

Transcendentality and crossing

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Abstract. We discuss possible phase factors for the S -matrix of planar $\mathcal{N} = 4$ gauge theory, leading to modifications at four-loop order as compared to an earlier proposal. While these result in a four-loop breakdown of perturbative BMN scaling, Kotikov–Lipatov transcendentality in the universal scaling function for large-spin twist operators may be preserved. One particularly natural choice, unique up to one constant, modifies the overall contribution of all terms containing odd-zeta functions in the earlier proposed scaling function based on a trivial phase. Excitingly, we present evidence that this choice is non-perturbatively related to a recently conjectured crossing-symmetric phase factor for perturbative string theory on $AdS_5 \times S^5$ once the constant is fixed to a particular value. Our proposal, if true, might therefore resolve the long-standing AdS/CFT discrepancies between gauge and string theory.

Keywords: integrable quantum field theory, integrable spin chains (vertex models), quantum integrability (Bethe ansatz)

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1. Introduction

Supersymmetric $\mathcal{N} = 4$ gauge theory is a conformal quantum field theory in four dimensions. As such, standard lore says that it does not possess an S -matrix since there are no asymptotic particles. Luckily this naive no-go theorem is wrong in two interesting ways.

Firstly, it turns out that an asymptotic ‘world-sheet’ S -matrix of this theory may be defined, in the planar limit, in an internal space [1]. This is possible since planar local composite operators may be interpreted as a one-dimensional ring on which eight elementary bosons and eight elementary fermions propagate [2]. Furthermore, this S -matrix appears to be *two-particle factorized*, since the particle model is integrable, at one-loop and beyond [3]. For closely related earlier work, see [4] and references therein. This allows us, under some assumptions, to largely construct the full S -matrix [5], up to an *a priori* unknown phase factor, and find the spectrum of the theory [6]. This phase factor is known to equal one up to three-loop order.

Secondly, it is very interesting to consider the set of all space–time on-shell n -gluon amplitudes of the $\mathcal{N} = 4$ gauge theory. While these are clearly not scattering amplitudes of asymptotic particles, they contain much physical information, and we can simply declare them to serve as a surrogate ‘space–time’ S -matrix. Fascinatingly, much evidence for an iterative ‘solvable’ structure in these amplitudes was discovered recently [7, 8].

The factorized world-sheet S -matrix and the iterative space–time S -matrix approaches are not uncorrelated. The conjectured ansatz for the all-loop gluon amplitudes contains a to-be-determined function $f(g)$, where

$$g = \frac{\sqrt{\lambda}}{4\pi} \tag{1.1}$$

is the gauge coupling constant. However, this function $f(g)$ may be extracted from the logarithmically divergent anomalous dimension of leading-twist operators in the gauge

theory [9]. As such, it may in principle be derived from the diagonalization of the world-sheet S -matrix. In this work we will call $f(g)$ the universal scaling function of $\mathcal{N} = 4$ gauge theory; it is also known as the cusp or soft anomalous dimension.

In practice, the comparison has so far been done reliably to three-loop order only. The space-time based approach was completed in [8], and the world-sheet method was exploited in [1]. The results agreed. The latter method was also successfully checked against rigorously known anomalous dimensions of twist operators at two loops, and brilliantly conjectured ones at three loops [10]. The conjecture was based on an intriguing *transcendentality principle*, which allowed us to extract the answer from a hard QCD computation [11].

Clearly, one would like to go to higher loops, and ideally compare the full universal scaling function. A four-loop computation by Bern, Czakon, Dixon, Kosower and Smirnov using the iterative gluon amplitude approach is underway [12]³. Furthermore, in [14] it was shown how to extract the universal scaling function from the Bethe equations which result from the diagonalized S -matrix. Assuming a trivial (i.e. equal to unity) phase factor, an all-loop equation determining the function was proposed. While the exact solution of the equation remains unknown, it allows us to generate perturbative results to any desired order with ease. In addition, the solution preserves the Kotikov–Lipatov transcendentality principle [15] to arbitrary loop order.

However, one troubling issue is whether the phase factor is really unity to all orders in perturbation theory. This leads to various vexing discrepancies [16, 17] with string theory results in the context of the AdS/CFT correspondence [18]. In fact, the S -matrix [5, 19] of string theory on $AdS_5 \times S^5$ definitely requires a phase factor [20] as can be seen by comparison to an integral equation describing classical solutions [21]. Furthermore, it was argued that the S -matrix should possess a *crossing symmetry* [22], which then necessarily calls for such a phase. In [23] it was shown that the known classical and one-loop part of the phase factor on the string side satisfies crossing symmetry, and in [24] a crossing-invariant all-order string phase factor was proposed.

In the weakly coupled gauge theory, the dressing phase could appear, at the earliest, at four-loop order [25]. The leading (four-loop) effect on the scaling function was already investigated in [14]. Here we will extend this analysis to all-loop order, and investigate the effects of the dressing phase on the universal scaling function.

2. Integral equations for the scaling function

Let us start in this section by considering the effect of the dressing phase on the universal scaling function. We first briefly recall the assumptions and main steps leading to an integral equation for the scaling function of leading twist operators of $\mathcal{N} = 4$ gauge theory in the large spin limit $S \rightarrow \infty$. Up to three loops, it is based on the asymptotic Bethe ansatz

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^S \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}. \quad (2.1)$$

³ While finalizing our manuscript we were informed that this computation [13] has been completed.

This ansatz was shown to properly work in the $\mathfrak{sl}(2)$ sector relevant to twist operators up to three-loop order in [1], and further tests were performed in [26]. It was constructed to possibly describe gauge theory at four loops and beyond in [6], in analogy with an earlier proposal for the $\mathfrak{su}(2)$ sector [27].

The large-spin computation is similar to a thermodynamic limit for a spin chain Bethe ansatz, as the number S of Bethe roots becomes very large, and their distribution may be described by a smooth density. Note, however, that the actual length L of this spin chain should remain quite short. Ideally it should be $L = 2$, but this is dangerous since the Bethe ansatz is *a priori* only asymptotic [17, 27, 1, 6]. A workaround was devised in [14], where it was argued that the scaling of the anomalous dimension of the lowest state of operators of finite twist- L is a universal function of the coupling constant g . This suggests that it should coincide with the scaling function of twist-two operators⁴. For a similar discussion, albeit at one loop, see [28].

Despite the similarity to a thermodynamic limit, the computation is quite subtle due to the singular distribution of the one-loop roots [29]. The problem was solved in [14] by splitting off the one-loop piece, and subsequently deriving an integral equation for the non-singular fluctuations around it. It was found that the density of fluctuations $\hat{\sigma}(t)$ is determined by the solution of a non-singular integral equation of the general form

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left[\hat{K}(2gt, 0) - 4g^2 \int_0^\infty dt' \hat{K}(2gt, 2gt') \hat{\sigma}(t') \right]. \quad (2.2)$$

The universal scaling function $f(g)$ is then given by

$$f(g) = 8g^2 - 64g^4 \int_0^\infty dt \hat{\sigma}(t) \frac{J_1(2gt)}{2gt}. \quad (2.3)$$

Here and in the following $J_r(t)$ denotes a standard Bessel function. Note that this expression can be reduced provided that the kernel satisfies the property (it will always be true in this paper)

$$\hat{K}(0, t) = \frac{J_1(t)}{t}. \quad (2.4)$$

The scaling function then simply takes the value of the fluctuation density at $t = 0$ [30]

$$f(g) = 16g^2 \hat{\sigma}(0). \quad (2.5)$$

For the above, particular Bethe ansatz (2.1) the kernel $\hat{K}(t, t') = \hat{K}_m(t, t')$ is given by (m stands for *main* scattering)

$$\hat{K}_m(t, t') = \frac{J_1(t) J_0(t') - J_0(t) J_1(t')}{t - t'}. \quad (2.6)$$

The inhomogeneous piece of the integral equation (2.2) with this kernel is thus

$$\hat{K}_m(t, 0) = \frac{J_1(t)}{t}. \quad (2.7)$$

⁴ At four loops the asymptotic Bethe ansätze are only fully reliable at twist three, or higher, and we have to assume the four-loop universality of the scaling function for the lowest state of twist operators. This universality may only be rigorously proved for twist ≥ 2 at one loop [28], for twist ≥ 2 at two and three loops, and for twist ≥ 3 at four loops [14]. It would clearly be most welcome to more directly confirm universality from the field theory.

For this specific kernel (2.6) derived from (2.1) the weak-coupling expansion of the scaling function is

$$\begin{aligned}
 f_m(g) = & 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - 16\left(\frac{73}{630}\pi^6 - 4\zeta(3)^2\right)g^8 \\
 & + 32\left(\frac{887}{14175}\pi^8 - \frac{4}{3}\pi^2\zeta(3)^2 - 40\zeta(3)\zeta(5)\right)g^{10} \\
 & - 64\left(\frac{136883}{3742200}\pi^{10} - \frac{8}{15}\pi^4\zeta(3)^2 - \frac{40}{3}\pi^2\zeta(3)\zeta(5)\right. \\
 & \left. - 210\zeta(3)\zeta(7) - 102\zeta(5)^2\right)g^{12} \\
 & + 128\left(\frac{7680089}{340540200}\pi^{12} - \frac{47}{189}\pi^6\zeta(3)^2 - \frac{82}{15}\pi^4\zeta(3)\zeta(5) - 70\pi^2\zeta(3)\zeta(7)\right. \\
 & \left. - 34\pi^2\zeta(5)^2 - 1176\zeta(3)\zeta(9) - 1092\zeta(5)\zeta(7) + 4\zeta(3)^4\right)g^{14} \\
 & \mp \dots
 \end{aligned} \tag{2.8}$$

Here we can observe the Kotikov–Lipatov transcendentality principle [15]: we attribute a degree of transcendentality k to the constants π^k as well as to $\zeta(k)$.

The ℓ -loop contribution to the universal scaling function $f(g)$ for $\mathcal{N} = 4$ gauge theory has a uniform degree of transcendentality $2\ell - 2$.

The function $f_m(g)$ derived from the integral equation (2.2) obeys this rule [14]. A rigorous proof to all orders in perturbation theory was given in [30].

Let us next generalize the universal scaling function to the case of an arbitrary weak-coupling dressing phase. The only restriction is that the phase should merely modify the gauge theory Bethe ansatz at four loops or beyond, as the Bethe ansatz (2.1) is firmly established up to three-loop order [31, 25]. The corrections of the higher-loop Bethe equations which are consistent with current knowledge are

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^S \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp(2i\theta(u_k, u_j)), \tag{2.9}$$

where the dressing phase θ is conjectured to be of the general form [20, 32, 25]

$$\theta(u_k, u_j) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g) (q_r(u_k) q_{r+1+2\nu}(u_j) - q_r(u_j) q_{r+1+2\nu}(u_k)). \tag{2.10}$$

The $q_r(u)$ are the eigenvalues of the conserved magnon charges, see [27]. The coefficient functions $\beta_{r,s}(g)$ expand in powers of the coupling constant

$$\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu} \beta_{r,r+1+2\nu}^{(r+\nu+\mu)} \tag{2.11}$$

with $\beta_{r,s}^{(\ell)}$ some numerical constants in the same notation as in [25]. We should eliminate from the start the only coefficient $\beta_{2,3}^{(2)} = 0$ contributing at three loops. Note that in [14]

only the leading four-loop correction

$$\exp\left(2i\theta(u_k, u_j)\right) = \exp\left(i8\beta g^6 (q_2(u_k) q_3(u_j) - q_3(u_k) q_2(u_j)) + \dots\right) \quad (2.12)$$

was treated. Thus the first constant in (2.10) is $\beta_{2,3}^{(3)} = 4\beta$.

The modified Bethe ansatz (2.9) with (2.10) may be treated in much the same fashion as the simpler ansatz (2.1), see [14] for the details. In particular, the singular one-loop distribution may be split off in the same way. In fact, for an arbitrary dressing phase the general form (2.2) of the integral equation as well as the properties (2.5) and (2.4) remain valid. What changes is the kernel (2.6), which generalizes to

$$\hat{K}(t, t') = \hat{K}_m(t, t') + \hat{K}_d(t, t') \quad (2.13)$$

with the dressing kernel

$$\begin{aligned} \hat{K}_d(t, t') &= \frac{4}{t t'} \sum_{\rho=1}^{\infty} \sum_{\nu=0}^{\infty} \sum_{\mu=\nu}^{\infty} g^{2\mu+1} (-1)^\nu (\beta_{2\rho, 2\rho+1+2\nu}^{(2\rho+\nu+\mu)} J_{2\rho+2\nu}(t) J_{2\rho-1}(t')) \\ &\quad + \beta_{2\rho+1, 2\rho+2\nu+2}^{(2\rho+1+\nu+\mu)} J_{2\rho}(t) J_{2\rho+1+2\nu}(t') \end{aligned} \quad (2.14)$$

and the inhomogeneous term of the integral equation (2.2) receives additional contributions

$$\hat{K}_d(t, 0) = \frac{J_1(t)}{t} + \frac{2}{t} \sum_{\nu=0}^{\infty} \sum_{\mu=\nu}^{\infty} g^{2\mu+1} (-1)^\nu \beta_{2, 3+2\nu}^{(2+\nu+\mu)} J_{2+2\nu}(t). \quad (2.15)$$

Notice that as promised (2.4) still holds for (2.13), (2.14) and arbitrary constants $\beta_{r,s}^{(\ell)}$, and that therefore also (2.5) remains valid.

Equation (2.2) is still just as suitable for a perturbative small- g expansion if we use the modified kernel (2.13) and (2.14) instead of (2.6). It was already stated in [14] that a four-loop dressing phase (2.12) leads to the following $\mathcal{O}(g^8)$ term in the scaling function $f(g)$:

$$f(g) = \dots - 16 \left(\frac{73}{630} \pi^6 - 4\zeta(3)^2 + 8\beta\zeta(3) \right) g^8 + \dots \quad (2.16)$$

This violates transcendentality for generic β . However, if β is a rational number times $\zeta(3)$ (or π^3) transcendentality is *preserved*. It is straightforward to extend this analysis to even higher loop order:

Interestingly, a study of the effect of the constants $\beta_{r,s}^{(\ell)}$ in (2.10) reveals that for arbitrary loop order transcendentality is preserved if and only if the degree of transcendentality of $\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$ is $2\mu + 1$, independently of r and ν . In other words

$$\beta_{r,s}^{(\ell)} \text{ should have degree of transcendentality } 2\ell + 2 - r - s. \quad (2.17)$$

A particularly curious case is $\beta = \frac{1}{2}\zeta(3)$, which cancels the term containing $\zeta(3)^2$, and thus leads to the much simpler four-loop answer $-(73/630)\pi^6 16g^8$. Beyond four loops, one finds that one can always choose the $\beta_{r,s}^{(\ell)}$ in many different ways such that *all* terms containing zeta functions of *odd* argument are cancelled from the expansion (2.8). What is truly remarkable, however, is that the constants $\beta_{r,s}^{(\ell)}$ are uniquely determined if

we impose a further restriction on them conjectured to hold for arbitrary long-range spin chains compatible with gauge theory⁵ in [25], cf equation (4.2) in that paper,

$$\beta_{r,r+1+2\nu}^{(r+\nu+\mu)} = 0 \quad \text{for } \mu < r + \nu - 1, \quad (2.18)$$

or, in a different notation, $\beta_{r,s}^{(\ell)} = 0$ for $\ell < r + s - 2$. One finds to the first few orders⁶

$$\begin{aligned} \beta_{2,3}^{(3)} &\rightarrow +2 \zeta(3), \\ \beta_{2,3}^{(4)} &\rightarrow -20 \zeta(5), \\ \beta_{2,3}^{(5)} &\rightarrow +210 \zeta(7), & \beta_{3,4}^{(5)} &\rightarrow +12 \zeta(5), & \beta_{2,5}^{(5)} &\rightarrow -4 \zeta(5), \\ \beta_{2,3}^{(6)} &\rightarrow -2352 \zeta(9), & \beta_{3,4}^{(6)} &\rightarrow -210 \zeta(7), & \beta_{2,5}^{(6)} &\rightarrow +84 \zeta(7). \end{aligned} \quad (2.19)$$

The expansion of the scaling function significantly simplifies as compared to (2.8):

$$\begin{aligned} f_0(g) = & 8g^2 - \frac{8}{3} \pi^2 g^4 + \frac{88}{45} \pi^4 g^6 - 16 \frac{73}{630} \pi^6 g^8 + 32 \frac{887}{14175} \pi^8 g^{10} \\ & - 64 \frac{136883}{3742200} \pi^{10} g^{12} + 128 \frac{7680089}{340540200} \pi^{12} g^{14} \mp \dots \end{aligned} \quad (2.20)$$

The even-zeta terms, and thus the parts containing only even powers of π , are unaffected.

However, (2.19) is not the only curious choice for the constants $\beta_{r,s}^{(\ell)}$ in the dressing phase (2.10). Another striking choice corresponds to doubling the just discussed special constants, e.g. to the first few orders (2.19) becomes:

$$\begin{aligned} \beta_{2,3}^{(3)} &= +4 \zeta(3), \\ \beta_{2,3}^{(4)} &= -40 \zeta(5), \\ \beta_{2,3}^{(5)} &= +420 \zeta(7), & \beta_{3,4}^{(5)} &= +24 \zeta(5), & \beta_{2,5}^{(5)} &= -8 \zeta(5), \\ \beta_{2,3}^{(6)} &= -4704 \zeta(9), & \beta_{3,4}^{(6)} &= -420 \zeta(7), & \beta_{2,5}^{(6)} &= +168 \zeta(7). \end{aligned} \quad (2.21)$$

Now the zeta functions of odd argument no longer cancel out. Instead, one finds to, for example, seven-loop order

$$\begin{aligned} f(g) = & 8g^2 - \frac{8}{3} \pi^2 g^4 + \frac{88}{45} \pi^4 g^6 - 16 \left(\frac{73}{630} \pi^6 + 4 \zeta(3)^2 \right) g^8 \\ & + 32 \left(\frac{887}{14175} \pi^8 + \frac{4}{3} \pi^2 \zeta(3)^2 + 40 \zeta(3) \zeta(5) \right) g^{10} \\ & - 64 \left(\frac{136883}{3742200} \pi^{10} + \frac{8}{15} \pi^4 \zeta(3)^2 + \frac{40}{3} \pi^2 \zeta(3) \zeta(5) \right. \\ & \left. + 210 \zeta(3) \zeta(7) + 102 \zeta(5)^2 \right) g^{12} \end{aligned}$$

⁵ This study applied to the $\mathfrak{su}(2)$ sector of the gauge theory, but the phase is universal for all sectors [1, 6, 5] and thus includes $\mathfrak{sl}(2)$.

⁶ Some of the coefficients appear in the expansion of the scaling function at higher orders than naively expected from the expansion of the kernel (2.13) and (2.14). They are nevertheless fixed. For example, $\beta_{3,4}^{(5)}$ does not appear at six loops as expected, but only at seven loops in $f(g)$. But at this order it is fixed to $12 \zeta(5)$ if we demand all odd-zeta terms to cancel.

$$\begin{aligned}
 & + 128 \left(\frac{7680\,089}{340\,540\,200} \pi^{12} + \frac{47}{189} \pi^6 \zeta(3)^2 + \frac{82}{15} \pi^4 \zeta(3) \zeta(5) + 70 \pi^2 \zeta(3) \zeta(7) \right. \\
 & \left. + 34 \pi^2 \zeta(5)^2 + 1176 \zeta(3) \zeta(9) + 1092 \zeta(5) \zeta(7) + 4 \zeta(3)^4 \right) g^{14} \\
 & \mp \dots
 \end{aligned} \tag{2.22}$$

Remarkably, the alternating sum (2.22) is identical to (2.8) for the case of a trivial dressing phase by multiplying all zeta functions with odd arguments by the imaginary unit i , i.e. the replacement $\zeta(2n+1) \rightarrow i\zeta(2n+1)$. After this operation, and in contradistinction to the earlier case as discussed in [14], now all relative signs of the terms in (2.22) are identical, but the terms are otherwise unchanged! A proof of this transformation will be given in appendix B.

To wrap up the above results, we would like to mention that the scaling functions $f_m(g)$, $f_0(g)$ and $f(g)$ are part of a one-parameter family $f_\kappa(g)$ interpolating between these three choices. The general function is obtained by multiplying the constants in (2.19) by an overall factor of $(1 + \kappa)$. The resulting universal scaling function $f_\kappa(g)$ is the same as the sign-synchronized scaling function $f(g)$ in (2.22), but with the replacement $\zeta(2n+1) \rightarrow \sqrt{\kappa} \zeta(2n+1)$.

We see that there are very interesting and seemingly natural ways to deform the scattering phase of [27, 6] while preserving Kotikov–Lipatov transcendentality. We will now argue in section 3 that the (conjectured) AdS/CFT correspondence, together with (conjectured) integrability and a (conjectured) crossing-symmetric strong-coupling phase factor, indeed appears to single out one of the above choices. We will continue in section 4 with the investigation of the kernels and scaling functions. There we will derive a closed form for the very same constants and a concise integral expression for the summed dressing kernel.

3. An analytic continuation of sorts

In the following section, we shall investigate the constants $\beta_{r,s}^{(\ell)}$ starting from string theory. For perturbative string theory it is useful to write the dressing phase $\theta_{k,j}$ in (2.9) and (2.10) as

$$\theta(u_k, u_j) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) (\tilde{q}_r(u_k) \tilde{q}_s(u_j) - \tilde{q}_s(u_k) \tilde{q}_r(u_j)). \tag{3.1}$$

Here the excitation charges $\tilde{q}_r(u)$ are normalized as $\tilde{q}_r(u) = g^{r-1} q_r(u)$ differently from (2.10). Consequently, the coefficient functions $c_{r,s}$ and $\beta_{r,s}$ are related by

$$c_{r,s}(g) = g^{2-r-s} \beta_{r,s}(g). \tag{3.2}$$

The strong-coupling expansion of $c_{r,s}$ within string theory is non-trivial [20], [33]–[35]

$$c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}. \tag{3.3}$$

A proposal for the all-order strong-coupling expansion based on available data [20], [33]–[35] and crossing symmetry [22] was made in section 5.2 of [24]

$$c_{r,s}^{(n)} = \frac{(1 - (-1)^{r+s})\zeta(n)}{2(-2\pi)^n \Gamma(n-1)} (r-1)(s-1) \frac{\Gamma[\frac{1}{2}(s+r+n-3)] \Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s+r-n+1)] \Gamma[\frac{1}{2}(s-r-n+3)]}. \quad (3.4)$$

This expression is formally 0/0 when setting $n = 0, 1$ so we have to use a proper regularization [24]: in these cases the correct expression is known from comparison to classical and one-loop string theory data [20], [33]–[35]

$$c_{r,s}^{(0)} = \delta_{r+1,s}, \quad c_{r,s}^{(1)} = -\frac{(1 - (-1)^{r+s})}{\pi} \frac{(r-1)(s-1)}{(s+r-2)(s-r)}. \quad (3.5)$$

The expression for $c_{r,s}^{(1)}$ is easily recovered from (3.4) through the limit $\zeta(n)/\Gamma(n-1) \rightarrow 1$ when $n \rightarrow 1$. For $n = 0$ we first set r, s to some integer values with $s > r$. If $s > r + 1$ the only singular term is $1/\Gamma(n-1)$ which guarantees $c_{r,s}^{(0)} = 0$. This explains the $\delta_{r+1,s}$ term in (3.5). For $s = r + 1$, however, the term $\Gamma[\frac{1}{2}(s-r+n-1)]$ diverges at $n = 0$. In this case we can rewrite the expression with arbitrary n as

$$c_{r,r+1}^{(n)} = \frac{2(n-1)\zeta(n)\Gamma(1+\frac{1}{2}n)}{(-2\pi)^n \Gamma(n+1)\Gamma(2-\frac{1}{2}n)} \frac{r(r-1)\Gamma(r-1+\frac{1}{2}n)}{\Gamma(r+1-\frac{1}{2}n)}. \quad (3.6)$$

Here we can easily set $n = 0$ and obtain $c_{r,r+1}^{(0)} = 1$ as required by (3.5).

In order to compare to gauge theory, we have to find an expansion of $c_{r,s}(g)$ at weak coupling according to (2.10). The main obstruction in the investigation of the function $c_{r,s}(g)$ is that the known expansion at strong coupling is merely asymptotic: for fixed r, s and even n the sequence of coefficients $c_{r,s}^{(n)}$ terminates at $n = s - r + 1$ [24]. However, the odd- n sequence does not terminate and grows factorially as

$$\frac{c_{r,s}^{(n+2)}}{c_{r,s}^{(n)}} = \frac{n^2}{64\pi^2} + \mathcal{O}(n). \quad (3.7)$$

Consequently, the series of $c_{r,s}^{(n)}$ is asymptotic and has zero radius of convergence around $g = \infty$. It is Borel summable, but the expression for the coefficients may be too complex to perform the sum in practice.

In order to extrapolate to weak coupling, let us consider a simple model function first. In [24] some similarities between the phase and the digamma function $\Psi(z) = \partial_z \log \Gamma(z)$ were observed. This function has the following asymptotic expansion for large and positive z

$$\Psi(1+z) = \log z + \sum_{n=1}^{\infty} \frac{c_n}{z^n}, \quad c_n = -\frac{B_n}{n} = (-1)^n \zeta(1-n), \quad (3.8)$$

where B_n are the Bernoulli numbers which can also be expressed through the zeta function as $B_n = (-1)^{n+1} n \zeta(1-n)$. Conversely, the expansion around $z = 0$ is

$$\Psi(1+z) = -\gamma_E + \sum_{k=1}^{\infty} \tilde{c}_k z^k, \quad \tilde{c}_k = -(-1)^k \zeta(1+k) \quad (3.9)$$

with γ_E being Euler's constant. The curious observation is that the expansion coefficients for large z and for small z are almost the same. They are related by

$$c_n = -\tilde{c}_{-n}. \tag{3.10}$$

In other words, we obtain the expansion coefficients by *analytic continuation* of the index parameter n to negative values.

Could it be that a similar relation holds also for the expansion of the coefficients $c_{r,s}(g)$ at strong and weak coupling, respectively? Admittedly, this is a very wild guess, but it turns out to be literally true. The expansion of $c_{r,s}(g)$ at weak coupling is

$$c_{r,s}(g) = - \sum_{n=1}^{\infty} c_{r,s}^{(-n)} g^{1+n}. \tag{3.11}$$

In the following we shall not only provide indications for the mathematical correctness of this statement, but also argue that it leads to a consistent physical picture.

As a first test, let us compute some $c_{r,s}^{(n)}$ with negative value of n . Straight evaluation turns out to yield 0/0 in most cases and the expressions need to be regularized. We will therefore use the same procedure as in the cases $n = 0, 1$ and set r, s to their expected values first. Afterwards we take the analytic continuation of $c_{r,s}^{(n)}$ for the expected value of n . In this way we obtain, e.g.

$$c_{2,3}^{(-1)} = 0, \quad c_{2,3}^{(-2)} = -4 \zeta(3). \tag{3.12}$$

The absence of a $n = -1$ contribution translates via $\ell = (r + s - n - 1)/2$ to the absence of a $\ell = 5/2$ contribution, where ℓ represents the number of loops in gauge theory. In general all contributions for odd negative n should be zero to guarantee absence of contributions at fractional orders. In that case the expansion coefficients at weak coupling would be

$$\beta_{r,s}^{(\ell)} = -c_{r,s}^{(r+s-2\ell-1)}. \tag{3.13}$$

The first non-trivial contribution appears at⁷

$$\beta_{2,3}^{(3)} = 4 \zeta(3). \tag{3.14}$$

This is encouraging for several reasons: firstly, it influences anomalous dimensions starting from four loops. Currently, we know $\beta_{2,3}^{(\ell)} = 0$ for $\ell = 0, 1, 2$ only, so this is perfectly consistent with available data from gauge theory. Moreover, $\beta_{2,3}^{(3)}$ is the first allowed contribution according to (2.18). In addition, the value $4\zeta(3)$ has transcendentality 3 in the counting scheme of [15]. This is precisely the right transcendentality required for a uniform transcendentality of the scaling function [14]. Finally, the value (3.14) agrees precisely with the first in the series of sign reversing constants (2.21)! Let us now rewrite

⁷ A curious observation is that the leading contribution to the function $\beta_{2,3} = \zeta(3) \lambda^3/1024\pi^6 + \dots$ resembles the value $3\zeta(3)\lambda^3/512\pi^4$ [36] of a certain X-shaped diagram [37] to a circular Maldacena–Wilson loop [38] (the conjectured value $\lambda^3/4!^3$ in [37] is almost true). If a contribution proportional to $\zeta(3)$ survives in the sum over all diagrams at this order, it would imply that the ladder approximation [39] is not complete. This may seem unlikely, as this approximation yields a consistent strong-coupling result [39, 40]. However, it is not yet excluded either that non-ladder contributions have a mild, but non-vanishing effect.

the coefficients in a form more convenient for negative n . We use the identities

$$\zeta(1-z) = 2(2\pi)^{-z} \cos(\frac{1}{2}\pi z) \Gamma(z) \zeta(z) \quad \text{and} \quad \Gamma(1-z) = \frac{\pi}{\sin(\pi z) \Gamma(z)} \quad (3.15)$$

and obtain for integer r, s a new form for the coefficients

$$c_{r,s}^{(n)} = \frac{(1 - (-1)^{r+s}) \cos(\frac{1}{2}\pi n) (-1)^{s-1-n} \zeta(1-n) \Gamma(2-n) \Gamma(1-n) (r-1)(s-1)}{\Gamma[\frac{1}{2}(5-n-r-s)] \Gamma[\frac{1}{2}(3-n+r-s)] \Gamma[\frac{1}{2}(3-n-r+s)] \Gamma[\frac{1}{2}(1-n+r+s)]}. \quad (3.16)$$

The factor $\cos(\frac{1}{2}\pi n)$ makes it apparent that n must be even and that there are no fractional-loop contributions. Furthermore the factor $1/\Gamma[\frac{1}{2}(5-n-r-s)]$ imposes the lower bound $-n \geq r+s-3$ for the loop order $\ell = (r+s-n-1)/2 \geq r+s-2$ consistent with (2.18). As we discussed already in section 2,

$$\beta_{r,s}^{(\ell)} \sim \zeta(2+2\ell-r-s), \quad (3.17)$$

this is precisely what is required for a uniform transcendentality of the scaling function.

Being convinced of the physical plausibility of the proposal, we shall now investigate the equivalence of the weak-and strong-coupling expansions for the choice $r=2, s=3$. The proposed weak-coupling expansion is

$$c_{2,3}(g) = - \sum_{n=1}^{\infty} c_{2,3}^{(-2n)} g^{1+2n} \quad (3.18)$$

with

$$c_{2,3}^{(-2n)} = \frac{4(-1)^n \zeta(2n+1) \Gamma(2n) \Gamma(2n-1)}{\Gamma(n) \Gamma(n+1) \Gamma(n+2) \Gamma(n+3)}. \quad (3.19)$$

We shall write $\zeta(1+2n)$ using its series representation

$$c_{2,3}(g) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{4(-1)^{n+1} (2n-1)! (2n)! (g/k)^{1+2n}}{(n-1)! n! (n+1)! (n+2)!}. \quad (3.20)$$

The sum over n can now be evaluated explicitly to

$$c_{2,3}(g) = \sum_{k=1}^{\infty} h_{2,3}(g/k), \quad h_{2,3}(z) = -\frac{1}{z} + \frac{1+16z^2}{6\pi z^3} K(4iz) - \frac{1-4z^2}{6\pi z^3} E(4iz), \quad (3.21)$$

where K and E are the elliptic integrals of the first and second kind, respectively⁸. The function $h_{2,3}$, some partial sums as well as the series expansion of $c_{2,3}$ are displayed in figure 1.

It is now possible to connect to strong coupling. The expansion of the function $h_{2,3}$ at large positive z is

$$h_{2,3}(z) = \frac{8}{3\pi} - \frac{1}{z} + \frac{3 \log(16z)}{4\pi z^2} - \frac{5}{8\pi z^2} + \frac{5 \log(16z)}{512\pi z^4} + \frac{1}{2048\pi z^4} + \dots \quad (3.22)$$

⁸ We use the convention that the modulus appears in a squared form as $(4iz)^2 = m$.

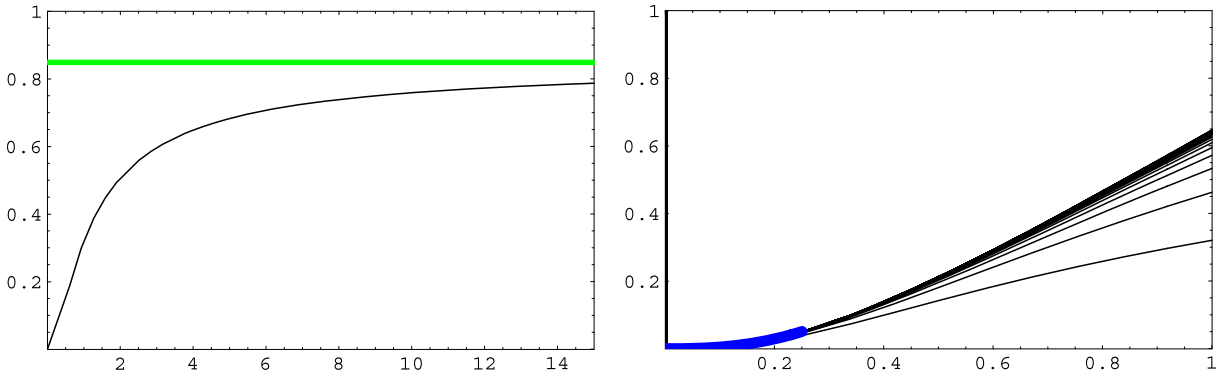


Figure 1. Left: plot of $h_{2,3}(z)$ with asymptotic value $8/3\pi$. Right: plot of the partial sums in $c_{2,3}(g)$ with weak coupling expansion up to radius of convergence $g = \frac{1}{4}$.

Zeta function regularization⁹ of the sum yields

$$\begin{aligned}
 \sum_{k=1}^{\infty} h_{2,3}(g/k) &= g \int_0^{\infty} h_{2,3}(1/z) dz + \frac{8\zeta(0)}{3\pi} - \frac{\zeta(-1)}{g} \\
 &+ \frac{3}{4\pi} \frac{\zeta'(-2)}{g^2} - \frac{5}{8\pi} \frac{\zeta(-2)}{g^2} + \frac{5}{512\pi} \frac{\zeta'(-4)}{g^4} - \frac{1}{2048\pi} \frac{\zeta(-4)}{g^4} + \dots \\
 &= g - \frac{4}{3\pi} + \frac{1}{12g} - \frac{3\zeta(3)}{16\pi^3 g^2} + \frac{15\zeta(5)}{2048\pi^5 g^4} + \dots \\
 &= \sum_{n=0}^{\infty} c_{2,3}^{(n)} g^{1-n}. \tag{3.23}
 \end{aligned}$$

Here the sum over k^n is regularized to $\zeta(-n)$ while the sum over $k^n \log(k)$ yields $-\zeta'(-n)$. We have verified the perfect agreement of the expansion with the strong-coupling coefficients $c_{2,3}^{(n)}$ to order g^{-100} leaving little room for error. In appendix C we present a proof of this statement to all orders. In the proof we observe some exponential corrections of the type $\mathcal{O}(e^{-g})$ to the series. These corrections are not unexpected because the series is asymptotic. A plot of the partial sums and asymptotic strong-coupling expansion of $c_{2,3}$ is given in figure 2.

We have also confirmed the same agreement for other values of (r, s) : we can always replace the $\zeta(1 + 2n)$ in $c_{r,s}$ by a new sum and write $c_{r,s}$ in terms of a function $h_{r,s}(z)$ without odd-zeta terms¹⁰

$$c_{r,s}(g) = \sum_{k=1}^{\infty} h_{r,s}(g/k). \tag{3.24}$$

⁹ Zeta function regularization is closely related to Euler–MacLaurin summation. However, the latter has to be adapted to deal correctly with terms $k^n \log(k)$ in the expansion of h , see [40]. We thank S Frolov for providing us with this reference.

¹⁰ This form of the phase may be suggestive of the presence of bound states [41] (or in general larger multiplets [42]) in intermediate channels during the scattering process. For instance, the dispersion relation as well as other characteristic quantities of a bound state of k particles is typically obtained by replacing $g \rightarrow g/k$ in the expressions for a single particle.

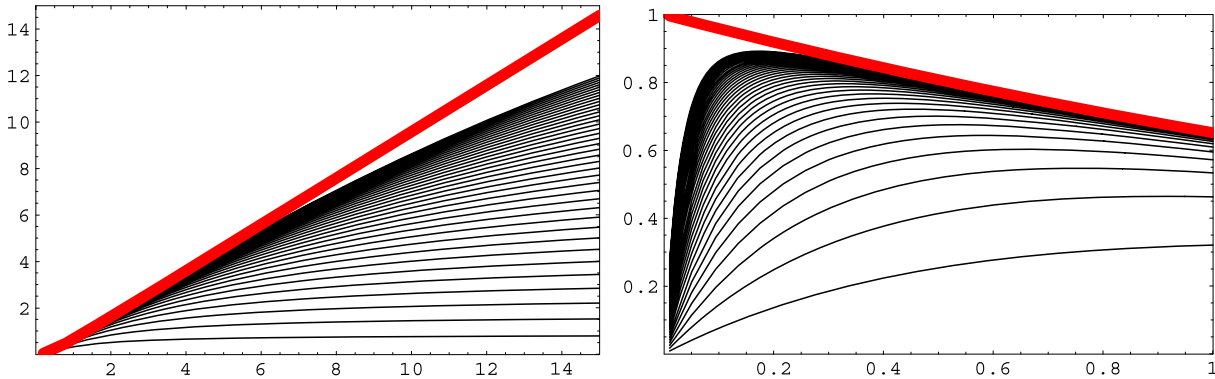


Figure 2. Left: plot of partial sums in $c_{2,3}(g)$ with asymptotic strong-coupling expansion. Right: we have plotted $x c_{2,3}(1/x)$ instead to improve the view of the strong-coupling behaviour.

The series in g can now be written formally as the hypergeometric function

$$h_{r,s}(z) = \frac{2 \cos[\frac{1}{2}\pi(s-r-1)] \Gamma(r+s-2)}{\Gamma(r-1) \Gamma(s-1)} z^{r+s-2} {}_4F_3(\vec{a}; \vec{b}; -16z^2) \quad (3.25)$$

with arguments $\vec{a} = [\frac{1}{2}(r+s-2), \frac{1}{2}(r+s-1), \frac{1}{2}(r+s-1), \frac{1}{2}(r+s)]$ and $\vec{b} = [r, s, r+s-1]$. This hypergeometric function can be evaluated to the general form using elliptic integrals

$$h_{r,s}(z) = \frac{P_{r,s}(z^2)}{z^{r+s-2}} + \frac{Q_{r,s}(z^2)}{\pi z^{r+s-2}} K(4iz) + \frac{R_{r,s}(z^2)}{\pi z^{r+s-2}} E(4iz). \quad (3.26)$$

Here P, Q, R are some polynomials of degree $(r+s-3)/2$ with rational coefficients. The polynomials P and R have special factors for all values of r, s , namely $P(z^2) \sim z^{2(r-2)}$ and $Q(z^2) \sim (1+16z^2)$. As in the above case of $(r, s) = (2, 3)$, we use zeta function regularization to evaluate the sums (3.24) at strong coupling¹¹. We have further expanded some of the functions (3.25) at strong coupling and they fully agreed with the proposal in [24] upon zeta function regularization in all cases tested (all pairs of r, s with $s \leq 10$ to 20 orders). In appendix C we will finally show for general (r, s) that $c_{r,s}(g)$ at large g yields the first two orders $c_{r,s}^{(0)}$ and $c_{r,s}^{(1)}$ in (3.5) which are known from perturbative string theory [20, 34].

It is therefore fair to claim that the strong-coupling expansion (3.3) and the weak-coupling expansion (3.11) describe one and the same function $c_{r,s}(g)$ for all (r, s) . However, we should stress that the weak-coupling expansion is more versatile than the strong-coupling one: the functions $h_{r,s}$ have singularities at $g = \pm \frac{1}{4}$, and thus the coefficients $c_{r,s}(g)$ have singularities at $g = \pm \frac{1}{4}k$ for all positive integers k .¹² Consequently, $c_{r,s}(g)$ has a finite radius of convergence around $g = 0$, and the weak-coupling series defines the function unambiguously. In contradistinction the strong-coupling expansion is merely asymptotic and does not fully define the function.

¹¹ The polynomial P generates even-loop orders in string theory, while Q, R generate the odd ones. Its coefficients are thus directly related to the coefficients $c_{r,s}^{(2n)}$.

¹² The singularities arise from the elliptic integrals in (3.26) and therefore appear to be related to a degeneration of the complex tori defining the kinematical space for the bound states in [41, 42].

4. Magic kernels

Now we will return to our study of the integral equation for the fluctuation density in the presence of a perturbative dressing phase $\theta \neq 0$. As noted in section 2, there appears to exist a very special choice for the constants $\beta_{r,s}^{(\ell)}$ in (2.10) such that the odd-zeta contributions to the scaling function based on a trivial phase $\theta = 0$ ($f_m(g)$, see (2.8)) either cancel ($f_0(g)$, see (2.20)), or synchronize their relative sign w.r.t. the even-zeta contributions ($f(g)$, see (2.22)). The constants of the latter case appeared to differ from the ones of the former case by a factor of two. Let us now prove these statements, and derive the relevant set of constants $\beta_{r,s}^{(\ell)}$.

An observation of central importance for this section is that the simplified scaling function $f_0(g)$ may be obtained from an effective kernel much simpler than (2.13) and (2.14). One just needs to decompose the ‘main scattering’ kernel (2.6) into two parts

$$\hat{K}_m(t, t') = \hat{K}_0(t, t') + \hat{K}_1(t, t'), \tag{4.1}$$

which are even and odd functions, respectively, under both $t \rightarrow -t$ and $t' \rightarrow -t'$. Explicitly these functions are

$$\begin{aligned} \hat{K}_0(t, t') &= \frac{t J_1(t) J_0(t') - t' J_0(t) J_1(t')}{t^2 - t'^2}, \\ \hat{K}_1(t, t') &= \frac{t' J_1(t) J_0(t') - t J_0(t) J_1(t')}{t^2 - t'^2}. \end{aligned} \tag{4.2}$$

It turns out that the component \hat{K}_1 is responsible for all odd-zeta contributions to f_m . If we eliminate it and set $\hat{K} = \hat{K}_0$ in (2.2) we obtain just the same functions $\hat{\sigma}_0$ and $f_0(g)$, see (2.20). Note that again (2.4) remains true under this symmetrization, and (2.7) holds with $\hat{K}_m \rightarrow \hat{K}_0$.

We can use this observation to obtain an almost closed expression for the dressing kernel. We start from the integral equation with a purely even kernel (4.2)

$$\hat{\sigma}_0(t) = \frac{t}{e^t - 1} \left[\hat{K}_0(2gt, 0) - 4g^2 \int_0^\infty dt' \hat{K}_0(2gt, 2gt') \hat{\sigma}_0(t') \right], \tag{4.3}$$

which leads to $f_0(g)$. Replacing \hat{K}_0 by $\hat{K}_m - \hat{K}_1$, using $\hat{K}_1(t, 0) = 0$, and substituting $\hat{\sigma}_0$ once into the ensuing second convolution only, we derive the relation

$$\begin{aligned} \hat{\sigma}_0(t) &= \frac{t}{e^t - 1} \left[\hat{K}_m(2gt, 0) + \hat{K}_c(2gt, 0) \right. \\ &\quad \left. - 4g^2 \int_0^\infty dt' \left(\hat{K}_m(2gt, 2gt') + \hat{K}_c(2gt, 2gt') \right) \hat{\sigma}_0(t') \right] \end{aligned} \tag{4.4}$$

with the function \hat{K}_c

$$\hat{K}_c(t, t') = 4g^2 \int_0^\infty dt'' \hat{K}_1(t, 2gt'') \frac{t''}{e^{t''} - 1} \hat{K}_0(2gt'', t'). \tag{4.5}$$

Now (4.4) is just the original integral equation (2.2), but with the kernel $\hat{K} = \hat{K}_m + \hat{K}_c$. Therefore we can interpret \hat{K}_c as a dressing kernel to generate the simplified dressing function $f_0(g)$. In fact, it agrees *precisely* with the expansion of the general dressing

kernel (2.13) and (2.14) using the first few coefficients found in (2.19)! The expression for the dressing kernel \hat{K}_c allows us to derive a closed expression for the constants $\beta_{r,s}^{(\ell)}$ leading to cancellation of odd-zeta contributions. The details of this derivation are presented in appendix A; the final result is

$$\beta_{r,r+1+2\nu}^{(r+\nu+\mu)} \rightarrow (-1)^{r+\mu+1} \frac{(r-1)(r+2\nu)}{2\mu+1} \binom{2\mu+1}{\mu-r-\nu+1} \binom{2\mu+1}{\mu-\nu} \zeta(2\mu+1). \quad (4.6)$$

Surprisingly, these coefficients agree, up to a global factor of $\frac{1}{2}$, with the ‘analytic continuation’ of the conjectured strong-coupling dressing phase constants (3.13) and (3.16)! The results of the previous section then suggest that the correct crossing-invariant choice of constants is just twice (4.6). Of course, these constants agree with those in (2.21). The corresponding kernel for the integral equation (2.2) of the universal scaling function $f(g)$ is, cf (2.6), (4.5)

$$\hat{K}(t, t') = \hat{K}_m(t, t') + 2 \hat{K}_c(t, t'). \quad (4.7)$$

What does this imply for the universal scaling function? We ‘experimentally’ observed in section 2, to rather high order, that the doubled constants (4.8) lead to sign reversals in those odd-zeta terms which do not have the same relative sign as the even-zeta terms in the original proposal (2.8). Up to this minor modification, the associated scaling function agrees with the one based on a trivial phase. We will present a proof of this statement in appendix B.

It is interesting to investigate the analytic structure of the three candidate scaling functions. In each case one would like to find the locations of all singularities in the complex g -plane, and determine whether they correspond to poles, branch points, or are of essential type. The convergence properties of the weak coupling expansion series are, in particular, clearly related to the singularity with smallest $|g|$.

The integral form of the dressing kernel K_c allows us to derive conveniently the first few orders in the expansion of the scaling function. The analytic expression is given (2.22). Beyond 20 orders, however, the analytic calculation slowed down substantially due to an exploding number of partitions for the odd-zeta contributions. This complexity can be reduced by reverting to a numerical computation. We were able to produce all three scaling functions $f_m(g)$, $f_0(g)$ and $f(g)$ up to 50 loops, $\mathcal{O}(g^{100})$, at 1000-digit accuracy (needed for intermediate steps). A plot of these three functions is shown in figure 3.

We have then used the quotient criterion to investigate the convergence properties of the series. Our results indicate that the radius of convergence in g is $\frac{1}{4}$ in all three cases. The related singularity is situated at $g = \pm \frac{i}{4}$ or, put differently, at $\lambda = -\pi^2$. The accuracy of this result was very high and the position matches with the singularity of the function in (3.21). The exponents of the singularity $f \simeq (\lambda + \pi^2)^\alpha$ appear to be $\alpha = +1, +\frac{2}{3}, +\frac{1}{2}$ for the three functions $f_m(g)$, $f_0(g)$ and $f(g)$, respectively. We were not able to produce a very accurate result in this case; we expect it to be accurate within a few per cent.

In conclusion, the final proposal for the, hopefully correct, expansion coefficients of the dressing phase (2.10) and (2.11) is

$$\beta_{r,r+1+2\nu}^{(r+\nu+\mu)} = 2(-1)^{r+\mu+1} \frac{(r-1)(r+2\nu)}{2\mu+1} \binom{2\mu+1}{\mu-r-\nu+1} \binom{2\mu+1}{\mu-\nu} \zeta(2\mu+1). \quad (4.8)$$

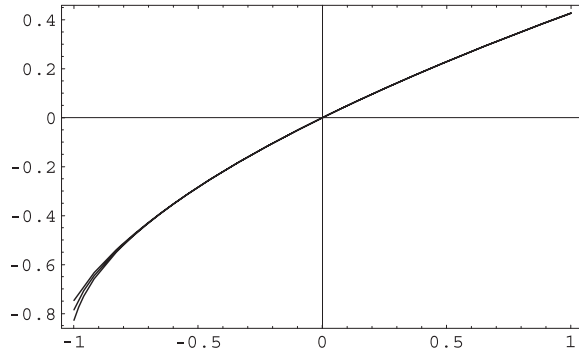


Figure 3. Top to bottom curves: plot of the scaling functions $f_m(g)$, $f_0(g)$ and $f(g)$ with $g = \frac{1}{4}\sqrt{x}$.

The resulting universal scaling function $f(g)$ in a weak-coupling expansion was presented in (2.22) to the first few orders. Its convergence properties seem to indicate a singularity at $\lambda = -\pi^2$ with exponent $+\frac{1}{2}$.

The strong-coupling expansion of the scaling function $f(g)$ remains to be evaluated within the Bethe ansatz framework. A comparison to available data [43, 44] from string theory would constitute an important test of our proposed dressing phase. The leading order at strong coupling based on the AFS phase [20] (which is consistent with our proposal) indeed agrees [45] with an explicit string theory calculation [43]. Conversely, in the case of the undressed function $f_m(g)$, the integral equation is troublesome and leads to a fluctuating solution [30]. Hopefully the situation is improved by the correct dressing phase in order to find agreement with string theory.

5. Conclusions and outlook

In this paper we have derived an expression (2.10), (2.11) and (4.8) for the complete weak-coupling expansion of the dressing phase θ in the Bethe ansatz [6] of planar $\mathcal{N} = 4$ gauge theory. It is based on a crossing-symmetric proposal in section 5.2 of [24] for the strong-coupling expansion of the dressing phase in conjunction with some curious structures of cancellations and replacements when a dressing phase modifies the scaling function $f_m(g)$ based on a trivial phase, as originally derived in [14]. Whether or not our proposal describes the AdS/CFT system accurately, we have shown that there exists a natural expression for the dressing phase which appears to interpolate between currently known spectral data at weak and strong coupling. We are hopeful, but have not shown, that any alteration of our proposal is likely to violate the transcendentality principle in gauge theory and/or crossing symmetry in string theory as well as leads to potential contradictions with the available data. In that sense our proposal lends support to an exact interpretation of the AdS/CFT correspondence [18], at least in the case of $\mathcal{N} = 4$ gauge theory in the 't Hooft limit and free IIB superstrings on $AdS_5 \times S^5$.

The dressing phase θ is given as a multiple expansion in excitation charges and the coupling constant. The radius of convergence w.r.t. the 't Hooft coupling λ around $\lambda = 0$ is $|\lambda| < \pi^2$. The singularities are situated at $\lambda = -n^2\pi^2$ for all non-zero integers n . Consequently, perturbative gauge theory appears to unambiguously define the phase

and its analytic continuation. Note, however, that we have not succeeded in summing over the modes r, s of $\beta_{r,s}(g)$ or even studying the analytic structure of the phase θ in general. Conversely, the strong-coupling expansion of the phase can merely be asymptotic because $\lambda = \infty$ is an accumulation point of singularities. The asymptotic string expansion proposed in section 5.2 of [24] seems to be in perfect agreement with our proposal, but it is not sufficient to *uniquely* define a complex function.

Two methods of testing our coefficients come to mind: the first is a direct computation of the scaling function $f(g)$ at four loops. Until recently, it was known up to three loops [7, 8], see also [10], but with methods based on the unitarity of scattering amplitudes it is within computational reach [12, 13]. Our prediction $\beta_{2,3}^{(3)} = 4\beta = 4\zeta(3)$ leads to a four-loop contribution to the universal scaling function

$$\begin{aligned} f(g) &= \dots - 16 \left(\frac{73}{630} \pi^6 + 4\zeta(3)^2 \right) g^8 + \dots \\ &\approx \dots - 3.01502 \times 10^{-6} \lambda^4 + \dots \end{aligned} \quad (5.1)$$

The final outcome of the four-loop calculation by Bern, Czakon, Dixon, Kosower and Smirnov [13], which was performed in parallel and completely independently from our analysis, is

$$\begin{aligned} f(g) &= \dots - 64 \times (29.335 \pm 0.052) g^8 + \dots \\ &= \dots - (3.0192 \pm 0.0054) \times 10^{-6} \lambda^4 + \dots \end{aligned} \quad (5.2)$$

The perfect agreement between the two numbers provides further confidence that our proposed dressing phase correctly describes the asymptotic spectrum of anomalous dimensions in planar $\mathcal{N} = 4$ gauge theory. At the least it ascertains that the dressing phase is non-trivial starting from $\mathcal{O}(g^6)$, and that the constant for the first correction in [14] takes the value $\beta = \frac{1}{4}\beta_{2,3}^{(3)} = \zeta(3)$. Note, however, that we have to assume the exact universality of the scaling function, i.e. that it is exactly the same for twist-two (as assumed in [13]) and for higher-twist (as required for the asymptotic Bethe ansatz).

A computation of the function $f(g)$ to even higher loop orders would provide more stringent tests of our proposal. Given the remarkable success of unitarity-based methods in [7, 8, 13] we do not dare call this impracticable. However, some yet-to-be-discovered iterative structure is likely to exist if our comparably simple integral equation holds true. With many perturbative orders available an extrapolation to strong coupling may be attempted and compared to string theory results [43, 44] via the AdS/CFT correspondence. Such an analysis was indeed performed in [13] using the KLV [46] and Padé approximations. It is based on their independent guess of the function $f(g)$ in (2.22) and how to obtain it from the earlier proposal (2.8) by synchronizing all signs. Strangely, their approximations, based on using the first seven or eight terms for example, appear to indicate that (2.22) is about 5% away from the expected strong-coupling value of [43]. Non-coincidence of the values would be rather puzzling with regard to the discussion at the end of sections 4 and 3. Would it imply that $f(g)$ at strong coupling does not describe the string states in [43, 44]? Are there some non-perturbative corrections? To that end, it would be worth investigating whether an adapted scheme which takes into account all singularities at $\lambda = -n^2\pi^2$ and their accumulation at $\lambda = \infty$ can produce an extrapolation of $f(g)$ closer to the expected string theory values [43, 44].

Another way of testing the proposal involves local operators of small classical dimension. A gauge Bethe ansatz modified by a dressing phase changes the anomalous dimensions Δ of all operators in all sectors. For example, in the $\mathfrak{su}(2)$ sector we would find for the length $L = 5$ operator $\text{Tr } X^2 Z^3 + \dots$ (this case is actually equivalent to the $\mathfrak{sl}(2)$ twist = length three operator $\text{Tr } D^2 Z^3 + \dots$) to four loops

$$\Delta = 5 + 8g^2 - 24g^4 + 136g^6 - 16 \left(\frac{115}{2} + 2\beta_{2,3}^{(3)} \right) g^8 + \dots \quad (5.3)$$

For the specific case of constants (4.8) we then have

$$\Delta = 5 + 8g^2 - 24g^4 + 136g^6 - 16 \left(\frac{115}{2} + 8\zeta(3) \right) g^8 + \dots \quad (5.4)$$

Intriguingly, this predicts the appearance of a non-rational term in the four-loop anomalous dimension of a finite length operator! Such contributions are impossible if the dressing phase is trivial, i.e. if $\theta = 0$. They are certainly not excluded, and *a priori* even likely, from the point of view of perturbative field theory. This would then also rule out the all-loop ‘BDS’ ansatz originally proposed in [27], and as a consequence also its finite-length description through the Lieb–Wu equations of the Hubbard model, cf (68) in [47]. It also eliminates the Hubbard Hamiltonian as an exact candidate for the $\mathfrak{su}(2)$ dilatation operator [47]. However, a gauge theory dressing phase (2.10) whose constants preserve transcendentality for twist operators, such as in the specific suggestion (4.8), would have the following intriguing property affecting operators of finite length and spin. Up to (at least) wrapping order, the BDS ansatz would give the correct ‘rational part’ of all anomalous dimension matrices, and the Hubbard Hamiltonian would emulate the ‘rational part’ of the $\mathfrak{su}(2)$ dilatation operator to (at least) wrapping order.

An interesting question concerns the computation of anomalous dimensions beyond wrapping order. The simplest case is the four-loop anomalous dimension of the Konishi field, i.e. the $\mathfrak{su}(2)$ length $L = 4$ operator $\text{Tr } X^2 Z^2 + \dots$, or, equivalently, in $\mathfrak{sl}(2)$ the lowest spin $S = 2$, length $L = 2$ operator $\text{Tr } D^2 Z^2 + \dots$. If we, *a priori* ‘illegally’, apply the asymptotic Bethe ansatz (2.9) and (2.10) we find the four-loop result

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 - 16 \left(\frac{705}{4} + \frac{9}{2}\beta_{2,3}^{(3)} \right) g^8 + \dots \quad (5.5)$$

Recall that $\beta_{2,3}^{(3)} = 0$ is the ‘BDS’ case with dressing phase $\theta = 0$. With our conjecture (4.8) we then have instead

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 - 16 \left(\frac{705}{4} + 18\zeta(3) \right) g^8 + \dots \quad (5.6)$$

However, as opposed to (5.4), this result is much less certain. It would only be true if our asymptotic Bethe equations turn out to be exact. Conversely, recall that the Hubbard model [47] yields for this state

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 - 16 \times 318g^8 + \dots \quad (5.7)$$

Could it be that the Hubbard model still properly yields the ‘rational part’ of fields even beyond wrapping order? This might then result in

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 - 16(318 + 18\zeta(3))g^8 + \dots, \quad (5.8)$$

but, in the absence of a yet-to-be-constructed rigorous Bethe ansatz for the finite system, this is of course a quite unfounded speculation.

Of course, there are more general questions: it would be very valuable to find a closed form for the proposed dressing phase θ and to prove its correctness. Likewise, the exact integrability of the planar AdS/CFT model still lacks a rigorous proof. Are the Bethe equations at finite coupling *and* finite length exact or is there a more fundamental description of the planar spectrum of which the Bethe equations are merely some limits? For instance, the exactness may be flawed by certain terms which are exponentially suppressed at long states [48]. The ‘covariant’ approaches started in [49, 47, 50] may provide an answer to this question as well as recover the dressing phase from a more fundamental (and simpler) description. Other promising approaches are the thermodynamical Bethe ansatz (see [51]) and Baxter equations (see [52]).

It is curious to see that in our proposal the deviations from a trivial phase start to contribute to anomalous dimensions at four loops, which is a rather high perturbative order. If true, this might indicate that intuition gained from some perturbative computations, even if they extend to two or three loops, may be deceiving. For instance, a non-zero value for $\beta_{2,3}^{(3)} = 4\beta$ leads to a four-loop breakdown of weak-coupling BMN scaling [2] in $\mathcal{N} = 4$ gauge theory. This possibility was discussed by several authors in the past in order to reconcile discrepancies between gauge and string theory structures at, respectively weak and strong coupling. See in particular [53, 17, 27, 54, 33, 55]. Our proposal suggests that this is indeed the case. The assumption of [2] that the BMN limit leads to a dilute gas with weakly interacting particles is not compatible with our phase. It should be noted that the perturbative dispersion relation obtained in [56] is indeed compatible with our proposal. However, it cannot be translated directly into anomalous dimensions of local operators as suggested in [2, 56].

The same applies for the spinning strings proposal [57]: the weak-coupling expansion in the effective coupling constant $\lambda' \sim \lambda/J^2$ leads to terms of the type $(\lambda')^4 J^2$ which diverge in the limit of large spin J . The investigation of spinning string solutions from the gauge theory side started in [58] will therefore fatally break down at the four-loop level. Nevertheless, the proposals [2, 57] have been of tremendous importance to initiate the study of stringy and integrable structures in $\mathcal{N} = 4$ gauge theory.

Finally, it is worth pointing out that $\beta_{2,3}(g) = \mathcal{O}(g^6)$ is reminiscent of the findings [59] within the plane wave (BMN) matrix model [2]. Also there, a non-trivial dressing phase was observed starting at the same order. The main difference is that the coefficient $\beta_{2,3}^{(3)}$ for the BMN matrix model is rational while ours is a rational multiple of $\zeta(3)$. This may potentially be related to the fact that the matrix model has only finitely many degrees of freedom while $\mathcal{N} = 4$ SYM is non-compact.

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Appendix A. Expansion of the magic kernel

In this appendix we present a weak-coupling expansion of the magic kernel \hat{K}_c in (4.5) leading to an expression for the coefficients $\beta_{r,s}^{(\ell)}$ of the dressing phase.

A useful integral representation for the even and odd parts was found in [14]:

$$\begin{aligned} \hat{K}_0(t, t') &= \int_0^1 d\lambda \lambda J_0(\lambda t) J_0(\lambda t'), \\ \hat{K}_1(t, t') &= \int_0^1 d\lambda \lambda J_1(\lambda t) J_1(\lambda t'). \end{aligned} \tag{A.1}$$

With their help, we may calculate \hat{K}_c to be

$$\begin{aligned} \hat{K}_c(t, t') &= 4g^2 \int_0^\infty dt'' \int_0^1 d\lambda_1 \lambda_1 J_1(\lambda_1 t) J_1(\lambda_1 2g t'') \frac{t''}{e^{t''} - 1} \\ &\quad \times \int_0^1 d\lambda_2 \lambda_2 J_0(\lambda_2 2g t'') J_0(\lambda_2 t') \\ &= -4 \sum_{\mu=1}^\infty (-1)^\mu g^{2\mu+1} \zeta(2\mu+1) (2\mu)! \sum_{n=0}^{\mu-1} \frac{1}{n! (n+1)! (\mu-1-n)! (\mu-1-n)!} \\ &\quad \times \int_0^1 d\lambda_1 \lambda_1^{2(n+1)} J_1(\lambda_1 t) \int_0^1 d\lambda_2 \lambda_2^{2(\mu-n)-1} J_0(\lambda_2 t') \end{aligned} \tag{A.2}$$

where we have first expanded the Bessel functions with argument $\lambda_{1,2} 2g t''$ in g and then executed the t'' integration. The two remaining parameter integrals in the last line may be found from a standard formula. In the first integral the power of λ_1 in front of $J_1(\lambda_1 t)$ is always even, and in the second one the power of λ_2 in front of $J_0(\lambda_2 t')$ is always odd. Both integrals therefore yield finite sums of Bessel functions:

$$\begin{aligned} \int_0^1 d\lambda_1 \lambda_1^{2m} J_1(\lambda_1 t) &= -\frac{(m-1)! m!}{(2m)!} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \frac{2k}{t} J_{2k}(t) \int_0^1 d\lambda_2 \lambda_2^{2m-1} J_0(\lambda_2 t') \\ &= -\frac{(m-1)! (m-1)!}{(2m-1)!} \sum_{l=1}^m (-1)^l \binom{2m-1}{m-l} \frac{2l-1}{t'} J_{2l-1}(t'). \end{aligned} \tag{A.3}$$

On substituting these into (A.2) we can cancel the factors which depend on n but not on k, l . We re-order the n, k, l sums to find:

$$\begin{aligned} \hat{K}_c(t, t') &= -\frac{4}{t t'} \sum_{\mu=1}^\infty (-1)^\mu g^{2\mu+1} \zeta(2\mu+1) (2\mu)! \\ &\quad \times \sum_{\substack{k+l \leq \mu+1 \\ k, l \geq 1}} (-1)^{k+l} (2k) J_{2k}(t) (2l-1) J_{2l-1}(t') \\ &\quad \times \sum_{n=k-1}^{\mu-l} \frac{1}{(n+1-k)! (n+1+k)! (\mu-l-n)! (\mu+l-1-n)!}. \end{aligned} \tag{A.4}$$

The sum over n can be done in closed form:

$$\begin{aligned} \sum_{n=k-1}^{\mu-l} \frac{1}{(n+1-k)!(n+1+k)!(\mu-l-n)!(\mu+l-1-n)!} \\ = \frac{1}{(2\mu+1)!} \binom{2\mu+1}{\mu+1-k-l} \binom{2\mu+1}{\mu+1+k-l}. \end{aligned} \quad (\text{A.5})$$

Collecting results, we get

$$\begin{aligned} \hat{K}_c(t, t') = -\frac{4}{tt'} \sum_{\mu=1}^{\infty} g^{2\mu+1} \sum_{\substack{k+l \leq \mu+1 \\ k, l \geq 1}} J_{2k}(t) J_{2l-1}(t') (-1)^{\mu+k+l} \\ \times \frac{(2k)(2l-1)}{2\mu+1} \binom{2\mu+1}{\mu+1-k-l} \binom{2\mu+1}{\mu+1+k-l} \zeta(2\mu+1). \end{aligned} \quad (\text{A.6})$$

This is *precisely* of the form of a kernel stemming from a non-trivial dressing phase θ , as we can see by comparing to (2.14)! We can even read off the constants $\beta_{r,s}^{(\ell)}$:

$$\beta_{r,r+1+2\nu}^{(r+\nu+\mu)} \rightarrow (-1)^{r+\mu+1} \frac{(r-1)(r+2\nu)}{2\mu+1} \binom{2\mu+1}{\mu-r-\nu+1} \binom{2\mu+1}{\mu-\nu} \zeta(2\mu+1). \quad (\text{A.7})$$

Finally, instead of using the integral representation (A.1), we may also cast the two kernels \hat{K}_0, \hat{K}_1 in the form of infinite sums¹³:

$$\begin{aligned} \hat{K}_0(t, t') &= \frac{2}{tt'} \sum_{n=0}^{\infty} (2n+1) J_{2n+1}(t) J_{2n+1}(t'), \\ \hat{K}_1(t, t') &= \frac{2}{tt'} \sum_{n=1}^{\infty} (2n) J_{2n}(t) J_{2n}(t'). \end{aligned}$$

It follows that

$$\hat{K}_c(t, t') = \frac{2}{tt'} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (-1)^{k+l} c_{2k+1, 2l+2}(g) J_{2k}(t) J_{2l+1}(t') \quad (\text{A.8})$$

with the coefficient functions

$$c_{r,s}(g) = 2 \cos \left[\frac{1}{2} \pi (s-r-1) \right] (r-1)(s-1) \int_0^{\infty} dt \frac{J_{r-1}(2gt) J_{s-1}(2gt)}{t(e^t-1)}. \quad (\text{A.9})$$

We can use this form to present an alternative proof. We firstly expand $1/(e^t-1)$ into a geometric series and secondly rescale the integration variable t by g such that the integral will manifestly depend on the ratio g/n only

$$\begin{aligned} c_{r,s}(g) &= \sum_{n=1}^{\infty} h_{r,s}(g/n), \\ h_{r,s}(g/n) &= 2 \cos \left[\frac{1}{2} \pi (s-r-1) \right] (r-1)(s-1) \\ &\quad \times \int_0^{\infty} \frac{dt}{t} \exp(-tn/g) J_{r-1}(2t) J_{s-1}(2t). \end{aligned} \quad (\text{A.10})$$

¹³ This result was first obtained by Serban [60].

The integral can now be performed and yields precisely the hypergeometric function derived earlier in (3.25).

The discussion of the strong-coupling behaviour of the dressing phase is tantamount to an analysis of the strong-coupling limit of the integrals $c_{r,s}(g)$. In appendix C we present, by way of example, the full expansion in the case $(r, s) = (2, 3)$. We also show that (A.10) reproduces the AFS phase [20] as well as the Hernández–López correction [34].

Appendix B. Flipping odd-zeta contributions

Here we show that the replacement of the kernel $\hat{K}_m \rightarrow \hat{K}_m + 2\hat{K}_c$ leads to the simple replacement $\zeta(2n + 1) \rightarrow i\zeta(2n + 1)$ in the scaling function $f_m(g) \rightarrow f(g)$.

Let

$$z(t) = \frac{t}{e^t - 1}. \tag{B.1}$$

For any kernel $\hat{K}(2gt, 2gt')$ and any function $f(t)$ we define the product

$$(\hat{K} * f)(t) = \int_0^\infty dt' \hat{K}(2gt, 2gt') z(t') f(t'). \tag{B.2}$$

The integral equation may be written as

$$\hat{\sigma}(t) = z(t) \left((\hat{K}_0 + \hat{K}_c)(2gt, 0) - 4g^2 \left((\hat{K}_0 + \hat{K}_1 + \hat{K}_c) * \frac{\hat{\sigma}}{z} \right) (t) \right). \tag{B.3}$$

In the following we need not consider the internal g dependence of the kernels. Within the radius of convergence of the iteration the solution of any Fredholm equation of the second type is the resolvent

$$\hat{\sigma}(t) = z(t) \sum_{n=0}^\infty (-4g^2)^n [(\hat{K}_0 + \hat{K}_1 + \hat{K}_c) *]^n P(t). \tag{B.4}$$

Here $P(t) = (\hat{K}_0 + \hat{K}_c)(2gt, 0)$ and $\hat{K}_c(2gt, 2gt') = 8g^2(\hat{K}_1 * \hat{K}_0)(2gt, 2gt')$. The Fredholm resolvent must have the form

$$\hat{\sigma}(t) = z(t) \sum_{\underline{n}} (-4g^2)^n \eta_{\underline{n}} (W_{\underline{n}} \hat{K}_0)(t) \tag{B.5}$$

because the dressing kernel \hat{K}_c consists itself of \hat{K}_0, \hat{K}_1 which are convoluted by the star operation (and it also carries the appropriate extra power of the coupling constant). By \underline{n} we denote a string built of 0, 1 with total length n , and $W_{\underline{n}}$ is the corresponding string of \hat{K}_0, \hat{K}_1 acting on the potential \hat{K}_0 by the $*$ product defined above in (B.2). The $\eta_{\underline{n}}$ are numerical coefficients. The integral equation (B.3) determines these numbers uniquely: We substitute (B.5) into the equation and pick the $\mathcal{O}(g^{2(n+2)})$ terms. All words are assumed independent; in particular the four double iterations of an arbitrary word $W_{\underline{n}}$:

$$W_{0,0,\underline{n}}, \quad W_{0,1,\underline{n}}, \quad W_{1,0,\underline{n}}, \quad W_{1,1,\underline{n}}. \tag{B.6}$$

This yields the four equations

$$\begin{aligned}\eta_{0,0,\underline{n}} &= -\eta_{0,\underline{n}}, \\ \eta_{0,1,\underline{n}} &= -\eta_{1,\underline{n}}, \\ \eta_{1,0,\underline{n}} &= -\eta_{0,\underline{n}} - 2\eta_{\underline{n}}, \\ \eta_{1,1,\underline{n}} &= -\eta_{1,\underline{n}}.\end{aligned}$$

The first two equations may be summarized as

$$\eta_{0,\underline{m}} = -\eta_{\underline{m}} \tag{B.7}$$

from which it follows that the third equation is:

$$\eta_{1,0,\underline{n}} = +\eta_{0,\underline{n}}. \tag{B.8}$$

Recursively, these equations imply that all the η coefficients are merely signs. Second, (B.8) implies that the operation of appending \hat{K}_1 after \hat{K}_0 is the only one that induces a sign flip as opposed to the iteration of the main scattering kernel $\hat{K}_m = \hat{K}_0 + \hat{K}_1$ alone (in that case the second term in the rhs of (B.9) would be absent so that $\eta_{1,\underline{m}} = -\eta_{\underline{m}}$, too).

Next, let

$$\mathcal{E} = \{t^{2l} : l \in \mathbb{N}_0\}, \quad \mathcal{O} = \{t^{2l+1} : l \in \mathbb{N}_0\}, \tag{B.9}$$

denote the even and odd powers of t , respectively. In order to obtain the scaling functions at weak coupling we will eventually expand the kernels $\hat{K}_{0,1}(2gt, 2gt')$ in the coupling constant; \hat{K}_0 (\hat{K}_1) is even (odd) in both arguments, respectively. Let us consider any one term in the expansion of a kernel convoluted on a power of t by the star product. We observe the graded structure

$$\begin{aligned}\hat{K}_0 : \mathcal{E} &\rightarrow \mathcal{E} : t^{2l} \mapsto \zeta(2m) t^{2l}, \\ &\mathcal{O} \rightarrow \mathcal{E} : t^{2l+1} \mapsto \zeta(2m+1) t^{2l}, \\ \hat{K}_1 : \mathcal{E} &\rightarrow \mathcal{O} : t^{2l} \mapsto \zeta(2m+1) t^{2l+1}, \\ &\mathcal{O} \rightarrow \mathcal{O} : t^{2l+1} \mapsto \zeta(2m) t^{2l+1}.\end{aligned} \tag{B.10}$$

The first integral in a chain of convolutions always acts on an element of \mathcal{E} because the potential is even. Likewise, the energy integral adds a final convolution on the even kernel $\hat{K}_0(0, 2gt)$. The contribution of a word $W_{\underline{n}}$ to the scaling function thus contains one odd-zeta function for each beginning and each end of a \hat{K}_1 string in $\hat{K}_0 * W_{\underline{n}} * \hat{K}_0$. In particular, if there are m such strings, the word gives a contribution with $2m$ odd-zeta functions.

Finally, the equations (B.7) and (B.8) say that each creation of a \hat{K}_1 string induces a sign change as opposed to the iteration of the main scattering kernel on its own. Our result is: the scaling functions for both cases contain terms with $2m$ odd-zeta functions. The relative sign of such terms is $(-1)^m$.

Note that the absolute sign of the terms with odd-zeta functions cannot easily be fixed without explicit computation. Words with n insertions of \hat{K}_1 pick up a factor $(-1)^n$ from the g^2 expansion relative to those with \hat{K}_0 alone. Presumably, words with the minimal number of \hat{K}_1 kernels give the dominating contribution: \hat{K}_1 starts at $\mathcal{O}(g^2)$, thereby restricting the number of ways one may choose the powers in the various kernels in a

chain, that is reducing the number of similar terms. If this is so, the iteration of the main scattering kernel should indeed produce a factor of $(-1)^m$ for a term with $2m$ odd-zeta functions relative to the leading purely even part. Consequently, the dressing kernel ought to align all signs.

Appendix C. Strong-coupling expansion

In this appendix we give an exact derivation of the strong-coupling expansion of $c_{2,3}(g)$. We modify a technique developed in [61] for the discussion of the ground state energy of the half-filled Hubbard model.

According to equation (A.10),

$$c_{2,3}(g) = 4 \sum_{n=1}^{\infty} \int_0^{\infty} dy \frac{J_1(2gy) J_2(2gy)}{y} e^{-ny} = \sum_{n=1}^{\infty} i f' \left(\frac{in}{4g} \right) \quad (C.1)$$

with

$$\begin{aligned} f'(z) &= 4z - 4z {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2} \right) - \frac{1}{2z} {}_2F_1 \left(\frac{1}{2}, \frac{3}{2}; 3; z^{-2} \right), \\ f(z) &= 2z^2 \left(1 - {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}; 2; z_n^{-2} \right) \right). \end{aligned} \quad (C.2)$$

We want to use the residue theorem to express the sum as a contour integral. For any holomorphic function $f(z)$

$$\text{res}_{in/4g} \frac{f(z)}{\sinh^2(4\pi gz)} = \frac{1}{(4\pi g)^2} f' \left(\frac{in}{4g} \right) \quad (C.3)$$

because the terms in which the derivative does not fall on $f(z)$ cancel out¹⁴. The integral is thus written as

$$c_{2,3}(g) = \frac{(4\pi g)^2}{2\pi} \int_C \frac{dz f(z)}{\sinh^2(4\pi gz)} = \int_C \frac{16\pi g^2 dz z^2}{\sinh^2(4\pi gz)} \left(1 - {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2} \right) \right). \quad (C.4)$$

The region encircled by the contour C contains all the points $in/4g$ up to $n < 4g/\epsilon$ and $f(z)$ must be holomorphic on it. Let $\epsilon > 0$ be infinitesimal. We shall consider integration over the closed contour, see figure C.1

$$\begin{aligned} C &= C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5, \\ C_1 &= (1/\epsilon)e^{i[0,\pi]}, \\ C_2 &= [-1/\epsilon, -1 - \epsilon] \cup [+1 + \epsilon, +1/\epsilon], \\ C_3 &= (-1 + \epsilon e^{i[0,\pi]}) \cup (+1 + \epsilon e^{i[0,\pi]}), \\ C_4 &= [-1 + \epsilon, -\epsilon] \cup [+ \epsilon, +1 - \epsilon], \\ C_5 &= \epsilon e^{i[0,\pi]}, \end{aligned} \quad (C.5)$$

Note that the hypergeometric function in $f(z)$ has singularities at $z = \pm 1$ and we shall deform the standard cut on the interval $[-1, 1]$ such that it lies below the contour C_4 .

¹⁴ The original article [61] uses a simple \sinh in the denominator. This would lead to an alternating sum of residues, though.

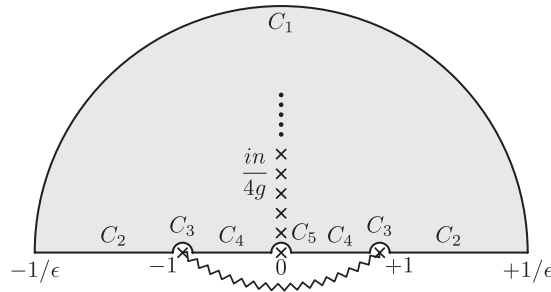


Figure C.1. Contour C and its decomposition (C.5).

Let us first consider the integral over the large semicircle C_1 . Here we have

$$f(z) = \frac{1}{4} + \mathcal{O}(1/z) \tag{C.6}$$

and for large enough $1/\epsilon$ we find (neglecting corrections in ϵ)

$$8\pi g^2 \int_{C_1} \frac{dz f(z)}{\sinh^2(4\pi g z)} = 2\pi g^2 \int_{C_1} \frac{dz}{\sinh^2(4\pi g z)} = g \coth(4\pi g/\epsilon) = g = c_{2,3}^{(0)} g. \tag{C.7}$$

The first term in $f(z)$ leads to no singularity on the real axis and can be integrated straightforwardly over the remaining contours

$$\lim_{m \rightarrow \infty} \int_{C_2 \cup C_3 \cup C_4 \cup C_5} \frac{16\pi g^2 dz z^2}{\sinh^2(4\pi g z)} = \int_{-\infty}^{\infty} \frac{16\pi g^2 dz z^2}{\sinh^2(4\pi g z)} = \frac{1}{12g} = \frac{c_{2,3}^{(2)}}{g}. \tag{C.8}$$

For very small z the remaining hypergeometric function goes as

$${}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2}\right) \sim \frac{4i}{3\pi z} \tag{C.9}$$

so that the semicircle C_5 at the origin contributes

$$-\int_{C_5} \frac{16\pi g^2 dz z^2}{\sinh^2(4\pi g z)} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2}\right) = -\frac{4}{3\pi} = c_{2,3}^{(1)}. \tag{C.10}$$

Since the integrand is an even function of z , the other three pieces add up to the real part of an integral along the positive semi-axis only:

$$\begin{aligned} H(g) &= -16\pi g^2 \int_{C_2 \cup C_3 \cup C_4} \frac{dz z^2}{\sinh^2(4\pi g z)} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2}\right) \\ &= -32\pi g^2 \operatorname{Re} \int_0^{\infty} \frac{dz z^2}{\sinh^2(4\pi g z)} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2}\right). \end{aligned} \tag{C.11}$$

The real part of the hypergeometric function is easily read off from its one-parameter representation

$$\operatorname{Re} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; z^{-2}\right) = \frac{4}{\pi} \int_0^{\min(z,1)} dt \sqrt{1-t^2} \sqrt{1-t^2 z^{-2}}. \tag{C.12}$$

In $H(g)$ we readily estimate the contribution from $z > 1$ to be of order $\mathcal{O}(e^{-g})$ due to the suppression by $1/\sinh(4\pi gz)^2$. The remainder is

$$H(g) = -128g^2 \int_0^1 \frac{dz z^2}{\sinh^2(4\pi gz)} \int_0^z dt \sqrt{1-t^2} \sqrt{1-t^2 z^{-2}} + \mathcal{O}(e^{-g}). \quad (\text{C.13})$$

To deal with these integrals we successively

- expand $\sqrt{1-t^2}$ into a series in t

$$\sqrt{1-t^2} = -\frac{1}{2\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{1}{2})}{\Gamma(m+1)} t^{2m}, \quad (\text{C.14})$$

- calculate the t integral in each term

$$\int_0^z dt t^{2m} \sqrt{1-t^2 z^{-2}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+2)} z^{1+2m}, \quad (\text{C.15})$$

- evaluate the z integral. Here we may extend the range to the whole positive semi-axis since the error is again seen to be $\mathcal{O}(e^{-g})$

$$\int_0^1 \frac{dz z^{2m+3}}{\sinh^2(4\pi gz)} = \int_0^{\infty} \frac{dz z^{2m+3}}{\sinh^2(4\pi gz)} + \mathcal{O}(e^{-g}) = \frac{4\Gamma(2m+4)\zeta(2m+3)}{(8\pi g)^{2m+4}} + \mathcal{O}(e^{-g}). \quad (\text{C.16})$$

On collecting terms:

$$\begin{aligned} H(g) &= \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{1}{2})\Gamma(m+\frac{1}{2})\Gamma(2m+4)}{\Gamma(m+1)\Gamma(m+2)} \frac{\zeta(2m+3)}{\pi^2(8\pi g)^{2m+2}} + \mathcal{O}(e^{-g}) \\ &= \sum_{n=3}^{\infty} \frac{2\zeta(n)}{(-2\pi)^n \Gamma(n-1)} \frac{\Gamma[\frac{1}{2}(n+2)]\Gamma[\frac{1}{2}n]}{\Gamma[\frac{1}{2}(6-n)]\Gamma[\frac{1}{2}(4-n)]} g^{1-n} + \mathcal{O}(e^{-g}) \\ &= \sum_{n=3}^{\infty} c_{2,3}^{(n)} g^{1-n} + \mathcal{O}(e^{-g}). \end{aligned} \quad (\text{C.17})$$

The second line is obtained by using some identities of gamma functions and setting $n = 2m + 3$. In conclusion:

$$c_{2,3}(g) = c_{2,3}^{(0)} g + c_{2,3}^{(1)} + \frac{c_{2,3}^{(2)}}{g} + H(g) = \sum_{m=0}^{\infty} c_{2,3}^{(m)} g^{1-m} + \mathcal{O}(e^{-g}). \quad (\text{C.18})$$

We proved the correctness of the strong-coupling expansion coefficients $c_{r,s}^{(m)}$ from equation (3.4) (see also (3.23)) for the case $(r, s) = (2, 3)$. The role of the zeta function regularization used in the main text is simply to dispose of the terms of order $\mathcal{O}(e^{-g})$ that become visible in this proof.

The argument presented in this appendix should remain applicable for generic values of (r, s) . We hope to return to this general case in future work. We can, however, perform the analysis for the first two orders. By Euler–MacLaurin summation (or zeta function

regularization), the leading two orders of $c_{r,s}(g)$ take the form

$$c_{r,s}^{(0)} = \int_0^\infty h_{r,s}(1/z) dz, \quad c_{r,s}^{(1)} = \zeta(0)h_{r,s}(\infty). \quad (\text{C.19})$$

In fact, one can show that the above argument yields the same answers. The integral of $h_{r,s}$ in the form (3.25) can be performed in closed form for all (r, s)

$$c_{r,s}^{(0)} = \frac{8(r-1)(s-1)\sin[\pi(r-s)]}{\pi(s-r-1)(s-r+1)(s+r-3)(s+r-1)} = \delta_{r+1,s} \quad (\text{C.20})$$

and one obtains full agreement with the AFS phase (3.5) [20] using that r, s are integers with $r \geq s + 1$. For the next order one can use the representation (A.10) of $h_{r,s}(z)$ at $z = \infty$

$$\begin{aligned} c_{r,s}^{(1)} &= \zeta(0)2 \cos\left[\frac{1}{2}\pi(s-r-1)\right] (r-1)(s-1) \int_0^\infty \frac{dt}{t} J_{r-1}(2t) J_{s-1}(2t) \\ &= -\cos\left[\frac{1}{2}\pi(s-r-1)\right] (r-1)(s-1) \frac{2 \cos[\frac{1}{2}\pi(s-r-1)]}{\pi(s-r)(s+r-2)} \\ &= -\frac{1 - (-1)^{r+s}}{\pi} \frac{(r-1)(s-1)}{(s-r)(s+r-2)}. \end{aligned} \quad (\text{C.21})$$

This is again in full agreement with the HL phase (3.5) [34].

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