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Pure type I supergravity and DE_{10}

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Abstract We establish a dynamical equivalence between the bosonic part of pure type I supergravity in $D = 10$ and a $D = 1$ non-linear σ-model on the Kac–Moody coset space $DE_{10}/K(DE_{10})$ if both theories are suitably truncated. To this end we make use of a decomposition of DE_{10} under its regular $SO(9,9)$ subgroup. Our analysis also deals partly with the fermionic fields of the supergravity theory and we define corresponding representations of the generalised spatial Lorentz group $K(DE_{10})$.

1 Introduction

Soon after the construction of the maximally supersymmetric $D = 11$ gravity theory [1] it was realised that this theory exhibits exceptional hidden symmetries E_7 and E_8 upon dimensional reduction from $D = 11$ to $D = 4$ and $D = 3$, respectively [2,3]. Much research has been devoted to this unexpected feature of maximal supergravity and its relevance for string theory [4–8]. However, already since the early days of the study of hidden symmetries it has been clear that also theories with non-maximal supersymmetry (or, in fact, no supersymmetry at all) can exhibit unexpected hidden symmetries upon dimensional reduction [9–13]. For the case of all simple and split symmetry groups G in $D = 3$ the question which higher-dimensional theories give rise to the hidden symmetry G upon dimensional reduction has been answered in [14] and there are only very few groups G for which the associated (oxidised) theory

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has supersymmetry. One example is the group $D_8 \equiv SO(8,8)$ which in $D=3$ is the hidden symmetry of the pure type I supergravity theory in $D=10$ [15,16] (which has half-maximal supersymmetry) after dimensional reduction. Further dimensional reduction to $D = 1$ was conjectured [10,12] to lead to an infinite-dimensional symmetry of hyperbolic Kac–Moody type, denoted DE_{10}, which will be defined below. As pointed out in [17] the conjectured symmetry DE_{10} is consistent with the embedding of pure type I into the maximal theory whose conjectured symmetry is E_{10} since DE_{10} is a proper subgroup of E_{10}.

In this paper, we revisit the hidden symmetries of pure type I supergravity motivated by recent results concerning such infinite-dimensional symmetries. Near a space-like singularity it was found that the effective and dominant dynamics of the model can be mapped to a so-called cosmological billiard system whose massless relativistic billiard ball bounces off the walls of a ten-dimensional (auxiliary) billiard table [18]. The location of these walls is identical to that of the bounding walls of the fundamental Weyl chamber of DE_{10} [18] (see also [19] for a review of cosmological billiards).

In analogy with the $E_{10}/K(E_{10})$ model developed in [20] for the maximally supersymmetric case, we here study a $D = 1$ geodesic model on the infinite-dimensional coset space $DE_{10}/K(DE_{10})$, extending the cosmological billiard dynamics. The dynamical behaviour of the σ-model will be related to that of the pure type I theory. $K(DE_{10})$ refers to the (formally) maximal compact subgroup of DE_{10} which plays the role of a generalised spatial Lorentz group. In the context of the E_{11} approach to Kac–Moody symmetries [21,22], the bosonic sectors of $D = 10$ type I theories (also with Abelian vector fields) have been investigated in [23] and the equations of motion were derived from a DE_{11} analysis in the pure type I case. The non-maximal pure $D = 5$, $N = 2$ supergravity has been studied from a Kac–Moody perspective in [24].

Our main result is that a truncated version of the bosonic pure type I equations of motion is dynamically equivalent to a truncation of the equations of the geodesic σ-model on $DE_{10}/K(DE_{10})$. The supergravity truncation roughly involves keeping only first-order spatial gradients but arbitrary time dependence, similar to the truncation in the E_{10} correspondence for the maximally supersymmetric theory [20,25]. Therefore we are *not* performing a dimensional reduction to $D = 1$. Along the way to demonstrating the correspondence we rewrite the relevant parts of the equations in a form which is manifestly $SO(9) \times SO(9)$ covariant (see [17] for an analysis of the maximal theory in an $SO(9) \times SO(9)$ formalism). For the Kac–Moody side of the correspondence the $SO(9) \times SO(9)$ covariance is straight-forward to obtain by taking a so-called level decomposition of DE_{10} with respect to its $SO(9,9)$ subgroup which, after the transition to compact subgroups, leads to $SO(9) \times SO(9) \subset K(DE_{10})$ covariance. On the supergravity side this requires more work and intricate redefinitions of the standard variables.[1]

[1] An $SO(n) \times SO(n)$ covariant formulation of the bosonic type I supergravity after strict dimensional reduction on an n-torus T^n, i.e. discarding all spatial gradients, was given in [26]. Our analysis goes beyond this since we keep spatial gradients. For completeness, we note that $SO(n,n;\mathbb{Z})$ also appears as the T-duality group of closed string theories compactified on T^n.

Besides the correspondence of the bosonic equations of motion we also study the fermionic fields of supergravity and show how they fit into consistent (albeit unfaithful) representations of the compact subgroup $K(DE_{10})$ of DE_{10}. This, together with the bosonic dictionary, allows us to rewrite the supersymmetry variations in a form which not only has manifest $SO(9) \times SO(9)$ covariance but also beginnings of a full $K(DE_{10})$ covariance.

Our paper is structured as followed. First we define type I supergravity in our conventions and introduce some field redefinitions in Sect. 2. In Sect. 3, we define the $DE_{10}/K(DE_{10})$ σ-model in one dimension and work out its equations of motion in an $SO(9,9)$ level decomposition. By comparing the two sets of equations of motion we will derive the dictionary relevant for the dynamical correspondence. In Sect. 4 we study the supersymmetric aspects and fermionic fields of type I and their relation to $K(DE_{10})$ before we close with some remarks and future prospects in Sect. 5.

2 Pure type I supergravity

2.1 Action and supersymmetry

The action of $D = 10$, $N = 1$ supergravity [15] in our conventions reads to lowest fermion order[2]

$$\begin{aligned} S_{\mathrm{I}} = \int \mathrm{d}^{10}x \Bigg[& \frac{\hat{E}}{4}\left(\hat{R} - \frac{1}{2}\partial_M\Phi\partial^M\Phi - \frac{1}{12}\mathrm{e}^{-\Phi}H_{MNP}H^{MNP}\right) \\ & - \frac{\mathrm{i}\hat{E}}{2}\left(\bar{\psi}_M\Gamma^{MNP}D_N\psi_P + \frac{1}{2}\bar{\lambda}\Gamma^M D_M\lambda + \frac{1}{2}\bar{\psi}_N\Gamma^M\partial_M\Phi\Gamma^N\lambda\right) \\ & + \frac{\mathrm{i}\hat{E}}{48}\mathrm{e}^{-\frac{1}{2}\Phi}H_{QRS}\left(\bar{\psi}_M\Gamma^{MNQRS}\psi_N + \bar{\psi}_N\Gamma^{QRS}\Gamma^N\lambda - 6\bar{\psi}^Q\Gamma^R\psi^S\right)\Bigg]. \end{aligned} \quad (2.1)$$

Here, $\hat{E} = \det(\hat{E}_M{}^A)$ is the zehnbein determinant and the curvature scalar $\hat{R}$ is defined in terms of the coefficients of anholonomy $\hat{\Omega}_{MN}{}^A$ and the spin connection $\hat{\omega}_M{}^{AB}$ via[3]

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[2] In what follows, we will always neglect higher order fermion contributions.

[3] Our index conventions are $A, B, \ldots = 0, \ldots, 9$ are *flat* space–time frame indices, $M, N, \ldots$ are *curved* space–time coordinate indices whereas lower case $a, b, \ldots = 1, \ldots, 9$ are flat spatial frame indices and $m, n, \ldots$ are curved spatial coordinate indices. Frame indices are raised and lowered with the flat Minkowski metric $\eta_{AB} = \mathrm{diag}(-1, +1, \ldots, +1)$ and we have chosen the Newton constant conveniently. In Sect. 2.2, we will introduce additional indices relevant for the $SO(9) \times SO(9)$ structure to be studied.

$$\begin{aligned}
\hat{\Omega}_{AB}{}^{C} &= 2\hat{E}_A{}^M \hat{E}_B{}^N \partial_{[M} \hat{E}_{N]}{}^C = \hat{\Omega}_{[AB]}{}^C, \\
\hat{\omega}_{A\,BC} &= \frac{1}{2}\left(\hat{\Omega}_{AB\,C} + \hat{\Omega}_{CA\,B} - \hat{\Omega}_{BC\,A}\right) = \hat{\omega}_{A\,[BC]}, \\
\hat{R}_{MN}{}^{AB} &= 2\partial_{[M}\hat{\omega}_{N]}{}^{AB} + 2\hat{\omega}_{[M}{}^{AC}\hat{\omega}_{N]\,C}{}^{B}, \\
\hat{R}_M{}^A &= \hat{E}_B{}^N \hat{R}_{MN}{}^{AB}, \\
\hat{R} &= \hat{E}_A{}^M \hat{R}_M{}^A.
\end{aligned} \tag{2.2}$$

The Lorentz covariant derivative acting on the spinors λ, ϵ and ψ_M is

$$D_N \psi_M = \partial_N \psi_M + \frac{1}{4}\hat{\omega}_{N\,AB}\Gamma^{AB}\psi_M, \tag{2.3}$$

and the supersymmetry variations leaving the action (2.1) invariant are

$$\delta_\epsilon \Phi = -\mathrm{i}\bar{\epsilon}\lambda, \tag{2.4a}$$

$$\delta_\epsilon \hat{E}_M{}^A = \mathrm{i}\bar{\epsilon}\left(\Gamma^A\psi_M + \frac{1}{12}\Gamma^A{}_M\lambda\right), \tag{2.4b}$$

$$\delta_\epsilon B_{MN} = -2\mathrm{i}\mathrm{e}^{\frac{1}{2}\Phi}\bar{\epsilon}\left(\Gamma_{[M}\psi_{N]} - \frac{1}{4}\Gamma_{MN}\lambda\right), \tag{2.4c}$$

$$\delta_\epsilon \lambda = -\frac{1}{2}\Gamma^M\epsilon\partial_M\Phi - \frac{1}{24}\mathrm{e}^{-\frac{1}{2}\Phi}\Gamma^{QRS}\epsilon H_{QRS}, \tag{2.4d}$$

$$\delta_\epsilon \psi_M = D_M\epsilon - \frac{1}{96}\mathrm{e}^{-\frac{1}{2}\Phi}\left(\Gamma_M{}^{QRS} - 9\delta_M^Q\Gamma^{RS}\right)\epsilon H_{QRS}. \tag{2.4e}$$

Note that the dilatino λ and the gravitino ψ_M have opposite spinor chirality as $SO(1,9)$ Majorana–Weyl spinors. As both can be derived from a single 11-dimensional gravitino,[4] we have used (32×32) Γ-matrices[5] Γ^A. The Γ-matrices Γ^M with curved indices appearing in (2.1) and below are obtained by conversion with the inverse zehnbein $\Gamma^M = \Gamma^A \hat{E}_A{}^M$. The spinors ψ_M and ϵ are understood as projected to one chiral half and λ to the other one. Spinor conjugation is defined by $\bar{\epsilon} = \epsilon^T\Gamma^0$. The field strength of the NSNS two-form B_{MN} is defined by $H_{MNP} = 3\partial_{[M}B_{NP]}$.

The purely bosonic equations of motion deduced from (2.1) are

$$\begin{aligned}
\hat{K}_{AB} := \hat{R}_{AB} - \frac{1}{2}\hat{\partial}_A\Phi\hat{\partial}_B\Phi - \frac{1}{4}\mathrm{e}^{-\Phi}\hat{H}_{ACD}\hat{H}_B{}^{CD} \\
+\frac{1}{48}\eta_{AB}\mathrm{e}^{-\Phi}\hat{H}_{CDE}\hat{H}^{CDE} = 0,
\end{aligned} \tag{2.5a}$$

$$\hat{M}_{AB} := \hat{D}^C(\mathrm{e}^{-\Phi}\hat{H}_{CAB}) = 0, \tag{2.5b}$$

$$\hat{S} := \hat{D}_A\hat{\partial}^A\Phi + \frac{1}{12}\mathrm{e}^{-\Phi}\hat{H}_{CDE}\hat{H}^{CDE} = 0. \tag{2.5c}$$

4 See e.g. [15] for the detailed derivation.

5 Our Γ-matrix conventions can be found in the Appendix.

The hats on the quantities denote their projection onto an orthonormal frame by using the zehnbein $\hat{E}_M{}^A$, e.g. $\hat{\partial}_A \equiv \hat{E}_A{}^M \partial_M$. The Lorentz covariant derivative $\hat{D}_A$ is defined with respect to the spin connection $\hat{\omega}_{A\,BC}$ such that for example $\hat{D}_A \hat{\partial}^A \Phi = \hat{\partial}_A \hat{\partial}^A \Phi + \hat{\omega}_A{}^{AC} \hat{\partial}_C \Phi$.

Finally, we have the Bianchi identities

$$\hat{D}_{[A} \hat{H}_{BCD]} = 0, \tag{2.6a}$$

$$\hat{R}_{[ABC]D} = 0, \tag{2.6b}$$

which are satisfied trivially if one substitutes in the definitions in terms of the zehnbein $\hat{E}_M{}^A$ and the potential B_{MN}. Here, it is more useful to keep them as separate equations since they will appear separately in the correspondence with the $DE_{10}/K(DE_{10})$ σ-model.

2.2 Redefinitions and gauge choices

In order to make the $SO(9) \times SO(9)$ structure manifest, we fix the following zero shift (pseudo-Gaussian) gauge for the zehnbein *à la* ADM:

$$\hat{E}_M{}^A = \begin{pmatrix} N & 0 \\ 0 & \hat{E}_m{}^a \end{pmatrix} \tag{2.7}$$

and then scale the spatial vielbein components and the dilaton field according to[6]

$$\begin{aligned} e_m{}^a &:= \mathrm{e}^{\frac{1}{4}\Phi} \hat{E}_m{}^a, \\ e^{\phi} &:= (\det(\hat{E}_m{}^a))^{-1} \mathrm{e}^{-\frac{1}{4}\Phi}. \end{aligned} \tag{2.8}$$

Furthermore, we define a new (densitised) lapse function by letting

$$\mathtt{n} := (\det(\hat{E}_m{}^a))^{-1} N = (\det(e_m{}^a))^{\frac{1}{8}} \mathrm{e}^{\frac{9}{8}\phi} N \tag{2.9}$$

and set

$$\mathtt{v}_a := \hat{\omega}_{bab} \mathrm{e}^{-\frac{1}{4}\Phi} = \partial_a \phi + e_a{}^m \partial_b e_m{}^b, \tag{2.10}$$

where we have used the abbreviation $\partial_a \equiv e_a{}^m \partial_m$. In general, unhatted quantities in flat indices have been projected using the new spatial neunbein $e_m{}^a$ instead of $\hat{E}_m{}^a$. Finally, we adopt a Coulomb-type gauge for the two-form potential by setting $B_{tm} = 0$.

[6] These redefinitions differ from the ones in [17] since we have made an additional conformal transformation to arrive at the action (2.1). The advantage of this is that also the new lapse of (2.9) below is the usual densitised lapse as in [20].

The bosonic fields are now combined into two sets of new variables. The first set contains only temporal derivatives and is given by

$$\mathrm{P}_{i\bar{\jmath}} := e_i{}^m e_{\bar{\jmath}}{}^n \left(\omega_{m\,nt} - \frac{1}{2}H_{tmn}\right), \tag{2.11a}$$

$$\mathrm{Q}_{ij} := e_i{}^m e_j{}^n \left(\omega_{t\,mn} + \frac{1}{2}H_{tmn}\right), \tag{2.11b}$$

$$\mathrm{Q}_{\bar{\imath}\bar{\jmath}} := e_{\bar{\imath}}{}^m e_{\bar{\jmath}}{}^n \left(\omega_{t\,mn} - \frac{1}{2}H_{t\,mn}\right), \tag{2.11c}$$

whereas the second set consists of combinations of spatial derivatives of the fields defined via

$$\mathrm{P}_{ijk} := -3!\mathrm{n}e^{-2\phi} e_i{}^m e_j{}^n e_k{}^p \left(\frac{1}{4}\omega_{[m\,np]} + \frac{1}{24}H_{mnp}\right), \tag{2.12a}$$

$$\mathrm{P}_{\bar{\imath}jk} := -2\mathrm{n}e^{-2\phi} e_{\bar{\imath}}{}^m e_j{}^n e_k{}^p \left(\frac{1}{4}\omega_{m[np]} + \frac{1}{8}H_{mnp}\right), \tag{2.12b}$$

$$\mathrm{P}_{i\bar{\jmath}\bar{k}} := +2\mathrm{n}e^{-2\phi} e_i{}^m e_{\bar{\jmath}}{}^n e_{\bar{k}}{}^p \left(\frac{1}{4}\omega_{m[np]} - \frac{1}{8}H_{mnp}\right), \tag{2.12c}$$

$$\mathrm{P}_{\bar{\imath}\bar{\jmath}\bar{k}} := +3!\mathrm{n}e^{-2\phi} e_{\bar{\imath}}{}^m e_{\bar{\jmath}}{}^n e_{\bar{k}}{}^p \left(\frac{1}{4}\omega_{[m\,np]} - \frac{1}{24}H_{mnp}\right), \tag{2.12d}$$

where ω is the spin connection with respect to the rescaled vielbein $e_m{}^a$. Its definition is analogous to (2.2), implying e.g. $\omega_{m\,nt} = e_{(m}{}^a \partial_t e_{n)a}$. Both the indices $i, j, \ldots$ and $\bar{\imath}, \bar{\jmath}, \ldots$ are frame indices taking values in the spatial directions $1, \ldots, 9$, where we identify $e_{\bar{\imath}}{}^m$ and $e_i{}^m$, so that for example $\partial_{\bar{\imath}} = \partial_i$. However, they will have different transformation properties under an $SO(9) \times SO(9)$ group we now introduce. To be more precise, the unbarred indices are $SO(9)$ vector indices of the first factor, whereas the barred indices are vector indices of the second factor. The spatial $SO(9)$ Lorentz group is the diagonal subgroup of $SO(9) \times SO(9)$ (see also [17]). The fields Q_{ij}, $\mathrm{Q}_{\bar{\imath}\bar{\jmath}}$, P_{ijk} and $\mathrm{P}_{\bar{\imath}\bar{\jmath}\bar{k}}$ are totally antisymmetric, whereas the mixed $\mathrm{P}_{i\bar{\jmath}}$, $\mathrm{P}_{\bar{\imath}jk}$ and $\mathrm{P}_{i\bar{\jmath}\bar{k}}$ are only antisymmetric in indices belonging to the same $SO(9)$ factor of $SO(9) \times SO(9)$. Repeated indices on the same level are summed over with δ^{ij} or $\delta^{\bar{\imath}\bar{\jmath}}$.

We note that the total number of components in P_{ijk}, $\mathrm{P}_{\bar{\imath}jk}$, $\mathrm{P}_{\bar{\imath}\bar{\jmath}k}$ and $\mathrm{P}_{\bar{\imath}\bar{\jmath}\bar{k}}$ is 816 whereas the number of independent components of the supergravity variables $\omega_{m\,np}$ and H_{mnp} involved in the redefinition is only 408 so that the redefinition is not one-to-one. Hence, there are equivalent ways of expressing a supergravity expression in these new variables. Our choice is such that it connects well to the DE_{10} analysis.

2.3 Supergravity dynamics

We now take certain combinations of the equations of motion (2.5) after separating the time index 0 from the spatial indices a. The independent components of the equations of motion then are

$$\hat{K}_{ab} = 0, \quad \hat{K}_{00} = 0, \quad \hat{K}_{a0} = 0,$$
$$\hat{M}_{ab} = 0, \quad \hat{M}_{a0} = 0, \quad \hat{S} = 0. \tag{2.13}$$

We combine the symmetric Einstein equation and the antisymmetric two-form equation into a single tensor equation with no definite symmetry

$$N^2 \left(\hat{K}_{ab} - \frac{1}{4}\delta_{ab}\hat{S} + \frac{1}{2}\mathrm{e}^{\frac{1}{2}\Phi}\hat{M}_{ab} \right) = 0. \tag{2.14}$$

In the new variables (2.11) and (2.12), Eq. (2.14) takes the form

$$\begin{aligned} &\mathtt{n} D_t \left(\mathtt{n}^{-1}\mathtt{P}_{i\bar{j}}\right) - \mathrm{e}^{2\phi}\left(\mathtt{P}_{ikl}\mathtt{P}_{\bar{j}kl} + 2\mathtt{P}_{\bar{k}il}\mathtt{P}_{l\bar{k}\bar{j}} + \mathtt{P}_{i\bar{k}\bar{l}}\mathtt{P}_{\bar{j}\bar{k}\bar{l}}\right) \\ &\quad = \mathtt{n}\partial_p\left[e_k{}^p\mathtt{P}_{\bar{j}ki} - e_{\bar{k}}{}^p\mathtt{P}_{i\bar{k}\bar{j}}\right] - \mathrm{e}^{2\phi}\left(\mathtt{P}_{ikl}\mathtt{P}_{\bar{j}kl} + \mathtt{P}_{i\bar{k}\bar{l}}\mathtt{P}_{\bar{j}\bar{k}\bar{l}} - 2\mathtt{P}_{\bar{k}il}\mathtt{P}_{l\bar{k}\bar{j}}\right) \\ &\qquad -\mathtt{n}^2\mathrm{e}^{-2\phi}\partial_{(i}\mathtt{v}_{\bar{j})} + \mathtt{n}\mathrm{e}^{-\phi}\partial_{(i}\left[\mathtt{n}\mathrm{e}^{-\phi}\left(\mathtt{n}^{-1}\partial_{\bar{j})}\mathtt{n} - \partial_{\bar{j})}\phi\right)\right] \end{aligned} \tag{2.15}$$

with the $SO(9) \times SO(9)$ covariant derivative D_t acting on $\mathtt{P}_{i\bar{j}}$ via

$$D_t\mathtt{P}_{i\bar{j}} := \partial_t\mathtt{P}_{i\bar{j}} + \mathtt{Q}_{ik}\mathtt{P}_{k\bar{j}} + \mathtt{Q}_{\bar{j}\bar{k}}\mathtt{P}_{i\bar{k}}. \tag{2.16}$$

The dilaton equation of motion (2.5c) can be combined with the spatial trace of the Einstein equation by

$$N^2\left(\frac{1}{4}\hat{S} - \delta^{ab}\hat{K}_{ab}\right) = 0 \tag{2.17}$$

to give the first scalar equation of motion

$$\begin{aligned} &\mathtt{n}\partial_t\left(\mathtt{n}^{-1}\partial_t\phi\right) + \frac{1}{6}\mathrm{e}^{2\phi}\left(\mathtt{P}_{ijk}\mathtt{P}_{ijk} + 3\mathtt{P}_{\bar{i}jk}\mathtt{P}_{\bar{i}jk} + 3\mathtt{P}_{i\bar{j}\bar{k}}\mathtt{P}_{i\bar{j}\bar{k}} + \mathtt{P}_{\bar{i}\bar{j}\bar{k}}\mathtt{P}_{\bar{i}\bar{j}\bar{k}}\right) \\ &\quad = \mathtt{n}^2\mathrm{e}^{-2\phi}\left[2\partial_a\mathtt{v}_a - \frac{1}{2}\,\Omega^{abc}\,\Omega_{acb} - \mathtt{v}_a\mathtt{v}_a\right] \\ &\qquad -\mathtt{n}\mathrm{e}^{-\phi}\partial_a\left(\mathtt{n}\mathrm{e}^{-\phi}\left[\mathtt{n}^{-1}\partial_a\mathtt{n} - \partial_a\phi\right]\right) + \mathtt{n}^2\mathrm{e}^{-2\phi}\mathtt{v}_a\left[\mathtt{n}^{-1}\partial_a\mathtt{n} - \partial_a\phi\right]. \end{aligned} \tag{2.18}$$

Furthermore, we have an independent second scalar equation

$$-N^2(\hat{K}_{00} + \delta^{ab}\hat{K}_{ab}) = 0, \tag{2.19}$$

which is proportional to the Hamiltonian constraint. In the new variables it reads

$$-(\partial_t\phi)^2 + \mathrm{P}_{i\bar{j}}\mathrm{P}_{i\bar{j}} + \frac{1}{6}\mathrm{e}^{2\phi}\left(\mathrm{P}_{ijk}\mathrm{P}_{ijk} + 3\mathrm{P}_{\bar{i}jk}\mathrm{P}_{\bar{i}jk} + 3\mathrm{P}_{i\bar{j}\bar{k}}\mathrm{P}_{i\bar{j}\bar{k}} + \mathrm{P}_{\bar{i}\bar{j}\bar{k}}\mathrm{P}_{\bar{i}\bar{j}\bar{k}}\right)$$
$$= \mathrm{n}^2\mathrm{e}^{-2\phi}\left[2\partial_a\mathrm{v}_a - \frac{1}{2}\Omega^{abc}\Omega_{acb} - \mathrm{v}_a\mathrm{v}_a\right]. \tag{2.20}$$

Finally, we have two vector equations stemming from the Gauss constraint on the two-form field, $\hat{M}_{a0} = 0$, and the diffeomorphism constraint, $\hat{K}_{a0} = 0$. We combine them by

$$N\mathrm{e}^{-\frac{1}{4}\Phi}\left(\hat{K}_{a0} \pm \frac{1}{2}\mathrm{e}^{\frac{1}{2}\Phi}\hat{M}_{a0}\right) = 0 \tag{2.21}$$

and get the two vector constraint equations

$$0 = \mathrm{n}\partial_m\left[e_{\bar{j}}{}^m\mathrm{n}^{-1}\mathrm{P}_{i\bar{j}}\right] + 2\mathrm{n}^{-1}\mathrm{e}^{2\phi}\mathrm{P}_{\bar{j}ki}\mathrm{P}_{k\bar{j}} + \mathrm{n}\mathrm{e}^{-\phi}\partial_i\left(\mathrm{n}^{-1}\mathrm{e}^{\phi}\partial_t\phi\right), \tag{2.22a}$$
$$0 = \mathrm{n}\partial_m\left[e_j{}^m\mathrm{n}^{-1}\mathrm{P}_{j\bar{i}}\right] - 2\mathrm{n}^{-1}\mathrm{e}^{2\phi}\mathrm{P}_{j\bar{k}\bar{i}}\mathrm{P}_{j\bar{k}} + \mathrm{n}\mathrm{e}^{-\phi}\partial_{\bar{i}}\left(\mathrm{n}^{-1}\mathrm{e}^{\phi}\partial_t\phi\right). \tag{2.22b}$$

The Eqs. (2.15)–(2.22) are completely equivalent to the set of bosonic equations of motion (2.5) upon substitution of the definitions (2.11) and (2.12).

We conclude this section by giving the Bianchi identities (2.6a) and (2.6b) in an appropriate form. Starting from the equations

$$\mathrm{n}\mathrm{e}^{-2\phi}D_t\left(\mathrm{n}^{-1}\mathrm{e}^{2\phi}\mathrm{P}_{ijk}\right) + 3\mathrm{P}_{[i|\bar{l}}\mathrm{P}_{\bar{l}|jk]} = -\frac{3}{2}\partial_{[i}\mathrm{Q}_{jk]}, \tag{2.23a}$$
$$\mathrm{n}\mathrm{e}^{-2\phi}D_t\left(\mathrm{n}^{-1}\mathrm{e}^{2\phi}\mathrm{P}_{\bar{i}jk}\right) + \mathrm{P}_{l\bar{i}}\mathrm{P}_{ljk} + 2\mathrm{P}_{[j|\bar{l}|}\mathrm{P}_{k]\bar{i}\bar{l}} = -\frac{1}{2}\partial_{\bar{i}}\mathrm{Q}_{jk} + \partial_{[j}\mathrm{P}_{k]\bar{i}}, \tag{2.23b}$$
$$\mathrm{n}\mathrm{e}^{-2\phi}D_t\left(\mathrm{n}^{-1}\mathrm{e}^{2\phi}\mathrm{P}_{i\bar{j}\bar{k}}\right) + \mathrm{P}_{i\bar{l}}\mathrm{P}_{\bar{l}\bar{j}\bar{k}} + 2\mathrm{P}_{l[\bar{j}}\mathrm{P}_{\bar{k}]il} = \frac{1}{2}\partial_i\mathrm{Q}_{\bar{j}\bar{k}} - \partial_{[\bar{j}}\mathrm{P}_{|i|\bar{k}]}, \tag{2.23c}$$
$$\mathrm{n}\mathrm{e}^{-2\phi}D_t\left(\mathrm{n}^{-1}\mathrm{e}^{2\phi}\mathrm{P}_{\bar{i}\bar{j}\bar{k}}\right) + 3\mathrm{P}_{l[\bar{i}}\mathrm{P}_{|l|\bar{j}\bar{k}]} = \frac{3}{2}\partial_{[\bar{i}}\mathrm{Q}_{\bar{j}\bar{k}]}, \tag{2.23d}$$

one recovers the Bianchi identities by taking suitable combinations. For the Bianchi identity $\hat{D}_{[0}\hat{H}_{abc]} = 0$ one has to sum (2.23a) and (2.23d), whereas $\hat{R}_{[0bcd]} = 0$ corresponds to the difference between (2.23a) and (2.23d). The difference between (2.23b) and (2.23c) gives the identity $R_{[0ab]c} = 0$. The Bianchi identities with purely spatial indices will not be discussed here but they can also be rewritten in the new variables of (2.11) and (2.12).

3 The geodesic $DE_{10}/K(DE_{10})$ coset model

3.1 Abstract derivation of the equations of motion

The abstract $D = 1$ σ-model on any group coset G/H is given in terms of a representative $\mathcal{V}(t) \in G/H$, where t is the parameter along the world-line. The velocity along this world-line pulled backed to the identity is the Lie(G) valued expression $\partial_t \mathcal{V}\,\mathcal{V}^{-1}$ that can be decomposed into generators along Lie(H) and Lie(G/H) as

$$\partial_t \mathcal{V}\,\mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P}, \tag{3.1}$$

where $\mathcal{Q} \in$ Lie(H) are the unbroken gauge connections in the language of non-linear realisations, and $\mathcal{P} \in$ Lie(G/H) correspond to the velocity components in the direction of the 'broken' generators. Using the invariant symmetric form ($\equiv$ Cartan–Killing form)[7] $\langle\cdot|\cdot\rangle$ on Lie(G), we define a Lagrange function that determines the dynamics of the bosonic $D = 1$ σ-model

$$L = \frac{1}{4} n^{-1} \langle \mathcal{P}|\mathcal{P}\rangle. \tag{3.2}$$

The Lagrange function is invariant under the standard non-linear transformation $\mathcal{V}(t) \to h(t)\mathcal{V}(t)g^{-1}$ for local $h(t) \in H$ and global $g \in G$. The factor $n(t)$ ensures reparametrisation invariance along the world-line and, since we have no mass term in L, the massless particle will move on a null trajectory.[8]

In order to derive the equations of motion from this Lagrange function we consider variations of the field $\mathcal{V}$ associated with a derivation δ which is assumed to commute with the time derivative ∂_t. Under this variation we get a similar decomposition $\delta\mathcal{V}\,\mathcal{V}^{-1} = \Sigma + \Lambda$ for $\Sigma \in$ Lie(H) and $\Lambda \in$ Lie(G/H). Substituting this variation into the Lagrange function (3.2) leads to the H covariant σ-model equations of motion

$$n\partial_t(n^{-1}\mathcal{P}) - [\mathcal{Q}, \mathcal{P}] = 0 \tag{3.3}$$

and the null constraint[9]

$$\langle \mathcal{P}|\mathcal{P}\rangle = 0. \tag{3.4}$$

Using the H covariant derivative

$$\mathcal{D} = \partial_t - \mathcal{Q}, \tag{3.5}$$

[7] This is the invariant trace in the adjoint representation in the finite-dimensional case.

[8] This is only possible if the invariant form $\langle\cdot|\cdot\rangle$ is *indefinite* as in our case.

[9] We assume here that $\mathcal{V}$ and n are independent.

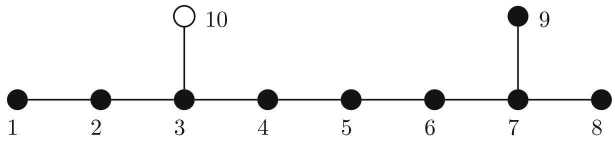

Fig. 1 Dynkin diagram of $\text{Lie}(DE_{10})$ with numbering of nodes. The solid nodes mark the $\mathfrak{so}(9,9) \equiv \text{Lie}(D_9)$ subalgebra

where $\mathcal{Q}$ acts on a H representation, here the algebraic coset $\text{Lie}(G/H)$, Eq. (3.3) can also be written as

$$\mathcal{D}(n^{-1}\mathcal{P}) = 0. \tag{3.6}$$

For any Kac–Moody algebra [27], there is a generalised transposition map $-\omega$ mapping the Chevalley generators e_i, f_i and h_i with $i = 1, \ldots, \text{rk}(\text{Lie}(G))$ to themselves by

$$-\omega(e_i) = f_i, \quad -\omega(f_i) = e_i, \quad -\omega(h_i) = h_i. \tag{3.7}$$

As $\omega^2 = \mathbf{1}_{\text{Lie}(G)}$, we can decompose any $\text{Lie}(G)$-valued object into eigenspaces $\mathcal{Q} \in \text{Lie}(H)$ and $\mathcal{P} \in \text{Lie}(G/H)$ via

$$-\omega(\mathcal{P}) = \mathcal{P}, \qquad -\omega(\mathcal{Q}) = -\mathcal{Q}. \tag{3.8}$$

H is then referred to as the maximal compact[10] subgroup of G, which we denote by $K(G)$. Now we study this general set-up for the case of $G = DE_{10}$ and $H = K(DE_{10})$.

3.2 The D_9 level decomposition of $\text{Lie}(DE_{10})$

The Lie algebra $\text{Lie}(DE_{10})$ is an infinite-dimensional hyperbolic Kac–Moody algebra [27] with Dynkin diagram given in Fig. 1 and we consider it in split real form. In order to analyse the dynamical equation (3.3), we need to know the structure constants of $\text{Lie}(DE_{10})$. However, a closed representation of $\text{Lie}(DE_{10})$ is not known. The only known presentation of $\text{Lie}(DE_{10})$ is in terms of simple generators e_i, f_i and h_i (for $i = 1, \ldots, 10$) and defining relations among them [27]. These simple generators and their independent multiple commutators form a basis of the vector spaces $\mathfrak{n}_+$, $\mathfrak{n}_-$ and the Cartan subalgebra $\mathfrak{h}$ respectively. Thus, one obtains the following decomposition of $\text{Lie}(DE_{10})$:

$$\text{Lie}(DE_{10}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{3.9}$$

Next, we define the level ℓ of a homogeneous element of $\mathfrak{n}_+$ with respect to $D_9 \equiv \text{Lie}(SO(9,9))$ to be the number of times e_{10} appears in the correspond-

[10] In the case of a finite-dimensional Lie group G, this is the standard notion of a compact manifold.

ing multiple commutator. This definition can be extended to the entire Kac–Moody algebra by counting f_{10} negatively and setting the degree of $\mathfrak{h}$ to zero. Thus, we have constructed an integer grading of Lie(DE_{10}), given by the level decomposition into subspaces labelled by levels $\ell \in \mathbb{Z}$. The level ℓ piece in this decomposition is finite-dimensional and is mapped to itself under the adjoint action of the $\ell = 0$ piece. Therefore any fixed level ℓ is a sum of irreducible representations of the $\ell = 0$ subalgebra of Lie(DE_{10}) and we first study $\ell = 0$.[11]

Following from this definition, the subspace with level $\ell = 0$ consists of all multiple generators of e_i, f_i for $i = 1, \ldots, 9$ and all ten Cartan subalgebra elements. Leaving out all e_{10} and f_{10} generators in commutators one arrives at a Lie(D_9) subalgebra of Lie(DE_{10}) as also evident from Fig. 1. A certain linear combination of the ten Cartan elements h_i is orthogonal to this $\mathfrak{so}(9,9)$ and therefore the resulting $\ell = 0$ subalgebra of Lie(DE_{10}) is the direct sum $\mathfrak{so}(9,9) \oplus \mathfrak{gl}(1)$.

We denote the $SO(9,9)$ generators by $M^{IJ} = -M^{JI}$ and take their commutation relation to be

$$\left[M^{IJ}, M^{KL}\right] = \eta^{KI} M^{JL} - \eta^{KJ} M^{IL} - \eta^{LI} M^{JK} + \eta^{LJ} M^{IK} \tag{3.10}$$

with

$$\eta^{IJ} = \mathrm{diag}(\mathbf{1}_9, -\mathbf{1}_9) \Longleftrightarrow \eta^{ij} = \delta^{ij} = -\eta^{\bar{\imath}\bar{\jmath}}; \quad \eta^{i\bar{\jmath}} = \eta^{\bar{\imath}j} = 0, \tag{3.11}$$

where we made use of the $SO(9) \times SO(9)$-indices $I \equiv (i, \bar{\imath})$.[12] We use η_{IJ} to raise and lower $SO(9,9)$ indices in the standard fashion. With the Cartan–Killing form

$$\left\langle M^{IJ} | M^{KL} \right\rangle = \eta^{KJ}\eta^{IL} - \eta^{KI}\eta^{JL}, \tag{3.12}$$

we can split the generators into compact and non-compact ones

$$J^{ij} := M^{ij}, \quad J^{\bar{\imath}\bar{\jmath}} := -M^{\bar{\imath}\bar{\jmath}}; \quad S^{i\bar{\jmath}} := M^{i\bar{\jmath}}, \tag{3.13}$$

where J^{ij} and $J^{\bar{\imath}\bar{\jmath}}$ generate the two $SO(9)$-groups of $SO(9) \times SO(9)$, whereas the symmetric $S^{i\bar{\jmath}}$ is a coset generator. The generator of $GL(1)$ will be denoted T and satifies

$$\begin{aligned} \left[T, M^{IJ}\right] &= 0, \\ \langle T | T \rangle &= -1. \end{aligned} \tag{3.14}$$

[11] Further details on this decomposition can be found in [17]; the general technique of level decompositions is explained for example in [20,28–30].

[12] The explicit mapping between the D_9 Chevalley operators and the M^{IJ} can be found in [17].

Restricting Lie(DE_{10}) to level $\ell = 1$ constitutes an irreducible representation of Lie($D_9 \times GL(1)$) which is in an antisymmetric three-tensor representation of $SO(9,9)$, denoted by E^{IJK}, and carries $GL(1)$ charge $+1$:

$$\left[T, E^{IJK}\right] = E^{IJK}, \tag{3.15a}$$

$$\left[M^{IJ}, E^{KLM}\right] = 3\eta^{I[K}E^{LM]J} - 3\eta^{J[K}E^{LM]I}. \tag{3.15b}$$

The commutation relations for the elements $F_{IJK} := -\omega(E^{IJK})$ on level $\ell = -1$ are obtained by using the generalised transposition $-\omega$ on (3.15a) and (3.15b) to give

$$\left[T, F^{IJK}\right] = -F^{IJK}, \tag{3.16a}$$

$$\left[M^{IJ}, F^{KLM}\right] = 3\eta^{I[K}F^{LM]J} - 3\eta^{J[K}F^{LM]I}. \tag{3.16b}$$

Here, indices have been raised with η^{IJ}. If we fix the normalisation by[13]

$$\left\langle E^{IJK} | F_{LMN} \right\rangle := 12\delta^{IJK}_{LMN}, \tag{3.17}$$

the invariance of the Cartan–Killing form yields the Lie(DE_{10}) commutation relation

$$\left[E^{IJK}, F_{LMN}\right] = -12\delta^{IJK}_{LMN}T - 36\delta^{[IJ}_{[LM}\eta_{N]P}M^{K]P}. \tag{3.18}$$

In summary, we have the following set of generators of DE_{10} on levels $|\ell| \leq 1$ in a decomposition with respect to D_9:

	$\ell = -1$	$\ell = 0$	$\ell = 1$
Generator	F_{IJK}	M^{IJ}, T	E^{IJK}
Dimension	816	153+1	816

As we will not use any further generators in this paper, we will not discuss the representation theory of the higher levels. More details and extensive tables up to $\ell = 5$ can be found in [17].

13 We use $\delta^{I_1\cdots I_l}_{L_1\cdots L_l} := \delta^{[I_1}_{L_1}\cdots\delta^{I_l]}_{L_l}$ and antisymmetrisations of strength one.

3.3 σ-model equation truncated at D_9 level 1

Given the knowledge of the generators up to level $\ell = 1$ we can parametrise the $DE_{10}/K(DE_{10})$ coset element $\mathcal{V}$ by

$$\mathcal{V}(t) = \mathrm{e}^{\varphi(t)\,T}\mathrm{e}^{\frac{1}{2}v_{IJ}(t)\,M^{IJ}}\mathrm{e}^{\frac{1}{3!}A_{IJK}(t)E^{IJK}}\cdots \tag{3.19}$$

in a Borel gauge consisting of only generators of levels $\ell \geq 0$. Singling out the $GL(1)$-generator T will introduce a factor $\mathrm{e}^{\ell\varphi}$ for the level ℓ term when evaluating the velocity (3.1), which we truncate for all $\ell \geq 2$.[14] Hence, we get the parametrisation $\partial_t\mathcal{V}\,\mathcal{V}^{-1} = \mathcal{P} + \mathcal{Q}$ with

$$\begin{aligned}\mathcal{P} &= \partial_t\varphi T + P_{i\bar{j}}S^{i\bar{j}} + \frac{1}{3!}\mathrm{e}^{\varphi}P_{IJK}S^{IJK},\\ \mathcal{Q} &= \frac{1}{2}Q_{ij}J^{ij} + \frac{1}{2}Q_{\bar{\imath}\bar{j}}J^{\bar{\imath}\bar{j}} + \frac{1}{3!}\mathrm{e}^{\varphi}P_{IJK}J^{IJK},\end{aligned} \tag{3.20}$$

where we have introduced a new notation for the coset generator S^{IJK} and the $K(DE_{10})$-generator J^{IJK}

$$S^{IJK} := \frac{1}{2}\left(E^{IJK} - \omega(E^{IJK})\right), \tag{3.21a}$$

$$J^{IJK} := \frac{1}{2}\left(E^{IJK} + \omega(E^{IJK})\right). \tag{3.21b}$$

The occurrence of the same coefficient P_{IJK} in $\mathcal{P}$ and $\mathcal{Q}$ in (3.20) is due to our Borel gauge condition. Although we could work out P_{IJK} explicitly in terms of the coset coordinate fields v_{IJ} and A_{IJK} we will leave it in this compact form since this will be sufficient for the comparison with the σ-model equations of motion (3.3).

In Sect. 3.1, we mentioned that the σ-model equations of motion (3.3) are only $H = K(DE_{10})$-covariant, whereas the $G = DE_{10}$-covariance is broken. However, $SO(9,9)$ is not a subgroup of $K(DE_{10})$, only its maximal compact subgroup $SO(9)\times SO(9)$ is. Hence for our truncation chosen, we can only expect to get $SO(9)\times SO(9)$ covariant equations of motion. This is already obvious from the definitions of the generators (3.21a) and (3.21b): S^{IJK} and J^{IJK} do not transform as $SO(9,9)$ tensors, but as $SO(9)\times SO(9)$ tensors. Therefore, we also decompose the $SO(9,9)$ tensor P_{IJK} into its irreducible $SO(9)\times SO(9)$ components P_{ijk}, $P_{\bar{\imath}jk}$, $P_{i\bar{j}\bar{k}}$ and $P_{\bar{\imath}\bar{j}\bar{k}}$ and write the σ-model equations of motion (3.3)

[14] In principle, we should add all higher level ℓ D_9-representations $E^{(\ell)}$ with appropriate coefficients $P^{(\ell)}$ to the parametrisation. However, in the present discussion we want to restrict the levels to $\ell \leq 1$, which is consistently achieved by setting all coefficients $P^{(\ell)} = 0$ initially for all $\ell \geq 2$ [25].

for the level $\ell = 0$, *i.e.* projected on the coset generators T and $S^{i\bar{j}}$, as

$$0 = n\partial_t \left(n^{-1}\partial_t\varphi\right) + \frac{1}{6}\mathrm{e}^{2\varphi}\left(P_{ijk}P_{ijk} + 3P_{\bar{i}jk}P_{\bar{i}jk} + 3P_{i\bar{j}\bar{k}}P_{i\bar{j}\bar{k}} + P_{\bar{i}\bar{j}\bar{k}}P_{\bar{i}\bar{j}\bar{k}}\right), \tag{3.22a}$$

$$0 = nD_t\left(n^{-1}P_{i\bar{j}}\right) - \mathrm{e}^{2\varphi}\left(P_{ikl}P_{\bar{j}kl} + 2P_{\bar{k}il}P_{l\bar{k}\bar{j}} + P_{i\bar{k}\bar{l}}P_{\bar{j}\bar{k}\bar{l}}\right). \tag{3.22b}$$

Here, D_t is the $SO(9) \times SO(9)$ covariant derivative of (2.16) with the connection $(Q_{ij}, Q_{\bar{i}\bar{j}})$. The σ-model equations of motion (3.3) for the level $\ell = 1$, *i.e.* projected on the generators S^{ijk}, $S^{\bar{i}jk}$, $S^{i\bar{j}\bar{k}}$ and $S^{\bar{i}\bar{j}\bar{k}}$, are

$$n\mathrm{e}^{-2\varphi}D_t\left(n^{-1}\mathrm{e}^{2\varphi}P_{ijk}\right) + 3P_{[i|\bar{l}}P_{\bar{l}|jk]} = 0, \tag{3.23a}$$

$$n\mathrm{e}^{-2\varphi}D_t\left(n^{-1}\mathrm{e}^{2\varphi}P_{\bar{i}jk}\right) + P_{l\bar{i}}P_{ljk} + 2P_{[j|\bar{l}}P_{k]\bar{i}\bar{l}} = 0, \tag{3.23b}$$

$$n\mathrm{e}^{-2\varphi}D_t\left(n^{-1}\mathrm{e}^{2\varphi}P_{i\bar{j}\bar{k}}\right) + P_{\bar{i}l}P_{\bar{l}\bar{j}\bar{k}} + 2P_{l[\bar{j}}P_{\bar{k}]il} = 0, \tag{3.23c}$$

$$n\mathrm{e}^{-2\varphi}D_t\left(n^{-1}\mathrm{e}^{2\varphi}P_{\bar{i}\bar{j}\bar{k}}\right) + 3P_{l[\bar{i}}P_{l|\bar{j}\bar{k}]} = 0. \tag{3.23d}$$

As we set $P^{(\ell)} = 0$ for $\ell \geq 2$ initially, the Eq. (3.3), describing its time evolution, preserves this setting. However, written in terms of the field $\mathcal{V}$ parametrising the coset $DE_{10}/K(DE_{10})$ this implies a non-trivial time evolution of the higher level fields. We stress that the terms extending the $SO(9) \times SO(9)$ covariant derivative D_t in (3.23) are the next terms in the full $K(DE_{10})$ covariant derivative $\mathcal{D}_t$ of (3.5).

We conclude this section with the null constraint $\langle\mathcal{P}|\mathcal{P}\rangle = 0$, cf. (3.4), in this parametrisation

$$0 = -(\partial_t\varphi)^2 + P_{i\bar{j}}P_{i\bar{j}} + \frac{1}{6}\mathrm{e}^{2\varphi}\left(P_{ijk}P_{ijk} + 3P_{\bar{i}jk}P_{\bar{i}jk} + 3P_{i\bar{j}\bar{k}}P_{i\bar{j}\bar{k}} + P_{\bar{i}\bar{j}\bar{k}}P_{\bar{i}\bar{j}\bar{k}}\right). \tag{3.24}$$

3.4 Comparison of the σ-model with supergravity

Now we turn to the comparison of the level $\ell = 0$ σ-model equations of motion (3.22) with the rewritten dynamical supergravity equations (2.15) and (2.18) and of the $\ell = 1$ equations (3.23) with the Bianchi constraints (2.23). We will also compare the Hamiltonian constraint (2.20) with the null constraint (3.24). All these equations contain the following objects:

$D = 10$ pure type I supergravity	$DE_{10}/K(DE_{10})$ σ-model
$\mathtt{P}_{i\bar{j}}(t,\mathbf{x})$, $\mathtt{Q}_{ij}(t,\mathbf{x})$, $\mathtt{Q}_{\bar{i}\bar{j}}(t,\mathbf{x})$	$P_{i\bar{j}}(t)$, $Q_{ij}(t)$, $Q_{\bar{i}\bar{j}}(t)$
$\mathtt{P}_{ijk}(t,\mathbf{x})$, $\mathtt{P}_{\bar{i}jk}(t,\mathbf{x})$, $\mathtt{P}_{i\bar{j}\bar{k}}(t,\mathbf{x})$, $\mathtt{P}_{\bar{i}\bar{j}\bar{k}}(t,\mathbf{x})$	$P_{ijk}(t)$, $P_{\bar{i}jk}(t)$, $P_{i\bar{j}\bar{k}}(t)$, $P_{\bar{i}\bar{j}\bar{k}}(t)$
$\phi(t,\mathbf{x})$	$\varphi(t)$
$\mathtt{n}(t,\mathbf{x})$	$n(t)$

Here, we have explicitly re-instated the dependence on the coordinates. Working locally in one coordinate chart $(t,\mathbf{x})$ and keeping the spatial point $\mathbf{x}$ fixed, the time coordinate t of supergravity can be identified with the parameter along the world-line of the coset model, as already anticipated in the table above.

By comparing the supergravity equations and the σ-model equations, we see that we can match large parts of the equations by demanding that the supergravity quantities evaluated at the fixed spatial point $\mathbf{x}$ correspond to the σ-model quantities. In other words, the *dynamical dictionary* which maps (parts of) the supergravity equations to the σ-model equations consists of letting

$$\mathtt{P}_{i\bar{j}}(t,\mathbf{x}) \leftrightarrow P_{i\bar{j}}(t) \quad \text{for all } t \text{ (and fixed } \mathbf{x}) \tag{3.25}$$

and similarly for the other objects in the table. To be more precise the σ-model equations (3.22)–(3.24) coincide with the left-hand sides of the equations of motion (2.18), (2.15) and (2.20) and the Bianchi identity (2.23) of $D = 10$ pure supergravity. However, the terms on the right-hand sides do not match in this correspondence which we now discuss in more detail, together with the vector constraint equations (2.22a) and (2.22b) which do not have corresponding σ-model equations.

We begin with the tensor equation (2.15) and the two vector constraints (2.22a) and (2.22b), where we want to show that, in some sense, we have only neglected spatial derivatives. Our identification fixed an arbitrary spatial position $\mathbf{x}$ in a coordinate chart and considered the evolution of the fields $\mathtt{P}$ in time only. However, direct spatial derivatives ∂_q of $\mathtt{P}$ (and $\mathtt{Q}$) are not expected to be represented in this truncated correspondence. They are thought to be represented by higher level fields $P^{(\ell>1)}$ [20] which we ignored in the σ-model and therefore this disagreement is not surprising. In the tensor equation (2.15), we have a term in the second line which seems not to be directly connected to a spatial derivative. However, if we assume that the $SO(9)\times SO(9)$ symmetry can be gauged and if we introduce an $SO(9)\times SO(9)$ valued zehnbein $e_K{}^p := (e_k{}^p, e_{\bar{k}}{}^p)$ analogously to the $SU(8)$ valued elfbein in [5], we can write the second line of the tensor equation (2.15) as follows.[15]:

$$\delta_i^I \delta_{\bar{j}}^J \left(\mathtt{n}\partial_p \left[e_K{}^p \delta^{KL} \mathtt{P}_{ILJ} \right] + \mathrm{e}^{2\phi} \mathtt{P}_I{}^{KL} \mathtt{P}_{JKL} \right), \tag{3.26}$$

[15] δ^{KL} denotes the second invariant tensor of $SO(9)\times SO(9)$ which is similiarly defined as the first one, η^{KL}, but with $\delta^{\bar{k}\bar{l}} = +\delta^{kl}$.

where indices have been raised with η^{KL}. Furthermore, the two vector equations combine to a single one

$$0 = -\mathrm{n}\partial_m \left[e_K{}^m \eta^{KL} \mathrm{n}^{-1} \mathrm{P}_{IL} \right] + \mathrm{e}^{2\phi} \mathrm{P}_I{}^{KL} \mathrm{n}^{-1} \mathrm{P}_{KL} \\ + \mathrm{n}\mathrm{e}^{-\phi} \partial_I \left(\mathrm{n}^{-1} \mathrm{e}^{\phi} \partial_t \phi \right), \quad (3.27)$$

if we set $P_{ij} = P_{\bar{\imath}\bar{\jmath}} = 0$. This looks like an $SO(9) \times SO(9)$-covariant derivative D_m with respect to *space* whereas before we only considered D_t. This would imply that the term[16]

$$\mathrm{e}^{2\phi} \mathrm{P}_i{}^{KL} \mathrm{P}_{\bar{\jmath}KL} = \mathrm{e}^{2\phi} \left(\mathrm{P}_{ikl} \mathrm{P}_{\bar{\jmath}kl} + \mathrm{P}_{i\bar{k}\bar{l}} \mathrm{P}_{\bar{\jmath}\bar{k}\bar{l}} - 2\mathrm{P}_{\bar{k}il} \mathrm{P}_{l\bar{k}\bar{\jmath}} \right) = -\frac{1}{2} \mathrm{n}^2 \mathrm{e}^{-2\phi} \Omega_{(i|kl} \Omega_{\bar{\jmath})lk}$$

should not be separated from the discussion of spatial derivatives. The third line in the tensor equation (2.15) is an explicit spatial derivative, which concludes the discussion of this equation.

Turning to the scalar equations we see that there are two different mismatches. The first one is the common term in the second lines of (2.18) and (2.20) whose value depends on the choice of coordinate system and local Lorentz gauge. These freedoms could be used, e.g., to let this term vanish or to fix $v_a(t, \mathbf{x}) = 0$[17].

The second mismatch in the scalar equations concerns the final line in (2.18). However, in order to account for this term we also have the densitised lapse n at our disposal which we can choose as convenient as long as it does not vanish. Evidently, choosing n suitably we can cause this line to vanish identically at the fixed spatial point **x**.

A fascinating possibility for taking the terms containing spatial gradients into account in the correspondence was proposed in [20], where it was suggested that they are related to some higher level fields of the σ-model which have been truncated in our analysis. The proper interpretation of these terms is still an open problem.

4 Fermions and supersymmetry

In this section, we extend our analysis to take into account the fermionic degrees of freedom. We will in particular check that the supersymmetry transformations (2.4) can be stated in an $SO(9) \times SO(9)$ covariant form, which is necessary for the $K(DE_{10})$ covariance that is conjectured to hold if all levels ℓ are fixed appropriately. We start with the discussion of the fermionic variations, before we move on to the bosonic fields.

16 The following equality shows that this term is precisely the term which is also problematic in the Einstein equation of the maximal theory [25].

17 By (2.10), this condition is equivalent to the vanishing of the spacial trace of the original $D = 10$ spin connection. A similar observation for the improvement of the equivalence of the E_{10} σ-model equations of motion and the ones for the maximal supersymmetric theory was made in [25].

4.1 $K(DE_{10})$ covariance of the fermionic transformations

The supersymmetry transformation of the fermions λ and ψ_M have been stated in Eqs. (2.4d) and (2.4e). In order to uncover the $SO(9)\times SO(9)$ covariance, we have to reparametrise the fermions as we have done with the bosons in (2.8) and (2.9), where we explicitly break the $SO(1,9)$ covariance again by treating the time component in a special way

$$\begin{aligned}
\varepsilon &:= (\det(\hat{E}_m{}^a))^{-\frac{1}{2}}\epsilon,\\
\chi_t &:= (\det(\hat{E}_m{}^a))^{-\frac{1}{2}}\left(\psi_t - N\Gamma_0\Gamma^a\hat{\psi}_a\right),\\
\chi_{\bar{\imath}} &:= (\det(\hat{E}_m{}^a))^{\frac{1}{2}}\left(\frac{1}{4}\Gamma_{\bar{\imath}}\lambda - \hat{\psi}_{\bar{\imath}}\right),\\
\chi &:= (\det(\hat{E}_m{}^a))^{\frac{1}{2}}\left(\Gamma^a\hat{\psi}_a - \frac{1}{4}\lambda\right).
\end{aligned}\tag{4.1}$$

The hats denote, as in Sect. 2, the projection onto the orthonormal frame $\hat{\psi}_a \equiv \hat{E}_a{}^m\,\psi_m$. The fermions in (4.1) can be assigned $SO(9)\times SO(9)$ transformation properties as follows: All spinors transform as 16-component Majorana spinors of the first $SO(9)$ factor and trivially under the second $SO(9)$ except for $\chi_{\bar{\imath}}$ which transforms as a vector. This is consistent with the different $SO(1,9)$ chiralities of the type I fermions since single (32×32) Γ-matrices, defined in the Appendix, intertwine between these two chiralities. The $SO(9)\times SO(9)$ representations considered here are the chiral half of the representations of [17,36].

Using the redefinitions (4.1), (2.8) and (2.9) in the supersymmetry variations (2.4d) and (2.4e), we arrive at the following results in leading fermion order:

$$\begin{aligned}
\delta_\varepsilon\chi_t &= \partial_t\varepsilon + \frac{1}{4}\Gamma^{ij}\varepsilon\mathrm{Q}_{ij} + \frac{1}{3!}\mathrm{e}^{\phi}\Gamma^{ijk}\Gamma^0\varepsilon\mathrm{P}_{ijk}\\
&\quad + \mathrm{n}\mathrm{e}^{-\phi}\Gamma^a\Gamma^0\left\{-\partial_a\varepsilon + \frac{1}{2}\varepsilon\partial_a\phi + \frac{1}{2}\varepsilon\mathrm{v}_a\right\}\\
&\quad + \frac{1}{2}\mathrm{n}\mathrm{e}^{-\phi}\Gamma^a\Gamma^0\varepsilon\left[\mathrm{n}^{-1}\partial_a\mathrm{n} - \partial_a\phi\right],
\end{aligned}\tag{4.2a}$$

$$\begin{aligned}
\delta_\varepsilon\chi_{\bar{\imath}} &= -\frac{1}{2}\mathrm{n}^{-1}\Gamma^j\Gamma^0\varepsilon\mathrm{P}_{j\bar{\imath}} + \frac{1}{2}\mathrm{n}^{-1}\mathrm{e}^{\phi}\Gamma^{jk}\varepsilon\mathrm{P}_{\bar{\imath}jk}\\
&\quad + \mathrm{e}^{-\phi}\left\{-\partial_{\bar{\imath}}\varepsilon + \frac{1}{2}\varepsilon\partial_{\bar{\imath}}\phi\right\},
\end{aligned}\tag{4.2b}$$

$$\begin{aligned}
\delta_\varepsilon\chi &= -\frac{1}{2}\mathrm{n}^{-1}\Gamma^0\varepsilon\partial_t\phi - \frac{1}{3!}\mathrm{n}^{-1}\mathrm{e}^{\phi}\Gamma^{ijk}\varepsilon\mathrm{P}_{ijk}\\
&\quad - \mathrm{e}^{-\phi}\Gamma^a\Gamma^0\left\{-\partial_a\varepsilon + \frac{1}{2}\varepsilon\partial_a\phi + \frac{1}{2}\varepsilon\mathrm{v}_a\right\},
\end{aligned}\tag{4.2c}$$

where we used the abbreviations defined in (2.11) and (2.12). We observe again that the $SO(9) \times SO(9)$ structure is preserved. From (4.2a) one can also read off the beginning of an extension of the $SO(9) \times SO(9)$ covariance to $K(DE_{10})$ covariance along the lines of [17,31–34] as we now discuss.

The key to unravelling the $K(DE_{10})$ structure is to assume that the bosonic σ-model can be extended to a $K(DE_{10})$ gauge invariant and locally *super-symmetric* $D = 1$ coset model. This requires the introduction of fermionic fields transforming in $K(DE_{10})$ representations. In such a model there will be a superpartner χ_t to the lapse n which acts as the one-dimensional gravitino and therefore should transform into a $K(DE_{10})$ covariant derivative of the super-symmetry parameter

$$\delta_\varepsilon \chi_t = \mathcal{D}_t \varepsilon = \Big(\partial_t - \frac{1}{2} Q_{ij} J^{ij} - \frac{1}{2} Q_{\bar{\imath}\bar{\jmath}} J^{\bar{\imath}\bar{\jmath}} - \frac{1}{3!} P_{ijk} J^{ijk} - \frac{1}{2!} P_{\bar{\imath}jk} J^{\bar{\imath}jk} \\ - \frac{1}{2!} P_{\bar{\imath}\bar{\jmath}k} J^{\bar{\imath}\bar{\jmath}k} - \frac{1}{3!} P_{\bar{\imath}\bar{\jmath}\bar{k}} J^{\bar{\imath}\bar{\jmath}\bar{k}} + \dots \Big) \varepsilon \tag{4.3}$$

in Borel gauge (3.20). By comparing this relation to (4.2a), we can read off the form the $K(DE_{10})$ generators take as a matrix representation on ε. On the first two 'levels', the result is

$$\begin{aligned} J^{ij}\varepsilon &= -\frac{1}{2}\Gamma^{ij}\varepsilon, \quad J^{\bar{\imath}\bar{\jmath}}\varepsilon = 0, \\ J^{ijk}\varepsilon &= -\Gamma^{ijk}\Gamma^0\varepsilon, \quad J^{\bar{\imath}jk}\varepsilon = 0, \quad J^{\bar{\imath}\bar{\jmath}k}\varepsilon = 0, \quad J^{\bar{\imath}\bar{\jmath}\bar{k}}\varepsilon = 0. \end{aligned} \tag{4.4}$$

This implies that only two generators are represented non-trivially. In [32,34] it was demonstrated in the maximally supersymmetric case that such restricted transformation rules can be sufficient to prove that χ_t is a consistent *unfaithful* representation of $K(DE_{10})$. It follows from (4.3) that ε has to transform in the same $K(DE_{10})$ representation. We now give the criterion for establishing such a consistent representation and show that it is satisfied in the present situation.

As shown in [34,35] the generators of the compact subgroup of a Kac–Moody group can be written in terms of simple generators $x_i = e_i - f_i$ (deduced from the Chevalley generators of Sect. 3.2) and defining relations induced by the relations satisfied by the generators e_i and f_i. Working in an $SO(9)\times SO(9)$ covariant formalism as we are doing here, the only consistency relation to check turns out to be

$$\big[x_{10}, [x_{10}, x_3]\big] + x_3 = 0, \tag{4.5}$$

where x_3 and x_{10} are given in terms of the antisymmetric generators (3.13) and (3.21b) via [17]

$$\begin{aligned} x_3 &= J^{34} + J^{\bar{3}\bar{4}}, \\ x_{10} &= \frac{1}{2}\Big(J^{123} + J^{\bar{1}23} + J^{1\bar{2}3} + J^{12\bar{3}} + J^{\bar{1}\bar{2}3} + J^{\bar{1}2\bar{3}} + J^{1\bar{2}\bar{3}} + J^{\bar{1}\bar{2}\bar{3}}\Big). \end{aligned} \tag{4.6}$$

By substituting these generators into the consistency relation (4.5) in a specific matrix representation acting on a vector space V, one can check whether V is a representation space of $K(DE_{10})$.[18]

The specific expressions for $J^{ij}, J^{\bar{\imath}\bar{\jmath}}$ and J^{IJK} found in (4.4) satisfy the relation (4.5) as can be checked by straight-forward Γ-algebra. In terms of (32×32) matrices the representation matrices Γ^{ij} and $\Gamma^{ijk}\Gamma^0$ are block-diagonal and so act consistently on the projected 16-dimensional (chiral) spinor χ_t. Naturally, there is also a representation on the other 16 components whose representation matrices are given by

$$J^{ij}\eta = 0, \qquad J^{\bar{\imath}\bar{\jmath}}\eta = -\frac{1}{2}\Gamma^{ij}\eta,$$
$$J^{ijk}\eta = 0, \qquad J^{\bar{\imath}jk}\eta = 0, \qquad J^{\bar{\imath}\bar{\jmath}k}\eta = 0, \quad J^{\bar{\imath}\bar{\jmath}\bar{k}}\eta = -\Gamma^{ijk}\Gamma^0\eta. \tag{4.7}$$

Therefore, there are two inequivalent 16-dimensional unfaithful spinor representations of $K(DE_{10})$ as already anticipated in [36].[19] One can write down a similar consistent unfaithful representation of $K(DE_{10})$ on $\chi_{\bar{\imath}}$ which has dimension 144; to deduce its transformation laws one needs to rewrite the fermionic equation of motion and interpret this as a $K(DE_{10})$ covariant derivative of $\chi_{\bar{\imath}}$ [32,34].

Having discussed the first few terms in (4.2a), we now briefly explore the remaining structure of Eqs. (4.2). If we compare the two scalar equations of motion (2.18) and (2.20) with the corresponding supersymmetry transformations (4.2a) and (4.2c), the similarities are striking: a part from the global factor of $-\mathtt{n}$, the second lines of (4.2a) and (4.2c) completely agree as in (2.18) and (2.20) and the third line of (4.2a) has the same structure as the one in (2.18). The final interpretation of these lines is still an open problem. However, the proposals which we discussed in Sect. 3.4 can be equally applied here.

4.2 Supersymmetry transformation of the bosons

The redefined variables $\mathtt{P}$ and $\mathtt{Q}$ defined in (2.11) and (2.12) have been assigned $SO(9) \times SO(9)$ transformation properties. Given the $SO(9) \times SO(9) \subset K(DE_{10})$ fermions of the preceding section it is natural to study the $SO(9) \times SO(9)$ and $K(DE_{10})$ transformation properties of $\mathtt{P}$ and $\mathtt{Q}$ after a supersymmetry transformation δ_ε.

We recall the definitions of $\mathtt{P}_{i\bar{\jmath}}$, $\mathtt{Q}_{ij}$ and $\mathtt{Q}_{\bar{\imath}\bar{\jmath}}$ from (2.11), where the latter two play the role of $SO(9) \times SO(9)$ gauge connections in the equations of motion (2.15) and (2.23). It will prove useful to define analogous quantities by replacing

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18 We reiterate that we assume $SO(9) \times SO(9)$ covariance of all generators—if this is not guaranteed there are additional relations which need to be verified.

19 The 32-dimensional unfaithful spinor representation of $K(E_{10})$ [31,32] decomposes into the sum of these two inequivalent 16-dimensional spinor representations of the $K(DE_{10})$ subgroup of $K(E_{10})$.

the time derivative ∂_t by the supersymmetry variation δ_ϵ in (2.11) and hence get[20]

$$\Lambda_{i\bar{j}} := e_{(i}{}^m \delta_\epsilon e_{m|\bar{j})} - \frac{1}{2} e_i{}^m e_{\bar{j}}{}^n \delta_\epsilon B_{mn}, \tag{4.8a}$$

$$\Sigma_{ij} := e_{[i}{}^m \delta_\epsilon e_{m|j]} + \frac{1}{2} e_i{}^m e_j{}^n \delta_\epsilon B_{mn}, \tag{4.8b}$$

$$\Sigma_{\bar{i}\bar{j}} := e_{[\bar{i}}{}^m \delta_\epsilon e_{m|\bar{j}]} - \frac{1}{2} e_{\bar{i}}{}^m e_{\bar{j}}{}^n \delta_\epsilon B_{mn}. \tag{4.8c}$$

Substituting in the supersymmetry variations (2.4) as well as the redefinitions of the bosons (2.8) and the fermions (4.2), we find explicitly

$$\Lambda_{i\bar{j}} = -\mathrm{i}\varepsilon\Gamma_i \chi_{\bar{j}}. \tag{4.9}$$

Furthermore, in analogy to the $SO(9) \times SO(9)$ covariant derivative D_t defined in (2.16), we define an $SO(9) \times SO(9)$ covariant supersymmetry transformation $\underline{\delta}_\varepsilon$ by adding a local (in time) and field dependent $SO(9) \times SO(9)$ gauge transformation δ_Σ to the supersymmetry variation δ_ε

$$\underline{\delta}_\varepsilon = \delta_\varepsilon + \delta_\Sigma, \tag{4.10}$$

as was done in the $SU(8)$ case in [5].[21] With the definitions (2.2) and (2.11), a short calculation yields the identities

$$\underline{\delta}_\varepsilon \mathrm{P}_{i\bar{j}} = \delta_\varepsilon \mathrm{P}_{i\bar{j}} + \Sigma_{ik} P_{k\bar{j}} + \Sigma_{\bar{j}\bar{k}} P_{i\bar{k}} = D_t \Lambda_{i\bar{j}}, \tag{4.11a}$$

$$\underline{\delta}_\varepsilon \mathrm{Q}_{ij} = \delta_\varepsilon \mathrm{Q}_{ij} + 2\Sigma_{[i|l} \mathrm{Q}_{l|j]} - \partial_t \Sigma_{ij} = 2\mathrm{P}_{[i|\bar{l}} \Lambda_{j]\bar{l}}, \tag{4.11b}$$

$$\underline{\delta}_\varepsilon \mathrm{Q}_{\bar{i}\bar{j}} = \delta_\varepsilon \mathrm{Q}_{\bar{i}\bar{j}} + 2\Sigma_{[\bar{i}|\bar{l}} \mathrm{Q}_{\bar{l}|\bar{j}]} - \partial_t \Sigma_{\bar{i}\bar{j}} = 2\mathrm{P}_{l[\bar{i}} \Lambda_{l|\bar{j}]}, \tag{4.11c}$$

where the gauge fields Q_{ij} and $\mathrm{Q}_{\bar{i}\bar{j}}$ transform with explicit time derivatives of the gauge transformation parameters Σ_{ij} and $\Sigma_{\bar{i}\bar{j}}$ as usual.

For the fields defined in (2.12), we get

$$\mathrm{n}e^{-2\phi} \underline{\delta}_\varepsilon \left(\mathrm{n}^{-1} e^{2\phi} \mathrm{P}_{ijk} \right) = -3\Lambda_{[i|\bar{l}} \mathrm{P}_{\bar{l}|jk]} - \frac{3}{2} \partial_{[i} \Sigma_{jk]}, \tag{4.12a}$$

$$\begin{aligned} \mathrm{n}e^{-2\phi} \underline{\delta}_\varepsilon \left(\mathrm{n}^{-1} e^{2\phi} \mathrm{P}_{\bar{i}jk} \right) &= -\Lambda_{l\bar{i}} \mathrm{P}_{ljk} - 2\Lambda_{[j|\bar{l}} \mathrm{P}_{k]\bar{i}\bar{l}} \\ &\quad - \frac{1}{2} \partial_{\bar{i}} \Sigma_{jk} + \partial_{[j} \Lambda_{k]\bar{i}}, \end{aligned} \tag{4.12b}$$

[20] As in Sect. 2.2, there is no distinction between barred and unbarred frame indices at this point.

[21] As $\Sigma = (\Sigma_{ij}, \Sigma_{\bar{i}\bar{j}})$ is of order two in fermions and as we have neglected higher order fermion contributions throughout the paper, the introduction of the covariant supersymmetry transformation $\underline{\delta}_\varepsilon$ does not affect the discussion of the fermions in Sect. 4.1.

$$\mathrm{n}e^{-2\phi}\underline{\delta}_\varepsilon\left(\mathrm{n}^{-1}e^{2\phi}\mathrm{P}_{i\bar{j}\bar{k}}\right) = -\Lambda_{i\bar{l}}\mathrm{P}_{\bar{l}\bar{j}\bar{k}} - 2\Lambda_{l[\bar{j}}\mathrm{P}_{\bar{k}]il} + \frac{1}{2}\partial_i\Sigma_{\bar{j}\bar{k}} - \partial_{[\bar{j}}\Lambda_{|i|\bar{k}]}, \quad (4.12c)$$

$$\mathrm{n}e^{-2\phi}\underline{\delta}_\varepsilon\left(\mathrm{n}^{-1}e^{2\phi}\mathrm{P}_{\bar{i}\bar{j}\bar{k}}\right) = -3\Lambda_{l[\bar{i}}\mathrm{P}_{|l|\bar{j}\bar{k}]} + \frac{3}{2}\partial_{[\bar{i}}\Sigma_{\bar{j}\bar{k}]}. \quad (4.12d)$$

Again, we observe the $SO(9) \times SO(9)$ covariance. The appearance of spatial derivatives of the $SO(9) \times SO(9)$ transformation parameter Σ indicates that we should gauge the symmetry group $SO(9) \times SO(9)$ with respect to space–time in fact by introducing also an $SO(9) \times SO(9)$ derivative D_m as discussed in Sect. 3.4. However, this is not the way we want to pursue; instead we now study supersymmetry transformations in the $DE_{10}/K(DE_{10})$ coset structure.

In the derivation of the coset equations of motion in Sect. 3, we started with an element $\mathcal{V}$ in the group coset $DE_{10}/K(DE_{10})$ and considered variations of $\mathcal{V}$ under a general variation δ. As the supersymmetry operator is also realised as a derivative operator δ_ε, which commutes with the time derivative ∂_t, we can use the same chain of arguments of Sect. 3 to derive the supersymmetry variation of $\mathcal{P}$ and $\mathcal{Q}$. This means we first decompose the Lie(DE_{10}) valued expression

$$\delta_\varepsilon\mathcal{V}\mathcal{V}^{-1} = \tilde{\Lambda} + \tilde{\Sigma} \quad (4.13)$$

into generators $\tilde{\Sigma} \in \mathrm{Lie}(K(DE_{10}))$ and $\tilde{\Lambda} \in \mathrm{Lie}(DE_{10}/K(DE_{10}))$. Then, we parametrise $\tilde{\Sigma}$ and $\tilde{\Lambda}$ similiarly to (3.20) in a level decomposition truncated for $\ell \geq 2$, i.e.

$$\begin{aligned}\tilde{\Lambda} &= \delta_\varepsilon\varphi T + \tilde{\Lambda}_{i\bar{j}}S^{i\bar{j}} + \frac{1}{3!}e^{\varphi}\tilde{\Lambda}_{IJK}S^{IJK},\\ \tilde{\Sigma} &= \frac{1}{2}\tilde{\Sigma}_{ij}J^{ij} + \frac{1}{2}\tilde{\Sigma}_{\bar{i}\bar{j}}J^{\bar{i}\bar{j}} + \frac{1}{3!}e^{\varphi}\tilde{\Lambda}_{IJK}J^{IJK},\end{aligned} \quad (4.14)$$

where we again work in Borel gauge. Finally, from the fact that both derivative operators commute it follows that we get the variations after projecting on the $\ell = 0$ generators $S^{i\bar{j}}$, T and the gauge orbit generators J^{ij} and $J^{\bar{i}\bar{j}}$

$$\underline{\delta}_\varepsilon(\partial_t\varphi) = \delta_\varepsilon(\partial_t\varphi) = \partial_t(\delta_\varepsilon\varphi), \quad (4.15a)$$

$$\underline{\delta}_\varepsilon P_{i\bar{j}} = \delta_\varepsilon P_{i\bar{j}} + \tilde{\Sigma}_{ik}P_{k\bar{j}} + \tilde{\Sigma}_{\bar{j}\bar{k}}P_{i\bar{k}} = D_t\tilde{\Lambda}_{i\bar{j}}, \quad (4.15b)$$

$$\underline{\delta}_\varepsilon Q_{ij} = \delta Q_{ij} + 2\tilde{\Sigma}_{[i|k}Q_{k|j]} - \partial_t\tilde{\Sigma}_{ij} = 2P_{[i|\bar{l}}\tilde{\Lambda}_{j]\bar{l}}, \quad (4.15c)$$

$$\underline{\delta}_\varepsilon Q_{\bar{i}\bar{j}} = \delta Q_{\bar{i}\bar{j}} + 2\tilde{\Sigma}_{[\bar{i}|\bar{k}}Q_{\bar{k}|\bar{j}]} - \partial_t\tilde{\Sigma}_{\bar{i}\bar{j}} = 2P_{k[\bar{i}}\tilde{\Lambda}_{k|\bar{j}]}, \quad (4.15d)$$

where we have used the covariant derivative D_t as in (3.22b). With the covariant supersymmetry transformation $\underline{\delta}_\varepsilon$ we can write the equations resulting from the projection onto the $\ell = 1$ generators S^{ijk}, $S^{\bar{i}jk}$, $S^{i\bar{j}\bar{k}}$ and $S^{\bar{i}\bar{j}\bar{k}}$ as

$$\underline{\delta}_\varepsilon P_{ijk} - 3\tilde{\Lambda}_{[i|\bar{l}}P_{\bar{l}|jk]} = D_t\tilde{\Lambda}_{ijk} - 3P_{[i|\bar{l}}\tilde{\Lambda}_{\bar{l}|jk]}, \tag{4.16a}$$

$$\underline{\delta}_\varepsilon P_{\bar{i}jk} - \tilde{\Lambda}_{l\bar{i}}P_{ljk} + 2\tilde{\Lambda}_{[j|\bar{l}}P_{k]\bar{i}\bar{l}} = D_t\tilde{\Lambda}_{\bar{i}jk} - P_{l\bar{i}}\tilde{\Lambda}_{ljk} + 2P_{[j|\bar{l}}\tilde{\Lambda}_{k]\bar{i}\bar{l}}, \tag{4.16b}$$

$$\underline{\delta}_\varepsilon P_{i\bar{j}\bar{k}} - 2\tilde{\Lambda}_{m[\bar{j}}P_{\bar{k}]im} + \tilde{\Lambda}_{i\bar{l}}P_{\bar{l}\bar{j}\bar{k}} = D_t\tilde{\Lambda}_{i\bar{j}\bar{k}} - 2P_{m[\bar{j}}\tilde{\Lambda}_{\bar{k}]im} + P_{i\bar{l}}\tilde{\Lambda}_{\bar{l}\bar{j}\bar{k}}, \tag{4.16c}$$

$$\underline{\delta}_\varepsilon P_{\bar{i}\bar{j}\bar{k}} - 3\tilde{\Lambda}_{l[\bar{i}}P_{|l|\bar{j}\bar{k}]} = D_t\tilde{\Lambda}_{\bar{i}\bar{j}\bar{k}} - 3P_{l[\bar{i}}\tilde{\Lambda}_{|l|\bar{j}\bar{k}]}. \tag{4.16d}$$

It should be noted that an inclusion of higher level terms $\ell \geq 2$ in the expansions (3.20) and (4.14) above does not alter Eqs. (4.15) and (4.16) in contradistinction to the equations of motion (3.22) and (3.23). This is a general property of the Borel gauge [34].

As we have identified the supergravity variables $\mathtt{P}$ and $\mathtt{Q}$ at a fixed spatial point $\mathbf{x}$ with the coset variables P and Q in Sect. 3.4, a comparison of Eq. (4.11) and (4.15) forces us to identify $\Lambda_{i\bar{j}}$ with $\tilde{\Lambda}_{i\bar{j}}$ and $(\Sigma_{ij}, \Sigma_{\bar{i}\bar{j}})$ with $(\tilde{\Sigma}_{ij}, \tilde{\Sigma}_{\bar{i}\bar{j}})$, respectively. Using the explicit form of $\tilde{\Lambda}_{i\bar{j}}$ in terms of fermion bilinears (4.9) and the $K(DE_{10})$ transformation rules for the fermions deduced in Sect. 4.1, we could in principle compute $\tilde{\Lambda}_{IJK}$ from this by comparing it with a $K(DE_{10})$ transformation of the coset representation. We can obtain an independent answer for $\tilde{\Lambda}_{IJK}$ by comparing with supergravity. It is not guaranteed that the two answers will agree. In [34] a similar analysis was carried out in the maximal $D = 11$ supergravity context and some but not all expressions for the analogues of $\tilde{\Lambda}_{IJK}$ agree. We take this as an indication that there is a disparity between the unfaithful, finite-dimensional fermionic $K(DE_{10})$ representation and the infinite-dimensional coset representation, which is in conflict with supersymmetry on the coset side.

5 Discussion

In this paper, we have rewritten the bosonic and fermionic fields of pure type I supergravity in terms of variables which we assigned to representations of $SO(9) \times SO(9)$ [cf. (2.11), (2.12) and (4.1)]. The relevant bosonic representations are identical to those that arise in the D_9 level decomposition of DE_{10} on the levels $\ell = 0, 1$ as shown in Sect. 3.2. In Sect. 4.1 we also showed that some of the relevant fermionic representations of $SO(9) \times SO(9)$ can be consistently extended to unfaithful representations of $K(DE_{10})$.

At the dynamical level we demonstrated that the bosonic equations of pure type I supergravity in this parametrisation (evaluated at a fixed spatial point) coincide with those derived from a simple $D = 1$ non-linear σ-model on $DE_{10}/K(DE_{10})$ truncated consistently beyond $\ell = 1$ up to a number of terms which can either be gauged away or can be argued to be of the form of (generalised) spatial gradients, see Sect. 3.4.[22] It would be very interesting to see

[22] Our analysis was always at the level of the equations of motion and not at the level of the action.

whether the full $D = 10$ equations can be cast in $SO(9) \times SO(9)$ covariant form, analogous to the treatment in [5]. Our focus was not on this question but rather if we can extend the (partial) $SO(9) \times SO(9)$ covariance to a (partial) $K(DE_{10})$ covariance to further test the ideas of [20]. As pointed out for example below (3.23) the truncated coset model equations of motion include terms which are part of a $K(DE_{10})$ covariant formulation and these terms agree with identical terms in the supergravity equations (2.23). A similar phenomenon was observed for the supersymmetry variation of certain fermionic fields, see (4.3).

We consider our results as evidence that $K(DE_{10})$ might be a dynamical symmetry of the pure type I theory. There are also a number of conundrums related to our analysis, some of which were already hinted at.

As is well known, the pure type I supergravity theory is not anomaly free. However, by adding appropriate vector multiplets [37] the anomalies can be cancelled [38,39]. The possibilities which are realised as string theory low energy effective theories are those which have vector multiplets transforming as Yang–Mills fields of either $SO(32)$ or $E_8 \times E_8$. The inclusion of these non-Abelian symmetries in the context of Kac–Moody symmetries is poorly understood. Augmenting the theory (2.1) by Abelian or non-Abelian vector fields changes the associated cosmological billiard from DE_{10} to a group called BE_{10} [18] but, at the level of coset model, BE_{10} is not appropriate for accommodating more than a single Abelian vector field. In [23] multiple Abelian vector fields were added to the pure type I theory by increasing the rank of the Kac–Moody symmetry and changing the real form.[23] A proper understanding of the non-Abelian symmetries from a Kac–Moody algebraic point of view is lacking at the moment.

Another interesting challenge is to extend the bosonic $DE_{10}/K(DE_{10})\sigma$-model of (3.2) to a locally supersymmetric model in $D = 1$. With the fermionic representations employed in this paper, it appears impossible to construct such a model. In fact, in the present situation there is no non-vanishing combination that can be constructed from terms bilinear in the fermions (4.1) of the form $\tilde{\Lambda}_{\bar{\imath}\bar{\jmath}k}$ whereas such an expression necessarily appears in the supersymmetry variation of the $\ell = 1$ field $P_{\bar{\imath}\bar{\jmath}k}$ due to (4.12) and (4.16).

Finally, it is crucial to bring the 'gradient conjecture' of [20] back into view. According to this conjecture the σ-model can capture the *full* dynamics in a neighbourhood of the fixed spatial point **x** by translating the information about all spatial gradients of the supergravity fields into higher level degrees of freedom of the σ-model. As discussed in Sect. 3.4, the concrete realisation of this translation is still an open problem.

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[23] See also [40] for a discussion of a non-maximal theory with vector fields obtained by orbifolding the maximal theory. In this analysis DE_{10} and further extensions of it similar to those of [23] appear.

Appendix: Conventions for Γ-matrices

We use the same conventions for Γ-matrices as [17] which we summarise for completeness.

The (32 × 32) real Γ-matrices Γ^A for $A = 0, \ldots, 10$ of $SO(1,10)$ are defined in terms of the real symmetric (16 × 16) γ-matrices γ^i ($i = 1, \ldots, 9$) of $SO(9)$ by

$$\Gamma^0 = \begin{pmatrix} 0 & -\mathbf{1}_{16} \\ \mathbf{1}_{16} & 0 \end{pmatrix}, \quad \Gamma^{10} = \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}. \quad \text{(A.1)}$$

The matrix Γ^0 is the charge conjugation matrix in $D = 11$. After descending to $SO(1,9)$ the matrix $\frac{1}{2}(\mathbf{1}_{32} \pm \Gamma^{10})$ serves as the projector on the two chiral spinors in $D = 10$. The type I fermions ψ_M and λ discussed in the paper have been projected from 32-component spinors to opposite chiralities.

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