

## LETTER TO THE EDITOR

## Negative Komar mass of single objects in regular, asymptotically flat spacetimes

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### Abstract

We study two types of axially symmetric, stationary and asymptotically flat spacetimes using highly accurate numerical methods. One type contains a black hole surrounded by a perfect fluid ring and the other a rigidly rotating disc of dust surrounded by such a ring. Both types of spacetime are regular everywhere (outside of the horizon in the case of the black hole) and fulfil the requirements of the positive energy theorem. However, it is shown that both the black hole and the disc can have a negative Komar mass. Furthermore, there exists a continuous transition from discs to black holes even when their Komar masses are negative.

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A great many definitions for mass in general relativity have been proposed. Many of them consider only the spacetime as a whole, whereas others assign a mass locally to a portion of the spacetime. Even the latter are not always applicable to a single object in a spacetime containing multiple ones.

In [1], Bardeen considers an axially symmetric, stationary spacetime containing a black hole surrounded by a perfect fluid matter distribution and assigns to each of the two objects a mass based on the Komar integral [2]. In a similar way, he assigns an angular momentum to each of the objects.

Various other local mass definitions include the Hawking mass [3], the Bartnik mass [4], the Christodoulou mass [5], and a related mass used for isolated horizons, which can be generalized to dynamical ones [6, 7]. For a review article see [8]. A mass definition applicable both to matter and black holes, and which can be used for single components of a many-body problem in stationary spacetimes, is that based on the Komar integral. We will refer to it here as the Komar mass even when applied to single objects.

In this letter, we study axially symmetric and stationary spacetimes containing either a black hole or a rigidly rotating disc of dust, each surrounded by a perfect fluid ring. We find that there exists a continuous transition from the disc to the black hole. If one is interested in being able to talk about the mass of the individual objects in such stationary spacetimes and since the transition from the disc to the black hole exists, then one is led to look for a definition that is applicable in either scenario. We thus choose to examine the behaviour of the Komar mass and find that it can become negative both for the black hole<sup>3</sup> and for the disc although the total mass of the spacetime is of course positive as guaranteed by the positive energy theorem. We find moreover that the continuous transition mentioned above exists even when the Komar mass of each of the two central objects is negative.

The Poisson equation in Newtonian gravity

$$\nabla^2 U = 4\pi \varepsilon \quad (1)$$

relates the potential  $U$  to the mass density  $\varepsilon$ .<sup>4</sup> The mass contained in any volume  $\mathcal{V}$  of space can be defined by integrating equation (1) over that volume and applying the divergence theorem

$$M = \int_{\mathcal{V}} \varepsilon d^3x = \frac{1}{4\pi} \oint_{\partial\mathcal{V}} \nabla U \cdot d\mathbf{F}. \quad (2)$$

Obviously, a region of space containing no matter has zero mass and the total mass can be found using

$$M_{\text{tot}} = - \lim_{r \rightarrow \infty} (rU).$$

Moreover, mass is additive, i.e. the mass contained in a region of space is simply the sum of the mass contained in each subregion of an arbitrary subdivision of that space.

A similar procedure can be used to define a relativistic mass in axially symmetric and stationary spacetimes. Such a spacetime containing black holes and perfect fluids with strictly azimuthal motion can be described in Weyl–Lewis–Papapetrou coordinates by the line element

$$ds^2 = e^{2\mu} (d\rho^2 + d\zeta^2) + \rho^2 B^2 e^{-2\nu} (d\varphi - \omega dt)^2 - e^{2\nu} dt^2,$$

where the metric functions depend only on  $\rho$  and  $\zeta$ . The energy–momentum tensor of the perfect fluid is

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu},$$

where  $\varepsilon$  is the energy density of the fluid,  $p$  its pressure and  $u^\mu$  its four-velocity.

The Komar integral for the mass can be constructed by integrating Einstein's equation

$$R_t^t = 8\pi (T_t^t - \frac{1}{2}T)$$

over any volume in the 3-space generated by taking  $t = \text{constant}$ . Applying the divergence theorem again then yields (see [1])

$$M = \int_{\mathcal{V}} \tilde{\varepsilon} d^3x = \frac{1}{4\pi} \oint_{\partial\mathcal{V}} \left( B\nabla\nu - \frac{1}{2}\rho^2 B^3 e^{-4\nu} \omega \nabla\omega \right) \cdot d\mathbf{F}, \quad (3)$$

with

$$\tilde{\varepsilon} := e^{2\mu} B \left[ (\varepsilon + p) \frac{1 + v^2}{1 - v^2} + 2p + 2\rho B e^{-2\nu} (\varepsilon + p) \omega \frac{v}{1 - v^2} \right]$$

$$v := \rho B e^{-2\nu} (\Omega - \omega),$$

<sup>3</sup> Negative horizon masses have also been observed for rotating black holes of Einstein–Maxwell–Chern–Simons theory [9].

<sup>4</sup> We use units in which the gravitational constant and speed of light are equal to one,  $G = c = 1$ .

where  $\Omega$  is the angular velocity of a fluid element with respect to infinity and the vector operators have the same meaning as in an Euclidean space in which  $(\varrho, \varphi, \zeta)$  are cylindrical coordinates.

The mass defined in the surface integral of equation (3) is the Komar mass (we use the term here even when  $\mathcal{V}$  is of finite extent). One can see that here too a *regular* volume containing no matter (i.e.  $\varepsilon = p = 0$ ) has zero mass. If one is considering a black hole spacetime, then the region interior to the horizon can be excised and the surface integral in equation (3) used to define the mass of the black hole. The mass of any single object in a stationary spacetime could thus be defined by calculating the surface integral in equation (3) over a surface containing that object and only that object. It can be used both for matter and for black holes and has the additive property familiar from the Newtonian theory that the sum of the masses of the single objects equals the total (ADM) mass of the spacetime:

$$M_{\text{tot}} = - \lim_{r \rightarrow \infty} (rv).$$

A quantity, which plays an important role in black hole thermodynamics and is constant over the horizon is the surface gravity. In coordinates, such as the ones chosen in this letter, in which the horizon is a sphere, it reads

$$\kappa = e^{-\mu} \frac{\partial}{\partial r} e^{\nu}, \quad r := \sqrt{\varrho^2 + \zeta^2}.$$

Smarr [10] showed that for the Komar mass of the black hole

$$M_{\text{h}} = \frac{\kappa A}{4\pi} + 2\Omega_{\text{h}} J_{\text{h}} \quad (4)$$

always holds, true even in the presence of a surrounding ring [1], see also [11]. Here  $\Omega_{\text{h}}$  is the angular velocity of the black hole (i.e. the constant value of the function  $\omega$  over the horizon),  $J_{\text{h}}$  its angular momentum and  $A$  its area.

A similar expression can be derived for the rigidly rotating disc of dust (cf III.15 in [12]), also valid in the presence of a surrounding ring. It turns out that the potential  $g'_{tt}$  in a coordinate system co-rotating with the disc (and denoted here by the prime) must be a constant along its 'surface':

$$-g'_{tt} = e^{2\nu} (1 - v^2) =: e^{2V_0^{\text{d}}}. \quad (5)$$

This constant is related to the relative redshift  $Z_0^{\text{d}}$  of photons with zero angular momentum emitted from the surface of the disc and observed at infinity via the equation

$$Z_0^{\text{d}} = e^{-V_0^{\text{d}}} - 1.$$

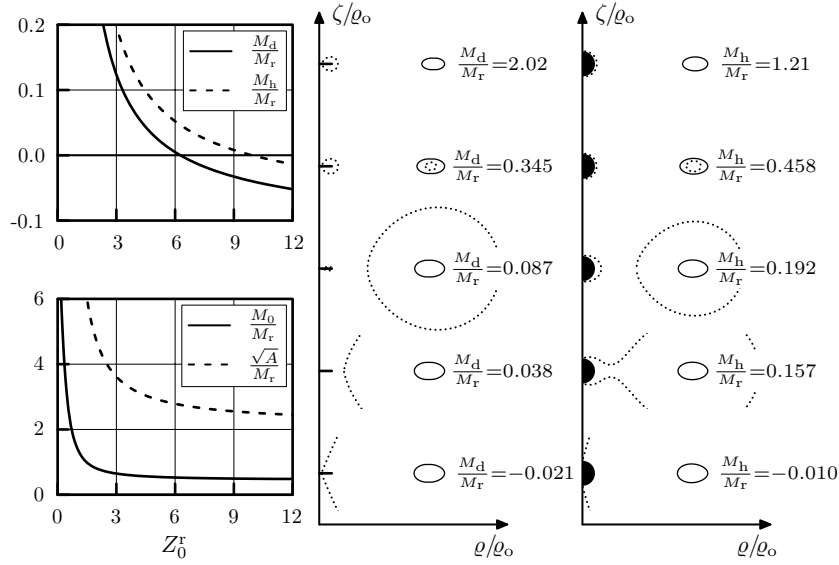
Making use of the definition for the baryonic mass

$$M_0 = \int \varepsilon u^t \sqrt{-g} \, d^3x,$$

we can write the disc's Komar mass as

$$M_{\text{d}} = e^{V_0^{\text{d}}} M_0 + 2\Omega_{\text{d}} J_{\text{d}}. \quad (6)$$

The first term on the right-hand side of equations (4) and (6) is non-negative. In the absence of the ring, the angular velocity and angular momentum must have the same sign, so that the second term is also non-negative. If however a ring is present, it can induce a frame-dragging effect which allows  $\Omega$  and  $J$  of the central object to have different signs. If moreover  $\kappa$  (or  $e^{V_0^{\text{d}}}$ ) becomes sufficiently small, then the expression for the Komar mass could become negative. Our results demonstrate that this indeed occurs.



**Figure 1.** On the upper left, the ratio of the Komar mass of the central object to that of the ring is plotted versus  $Z_0^r$  for a sequence with  $C_i/C_o = 0.85$ ,  $\beta_o = 0.3$  and  $\varrho_d/\varrho_o = 0.1$  for the disc or  $r_h/\varrho_o = 0.1$  for the black hole (see text for an explanation of the symbols). Knowing that  $M_r$  remains positive, one can see that  $M_{d/h}$  becomes negative. The lower left shows a similar plot, but containing the disc's baryonic mass and the square root of the horizon area. On the right, the coordinate shape of the ring and central object (solid lines) and their ergospheres (dotted lines) are drawn for these sequences.

In order to study spacetimes containing a black hole surrounded by a rigidly rotating ring with constant energy density, we make use of the multi-domain pseudo-spectral method described in [13]. To study the scenario containing a disc as opposed to a black hole, we made appropriate modifications to the program.

In the upper plot of figure 1 we consider sequences of configurations that demonstrate the existence of negative Komar masses for discs and black holes surrounded by rings. In the absence of a ring, the analytic solution for the disc is known [14] and depends on one 'physical' and one scaling parameter as does the Kerr metric. When a ring is present, the configurations are described by four physical parameters, so that a sequence can be specified by holding three parameters constant and varying a fourth. Along the sequences in the plot, the ratio of proper inner to outer circumference of the ring was held at a value of  $C_i/C_o = 0.85$  and a mass-shed parameter<sup>5</sup> for the outer edge of the ring was held at a value  $\beta_o = 0.3$ . For the disc sequence, the ratio of its coordinate radius to that of the outer edge of the ring was chosen to be  $\varrho_d/\varrho_o = 0.1$ , whereas for the black hole sequence  $r_h/\varrho_o = 0.1$  was chosen. This last parameter choice, which refers to the radius of the black hole  $r_h$ , is possible since coordinates have been chosen in which the black hole is always a sphere. In the figure, various quantities are plotted versus the relative redshift  $Z_0^r$  of the ring, which is defined as in equation (5). This quantity is also discussed in [15], in which relativistic rings without a central object are examined. The figure shows that the ratio of the central mass to the ring

<sup>5</sup> The definition of  $\beta_o$  is  $\frac{2\varrho_o}{\varrho_o - \varrho_i} \frac{d(\zeta_B^2)}{d(\varrho^2)} \Big|_{\varrho=\varrho_o}$  where  $\zeta_B = \zeta_B(\varrho)$  is a parametric representation of the surface of the ring. The value  $\beta_o = 0$  corresponds to the outer mass-shedding limit.

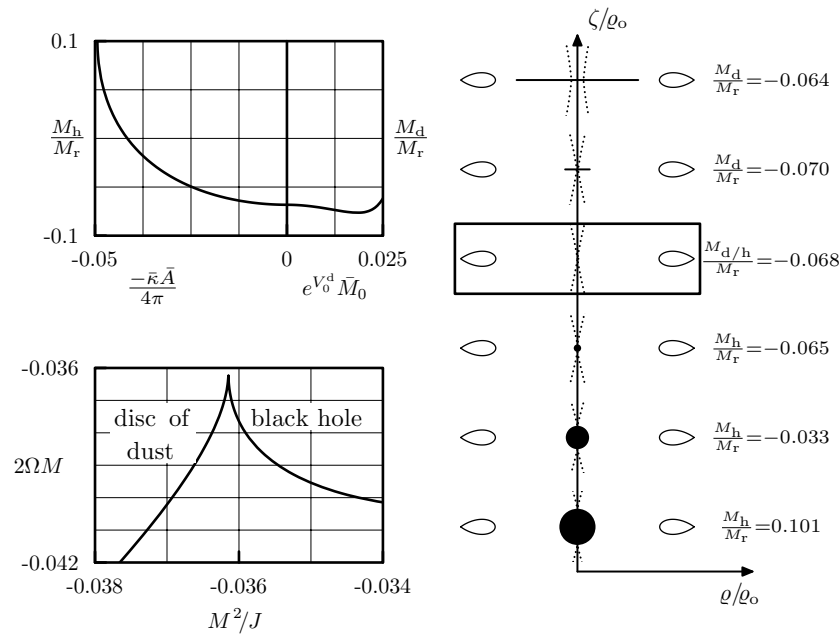
mass  $M_{d/h}/M_r$  indeed becomes negative ( $M_r$  remains positive throughout). It is important to emphasize that  $M_{d/h} = 0$  in this plot does not correspond to a vanishing central object. This fact is demonstrated in the lower plot of figure 1 which shows that neither the disc's baryonic mass  $M_0$  nor the black hole's horizon area  $A$  tends to zero when compared to the ring's Komar mass for large  $Z_0^r$ .

The evolution of the coordinate shape of the ring, the central object and their ergospheres can be followed on the right of figure 1. Looking first at the series of pictures on the left, we begin with a fairly 'Newtonian' ring (i.e. with small  $Z_0^r$ ) and can see that only the disc possesses an ergosphere. The ring and the disc are counter-rotating, meaning their angular velocities have opposite signs. As the ring becomes increasingly relativistic and develops an ergosphere, its frame-dragging effect on the disc becomes more pronounced, causing the disc's angular velocity to decrease, whence its ergosphere shrinks and finally vanishes. As  $Z_0^r$  increases further, the frame dragging finally forces the disc to co-rotate with the ring, although its angular momentum still has the opposite sign. Relative to the size of the ring, the ergosphere grows very large (hence we show only a portion of its boundary in the last two pictures in the sequence). From an outside observer's perspective, the configuration is shrinking towards the centre and the outside metric beginning to resemble that of the extreme Kerr metric. The ring's ergosphere continues to grow, finally engulfing the disc. After a good portion of the disc finds itself inside the ergosphere, the frame dragging becomes significant enough that the magnitude of  $\Omega_d J_d$  is sufficiently large to result in a negative mass. The series of pictures on the right is the counterpart for a black hole and shows similar behaviour to the disc case. A black hole with non-vanishing  $\Omega_h$  always has an ergosphere surrounding it however. Its sense of rotation must agree with that of the ring before their ergospheres merge, independent of the sign of  $J_h$ .

The parametric transition from a rigidly rotating disc (without a surrounding ring) to a black hole has been studied numerically [12] and analytically [16]. A generalization of the proofs in [17, 18] implies that such a transition also exists for a disc surrounded by a ring if and only if  $V_0^d$  of the disc tends to  $-\infty$ , which in turn implies that  $M_d = 2\Omega_d J_d$  must hold<sup>6</sup>. The equality of equations (6) and (4) then requires for non-vanishing black holes that  $\kappa = 0$ . The upper plot on the left of figure 2 shows that such transitions do indeed exist. Here the mass ratio was chosen as an exemplary parameter and plotted versus a measure of the distance to the transition point representing a degenerate black hole surrounded by a ring. Both sequences are defined by  $\beta_0 = 0$ ,  $V_0^r = -2.7$  and  $C_i/C_o = 0.85$ . A bar over a quantity indicates that it has been made dimensionless through multiplication with the appropriate power of the ring's density  $\varepsilon$ . The lower plot suggests a very similar transition to that known analytically for the rigidly rotating disc of dust without a ring as can be seen by comparing it to figure 2 in [19], in which an interpretation in terms of a phase transition was considered. The picture sequence on the right of figure 2 shows the evolution of the coordinate shapes of these configurations.

References to 'the mass' of a single constituent of a many-body system are accepted by convention in some branches of general relativity, such as within the binary black hole community. In stationary spacetimes, one may also wish to be able to refer to individual masses and, indeed, the Komar mass can be defined rigorously and has various attractive features, perhaps the most important one being that it can be used on either side of the transition from matter to a black hole. The fact that it can become negative is related to the fact that the definition involves both local quantities and a reference through  $\Omega$  to asymptotic infinity.

<sup>6</sup> The generalization of the arguments in [18] requires the assumption that  $-\xi^i u_i$  as defined there is bounded from below. In the presence of a surrounding ring,  $\eta^i u_i$  need not have the same sign as  $\Omega_d$ .



**Figure 2.** On the upper left, the ratio of the Komar mass of the central object to that of the ring is plotted versus a measure of the distance to the degenerate black hole solution. On the lower left, a similar plot to figure 2 in [19] is shown for the transition. On the right, the coordinate shape of the ring and central object (solid lines) and their ergospheres (dotted lines) are drawn. The framed picture indicates the transition point from the disc to the black hole.

It thus seems unlikely that locally unusual properties will be observed, but an investigation of geodesic motion in the vicinity of such objects would be interesting. Other interesting questions such as the minimal attainable mass ratio (i.e. how close it can come to  $-1$ ) will be the topic of later work. Finally, we want to point out that various authors (e.g. [20]) have considered negative Komar masses to be unphysical and we hope that the present work shows that this need not be the case.

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