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Experimentally feasible purification of continuous-variable entanglement

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We propose a scheme for purification and distillation of squeezed and entangled continuous-variable states of light transmitted through a channel exhibiting phase fluctuations. The outstanding advantage of the suggested protocol is its experimental feasibility since it only involves an interference of two copies of the decohered state on a balanced beam splitter, a homodyne measurement on one of the output beams, and a conditioning on the measurement outcome. The purification can counteract the detrimental effects of phase fluctuations in optical quantum-communication networks.

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Quantum-information processing with continuous variables [1] exploits the continuous degrees of freedom of physical systems such as quadratures of a mode of the electromagnetic field or a collective spin of an atomic ensemble to encode, transmit, and process quantum information. Continuous variables offer several important advantages because many tasks such as preparation of entangled states [2], Bell measurement, quantum teleportation [3], entanglement swapping [4], and dense coding [5] can be implemented in a deterministic unconditional way using squeezed Gaussian states and Gaussian operations which can be easily implemented with the help of optical parametric oscillators, beam splitters, phase shifters, and homodyne detectors. However, this approach also suffers from some limitations. Most importantly, it is not possible to purify and distill the entanglement of Gaussian states by means of the experimentally feasible local Gaussian operations [6]. The purpose of entanglement purification and distillation is to extract from many copies of weakly entangled mixed states a single copy of a highly entangled state by means of local operations and classical communication between the two parties which share the entangled states [7,8]. This procedure is crucial for suppression of losses, noise, and decoherence which inevitably arise in the distribution of the entanglement and squeezing over realistic channels such as optical fibers. Several methods were proposed to purify entanglement using non-Gaussian operations which yield non-Gaussian states from the initial Gaussian ones [9–13]. One can then drive the resulting non-Gaussian states to Gaussian states via Gaussian operations [12,13]. A first pioneering experimental step in this direction has been achieved by degaussifying a single-mode [14,15] and two-mode [16] squeezed vacuum by subtracting a single photon from it.

In this paper we propose a simple scheme for purification of mixed *non-Gaussian* squeezed and entangled continuous variable states. The protocol relies on Gaussian operations utilizing well-established technology, and will overcome random phase fluctuations in envisioned long-distance quantum-communication networks. Moreover, sources of squeezed and entangled states may suffer from *inherent* phase fluctuations which may be suppressed by our protocol, whereby improving the properties of nonclassical sources.

Our purification procedure exploits the interference of two copies of the state on a balanced beam splitter [12,13] followed by balanced homodyne detection on one of the output beams and conditioning on the measurement outcome. The scheme does not require any non-Gaussian operations or measurements which renders the method experimentally feasible [17]. Our protocol particularly differs from the Gaussification scheme recently proposed by Browne *et al.* [12]. They assumed conditioning on projection onto the vacuum state, while we suggest to condition on outcomes of quadrature measurements, which is much easier to implement and is particularly well suited for the purpose of purification of phase-diffused squeezed states [17]. Our protocol can actually purify a wide class of non-Gaussian states and we choose here the phase-diffused states as a simple yet interesting example to illustrate the performance of the scheme. Our results reveal that for certain decoherence scenarios there can exist alternative simpler purification procedures which do not require non-Gaussian operations.

We will first describe the protocol for the purification of phase-diffused single-mode squeezed states and then we will extend it in a straightforward manner to the entangled phase-diffused two-mode squeezed vacuum states distributed to two distant parties. Consider a single-mode squeezed state whose Wigner function reads $W_{\mathrm{SMS}}(x,p)=\frac{1}{2\pi\sqrt{V_xV_p}}\exp\left[-\frac{x^2}{2V_x}-\frac{p^2}{2V_p}\right]$, where V_x and V_p are variances of the x and p quadratures, respectively, with $V_xV_p\geq\frac{1}{4}$. Assuming $V_x\leq V_p$, the state is squeezed if $V_x<1/2$. Under the influence of the random phase fluctuations this state will evolve into a mixed generally non-Gaussian state with the Wigner function given by

$$W(x,p)=\frac{1}{2\pi\sqrt{V_xV_p}}\int_{-\infty}^{\infty}\exp\left[-\frac{x_\phi^2}{2V_x}-\frac{p_\phi^2}{2V_p}\right]\Phi(\phi)d\phi, \tag{1}$$

where $x_\phi=x\cos\phi+p\sin\phi$, $p_\phi=p\cos\phi-x\sin\phi$, and $\Phi(\phi)$ denotes the probability distribution of the random phase shift, $\int\Phi(\phi)d\phi=1$. In the following we will assume Gaussian fluctuations of ϕ, $\Phi(\phi)=(2\pi\sigma^2)^{-1/2}\exp\left(-\frac{\phi^2}{2\sigma^2}\right)$, but our

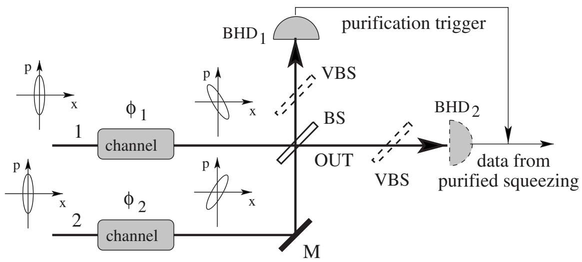

FIG. 1. Proposed scheme for purification of dephased single-mode squeezed vacuum states. BS—balanced beam splitter; BHD—balanced homodyne detectors; VBS—virtual beam splitters modeling imperfect homodyning. Note that an alternative similar but less-practical scheme where conditioning on projections onto vacuum state is employed instead of balanced homodyning is considered in Refs. [12,13].

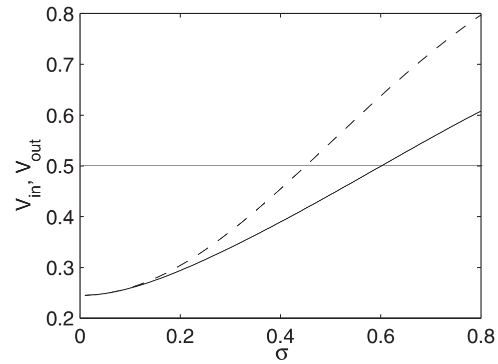

FIG. 2. Variances of the quadrature x versus strength of the phase noise before (V_{in}, dashed line) and after purification (V_{out}, solid line); $V_x=0.2$, $V_p=2$, $\eta=0.85$, and $X=0.45$; the vacuum noise reference is given by the horizontal line.

results are largely independent of the particular form of the fluctuations and qualitatively hold for any $\Phi(\phi)$.

The proposed setup for the purification of the state in Eq. (1) is depicted in Fig. 1. The purification requires two copies of the state which are combined on a balanced beam splitter (BS). Subsequently, the x quadrature of one output mode is measured in a balanced homodyne detector (BHD_1). The purification is a probabilistic operation which succeeds if the absolute value of the measured quadrature falls below certain trigger threshold X. We shall show that this procedure improves the squeezing of the x quadrature of the beam in the second output port of BS. To verify this improvement, we suggest to measure the x quadrature of the second beam by another balanced homodyne detector (BHD_2). For phase-diffused states (1) the purified state can again be expressed as a statistical mixture of Gaussian states and its Wigner function is positive definite.

Assume for a moment that the random phase shifts of each copy attained some particular values ϕ_1 and ϕ_2. In this case, the initial joint probability distribution of the input quadratures x_1^{in} and x_2^{in} before mixing on a BS can be expressed as

$$P(x_1^{\mathrm{in}},x_2^{\mathrm{in}}) = \frac{1}{2\pi\sqrt{V_1V_2}}\exp\left[-\frac{(x_1^{\mathrm{in}})^2}{2V_1}-\frac{(x_2^{\mathrm{in}})^2}{2V_2}\right], \tag{2}$$

where $V_1=V_x\cos^2\phi_1+V_p\sin^2\phi_1$, $V_2=V_x\cos^2\phi_2+V_p\sin^2\phi_2$. The x quadratures after combining the modes on a BS read $x_1=\frac{1}{\sqrt{2}}(x_1^{\mathrm{in}}+x_2^{\mathrm{in}})$, $x_2=\frac{1}{\sqrt{2}}(x_2^{\mathrm{in}}-x_1^{\mathrm{in}})$. In our model we take into account the imperfect homodyning with efficiency $\eta<1$ by introducing a virtual beam splitter of transmittance η into the path of each beam impinging on a homodyne detector, see Fig. 1. The joint probability distribution of the x quadratures after mixing on BS and after passing through the virtual beam splitters has the form

$$\widetilde{P}(x_1,x_2) = \frac{1}{2\pi\sqrt{A^2-B^2}}\exp\left[-\frac{A(x_1^2+x_2^2)-2Bx_1x_2}{2(A^2-B^2)}\right], \tag{3}$$

where $A=\frac{\eta}{2}(V_1+V_2)+\frac{1-\eta}{2}$, $B=\frac{\eta}{2}(V_2-V_1)$. The (unnormalized) probability distribution of the quadrature x_2 conditional on $|x_1|<X$ is given by $P_{\mathrm{cond}}(x_2)=\int_{-X}^{X}\widetilde{P}(x_1,x_2)dx_1$. The resulting distribution of the quadrature x_2 of the conditionally purified state can be obtained by averaging $P_{\mathrm{cond}}(x_2)$ over the random phase shifts ϕ_1 and ϕ_2 (we assume that these phase shifts are independent),

$$P_{\mathrm{out}}(x_2) = \frac{1}{\mathcal{P}}\int_{\phi_1}\int_{\phi_2}P_{\mathrm{cond}}(x_2)\Phi(\phi_1)\Phi(\phi_2)d\phi_1d\phi_2. \tag{4}$$

Here the normalization factor $\mathcal{P}$ denotes the probability of positive trigger events from BHD_1,

$$\mathcal{P} = \int_{\phi_1}\int_{\phi_2}\mathrm{erf}\left(\frac{X}{\sqrt{2A}}\right)\Phi(\phi_1)\Phi(\phi_2)d\phi_1d\phi_2. \tag{5}$$

The squeezing of the output quadrature x_2 is most conveniently characterized by its variance. It can be easily seen from Eqs. (3) and (4) that $\langle x_2\rangle=0$, hence the variance V_{out} measured with the use of BHD_2 is simply equal to $\langle x_2^2\rangle$ and after some algebra we obtain

$$V_{\mathrm{out}} = \frac{1}{\mathcal{P}}\int_{\phi_1}\int_{\phi_2}\left[A\,\mathrm{erf}\left(\frac{X}{\sqrt{2A}}\right)-\sqrt{\frac{2}{\pi}}\frac{B^2X}{A^{3/2}}e^{-X^2/2A}\right] \times\Phi(\phi_1)\Phi(\phi_2)d\phi_1d\phi_2. \tag{6}$$

This should be compared with the variance of x quadrature of the initial dephased state (1) before purification which would be observed in a homodyne detection with efficiency η,

$$V_{\mathrm{in}} = \eta\int_{\phi}(V_x\cos^2\phi+V_p\sin^2\phi)\Phi(\phi)d\phi+\frac{1}{2}(1-\eta). \tag{7}$$

The results of numerical calculations are shown in Fig. 2 for experimentally achievable set of parameters $V_x=0.2$, corresponding to 4 dB squeezing, $V_p=2$, corresponding to 6 dB antisqueezing, and $\eta=0.85$ [18]. Note that the squeezed state is mixed accounting for optical loss during squeezed-state generation. We can see that the purification clearly enhances the squeezing and $V_{\mathrm{out}}<V_{\mathrm{in}}$. We can also see that the purification effect becomes more pronounced for stronger phase

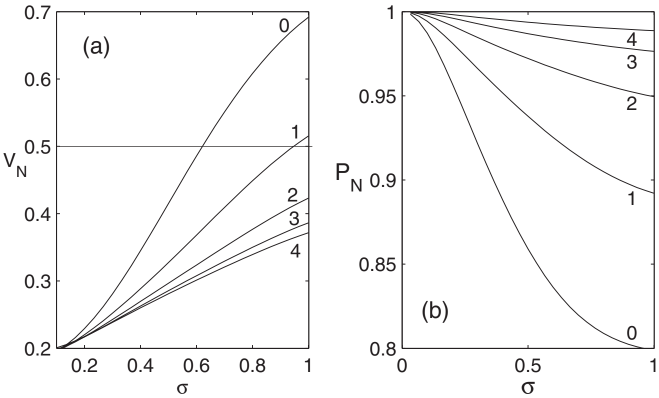

FIG. 3. Iterative purification protocol. Variance (a) and purity (b) of the state after N purification steps are plotted versus σ ($N=0$ corresponds to the original dephased state). The results were obtained for $V_x=e^{-2r}/2$, $V_p=e^{2r}/2$, $r=0.5$, $\eta=1$, and conditioning on $x_1=0$. This corresponds to the limit of very a narrow acceptance window, $X\to 0$, where the protocol exhibits optimum performance.

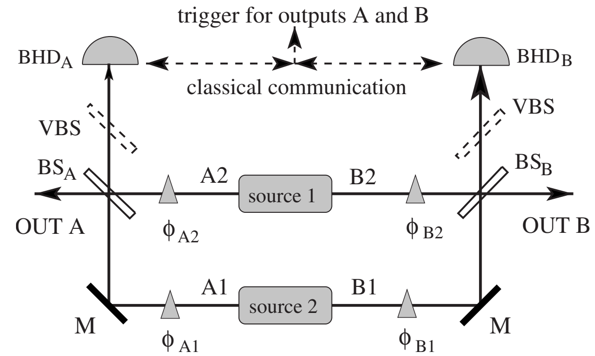

FIG. 4. Scheme for the purification of dephased continuous variable entangled states by means of local operations and classical communication.

fluctuations and in our example the quadrature fluctuations can be reduced by more than 1 dB. Note that the present purification procedure cannot reduce V_{out} below the squeezed variance V_x of the original Gaussian state before transmission through the noisy channel.

The main mechanism of the purification can be understood by realizing that by measuring $x_1=\frac{1}{\sqrt{2}}(x_1^{\mathrm{in}}+x_2^{\mathrm{in}})$ we probe the phase fluctuations in the two channels. If x_1 is close to zero then with high probability both x_1^{in} and x_2^{in} were squeezed and, consequently, the random phase shifts ϕ_1 and ϕ_2 were small. On the other hand, large values of x_1 indicate that a large fraction of noisy antisqueezed quadrature p_1^{in} (or p_2^{in}) was admixed to x_1^{in} (x_2^{in}) due to a large phase shift ϕ_1 (ϕ_2). These events, which reduce the squeezing, are (partially) suppressed by the conditioning on $|x_1|\leq X$. Our numerical analysis suggests that it is suitable to choose $X\approx\sqrt{V_x}$, which results in good squeezing enhancement while at the same time the probability of success is of the order of 50%.

The purification procedure can be, similarly as other such methods [13], applied iteratively in order to distill from many copies of mixed phase-diffused and weakly squeezed state a single copy of strongly squeezed almost pure state. In the iterative purification, two output states produced by two successful purifications are used as an input in the next purification step, which can be formally described by a map $\rho_{N+1}=\mathcal{E}_P(\rho_N^{\otimes 2})$. We have numerically simulated the iterative purification and the results are shown in Fig. 3. We can see that each iteration increases the squeezing of the state. Moreover, the purification also improves the purity of the state $P=\mathrm{Tr}[\rho^2]$. A highly pure state is obtained after few iteration steps, see Fig. 3(b). Our procedure thus meets all criteria imposed on a proper purification protocol. A detailed analysis of the full iterative purification procedure including iterative purification of entangled two-mode squeezed states is beyond the scope of the present paper and will be reported in a separate publication.

Let us now consider the entanglement purification of two-mode squeezed vacuum state, $|\psi_{\mathrm{TMS}}\rangle_{AB}=\sqrt{1-\lambda^2}\Sigma_{n=0}^{\infty}\lambda^n|n,n\rangle_{A,B}$, where n denotes photon number, $\lambda=\tanh r$, and r the squeezing constant. We assume that during the distribution of the state $|\psi_{TMS}\rangle_{AB}$ from the source to the two parties Alice and Bob each mode undergoes random phase shift ϕ_A and ϕ_B, see Fig. 4. However, due to the peculiar structure of the pure state $|\psi_{\mathrm{TMS}}\rangle_{AB}$ it is only the phase sum $\phi=\phi_A+\phi_B$ which matters. The purification procedure which we suggest requires only local operations on two copies of the decohered state and classical communication. Alice and Bob each combine the two modes which they possess (A_1, A_2 and B_1, B_2, respectively) on a balanced beam splitter and they measure the x quadrature of output mode, i.e., Alice measures $X_A=\frac{1}{\sqrt{2}}(x_{A1}^{\mathrm{in}}-x_{A2}^{\mathrm{in}})$ and Bob observes $X_B=\frac{1}{\sqrt{2}}(x_{B1}^{\mathrm{in}}-x_{B2}^{\mathrm{in}})$. Alice and Bob exchange their measurement results via classical communication channel and evaluate the difference $\delta X=\frac{1}{\sqrt{2}}(X_A-X_B)$. The purification succeeds if $|\delta X|\leq X$ and fails otherwise.

The two-mode squeezed vacuum state exhibits squeezing of the two commuting quadratures $x_-=\frac{1}{\sqrt{2}}(x_A-x_B)$ and $p_+=\frac{1}{\sqrt{2}}(p_A+p_B)$. The entanglement of the Gaussian state $|\psi_{\mathrm{TMS}}\rangle_{AB}$ can be fully quantified by the Einstein-Podolsky-Rosen (EPR) uncertainty

$$\Delta_{\mathrm{EPR}}=\langle(\Delta x_-)^2\rangle+\langle(\Delta p_+)^2\rangle. \tag{8}$$

If $\Delta_{\mathrm{EPR}}<1$ then the state is entangled [19]. Although for non-Gaussian states Δ_{EPR} is strictly speaking not an entanglement measure, it quantifies the degree of nonlocal correlations between modes A and B and can be easily measured experimentally. In our model we assume that Δ_{EPR} is measured using balanced homodyning with efficiency η and we evaluate Δ_{EPR} that would be observed in the experiment.

In the course of purification, both variances $\langle(\Delta x_-)^2\rangle$ and $\langle(\Delta p_+)^2\rangle$ are reduced. Let us first consider x_-. The quadratures $x_{-,1}$ and $x_{-,2}$ of the two copies of the phase-diffused two-mode squeezed vacuum nonlocally interfere on the two balanced beam splitters BS_A and BS_B and the measured difference δX can be interpreted as the measurement of $x_{-,1}^{\mathrm{in}}-x_{-,2}^{\mathrm{in}}$. Hence for the quadrature x_- the scheme is totally equivalent to the purification of the single-mode squeezed state as shown in Fig. 1 and the variance of x_- after the purification can be directly calculated from Eq. (6) where the two independent phase shifts now read $\phi_1=(\phi_{A1}+\phi_{B1})/2$, $\phi_2=(\phi_{A2}+\phi_{B2})/2$, $V_x=\frac{1}{2}e^{-2r}$, and $V_p=\frac{1}{2}e^{2r}$.

Let us now consider the output quadrature p_+. For fixed phase shifts ϕ_1 and ϕ_2, both p_+ and the measured δX exhibit Gaussian distribution centered on origin with variance A. Due to the peculiar structure of the state $|\psi_{\mathrm{TMS}}\rangle_{AB}$ the distributions of δX and p_+ are not correlated. The condition $|\delta X| \leq X$ will be satisfied with probability $\mathrm{erf}(X/\sqrt{2A})$ and the resulting variance of p_+ after the purification can be obtained as a properly normalized average of A over the probability distributions of the random phase shifts,

$$\langle(\Delta p_+^{\mathrm{out}})^2\rangle = \frac{1}{\mathcal{P}}\int_{\phi_1}\int_{\phi_2} A\,\mathrm{erf}\left(\frac{X}{\sqrt{2A}}\right)\Phi(\phi_1)\Phi(\phi_2)d\phi_1 d\phi_2, \quad (9)$$

where the probability of successful entanglement purification $\mathcal{P}$ is given by Eq. (5). Note that the function $\mathrm{erf}(X/\sqrt{2A})$ in Eq. (9) acts as a filter which suppresses large A contributions to $\langle(\Delta p_+^{\mathrm{out}})^2\rangle$ and consequently the purification reduces the variance of p_+.

The reduction of the EPR uncertainty by our entanglement purification protocol is illustrated in Fig. 5(a). The results are similar to the improvement of squeezing in the single-mode version of our protocol. To rigorously prove that the purification increases the entanglement we have also numerically evaluated the logarithmic negativity $E_{\mathcal{N}}=\log_2\|\rho_{AB}^{T_B}\|_1$ where $\|\cdot\|_1$ denotes the trace norm and T_B indicates partial transposition with respect to the subsystem B [20]. $E_{\mathcal{N}}$ is an entanglement measure that provides an upper bound on the distillable entanglement [20] and a lower bound on the PPT-entanglement cost for the exact preparation of the state [21]. The calculations were done in the Fock basis similarly as in Ref. [22]. The results are shown in Fig. 5(b) and clearly confirm that the purification increases the entanglement.

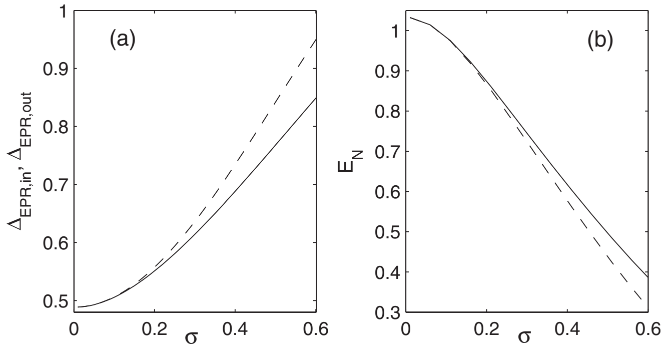

FIG. 5. (a) The EPR uncertainty versus strength of phase fluctuations, before purification ($\Delta_{\mathrm{EPR,in}}$, dashed line) and after ($\Delta_{\mathrm{EPR,out}}$, solid line); $r=0.46$, $\eta=0.85$; (b) logarithmic negativity of the dephased state (dashed line) and purified state (solid line).

To conclude, we note that our proposed scheme cannot counteract the effect of losses which is one of the dominating decoherence sources in optical fibers. However, if combined with a single de-Gaussifying operation such as recently demonstrated photon subtraction [14–16], it would provide a generic continuous variable (CV) entanglement purification and distillation scheme capable of also suppressing the effect of losses [12,13]. We therefore expect that our method will play an important role in the further development of the continuous variable quantum-communication networks.

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