\left[J_{(3n+1)}^{\alpha_1\alpha_2\alpha_3}, J_{(3m+2)}^{\beta_1\dots\beta_6} \right] &= 3\epsilon^{\gamma\beta_1\dots\beta_6\alpha_1\alpha_2} \left(J_{(3(m+n)+3)}^{\alpha_3\gamma} + S_{(3(m+n)+3)}^{\alpha_3\gamma} \right) \\
&\quad - 120\delta_{\alpha_1\alpha_2\alpha_3}^{\beta_1\beta_2\beta_3} J_{(3(m-n)+1)}^{\beta_4\beta_5\beta_6}, \\
\left[J_{(3m+2)}^{\alpha_1\dots\alpha_6}, J_{(3n+2)}^{\beta_1\dots\beta_6} \right] &= -6 \cdot 6! \delta^{\alpha_1\beta_1} \dots \delta^{\alpha_5\beta_5} \left(J_{(3(m-n))}^{\alpha_6\beta_6} + S_{(3(m-n))}^{\alpha_6\beta_6} \right) \\
&\quad - 400\delta^{\alpha_1\beta_1} \dots \delta^{\alpha_3\beta_3} \epsilon^{\alpha_4\dots\alpha_6\beta_4\dots\beta_5\gamma_1\gamma_2\gamma_3} J_{(3(m+n)+4)}^{\gamma_1\gamma_2\gamma_3}, \tag{3.5}
\end{aligned}$$

with implicit (anti-)symmetrizations on the r.h.s. according to the symmetries of the l.h.s. and with the understanding that the level zero symmetric piece vanishes: $S_{(0)}^{\alpha\beta} = 0$. Note that in some relations a level index become negative for $m = n$; in those cases one has to use the formula in the next row for which this does not happen. Let us emphasize once

more that these formulas were deduced by making use of the identifications found in the previous section, and by exploiting the fact that the affine E_9 commutators are known for all levels, whereas we have no complete knowledge of the higher level commutation relations for E_{10} . From the above commutation relations, one readily verifies that the Lie algebra $K(E_9)$ indeed possesses a ‘filtered’ structure, with

$$[J_{(k)}, J_{(l)}] = J_{(k+l)} + J_{(|k-l|)} \quad (k, l \geq 0). \quad (3.6)$$

3.1 Dirac-spinor ideal

Under $K(E_{10})$ the 32-dimensional Dirac-spinor ε transforms as follows on the first four levels [8, 9, 11],

$$\begin{aligned} J_{(0)}^{ab} \varepsilon &= \frac{1}{2} \Gamma^{ab} \varepsilon, \\ J_{(1)}^{a_1 a_2 a_3} \varepsilon &= \frac{1}{2} \Gamma^{a_1 a_2 a_3} \varepsilon, \\ J_{(2)}^{a_1 \dots a_6} \varepsilon &= \frac{1}{2} \Gamma^{a_1 \dots a_6} \varepsilon, \\ J_{(3)}^{a_0 | a_1 \dots a_8} \varepsilon &= 4 \delta^{a_0 [a_1} \Gamma^{a_2 \dots a_8]} \varepsilon, \end{aligned} \quad (3.7)$$

where Γ^a are the ten real, symmetric (32×32) Γ -matrices of $SO(10) \subset GL(10)$ (see appendix A) and $\Gamma^{ab} = \Gamma^{[a} \Gamma^{b]}$ etc. denote their anti-symmetrised products. Note that only the $SO(10)$ trace part of $J_{(3)}^{a_0 | a_1 \dots a_8}$ is realised non-trivially, in accordance with the fact that no Young tableaux other than fully antisymmetric ones can be built with Γ -matrices. Furthermore, we have rescaled the ‘level’ 3 generator by a factor 1/3 relative to [3, 9, 11]. As emphasized in [9–11], the above representation is *unfaithful* as the infinite-dimensional group is realized on a finite number of spinor components.

Before proceeding it is useful to define the matrix

$$\Gamma^* := \Gamma^1 \Gamma^0, \quad (3.8)$$

in terms of which the following relation holds for the (32×32) Γ -matrices

$$\Gamma^{\alpha_1 \dots \alpha_9} = \epsilon^{\alpha_1 \dots \alpha_9} \Gamma^* \quad \Rightarrow \quad \Gamma^{\alpha_1 \dots \alpha_k} = \frac{(-1)^{k(k-1)/2}}{(9-k)!} \epsilon^{\alpha_1 \dots \alpha_k \beta_{k+1} \dots \beta_9} \Gamma_{\beta_{k+1} \dots \beta_9} \Gamma^* \quad (3.9)$$

with the $SO(9)$ invariant tensor $\epsilon^{\alpha_1 \dots \alpha_9}$. The matrix Γ^* satisfies $(\Gamma^*)^2 = 1$ and commutes with all Γ^α for $\alpha = 2, \dots, 10$, but anticommutes with Γ^0 and Γ^1 , and hence should be identified with the chirality (helicity) matrix in $(1+1)$ space-time dimensions. By defining $\chi_\pm = \frac{1}{2}(1 \pm \Gamma^*)\chi$ for any 32-component spinor, it therefore serves to split any such χ into two sets of 16-component objects, which can be viewed as the right- and left-handed components, respectively, of a spinor in $(1+1)$ dimensions, and whose 16 ‘internal’ components transform as spinors under $SO(9) = K(SL(9)) \subset K(E_9)$.

The (unfaithful) action of $K(E_9)$ on a Dirac-spinor ε is obtained from (3.7) by restricting the range of the indices, as described before. From the construction of the consistent

representation we can in this case derive a closed formula for the action of *all* $K(E_9)$ generators by repeated commutation of the low level elements (3.7) and use of (3.9), and finally comparison with (3.5). The result is⁸

$$\begin{aligned}
 J_{(3k)}^{\alpha\beta} &= \frac{1}{2}\Gamma^{\alpha\beta}(\Gamma^*)^k, \\
 J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= \frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}(\Gamma^*)^k, \\
 J_{(3k+2)}^{\alpha_1\dots\alpha_6} &= \frac{1}{2}\Gamma^{\alpha_1\dots\alpha_6}(\Gamma^*)^k, \\
 S_{(3k+3)}^{\alpha\beta} &= 0,
 \end{aligned}
 \tag{3.10}$$

where, of course, $k \geq 0$. It follows from (3.10) in particular that $S_{(3k+3)}^{\alpha\beta}$ is represented trivially on the Dirac spinor, and likewise that the relations involving $S_{(3k+3)}^{\alpha\beta}$ all trivialise, as it should be. For the (reducible) Dirac representation, we thus read off the relations (again for $k \geq 0$)

$$\begin{aligned}
 J_{(3k)}^{\alpha\beta} &= J_{(3k+6)}^{\alpha\beta}, & S_{(3k+3)}^{\alpha\beta} &= 0, \\
 J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= -\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6}J_{(3k+5)}^{\beta_1\dots\beta_6}, \\
 J_{(3k+2)}^{\alpha_1\dots\alpha_6} &= -\frac{1}{3!}\epsilon^{\alpha_1\dots\alpha_6\beta_1\beta_2\beta_3}J_{(3k+4)}^{\beta_1\beta_2\beta_3}.
 \end{aligned}
 \tag{3.11}$$

The existence of a 32-dimensional unfaithful representation of $K(E_9)$ (derived from the 32-dimensional irreducible Dirac spinor of $K(E_{10})$) is thus reflected by the existence of a non-trivial ideal within the Lie algebra $K(E_9)$, via (1.1). For obvious reasons, we will refer to this ideal as the *Dirac ideal* and designate it by $\mathfrak{i}_{\text{Dirac}}$. To be completely precise, the latter is defined as the linear span within $K(E_9)$ of the relations (3.11). It is straightforward to check that $\mathfrak{i}_{\text{Dirac}}$ is indeed an ideal, i.e. $[K(E_9), \mathfrak{i}_{\text{Dirac}}] \subset \mathfrak{i}_{\text{Dirac}}$. Furthermore, since by (3.11) all generators of level greater than three can be expressed in terms of lower level generators, the codimension of this ideal within $K(E_9)$ is finite, and equal to the number of independent non-zero elements up to level three, which is $2 \times (36 + 84)$. The resulting quotient is a finite-dimensional subalgebra of $\mathfrak{gl}(32)$ and has the structure

$$\mathfrak{q}_{\text{Dirac}} = K(E_9)/\mathfrak{i}_{\text{Dirac}} = \mathfrak{so}(16)_+ \oplus \mathfrak{so}(16)_-.
 \tag{3.12}$$

To see that the Lie algebra on the r.h.s. has been correctly identified, recall from [9, 11] that the quotient algebra associated with the unfaithful Dirac-spinor in $K(E_{10})$ is $\mathfrak{so}(32)$; according to (3.12) this splits into $\mathfrak{so}(16)_+ \oplus \mathfrak{so}(16)_-$, since all anti-symmetric (16×16) matrices are contained in the list (3.10).

Since Γ^* commutes with all these representation matrices, we can decompose the 32-dimensional $K(E_9)$ representation space further into eigenspaces of Γ^* which are invariant under the $K(E_9)$ action. These are projected out by $\frac{1}{2}(1 \pm \Gamma^*)$, and we have the branching

$$\mathbf{32} \quad \rightarrow \quad \mathbf{16}_+ \oplus \mathbf{16}_-
 \tag{3.13}$$

⁸The rescaling of the level $\ell = 3$ generators by $1/3$ in comparison with [3] is needed to ensure that the level $(3k)$ generators are uniformly normalised, cf. also (3.16).

into two inequivalent spinor representations of $K(E_9)$. On the $\mathbf{16}_\pm$ representations of $K(E_9)$, one can thus replace Γ^* by ± 1 . This allows us to enlarge the Dirac ideal (3.11) in two possible ways by replacing the relations (3.11) by

$$\begin{aligned} J_{(3k)}^{\alpha\beta} &= \pm J_{(3k+3)}^{\alpha\beta}, \\ J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= \mp \frac{1}{6!} \epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6} J_{(3k+2)}^{\beta_1\dots\beta_6}, \\ S_{(3k)}^{\alpha\beta} &= 0, \end{aligned} \quad (3.14)$$

for the $\mathbf{16}_\pm$ representations, thereby defining two new ideals $\mathfrak{i}_{\text{Dirac}}^\pm \supset \mathfrak{i}_{\text{Dirac}}$. The quotient algebras are easily seen to be

$$\mathfrak{q}_{\text{Dirac}}^\pm = K(E_9)/\mathfrak{i}_{\text{Dirac}}^\pm = \mathfrak{so}(16)_\pm. \quad (3.15)$$

Let us now study in a bit more detail the ideal associated with the $\mathbf{16}_\pm$ Dirac-spinors of $K(E_9)$ determined by (3.14) and, in particular, its orthogonal complement with respect to the $K(E_9)$ (and E_9 [36]) invariant form $\langle \cdot | \cdot \rangle$ under which

$$\begin{aligned} \left\langle J_{(3k)}^{\alpha\beta} \middle| J_{(3k)}^{\gamma\delta} \right\rangle &= -2 \cdot 2! \delta_{\gamma\delta}^{\alpha\beta} \quad \left[= \frac{1}{16} \text{Tr} \left(\Gamma^{\alpha\beta} \Gamma^{\gamma\delta} \right) \right], \\ \left\langle J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} \middle| J_{(3k+1)}^{\beta_1\beta_2\beta_3} \right\rangle &= -2 \cdot 3! \delta_{\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} \quad \left[= \frac{1}{16} \text{Tr} \left(\Gamma^{\alpha_1\alpha_2\alpha_3} \Gamma^{\beta_1\beta_2\beta_3} \right) \right], \\ \left\langle J_{(3k+2)}^{\alpha_1\dots\alpha_6} \middle| J_{(3k+2)}^{\beta_1\dots\beta_6} \right\rangle &= -2 \cdot 6! \delta_{\beta_1\dots\beta_6}^{\alpha_1\dots\alpha_6} \quad \left[= \frac{1}{16} \text{Tr} \left(\Gamma^{\alpha_1\dots\alpha_6} \Gamma^{\beta_1\dots\beta_6} \right) \right]. \end{aligned} \quad (3.16)$$

We also have the consistency of orthogonality relations

$$\left\langle J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} \middle| J_{(3k+2)}^{\beta_1\dots\beta_6} \right\rangle = 0 \quad \left[= \frac{1}{16} \text{Tr} \left(\Gamma^{\alpha_1\alpha_2\alpha_3} \Gamma^{\beta_1\dots\beta_6} \right) \right]. \quad (3.17)$$

Note that the invariant inner product Tr on the 32-dimensional representation agrees with the one on the algebra for the $J_{(m)}$ generators. Evaluated on $S_{(3k)}^{\alpha\beta}$ it vanishes in contrast with the non-vanishing inner product in $K(E_9)$. This is no contradiction since we are dealing with an unfaithful representation.

Defining the infinite linear combinations

$$\begin{aligned} \mathcal{J}_\pm^{\alpha\beta} &= \sum_{n \geq 0} (\pm 1)^n J_{(3n)}^{\alpha\beta}, \\ \mathcal{J}_\pm^{\alpha\beta\gamma} &= \sum_{n \geq 0} (\pm 1)^n \left(J_{(3n+1)}^{\alpha\beta\gamma} \pm \epsilon^{\alpha\beta\gamma\delta_1\dots\delta_6} J_{(3n+2)}^{\delta_1\dots\delta_6} \right), \end{aligned} \quad (3.18)$$

one checks that w.r.t. (3.1),

$$\left(\mathcal{J}_\pm^{\alpha\beta} \middle| Z \right) = \left(\mathcal{J}_\pm^{\alpha\beta\gamma} \middle| Z \right) = 0 \quad \text{for all } Z \in \mathfrak{i}_{\text{Dirac}}^\pm, \quad (3.19)$$

and so $\mathcal{J}_\pm^{\alpha\beta}$ and $\mathcal{J}_\pm^{\alpha\beta\gamma}$ formally belong to the orthogonal complement of $\mathfrak{i}_{\text{Dirac}}^\pm$.⁹ Thus, the elements (3.18) are not proper elements of the vector space underlying the Lie algebra

⁹Where the elements of $Z \in \mathfrak{i}_{\text{Dirac}}^\pm$ are understood to be *finite* linear combinations of (3.14).

$K(E_9)$ because the infinite series (3.18) do not converge in the (Hilbert space) completion of $K(E_9)$ w.r.t. the norm (3.1). However, they do exist as *distributions*, that is, as elements of the dual of the space of finite linear combinations of basis elements (3.10) (which is dense in the Hilbert space completion of $K(E_9)$). This is also the reason why the elements $\{\mathcal{J}_\pm^{\alpha\beta}, \mathcal{J}_\pm^{\alpha\beta\gamma}\}$ do not close into a proper subalgebra of $K(E_9)$, as would be the case for the orthogonal complement of an ideal in a finite-dimensional Lie algebra. Nevertheless, as we saw above, there is a way to make sense of (3.18) as defining a Lie algebra by passing to the quotient algebras (3.12) and (3.15). In section 4.2 we will see that these quotient algebras correspond to *generalised evaluation maps* in terms of a loop algebra description. The *distributional nature* of these objects is also evident from the fact that formal commutation of the elements (3.18) leads to infinite factors $\sim \sum_{k=1}^{\infty} 1$. Whereas for $K(E_9)$ the distributional nature can be made precise in terms of usual δ -functions on the spectral parameter plane (see section 4.2), such a description is not readily available for $K(E_{10})$. Giving a more precise definition of the space of distributions for $K(E_{10})$ could prove helpful in understanding the $K(E_{10})$ structure better.

It may seem surprising that $K(E_9)$ admits non-trivial ideals, whereas E_9 does not (except for the one-dimensional center). One reason that E_9 does not admit any other ideals is the presence of the derivation d as an element of E_9 (or any other affine) Lie algebra: because relations such as (3.11) and (3.14) involve *different* affine levels (even within generators $J_{(n)}$ of fixed $\mathfrak{sl}(9)$ level n , as we saw), commutation with d will change the relative coefficients between the terms defining the ideal by (2.7), hence will force the individual terms to vanish also, thus leading to the trivial ideal $\mathfrak{i} = 0$. The existence of non-trivial ideals in $K(E_9)$ is thus due in particular to the fact that d is *not* an element of $K(E_9)$. In the section 4.2 we shall give a loop algebra interpretation of this result.

3.2 Vector-spinor ideal

The $K(E_{10})$ transformation of the 320-component vector-spinor ψ_a can also be written in terms of $SO(10)$ Γ -matrices [9, 10]. For the first three $SO(10)$ ‘levels’ the $K(E_{10})$ expressions are¹⁰

$$\begin{aligned}
 (J_{(0)}^{ab}\psi)_c &= \frac{1}{2}\Gamma^{ab}\psi_c + 2\delta_c^{[a}\psi^{b]}, \\
 (J_{(1)}^{a_1a_2a_3}\psi)_b &= \frac{1}{2}\Gamma^{a_1a_2a_3}\psi_b + 4\delta_b^{[a_1}\Gamma^{a_2}\psi^{a_3]} - \Gamma_b^{[a_1a_2}\psi^{a_3]}, \\
 (J_{(2)}^{a_1\dots a_6}\psi)_b &= \frac{1}{2}\Gamma^{a_1\dots a_6}\psi_b - 10\delta_b^{[a_1}\Gamma^{a_2\dots a_5}\psi^{a_6]} + 4\Gamma_b^{[a_1\dots a_5}\psi^{a_6]}, \\
 (J_{(3)}^{a_0|a_1\dots a_8}\psi)_b &= \frac{16}{9}\left(\Gamma_b^{a_1\dots a_8}\psi^{a_0} - \Gamma_b^{a_0[a_1\dots a_7}\psi^{a_8]}\right) \\
 &\quad + 4\delta^{a_0[a_1}\Gamma^{a_2\dots a_8]}\psi_b - 56\delta^{a_0[a_1}\Gamma_b^{a_2\dots a_7}\psi^{a_8]} \\
 &\quad + \frac{16}{9}\left(8\delta_b^{a_0}\Gamma^{[a_1\dots a_7}\psi^{a_8]} - \delta_b^{[a_1}\Gamma^{a_2\dots a_8]}\psi^{a_0} + 7\delta_b^{[a_1}\Gamma_{a_0}^{a_2\dots a_7}\psi^{a_8]}\right).
 \end{aligned} \tag{3.20}$$

¹⁰When comparing these expressions to [9] we recall once more that we have re-scaled $J_{(3)}$ by $1/3$ as for the Dirac-spinor.

Reducing these transformations to $K(E_9)$ one decomposes the gravitino field ψ_a into an $SO(9)$ vector spinor ψ_α , and in addition the component ψ_1 entering via

$$\eta := \Gamma^1 \psi_1, \quad (3.21)$$

which transforms in the spinor representation of the two-dimensional Lorentz group $SO(1,1)$ and $SO(9) \subset K(E_9)$. The correspondence of the fields ψ_α and η with the fermionic fields used in [1] will be explained in section 4.2.

Computing the $K(E_9)$ transformations for ‘levels’ 0 up to 3 on the components ψ_α one obtains

$$\begin{aligned} (J_{(0)}^{\alpha\beta} \psi)_\gamma &= \frac{1}{2} \Gamma^{\alpha\beta} \psi_\gamma + 2\delta_\gamma^{[\alpha} \psi^{\beta]}, \\ (J_{(1)}^{\alpha_1\alpha_2\alpha_3} \psi)_\beta &= \frac{1}{2} \Gamma^{\alpha_1\alpha_2\alpha_3} \psi_\beta + 4\delta_\beta^{[\alpha_1} \Gamma^{\alpha_2} \psi^{\alpha_3]} - \Gamma_\beta^{[\alpha_1\alpha_2} \psi^{\alpha_3]}, \\ (J_{(2)}^{\alpha_1\dots\alpha_6} \psi)_\beta &= \frac{1}{2} \Gamma^{\alpha_1\dots\alpha_6} \psi_\beta - 10\delta_\beta^{[\alpha_1} \Gamma^{\alpha_2\dots\alpha_5} \psi^{\alpha_6]} + 4\Gamma_\beta^{[\alpha_1\dots\alpha_5} \psi^{\alpha_6]}, \\ (J_{(3)}^{\alpha\beta} \psi)_\gamma &= -\Gamma^* \left[\frac{1}{2} \Gamma^{\alpha\beta} \psi_\gamma + 2\delta_\gamma^{[\alpha} \psi^{\beta]} \right], \\ (S_{(3)}^{\alpha\beta} \psi)_\gamma &= -\Gamma^* \left[\frac{2}{9} \delta^{\alpha\beta} \Gamma_\gamma - 2\delta_\gamma^{(\alpha} \Gamma^{\beta)} \right] \Gamma^\delta \psi_\delta. \end{aligned} \quad (3.22)$$

Note that the transformations on ψ_α close on themselves. Extending the action (3.22) by the commutation relations (3.5) we deduce the general action on ψ_α ,

$$\begin{aligned} (J_{(3k)}^{\alpha\beta} \psi)_\gamma &= (-\Gamma^*)^k \left[\frac{1}{2} \Gamma^{\alpha\beta} \psi_\gamma + 2\delta_\gamma^{[\alpha} \psi^{\beta]} \right], \\ (J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} \psi)_\beta &= (-\Gamma^*)^k \left[\frac{1}{2} \Gamma^{\alpha_1\alpha_2\alpha_3} \psi_\beta + 4\delta_\beta^{[\alpha_1} \Gamma^{\alpha_2} \psi^{\alpha_3]} - \Gamma_\beta^{[\alpha_1\alpha_2} \psi^{\alpha_3]} \right. \\ &\quad \left. + k \left(\frac{1}{3} \Gamma^{\alpha_1\alpha_2\alpha_3\beta} + 2\delta^\beta_{[\alpha_1} \Gamma^{\alpha_2\alpha_3]} \right) \Gamma^\gamma \psi_\gamma \right], \\ (J_{(3k+2)}^{\alpha_1\dots\alpha_6} \psi)_\beta &= (-\Gamma^*)^k \left[\frac{1}{2} \Gamma^{\alpha_1\dots\alpha_6} \psi_\beta - 10\delta_\beta^{[\alpha_1} \Gamma^{\alpha_2\dots\alpha_5} \psi^{\alpha_6]} + 4\Gamma_\beta^{[\alpha_1\dots\alpha_5} \psi^{\alpha_6]} \right. \\ &\quad \left. + k \left(\frac{2}{3} \Gamma^{\alpha_1\dots\alpha_6\beta} - 2\delta^\beta_{[\alpha_1} \Gamma^{\alpha_2\dots\alpha_6]} \right) \Gamma^\gamma \psi_\gamma \right], \\ (S_{(3k)}^{\alpha\beta} \psi)_\gamma &= (-\Gamma^*)^k k \left[\frac{2}{9} \delta^{\alpha\beta} \Gamma_\gamma - 2\delta_\gamma^{(\alpha} \Gamma^{\beta)} \right] \Gamma^\delta \psi_\delta. \end{aligned} \quad (3.23)$$

Similar to (3.11) we immediately find the following relations which are valid on the ψ_α components,

$$\begin{aligned} J_{(3k)}^{\alpha\beta} &= J_{(3k+6)}^{\alpha\beta}, \\ J_{(3k+7)}^{\alpha_1\alpha_2\alpha_3} - J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= \frac{1}{6!} \epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6} \left(J_{(3k+5)}^{\beta_1\dots\beta_6} - J_{(3k-1)}^{\beta_1\dots\beta_6} \right), \\ (3k+1)J_{(3k+7)}^{\alpha_1\alpha_2\alpha_3} - (3k+7)J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= -\frac{1}{6!} \epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6} \\ &\quad \times \left((3k-1)J_{(3k+5)}^{\beta_1\dots\beta_6} - (3k+5)J_{(3k-1)}^{\beta_1\dots\beta_6} \right), \\ (k+2)S_{(3k)}^{\alpha\beta} &= k S_{(3k+6)}^{\alpha\beta}. \end{aligned} \quad (3.24)$$

The first two relations arise from considering the ψ_α pieces of the transformed spinor (3.23), the latter two can be derived by focussing on the trace parts in the transformed spinor (3.23) and are evidently k -dependent. Note also that the relations in the middle involve *four* different $\mathfrak{sl}(9)$ levels.

Just as in the Dirac case it follows immediately from the form of the transformations (3.23) that Γ^* commutes with all representation matrices and therefore one can restrict to the $\Gamma^* = \pm \mathbf{1}$ eigenspaces. Hence, on the $\Gamma^* = \pm \mathbf{1}$ eigenspaces the relations (3.24) simplify in analogy with (3.14) to

$$\begin{aligned}
 J_{(3k)}^{\alpha\beta} &= \mp J_{(3k+3)}^{\alpha\beta}, \\
 J_{(3k+4)}^{\alpha_1\alpha_2\alpha_3} \pm J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= \mp \frac{1}{6!} \epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6} \left(J_{(3k+5)}^{\beta_1\dots\beta_6} \pm J_{(3k+2)}^{\beta_1\dots\beta_6} \right), \\
 (3k+1)J_{(3k+4)}^{\alpha_1\alpha_2\alpha_3} \pm (3k+4)J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3} &= \pm \frac{1}{6!} \epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6} \\
 &\quad \times \left((3k+2)J_{(3k+5)}^{\beta_1\dots\beta_6} \pm (3k+5)J_{(3k+2)}^{\beta_1\dots\beta_6} \right), \\
 S_{(3k)}^{\alpha\beta} &= (\pm 1)^{k+1} k S_{(3)}^{\alpha\beta}. \tag{3.25}
 \end{aligned}$$

We stress that these and (3.24) are valid only on the ψ_α components.

The transformation properties of the remaining component $\eta = \Gamma^1\psi_1$ are more complicated. At the first three levels, they read

$$\begin{aligned}
 J_{(0)}^{\alpha\beta}\eta &= \frac{1}{2}\Gamma^{\alpha\beta}\eta, \\
 J_{(1)}^{\alpha_1\alpha_2\alpha_3}\eta &= -\frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}\eta - \Gamma^{[\alpha_1\alpha_2}\psi^{\alpha_3]}, \\
 J_{(2)}^{\alpha_1\dots\alpha_6}\eta &= \frac{1}{2}\Gamma^{\alpha_1\dots\alpha_6}\eta + 4\Gamma^{[\alpha_1\dots\alpha_5}\psi^{\alpha_6]}, \\
 J_{(3)}^{\alpha\beta}\eta &= -\Gamma^* \left[\frac{1}{2}\Gamma^{\alpha\beta}\eta + \Gamma^{\alpha\beta}\Gamma^\gamma\psi_\gamma \right], \\
 S_{(3)}^{\alpha\beta}\eta &= -\Gamma^* \left[-2\Gamma^{(\alpha}\psi^{\beta)} + \frac{2}{9}\delta^{\alpha\beta}\Gamma^\gamma\psi_\gamma \right]. \tag{3.26}
 \end{aligned}$$

where the mixing of ψ_α into η is manifest. We can again use the $K(E_9)$ commutation relations (3.5) to deduce the action for all generators from (3.26),

$$\begin{aligned}
 J_{(3k)}^{\alpha\beta}\eta &= (-\Gamma^*)^k \left[\frac{1}{2}\Gamma^{\alpha\beta}\eta + k^2\Gamma^{\alpha\beta}\Gamma^\gamma\psi_\gamma \right] \\
 J_{(3k+1)}^{\alpha_1\alpha_2\alpha_3}\eta &= (-\Gamma^*)^k \left[-\frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}\eta - (3k+1)\Gamma^{[\alpha_1\alpha_2}\psi^{\alpha_3]} \right. \\
 &\quad \left. - \frac{1}{3}k(3k+1)\Gamma^{\alpha_1\alpha_2\alpha_3}\Gamma^\beta\psi_\beta \right], \\
 J_{(3k+2)}^{\alpha_1\dots\alpha_6}\eta &= (-\Gamma^*)^k \left[\frac{1}{2}\Gamma^{\alpha_1\dots\alpha_6}\eta + 2(3k+2)\Gamma^{[\alpha_1\dots\alpha_5}\psi^{\alpha_6]} \right. \\
 &\quad \left. + \frac{1}{3}k(3k+2)\Gamma^{\alpha_1\dots\alpha_6}\Gamma^\beta\psi_\beta \right], \\
 S_{(3k)}^{\alpha\beta}\eta &= (-\Gamma^*)^k k \left[-2\Gamma^{(\alpha}\psi^{\beta)} + \frac{2}{9}\delta^{\alpha\beta}\Gamma^\gamma\psi_\gamma \right]. \tag{3.27}
 \end{aligned}$$

Using (3.23) and (3.27) we can now deduce relations analogous to (3.24) valid on both the ψ_α and the η components of ψ_a and hence on the full representation. These define the vector-spinor ideal. Since the k -dependence in (3.27) is *quadratic*, they will be more complicated than (3.24) and involve up to *six* different $\mathfrak{sl}(9)$ levels. We will discuss their structure at the end of this section and give them explicitly in a simplifying ‘gauge’ which we now present.

From the transformations (3.23) it can be shown that the gamma-trace $\Gamma^\alpha \psi_\alpha$ transforms into itself. For this reason, we can consistently consider the tracelessness condition

$$\Gamma^\alpha \psi_\alpha = 0, \tag{3.28}$$

which, as we will recall in section 4.2, corresponds to a supersymmetric gauge choice for the dilatino in the reduction from three to two dimensions. As shown in [12] and [11], cf. eq. (2.29), this condition is compatible with $K(E_n)$ only for $n = 9$, as required. In fact it follows from (3.23) that $\Gamma^\alpha \psi_\alpha$ transforms just as a Dirac-spinor. With the tracelessness condition (3.28), the k -dependence in (3.23) vanishes, and in particular $S_{(3k)}^{\alpha\beta}$ acts trivially on ψ_α for all k . The corresponding ideal would then be the same as in (3.11). That is, we have the same $\text{SO}(16)_+ \times \text{SO}(16)_-$ acting on this part of the gravitino. The action for $J_{(3k+2)}^{\alpha_1 \dots \alpha_6}$ can be written in a dual form as shown above. Moreover, we see that we can again specialise to the $\Gamma^* = \pm 1$ subspaces. There it is easiest to deduce the following relations for the vector-spinor components ψ_α in the traceless gauge,

$$\begin{aligned} J_{(3k+3)}^{\alpha\beta} &= \mp J_{(3k)}^{\alpha\beta}, \\ J_{(3k+1)}^{\alpha_1 \alpha_2 \alpha_3} &= \pm \frac{1}{6!} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \beta_1 \dots \beta_6} J_{(3k+2)}^{\beta_1 \dots \beta_6}, \\ S_{(3k)}^{\alpha\beta} &= 0 \end{aligned} \tag{3.29}$$

in analogy with (3.14) (except that Γ^* is replaced by $(-\Gamma^*)$). By the arguments of the preceding sections the relevant ideal on the components ψ_α gives a quotient isomorphic to $\mathfrak{so}(16)_\pm$. However, as noted above, the component η *mixes* with the ψ_α components and one can show that they cannot be decoupled by a change of basis. Therefore the relations (3.29) have to be weakened in order to describe the full vector-spinor ideal. In the gauge (3.28) the transformations (3.27) simplify and the k -dependence becomes linear instead of quadratic. Then it is easy to check that the vector-spinor ideal relations are identical to (3.24).

Let us now summarize our findings and write out the branching of the **320** representation of $K(E_{10})$ into representations of its $K(E_9)$ subalgebra. In comparison with the Dirac representation, the vector-spinor representation exhibits a curious new feature in the branching. Namely, the transformations on η contain contributions also involving ψ_α . On the other hand the ψ_α components transform solely among themselves. This means that the ψ_a representation of $K(E_{10})$ *does not completely reduce into irreducible representations of $K(E_9)$* as one might have expected, rather we have a triangular structure

$$\mathbf{320} \rightarrow (\mathbf{16}_+ \oplus \mathbf{16}_-) + (\mathbf{128}_+ \oplus \mathbf{128}_-) + (\mathbf{16}_+ \oplus \mathbf{16}_-) \tag{3.30}$$

where the plus signs between the parantheses denote a *semidirect* sum of, from right to left, the trace components $\Gamma^\alpha\psi_\alpha$, the traceless part of ψ_α , and the η components: only the trace components transform among themselves, the other two summands mix with those to the left. These results are in accordance with the results of [1], see eqs. (5.12) there, as we will discuss in more detail below. The triangular structure can, for each chirality, be pictured by $K(E_9)$ representation matrices of block form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}. \quad (3.31)$$

The blocks are of dimensions 16×16 , 128×128 and 16×16 , respectively, and correspond to the summands in the decomposition (3.30) in reverse order. In this manner, the lower right block corresponds to the transformation of the gamma-trace $\Gamma^\alpha\psi_\alpha$ into itself. The non-reducibility of the **320** is tantamount to saying that the representation matrix cannot be block-diagonalised.

The structure of the ideal in $K(E_9)$ associated with this representation can be revealed by starting with the ‘innermost’ layer of the triangular structure, namely $\Gamma^\alpha\psi_\alpha$. As stated above this transforms as a Dirac-spinor so the associated quotient algebra (projected onto the two Γ^* chiralities) is $\mathfrak{so}(16)_\pm$, cf. (3.15). This gets enlarged since the ideal relations are weakened due to the appearance of the gamma-trace in the $K(E_9)$ action on ψ_α , cf. (3.23), and even more due to (3.27). The expected structure is

$$\mathfrak{q}_{\text{vs}}^\pm = \mathfrak{so}(16)_\pm + \mathfrak{p}_\pm^{(1)} + \mathfrak{p}_\pm^{(2)} \subset \mathfrak{gl}(160) \quad (3.32)$$

as a semi-direct sum with actions from left to right as before, so that $\mathfrak{so}(16)_\pm$ acts on the pieces $\mathfrak{p}_\pm^{(1)}$ and $\mathfrak{p}_\pm^{(2)}$ via some representation, $[\mathfrak{p}_\pm^{(1)}, \mathfrak{p}_\pm^{(1)}] \subset \mathfrak{p}_\pm^{(2)}$, and $\mathfrak{p}_\pm^{(2)}$ is abelian. In the traceless case (3.28) this can be evaluated further and we find

$$\mathfrak{q}_{\text{vs}}^\pm = \mathfrak{so}(16)_\pm + \mathfrak{p}_\pm \subset \mathfrak{gl}(144) \quad (\Gamma^\alpha\psi_\alpha = 0) \quad (3.33)$$

where \mathfrak{p}_\pm are 128 abelian translations and the whole ideal has codimension 248 as can be counted from (3.24): The action of all $K(E_9)$ generators in the vector-spinor representation can be reduced to that of $J_{(0)}^{\alpha\beta}$, $J_{(1)}^{\alpha_1\alpha_2\alpha_3}$, $J_{(2)}^{\alpha_1\dots\alpha_6}$ and $S_{(3)}^{\alpha\beta}$ which amount to $(40+80) + (80+48) = 120+128$ independent generators. Via the relations (3.24), all higher level generators can thus be expressed as linear combinations of these 248 basic ones. This discussion shows that the structure of the ideals in the vector-spinor case is far richer than that of the Dirac-spinor.

4. Relation to current algebra realisation

In previous work [1], $K(E_9)$ transformations of unfaithful fermion representations were derived starting from the linear system description of $N = 16$ supergravity in $D = 2$ [21, 23]. In the present section we will show that the transformations (3.23) and (3.27) we deduced from the dimensionally reduced theory above are completely equivalent to those in the linear system.

4.1 $\mathfrak{so}(16) \subset E_{8(8)}$

Since the linear system transformations are written using the spectral parameter presentation of $K(E_9)$ in the $K(E_8) \equiv \mathfrak{so}(16)$ decomposition of E_8 we first need to briefly recall some notation necessary for the comparison; in particular, we require the E_8 commutation relations adapted to the compact $\mathfrak{so}(16)$ subalgebra. In this basis, $E_{8(8)}$ decomposes into the adjoint **120** of $\mathfrak{so}(16)$ (corresponding to the anti-symmetric compact generators) and the $\mathfrak{so}(16)$ spinor representation **128_s** (corresponding to the symmetric non-compact generators) which can be further decomposed as

$$\begin{aligned} X^{IJ} \in \mathbf{120} &\rightarrow (\mathbf{28}, \bar{\mathbf{1}}) \oplus (\mathbf{1}, \bar{\mathbf{28}}) \oplus (\mathbf{8}_s, \bar{\mathbf{8}}_c) \rightarrow \mathbf{28} \oplus \mathbf{28} \oplus \mathbf{56}_v \oplus \mathbf{8}_v, \\ Y^A \in \mathbf{128}_s &\rightarrow (\mathbf{8}_v, \bar{\mathbf{8}}_v) \oplus (\mathbf{8}_s, \bar{\mathbf{8}}_c) \rightarrow \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v \oplus \mathbf{8}_v \oplus \mathbf{56}_v, \end{aligned} \quad (4.1)$$

with the chain of embeddings $\mathfrak{so}(16) \supset \mathfrak{so}(8) \oplus \mathfrak{so}(8) \supset \mathfrak{so}(8)$, where the indices v, s, c (= vector, spinor, and conjugate spinor) label the three inequivalent eight-dimensional representations of the various $\text{SO}(8)$ groups. The diagonal subalgebra $\mathfrak{so}(8)$ is to be identified with the $\mathfrak{so}(8) \subset \mathfrak{sl}(8)$ of the preceding sections. Furthermore, we here take over the notation from [1]: $I, J = 1, \dots, 16$ are $\text{SO}(16)$ vector indices and $A = 1, \dots, 128$ labels the components of a chiral $\text{SO}(16)$ spinor. Evidently, the first line in (4.1) corresponds to the $\mathfrak{so}(8)$ representations inherited from table 2. The formulas relating the $\text{SO}(9)$ and $\text{SO}(16)$ bases are spelled out in appendix B. From (4.1) we also recover the decompositions of $\text{SO}(16)$ under its $\text{SO}(9)$ subgroup, viz.

$$\mathbf{120} \rightarrow \mathbf{36} \oplus \mathbf{84}, \quad \mathbf{128}_s \rightarrow \mathbf{44} \oplus \mathbf{84}. \quad (4.2)$$

In the conventions of [35], the E_8 commutation relations read

$$\begin{aligned} [X^{IJ}, X^{KL}] &= 2\delta^{I[K} X^{L]J} - 2\delta^{J[K} X^{L]I}, \\ [X^{IJ}, Y^A] &= -\frac{1}{2}\Gamma_{AB}^{IJ} Y^B, \quad [Y^A, Y^B] = \frac{1}{4}\Gamma_{AB}^{IJ} X^{IJ}. \end{aligned} \quad (4.3)$$

With the current algebra generators (for $m \in \mathbb{Z}$)

$$X^{(m)IJ} \equiv X^{IJ} \otimes t^m, \quad Y^{(m)A} \equiv Y^A \otimes t^m, \quad (4.4)$$

the $K(E_9)$ generators can be represented in the form (for $m \geq 0$)

$$A^{(m)IJ} := \frac{1}{2} \left(X^{(m)IJ} + X^{(-m)IJ} \right), \quad S^{(m)A} := \frac{1}{2} \left(Y^{(m)A} - Y^{(-m)A} \right), \quad (4.5)$$

implying $S^{(0)A} \equiv 0$. The $K(E_9)$ commutation relations then read

$$\begin{aligned} [A^{(m)IJ}, A^{(n)KL}] &= 2\delta^{I[K} \left(A^{(m+n)L]J} + A^{(|m-n|L]J} \right) \\ [A^{(m)IJ}, S^{(n)A}] &= -\frac{1}{4}\Gamma_{AB}^{IJ} \left(S^{(m+n)B} - \text{sgn}(m-n)S^{(|m-n|)B} \right) \\ [S^{(m)A}, S^{(n)B}] &= \frac{1}{8}\Gamma_{AB}^{IJ} \left(A^{(m+n)IJ} - A^{(|m-n|)IJ} \right) \end{aligned} \quad (4.6)$$

for $m, n \geq 0$ (recall that the central term drops out).

In the formulation (4.6) we can immediately look for ideals of $K(E_9)$. The Dirac ideals i_{Dirac}^\pm are now defined by the relations

$$A^{(m)IJ} - (\pm 1)^m A^{(0)IJ} = 0 \quad , \quad S^{(m)A} = 0. \tag{4.7}$$

That is, the ideals are defined as the linear span of the expressions on the l.h.s., and it is then straightforward to verify the ideal property, namely that these subspaces are mapped onto themselves under the adjoint action of $K(E_9)$. The quotient algebras obtained by division of $K(E_9)$ by these ideals are obviously isomorphic to $\mathfrak{so}(16)$ for both choices of signs.

The vector-spinor ideals i_{vs}^\pm , on the other hand, can be defined by the relations (for $m \geq 1$)

$$A^{(m)IJ} - (\pm 1)^m A^{(0)IJ} = 0 \quad , \quad S^{(m)A} \mp (\pm 1)^m m S^{(1)A} = 0. \tag{4.8}$$

They define smaller ideals of codimension 248 since everything is determined by $A^{(0)IJ}$ and $S^{(1)A}$. The part of the above relations involving $A^{(m)IJ}$ is identical to that of the Dirac-spinor (4.7) indicating that there is some relation of the associated quotient to $\mathfrak{so}(16)$ with an additional part arising from the $S^{(m)A}$ relations. We will see this in more detail below.

The vector-spinor ideals i_{vs}^\pm can be generated from $A^{(1)IJ} \mp A^{(0)IJ} = 0$ since for example

$$\left[A^{(1)IJ} \mp A^{(0)IJ}, S^{(1)A} \right] = -\frac{1}{4} \Gamma_{AB}^{IJ} \left(S^{(2)B} \mp 2S^{(1)B} \right) \tag{4.9}$$

implies by the ideal property that $S^{(2)B} \mp 2S^{(1)B}$ has to vanish. Similar calculations show that $A^{(1)IJ} \mp A^{(0)IJ} = 0$ generates all ideal relations.

In this basis it is not hard to construct further ideals. One example is obtained by starting from the relation $S^{(2)A} \mp 2S^{(1)A} = 0$, without requiring that $A^{(1)IJ} \mp A^{(0)IJ} = 0$. Commuting with $S^{(1)B}$ and demanding that the resulting expression also belongs to the ideal leads to

$$A^{(3)IJ} - A^{(1)IJ} \mp 2A^{(1)IJ} \pm A^{(0)IJ} = 0, \tag{4.10}$$

a relation involving four affine levels. In the case of the vector-spinor these vanish by taking pairwise combinations, here they define a new ideal which is strictly smaller than the vector-spinor ideal.

In section 3.1 we explained that the absence of non-trivial ideals in E_9 (other than the one-dimensional center) can be interpreted as a consequence of the presence of the derivation d . In the current algebra realization, d acts by differentiation: $d \equiv \partial_t$. Setting $X(t_0) = 0$ for some fixed t_0 would then force all higher repeated commutators of this element with d to vanish at $t = t_0$ by consistency. This, in turn, would imply the vanishing of all derivatives $\partial_t^n X(t_0)$, hence would force $X(t) = 0$ (assuming analyticity in t). This confirms again that the existence of non-trivial ideals in $K(E_9)$ is thus due in particular to the fact that d is not an element of $K(E_9)$. The orthogonal complement of the ideal, given formally by (3.18), corresponds to distributions $X(t) = X_0 \delta(t - t_0)$ where, as we will see presently, $t_0 = \pm 1$. The associated ideal then consists of all elements of the loop algebra

which vanish at those points. We stress that this requires studying a distribution space outside of $K(E_9)$ and that this could prove a useful strategy also for further investigations of $K(E_{10})$.

4.2 Current algebra fermion transformations

In [1] it was realised that in the linear systems approach to two-dimensional $N = 16$ supergravity the transformation rules for the fermions can be written succinctly in terms of a current algebra description with a current parameter t . The non-propagating fermions are the gravitino φ^I and the dilatino φ_2^I , coming from the gravitino in three dimensions.¹¹ They both transform in the vector representation of $SO(16)$, while the field χ^A accomodates the 128 physical fermions and transforms in the conjugate spinor representation of $SO(16)$. The dilatino φ_2^I can be gauged away by use of local supersymmetry [1], corresponding to the tracelessness condition (3.28). It follows from a comparison with the reduction of 11-dimensional supergravity to three dimensions [37] that the correspondence between these $SO(16)$ representations and those used in the foregoing sections is (modulo a factor 2 in the relative normalisation of χ^A compared to φ_2^I and φ^I , required for the canonical normalisation of the Dirac term)

$$\begin{aligned} \chi^A &\leftrightarrow \psi_i - \frac{1}{2}\Gamma_i\Gamma^j\psi_j, \\ \varphi_2^I &\leftrightarrow \Gamma^*(\Gamma^2\psi_2 + \Gamma^i\psi_i), \\ \varphi^I &\leftrightarrow -\Gamma^1\psi_1 - \Gamma^i\psi_i, \end{aligned} \tag{4.11}$$

thus breaking $SO(9)$ covariance down to $SO(8)$. We have suppressed the two-dimensional Dirac-spinor indices on the l.h.s (which take two values, so that e.g. the φ^I stands for 2×16 components φ_{\pm}^I), and the $SO(9)$ spinor indices on the r.h.s (of which there are 2×16 , giving 2×128 components for the first line, see also (A.11). Thus the number of components on both sides matches.

The most general $K(E_9)$ Lie algebra element can be written in the form [1]¹²

$$\begin{aligned} h(t) &= \frac{1}{2} \sum_{n=0}^{\infty} h_n^{IJ} X^{IJ} \otimes (t^{-n} + t^n) + \sum_{n=1}^{\infty} h_n^A Y^A \otimes (t^{-n} - t^n) \\ &\equiv \frac{1}{2} h^{IJ}(t) X^{IJ} + h^A(t) Y^A. \end{aligned} \tag{4.12}$$

It can then be shown that $K(E_9)$ acts on the chiral components of the fermions via evalu-

¹¹These are called ψ^I and ψ_2^I in [1], but we choose a different notation here to avoid confusion with the gravitino in 11-dimensional supergravity.

¹²Since we are interested for the moment in the purely algebraic aspects of the transformation we suppress the space-time dependence throughout. (The spectral parameter t also depends on two-dimensional space-time.)

ation at the points $t = \pm 1$ in the spectral parameter plane (cf. eq. (5.12) of [1])¹³ as

$$\begin{aligned}
 \delta_h \varphi_{2\pm}^I &= \varphi_{2\pm}^J h^{IJ}|_{t=\mp 1}, \\
 \delta_h \chi_{\pm}^{\dot{A}} &= \frac{1}{4} \Gamma_{\dot{A}\dot{B}}^{IJ} \chi_{\pm}^{\dot{B}} h^{IJ}|_{t=\mp 1} - \Gamma_{A\dot{A}}^I \varphi_{2\pm}^I \partial_t h^A|_{t=\mp 1}, \\
 \delta_h \varphi_{\pm}^I &= \varphi_{\pm}^J h^{IJ}|_{t=\mp 1} \pm \Gamma_{A\dot{B}}^I \chi_{\pm}^{\dot{B}} \partial_t h^A|_{t=\mp 1} \mp 2\varphi_{2\pm}^J \partial_t^2 h^{IJ}|_{t=\mp 1}.
 \end{aligned}
 \tag{4.13}$$

Thus, from the point of view [1] the action of $K(E_9)$ on the fermions can be viewed as an *evaluation map* of the $K(E_9)$ elements, not at the origin in spectral parameter space $t = 0$ but at $t = \pm 1$. In fact, we are dealing with a *generalised* evaluation map in that the transformations depend on up to second derivatives in the spectral parameter at the points $t = \pm 1$.

Now we compare (4.13) to (3.23) and (3.27). Writing

$$h^{IJ}(t)|_{t=\pm 1} = 2 \sum_{n=0}^{\infty} h_n^{IJ} (\pm 1)^n
 \tag{4.14}$$

suggests the structure of an $\mathfrak{so}(16)$, so we see that the Taylor expansion (4.14) (considered as a formal power series) should indeed be identified with the formal infinite sum in (3.18). Considering also the parameters $\partial_t h^A$ and $\partial_t^2 h^{IJ}$, we can see that there is a structural agreement between the transformations (4.13) and those of the vector-spinor (ψ_α, η) in section 3.2. To make the agreement exact, we rewrite (4.13) using the basis given in the preceding section, and Γ^* as the chirality (helicity) matrix in (1+1) spacetime dimensions,

$$\begin{aligned}
 (A^{(m)KL} \varphi_2)^I &= 2(-\Gamma^*)^m \delta^{I[K} \varphi_2^{L]}, \\
 (A^{(m)KL} \varphi)^I &= 2(-\Gamma^*)^m \delta^{I[K} \varphi^{L]} + 4m^2 (-\Gamma^*)^{m-1} \delta^{I[K} \varphi_2^{L]}, \\
 (A^{(m)KL} \chi)^{\dot{A}} &= \frac{1}{2} (-\Gamma^*)^m \Gamma_{\dot{A}\dot{B}}^{KL} \chi^{\dot{B}}, \\
 (S^{(m)B} \varphi_2)^I &= 0, \\
 (S^{(m)B} \varphi)^I &= m(-\Gamma^*)^m \Gamma_{B\dot{B}}^I \chi^{\dot{B}}, \\
 (S^{(m)B} \chi)^{\dot{A}} &= m(-\Gamma^*)^m \Gamma_{B\dot{A}}^I \varphi_2^I.
 \end{aligned}
 \tag{4.15}$$

Using instead the definition (3.8) of Γ^* as the (32×32) matrix $\Gamma^1 \Gamma^0$ means that we consider the $\text{SO}(16)$ vectors as $\text{SO}(9)$ spinors, and the $\text{SO}(16)$ spinor $\chi^{\dot{A}}$ as eight vector components of a $\text{SO}(9)$ vector-spinor. We can thus relate them to the gravitino in section 3.2. This is done by splitting the vector, spinor and conjugate spinor indices of $\text{SO}(16)$ into those of $\text{SO}(8)$, and relating the corresponding gamma matrices to each other, as described in appendix A. In appendix B, finally, we explain how to express the generators (3.3) of $K(E_9)$ in the basis $(S^{(m)IJ}, A^{(m)A})$. We can then act with the generators (3.3) on the fields $(\chi^{\dot{A}}, \varphi^I, \varphi_2^I)$ according to (4.15) and require that the result, expressed in (ψ_α, η) , coincide with the transformations of these expressions under $K(E_9)$ according to (3.23)

¹³We note that in [1] it was also shown that, considering only induced $K(E_9)$ transformations, there is a non-linear combination of the fermionic and bosonic fields that reduces this action to an action of $\text{SO}(16)_+ \times \text{SO}(16)_-$.

and (3.27). It turns out that this requirement uniquely fixes the correspondance (4.11), in agreement with the dimensional reduction [37], up to a constant factor multiplying all fields, and an arbitrary multiple of φ_2^I that can be added to φ^I . In this fashion, we have recovered precisely the results of [1].

Acknowledgments

We are very grateful to T. Damour for numerous enlightening discussions.

A. Gamma matrix conventions

In this appendix, and the following one, we will no longer follow the index convention for α, β, \dots , introduced in section 2. Instead we will use α and $\dot{\alpha}$ as SO(8) spinor and conjugate spinor indices, respectively, while the indices i, j, \dots still take the values 3, \dots , 10 as SO(8) vector indices. The chiral (8×8) SO(8) gamma-matrices will be denoted by $\gamma_{\alpha\dot{\beta}}^i$.

Then eight real, symmetric (16×16) gamma matrices of SO(9) can be written

$$\gamma_{IJ}^i = \begin{pmatrix} 0 & \gamma_{\alpha\dot{\beta}}^i \\ \gamma_{\dot{\alpha}\beta}^i & 0 \end{pmatrix}, \tag{A.1}$$

where $\gamma_{\dot{\alpha}\beta}^i$ is the transpose of $\gamma_{\alpha\dot{\beta}}^i$. The first eight SO(9) gamma matrices square to one, anticommute, and define the ninth matrix by

$$\gamma^3 \dots \gamma^{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \gamma^2. \tag{A.2}$$

Thus γ^2 also squares to one, and anticommutes with γ^i . The SO(9) gamma matrices can be extended to the ten, real, symmetric (32×32) gamma matrices of SO(10), introduced in section 3.1, via

$$\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}. \tag{A.3}$$

In these conventions, the decomposition under Γ^2, Γ^i of a 32 component spinor into two chiral spinors is manifest. The SO(10) gamma matrices satisfy

$$\Gamma^1 \dots \Gamma^{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \Gamma^0 \tag{A.4}$$

and then we get

$$\Gamma^* \equiv \Gamma^1 \Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.5}$$

Triality implies that the matrices $\gamma_{i\dot{\beta}}^\alpha$ and $\gamma_{i\alpha}^{\dot{\beta}}$ have the same properties as $\gamma_{\dot{\alpha}\beta}^i$, and can also be extended to SO(9) matrices as in (A.1). Thus we can take as SO(16) gamma matrices the tensor products

$$\begin{aligned}\Gamma^\alpha &= \mathbf{1} \otimes \gamma^\alpha, \\ \Gamma^{\dot{\alpha}} &= \gamma^{\dot{\alpha}} \otimes \gamma^2,\end{aligned}\tag{A.6}$$

with the components

$$\begin{aligned}\Gamma_{\beta\dot{\gamma},\delta j}^\alpha &= \delta_{\beta\delta}\gamma_{\alpha\dot{\gamma}}^j, & \Gamma_{ij,k\dot{\delta}}^\alpha &= \delta_{ik}\gamma_{\alpha\dot{\delta}}^j, \\ \Gamma_{ij,\delta k}^{\dot{\alpha}} &= \delta_{jk}\gamma_{\delta\dot{\alpha}}^i, & \Gamma_{\beta\dot{\gamma},i\dot{\delta}}^{\dot{\alpha}} &= -\delta_{\dot{\gamma}\dot{\delta}}\gamma_{\beta\dot{\alpha}}^i,\end{aligned}\tag{A.7}$$

as in [37], and all other components are zero. From this one can compute the following non-trivial anti-symmetric products Γ_{AB}^{IJ} of gamma matrices,

$$\begin{aligned}\Gamma_{ij,kl}^{\alpha\beta} &= \delta_{ik}\gamma_{\alpha\beta}^{jl}, & \Gamma_{\gamma\dot{\alpha},\delta\dot{\beta}}^{\alpha\beta} &= \delta_{\gamma\delta}\gamma_{\dot{\alpha}[\alpha}\gamma_{\dot{\beta}]\beta}^k, \\ \Gamma_{ij,\gamma\dot{\delta}}^{\alpha\dot{\beta}} &= -\gamma_{\gamma\dot{\beta}}^i\gamma_{\alpha\dot{\delta}}^j, & \Gamma_{\gamma\dot{\delta},ij}^{\alpha\dot{\beta}} &= \gamma_{\gamma\dot{\beta}}^i\gamma_{\alpha\dot{\delta}}^j, \\ \Gamma_{ij,kl}^{\dot{\alpha}\dot{\beta}} &= \delta_{jl}\gamma_{\dot{\alpha}\dot{\beta}}^{ik}, & \Gamma_{\alpha\dot{\gamma},\beta\dot{\delta}}^{\dot{\alpha}\dot{\beta}} &= \delta_{\dot{\gamma}\dot{\delta}}\gamma_{\dot{\alpha}[\alpha}\gamma_{\dot{\beta}]\beta}^k.\end{aligned}\tag{A.8}$$

In (A.7), we see that the vector, spinor and conjugate spinor indices (I, A, \dot{A}) of SO(16) split into those of SO(8) as

$$\begin{aligned}I &= (\alpha, \dot{\alpha}), \\ A &= (\alpha\dot{\alpha}, ij), \\ \dot{A} &= (\alpha i, j\dot{\alpha}),\end{aligned}\tag{A.9}$$

according to the decompositions

$$\begin{aligned}\mathbf{16} &\rightarrow (\mathbf{8}_c, \bar{\mathbf{1}}) \oplus (\mathbf{1}, \bar{\mathbf{8}}_s) \rightarrow \mathbf{8}_s \oplus \mathbf{8}_c, \\ \mathbf{128}_s &\rightarrow (\mathbf{8}_v, \bar{\mathbf{8}}_v) \oplus (\mathbf{8}_s, \bar{\mathbf{8}}_c) \rightarrow \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v \oplus \mathbf{8}_v \oplus \mathbf{56}_v, \\ \mathbf{128}_c &\rightarrow (\mathbf{8}_v, \bar{\mathbf{8}}_c) \oplus (\mathbf{8}_s, \bar{\mathbf{8}}_v) \rightarrow \mathbf{8}_s \oplus \mathbf{56}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_c\end{aligned}\tag{A.10}$$

of these $\mathfrak{so}(16)$ representations under $\mathfrak{so}(8) \oplus \mathfrak{so}(8)$, and then under the diagonal $\mathfrak{so}(8)$ subalgebra. For example, the first line in (4.11) then reads

$$\begin{aligned}\chi_\pm^{i\dot{\alpha}} &= (\psi_\pm^i)^{\dot{\alpha}} - \frac{1}{2}(\gamma^i\gamma^j)_{\dot{\alpha}\dot{\beta}}(\psi_\pm^j)^{\dot{\beta}}, \\ \chi_\pm^{\alpha i} &= (\psi_\pm^i)^\alpha - \frac{1}{2}(\gamma^i\gamma^j)_{\alpha\beta}(\psi_\pm^j)^\beta.\end{aligned}\tag{A.11}$$

B. Relation between the two E_8 bases

We have in this article used two different bases of E_8 . The first one arose in the A_7 level decomposition described in section 2.1 (table 2), and for the compact generators it was

generalized to E_9 in section 3. The second one, covariant under the maximal compact subalgebra $\mathfrak{so}(16)$, was introduced in section 4, and extended to E_9 via the current algebra construction. We will now explain the relation between these two bases, which was also given in [35] but in different conventions. First, as for E_9 in section 3, we consider the compact linear combinations

$$\begin{aligned}
 J^{ij} &= G_j^i - G_i^j, \\
 J^{ijk} &= Z^{ijk} - Z_{ijk}, \\
 J^{i_1 \dots i_6} &= Z^{i_1 \dots i_6} - Z_{i_1 \dots i_6}, \\
 J^i &= Z^i - Z_i
 \end{aligned} \tag{B.1}$$

of the basis elements in table 2. These can now be expressed in X^{IJ} by the SO(9) or SO(8) gamma matrices as

$$\begin{aligned}
 J^{ij} &= \frac{1}{4} \gamma_{IJ}^{ij} X^{IJ} = \frac{1}{4} \gamma_{\alpha\beta}^{ij} X^{\alpha\beta} + \frac{1}{4} \gamma_{\dot{\alpha}\dot{\beta}}^{ij} X^{\dot{\alpha}\dot{\beta}}, \\
 J^{i_1 i_2 i_3} &= -\frac{1}{4} \gamma_{IJ}^{i_1 i_2 i_3} X^{IJ} = -\frac{1}{2} \gamma_{\alpha\beta}^{i_1 i_2 i_3} X^{\alpha\beta}, \\
 J^{i_1 \dots i_6} &= \frac{1}{4} \gamma_{IJ}^{i_1 \dots i_6} X^{IJ} = \frac{1}{4} \gamma_{\alpha\beta}^{i_1 \dots i_6} X^{\alpha\beta} + \frac{1}{4} \gamma_{\dot{\alpha}\dot{\beta}}^{i_1 \dots i_6} X^{\dot{\alpha}\dot{\beta}}, \\
 J^i &= -\frac{1}{4} (\gamma^i \gamma^2)_{IJ} X^{IJ} = -\frac{1}{2} (\gamma^i \gamma^2)_{\alpha\beta} X^{\alpha\beta},
 \end{aligned} \tag{B.2}$$

where a sign ambiguity in the derivation has been fixed by demanding that the generalised evaluation map (4.13) and the representation given by (3.23) and (3.27) agree. For the remaining generators

$$\begin{aligned}
 S^{ij} &= G_j^i + G_i^j, \\
 S^{i_1 i_2 i_3} &= Z^{i_1 i_2 i_3} + Z_{i_1 i_2 i_3}, \\
 S^{i_1 \dots i_6} &= Z^{i_1 \dots i_6} + Z_{i_1 \dots i_6}, \\
 S^i &= Z^i + Z_i,
 \end{aligned} \tag{B.3}$$

it is necessary to break SO(9) covariance, and split the SO(16) spinor indices. Then we get

$$\begin{aligned}
 S^{ij} &= 2Y^{(ij)} - \delta^{ij} Y^{kk}, \\
 S^{i_1 i_2 i_3} &= -\frac{1}{2} \gamma_{\alpha\beta}^{i_1 i_2 i_3} Y^{\alpha\beta}, \\
 S^{i_1 \dots i_6} &= \epsilon^{i_1 \dots i_6 k_1 k_2} Y^{k_1 k_2}, \\
 S^i &= -\frac{1}{2} \gamma_{\alpha\beta}^i Y^{\alpha\beta},
 \end{aligned} \tag{B.4}$$

where an overall sign ambiguity in the definition of Y^A has been fixed again by equivalence between the representations. Combining these formulas with (2.17), (2.18), (2.22) and (2.23), we can easily express the generators (3.3) in the basis $(S^{(m)IJ}, A^{(m)A})$ introduced in section 4.1. For example, we have

$$\begin{aligned}
 J_{(3m+1)}^{ijk} &= -\gamma_{\alpha\beta}^{ijk} S^{(m)\alpha\beta} - \gamma_{\alpha\beta}^{ijk} A^{(m)\alpha\beta}, \\
 \frac{1}{5!} \epsilon^{ijkl_1 \dots l_5} J_{(3m+2)}^{l_1 \dots l_5 2} &= \gamma_{\alpha\beta}^{ijk} S^{(m)\alpha\beta} - \gamma_{\alpha\beta}^{ijk} A^{(m)\alpha\beta},
 \end{aligned} \tag{B.5}$$

and in the same way we obtain the remaining, non-compact, generators of E_9 .

References

- [1] H. Nicolai and H. Samtleben, *On $K(E_9)$* , *Q. J. Pure Appl. Math.* **1** (2005) 180 [[hep-th/0407055](#)].
- [2] T. Damour, M. Henneaux and H. Nicolai, *E_{10} and a 'small tension expansion' of M-theory*, *Phys. Rev. Lett.* **89** (2002) 221601 [[hep-th/0207267](#)].
- [3] T. Damour and H. Nicolai, *Eleven dimensional supergravity and the $E_{10}/K(E_{10})$ σ -model at low A_9 levels*, [hep-th/0410245](#).
- [4] P.C. West, *Hidden superconformal symmetry in M-theory*, *JHEP* **08** (2000) 007 [[hep-th/0005270](#)].
- [5] P.C. West, *E_{11} and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [[hep-th/0104081](#)].
- [6] F. Englert and L. Houart, *\mathcal{G}^{+++} invariant formulation of gravity and M-theories: exact BPS solutions*, *JHEP* **01** (2004) 002 [[hep-th/0311255](#)].
- [7] A. Kleinschmidt and H. Nicolai, *Gradient representations and affine structures in AE_n* , *Class. and Quant. Grav.* **22** (2005) 4457 [[hep-th/0506238](#)].
- [8] S. de Buyl, M. Henneaux and L. Paulot, *Hidden symmetries and Dirac fermions*, *Class. and Quant. Grav.* **22** (2005) 3595 [[hep-th/0506009](#)].
- [9] T. Damour, A. Kleinschmidt and H. Nicolai, *Hidden symmetries and the fermionic sector of eleven-dimensional supergravity*, *Phys. Lett.* **B 634** (2006) 319 [[hep-th/0512163](#)].
- [10] S. de Buyl, M. Henneaux and L. Paulot, *Extended E_8 invariance of 11-dimensional supergravity*, *JHEP* **02** (2006) 056 [[hep-th/0512292](#)].
- [11] T. Damour, A. Kleinschmidt and H. Nicolai, *$K(E_{10})$, supergravity and fermions*, *JHEP* **08** (2006) 046 [[hep-th/0606105](#)].
- [12] A. Kleinschmidt and H. Nicolai, *IIA and IIB spinors from $K(E_{10})$* , *Phys. Lett.* **B 637** (2006) 107 [[hep-th/0603205](#)].
- [13] A. Kleinschmidt and H. Nicolai, *E_{10} and $SO(9,9)$ invariant supergravity*, *JHEP* **07** (2004) 041 [[hep-th/0407101](#)].
- [14] A. Kleinschmidt and H. Nicolai, *IIB supergravity and E_{10}* , *Phys. Lett.* **B 606** (2005) 391 [[hep-th/0411225](#)].
- [15] I. Schnakenburg and P.C. West, *Massive IIA supergravity as a non-linear realisation*, *Phys. Lett.* **B 540** (2002) 137 [[hep-th/0204207](#)].
- [16] I. Schnakenburg and P.C. West, *Kac-Moody symmetries of IIB supergravity*, *Phys. Lett.* **B 517** (2001) 421 [[hep-th/0107181](#)].
- [17] P. West, *The IIA, IIB and eleven dimensional theories and their common E_{11} origin*, *Nucl. Phys.* **B 693** (2004) 76 [[hep-th/0402140](#)].
- [18] C. Hillmann and A. Kleinschmidt, *Pure type-I supergravity and DE_{10}* , *Gen. Rel. Grav.* **38** (2006) 1861 [[hep-th/0608092](#)].
- [19] I. Schnakenburg and P. West, *Kac-moody symmetries of ten-dimensional non-maximal supergravity theories*, *JHEP* **05** (2004) 019 [[hep-th/0401196](#)].

- [20] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, in M. Flato, P. Sally and G. Zuckerman (eds.), *Applications of group theory in physics and mathematical physics*, Providence, U.S.A., Am. Math. Soc. (1985), LPTENS 82/22 (Lectures in Applied Mathematics **21**).
- [21] H. Nicolai, *The integrability of $N = 16$ supergravity*, *Phys. Lett.* **B 194** (1987) 402.
- [22] H. Nicolai and N.P. Warner, *The structure of $N = 16$ supergravity in two-dimensions*, *Commun. Math. Phys.* **125** (1989) 369.
- [23] H. Nicolai and H. Samtleben, *Integrability and canonical structure of $D = 2$, $N = 16$ supergravity*, *Nucl. Phys.* **B 533** (1998) 210 [[hep-th/9804152](#)].
- [24] R. Geroch, *A method for generating solutions of Einstein's equations*, *J. Math. Phys.* **12** (1971) 918;
A method for generating new solutions of Einstein's equation. 2, *J. Math. Phys.* **13** (1972) 394.
- [25] P. Breitenlohner and D. Maison, *On the Geroch group*, *Ann. Poincaré Phys. Theor.* **46** (1987) 215.
- [26] E. Cremmer and B. Julia, *The $SO(8)$ supergravity*, *Nucl. Phys.* **B 159** (1979) 141.
- [27] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities. I*, *Nucl. Phys.* **B 523** (1998) 73 [[hep-th/9710119](#)].
- [28] H. Nicolai and T. Fischbacher, *Low level representations for E_{10} and E_{11}* , [hep-th/0301017](#).
- [29] P. West, *Very extended E_8 and A_8 at low levels, gravity and supergravity*, *Class. and Quant. Grav.* **20** (2003) 2393 [[hep-th/0212291](#)].
- [30] M.J. Duff and J.T. Liu, *Hidden spacetime symmetries and generalized holonomy in M-theory*, *Nucl. Phys.* **B 674** (2003) 217 [[hep-th/0303140](#)].
- [31] C. Hull, *Holonomy and symmetry in M-theory*, [hep-th/0305039](#).
- [32] M.J. Duff and K.S. Stelle, *Multi-membrane solutions of $D = 11$ supergravity*, *Phys. Lett.* **B 253** (1991) 113.
- [33] A. Batrachenko, M.J. Duff, J.T. Liu and W.Y. Wen, *Generalized holonomy of M-theory vacua*, *Nucl. Phys.* **B 726** (2005) 275 [[hep-th/0312165](#)].
- [34] A. Keurentjes, *The topology of U-duality (sub-)groups*, *Class. and Quant. Grav.* **21** (2004) 1695 [[hep-th/0309106](#)].
- [35] K. Koepsell, H. Nicolai and H. Samtleben, *An exceptional geometry for $D = 11$ supergravity?*, *Class. and Quant. Grav.* **17** (2000) 3689 [[hep-th/0006034](#)].
- [36] V.G. Kac, *Infinite dimensional Lie algebras*, 3rd edition, Cambridge University Press, Cambridge (1990).
- [37] H. Nicolai, *$D = 11$ supergravity with local $SO(16)$ invariance*, *Phys. Lett.* **B 187** (1987) 316.