

# Pure type I supergravity and $DE_{10}$

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**Abstract:** We establish a dynamical equivalence between the bosonic part of pure type I supergravity in  $D = 10$  and a  $D = 1$  non-linear  $\sigma$ -model on the Kac–Moody coset space  $DE_{10}/K(DE_{10})$  if both theories are suitably truncated. To this end we make use of a decomposition of  $DE_{10}$  under its regular  $SO(9,9)$  subgroup. Our analysis also deals partly with the fermionic fields of the supergravity theory and we define corresponding representations of the generalized spatial Lorentz group  $K(DE_{10})$ .

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## 1 Introduction

Soon after the construction of the maximally supersymmetric  $D = 11$  gravity theory [1] it was realised that this theory exhibits exceptional hidden symmetries  $E_7$  and  $E_8$  upon dimensional reduction from  $D = 11$  to  $D = 4$  and  $D = 3$ , respectively [2, 3]. Much research has been devoted to this unexpected feature of maximal supergravity and its relevance for string theory [4, 5, 6, 7, 8]. However, already since the early days of the study of hidden symmetries it has been clear that also theories with non-maximal supersymmetry (or, in fact, no supersymmetry at all) can exhibit unexpected hidden symmetries upon dimensional reduction [9, 10, 11, 12, 13]. For the case of

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all simple and split symmetry groups  $G$  in  $D = 3$  the question which higher-dimensional theories give rise to the hidden symmetry  $G$  upon dimensional reduction has been answered in [14] and there are only very few groups  $G$  for which the associated (oxidised) theory has supersymmetry. One example is the group  $D_8 \equiv SO(8, 8)$  which in  $D = 3$  is the hidden symmetry of the pure type I supergravity theory in  $D = 10$  [15, 16] (which has half-maximal supersymmetry) after dimensional reduction. Further dimensional reduction to  $D = 1$  was conjectured [10, 12] to lead to an infinite-dimensional symmetry of hyperbolic Kac–Moody type, denoted  $DE_{10}$ , which will be defined below. As pointed out in [17] the conjectured symmetry  $DE_{10}$  is consistent with the embedding of pure type I into the maximal theory whose conjectured symmetry is  $E_{10}$  since  $DE_{10}$  is a proper subgroup of  $E_{10}$ .

In this paper we revisit the hidden symmetries of pure type I supergravity motivated by recent results concerning such infinite-dimensional symmetries. Near a space-like singularity it was found that the effective and dominant dynamics of the model can be mapped to a so-called cosmological billiard system whose massless relativistic billiard ball bounces off the walls of a ten-dimensional (auxiliary) billiard table [18]. The location of these walls is identical to that of the bounding walls of the fundamental Weyl chamber of  $DE_{10}$  [18] (see also [19] for a review of cosmological billiards).

In analogy with the  $E_{10}/K(E_{10})$  model developed in [20] for the maximally supersymmetric case, we here study a  $D = 1$  geodesic model on the infinite-dimensional coset space  $DE_{10}/K(DE_{10})$ , extending the cosmological billiard dynamics. The dynamical behaviour of the  $\sigma$ -model will be related to that of the pure type I theory.  $K(DE_{10})$  refers to the (formally) maximal compact subgroup of  $DE_{10}$  which plays the role of a generalised spatial Lorentz group. In the context of the  $E_{11}$  approach to Kac–Moody symmetries [21, 22], the bosonic sectors of  $D = 10$  type I theories (also with abelian vector fields) have been investigated in [23] and the equations of motion were derived from a  $DE_{11}$  analysis in the pure type I case. The non-maximal pure  $D = 5$ ,  $N = 2$  supergravity has been studied from a Kac–Moody perspective in [24].

Our main result is that a truncated version of the bosonic pure type I equations of motion is dynamically equivalent to a truncation of the equations of the geodesic  $\sigma$ -model on  $DE_{10}/K(DE_{10})$ . The supergravity truncation roughly involves keeping only first order spatial gradients but arbitrary time-dependence, similar to the truncation in the  $E_{10}$  correspondence for the maximally supersymmetric theory [20, 25]. Therefore we are *not* performing a dimensional reduction to  $D = 1$ . Along the way to demonstrating the correspondence we rewrite the relevant parts of the equations in a form which is manifestly  $SO(9) \times SO(9)$  covariant (see [17] for an analysis of the maximal theory in an  $SO(9) \times SO(9)$  formalism). For the Kac–Moody side

of the correspondence the  $SO(9) \times SO(9)$  covariance is straight-forward to obtain by taking a so-called level decomposition of  $DE_{10}$  with respect to its  $SO(9,9)$  subgroup which, after the transition to compact subgroups, leads to  $SO(9) \times SO(9) \subset K(DE_{10})$  covariance. On the supergravity side this requires more work and intricate redefinitions of the standard variables.<sup>1</sup>

Besides the correspondence of the bosonic equations of motion we also study the fermionic fields of supergravity and show how they fit into consistent (albeit unfaithful) representations of the compact subgroup  $K(DE_{10})$  of  $DE_{10}$ . This, together with the bosonic dictionary, allows us to rewrite the supersymmetry variations in a form which not only has manifest  $SO(9) \times SO(9)$  covariance but also beginnings of a full  $K(DE_{10})$  covariance.

Our paper is structured as followed. First we define type I supergravity in our conventions and introduce some field redefinitions in section 2. In section 3, we define the  $DE_{10}/K(DE_{10})$   $\sigma$ -model in one dimension and work out its equations of motion in an  $SO(9,9)$  level decomposition. By comparing the two sets of equations of motion we will derive the dictionary relevant for the dynamical correspondence. In section 4 we study the supersymmetric aspects and fermionic fields of type I and their relation to  $K(DE_{10})$  before we close with some remarks and future prospects in section 5.

## 2 Pure type I supergravity

### 2.1 Action and supersymmetry

The action of  $D = 10$ ,  $N = 1$  supergravity [15] in our conventions reads to lowest fermion order<sup>2</sup>

$$\begin{aligned}
S_I = & \int d^{10}x \left[ \frac{\hat{E}}{4} \left( \hat{R} - \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{12} e^{-\Phi} H_{MNP} H^{MNP} \right) \right. \\
& - \frac{i\hat{E}}{2} \left( \bar{\psi}_M \Gamma^{MNP} D_N \psi_P + \frac{1}{2} \bar{\lambda} \Gamma^M D_M \lambda + \frac{1}{2} \bar{\psi}_N \Gamma^M \partial_M \Phi \Gamma^N \lambda \right) \\
& \left. + \frac{i\hat{E}}{48} e^{-\frac{1}{2}\Phi} H_{QRS} (\bar{\psi}_M \Gamma^{MNQRS} \psi_N + \bar{\psi}_N \Gamma^{QRS} \Gamma^N \lambda - 6 \bar{\psi}^Q \Gamma^R \psi^S) \right]. \tag{2.1}
\end{aligned}$$

Here,  $\hat{E} = \det(\hat{E}_M^A)$  is the zehnbein determinant and the curvature scalar  $\hat{R}$  is defined in terms of the coefficients of anholonomy  $\hat{\Omega}_{MN}^A$  and the spin

<sup>1</sup>An  $SO(n) \times SO(n)$  covariant formulation of the bosonic type I supergravity after strict dimensional reduction on an  $n$ -torus  $T^n$ , i.e. discarding all spatial gradients, was given in [26]. Our analysis goes beyond this since we keep spatial gradients. For completeness, we note that  $SO(n, n; \mathbb{Z})$  also appears as the T-duality group of closed string theories compactified on  $T^n$ .

<sup>2</sup>In what follows, we will always neglect higher order fermion contributions.

connection  $\hat{\omega}_M{}^{AB}$  via<sup>3</sup>

$$\begin{aligned}
\hat{\Omega}_{AB}{}^C &= 2\hat{E}_A{}^M\hat{E}_B{}^N\partial_{[M}\hat{E}_{N]}{}^C = \hat{\Omega}_{[AB]}{}^C, \\
\hat{\omega}_{ABC} &= \frac{1}{2}\left(\hat{\Omega}_{ABC} + \hat{\Omega}_{CAB} - \hat{\Omega}_{BCA}\right) = \hat{\omega}_{A[BC]}, \\
\hat{R}_{MN}{}^{AB} &= 2\partial_{[M}\hat{\omega}_{N]}{}^{AB} + 2\hat{\omega}_{[M}{}^{AC}\hat{\omega}_{N]C}{}^B, \\
\hat{R}_M{}^A &= \hat{E}_B{}^N\hat{R}_{MN}{}^{AB}, \\
\hat{R} &= \hat{E}_A{}^M\hat{R}_M{}^A.
\end{aligned} \tag{2.2}$$

The Lorentz covariant derivative acting on the spinors  $\lambda$ ,  $\epsilon$  and  $\psi_M$  is

$$D_N\psi_M = \partial_N\psi_M + \frac{1}{4}\hat{\omega}_{NAB}\Gamma^{AB}\psi_M. \tag{2.3}$$

and the supersymmetry variations leaving the action (2.1) invariant are

$$\delta_\epsilon\Phi = -i\bar{\epsilon}\lambda, \tag{2.4a}$$

$$\delta_\epsilon\hat{E}_M{}^A = i\bar{\epsilon}\left(\Gamma^A\psi_M + \frac{1}{12}\Gamma^A{}_M\lambda\right), \tag{2.4b}$$

$$\delta_\epsilon B_{MN} = -2ie^{\frac{1}{2}\Phi}\bar{\epsilon}\left(\Gamma_{[M}\psi_{N]} - \frac{1}{4}\Gamma_{MN}\lambda\right), \tag{2.4c}$$

$$\delta_\epsilon\lambda = -\frac{1}{2}\Gamma^M\epsilon\partial_M\Phi - \frac{1}{24}e^{-\frac{1}{2}\Phi}\Gamma^{QRS}\epsilon H_{QRS}, \tag{2.4d}$$

$$\delta_\epsilon\psi_M = D_M\epsilon - \frac{1}{96}e^{-\frac{1}{2}\Phi}\left(\Gamma_M{}^{QRS} - 9\delta_M^Q\Gamma^{RS}\right)\epsilon H_{QRS}. \tag{2.4e}$$

Note that the dilatino  $\lambda$  and the gravitino  $\psi_M$  have opposite spinor chirality as  $SO(1,9)$  Majorana–Weyl spinors. As both can be derived from a single eleven-dimensional gravitino,<sup>4</sup> we have used  $(32 \times 32)$   $\Gamma$ -matrices<sup>5</sup>  $\Gamma^A$ . The  $\Gamma$ -matrices  $\Gamma^M$  with curved indices appearing in (2.1) and below are obtained by conversion with the inverse zehnbein  $\Gamma^M = \Gamma^A\hat{E}_A{}^M$ . The spinors  $\psi_M$  and  $\epsilon$  are understood as projected to one chiral half and  $\lambda$  to the other one. Spinor conjugation is defined by  $\bar{\epsilon} = \epsilon^T\Gamma^0$ . The field strength of the NSNS two-form  $B_{MN}$  is defined by  $H_{MNP} = 3\partial_{[M}B_{NP]}$ .

<sup>3</sup>Our index conventions are:  $A, B, \dots = 0, \dots, 9$  are *flat* space-time frame indices,  $M, N, \dots$  are *curved* space-time coordinate indices whereas lower case  $a, b, \dots = 1, \dots, 9$  are flat spatial frame indices and  $m, n, \dots$  are curved spatial coordinate indices. Frame indices are raised and lowered with the flat Minkowski metric  $\eta_{AB} = \text{diag}(-1, +1, \dots, +1)$  and we have chosen the Newton constant conveniently. In section 2.2, we will introduce additional indices relevant for the  $SO(9) \times SO(9)$  structure to be studied.

<sup>4</sup>See e.g. [15] for the detailed derivation.

<sup>5</sup>Our  $\Gamma$ -matrix conventions can be found in the appendix A.

The purely bosonic equations of motion deduced from (2.1) are

$$\begin{aligned} \hat{K}_{AB} := & \hat{R}_{AB} - \frac{1}{2}\hat{\partial}_A\Phi\hat{\partial}_B\Phi - \frac{1}{4}e^{-\Phi}\hat{H}_{ACD}\hat{H}_B{}^{CD} \\ & + \frac{1}{48}\eta_{AB}e^{-\Phi}\hat{H}_{CDE}\hat{H}{}^{CDE} = 0, \end{aligned} \quad (2.5a)$$

$$\hat{M}_{AB} := \hat{D}^C(e^{-\Phi}\hat{H}_{CAB}) = 0, \quad (2.5b)$$

$$\hat{S} := \hat{D}_A\hat{\partial}^A\Phi + \frac{1}{12}e^{-\Phi}\hat{H}_{CDE}\hat{H}{}^{CDE} = 0. \quad (2.5c)$$

The hats on the quantities denote their projection onto an orthonormal frame by using the zehnbain  $\hat{E}_M{}^A$ , e.g.  $\hat{\partial}_A \equiv \hat{E}_A{}^M \partial_M$ . The Lorentz covariant derivative  $\hat{D}_A$  is defined with respect to the spin connection  $\hat{\omega}_{ABC}$  such that for example  $\hat{D}_A\hat{\partial}^A\Phi = \hat{\partial}_A\hat{\partial}^A\Phi + \hat{\omega}_A{}^{AC}\hat{\partial}_C\Phi$ .

Finally, we have the Bianchi identities

$$\hat{D}_{[A}\hat{H}_{BCD]} = 0, \quad (2.6a)$$

$$\hat{R}_{[ABC]D} = 0, \quad (2.6b)$$

which are satisfied trivially if one substitutes in the definitions in terms of the zehnbain  $\hat{E}_M{}^A$  and the potential  $B_{MN}$ . Here, it is more useful to keep them as separate equations since they will appear separately in the correspondence with the  $DE_{10}/K(DE_{10})$   $\sigma$ -model.

## 2.2 Redefinitions and gauge choices

In order to make the  $SO(9) \times SO(9)$  structure manifest, we fix the following zero shift (pseudo-Gaussian) gauge for the zehnbain *à la* ADM

$$\hat{E}_M{}^A = \begin{pmatrix} N & 0 \\ 0 & \hat{E}_m{}^a \end{pmatrix} \quad (2.7)$$

and then scale the spatial vielbein components and the dilaton field according to<sup>6</sup>

$$\begin{aligned} e_m{}^a & := e^{\frac{1}{4}\Phi}\hat{E}_m{}^a, \\ e^\phi & := (\det(\hat{E}_m{}^a))^{-1}e^{-\frac{1}{4}\Phi}. \end{aligned} \quad (2.8)$$

Furthermore, we define a new (densitised) lapse function by letting

$$\mathbf{n} := (\det(\hat{E}_m{}^a))^{-1}N = (\det(e_m{}^a))^{\frac{1}{8}}e^{\frac{9}{8}\phi}N \quad (2.9)$$

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<sup>6</sup>These redefinitions differ from the ones in [17] since we have made an additional conformal transformation to arrive at the action (2.1). The advantage of this is that also the new lapse of (2.9) below is the usual densitised lapse as in [20].

and set

$$\mathbf{v}_a := \hat{\omega}_{bab} e^{-\frac{1}{4}\Phi} = \partial_a \phi + e_a^m \partial_b e_m^b, \quad (2.10)$$

where we have used the abbreviation  $\partial_a \equiv e_a^m \partial_m$ . In general, unhatted quantities in flat indices have been projected using the new spatial neunbein  $e_m^a$  instead of  $\hat{E}_m^a$ . Finally, we adopt a Coulomb-type gauge for the two-form potential by setting  $B_{tm} = 0$ .

The bosonic fields are now combined into two sets of new variables. The first set contains only temporal derivatives and is given by

$$\mathbf{P}_{i\bar{j}} := e_i^m e_{\bar{j}}^n \left( \omega_{mnt} - \frac{1}{2} H_{tmn} \right), \quad (2.11a)$$

$$\mathbf{Q}_{ij} := e_i^m e_j^n \left( \omega_{tmn} + \frac{1}{2} H_{tmn} \right), \quad (2.11b)$$

$$\mathbf{Q}_{i\bar{j}} := e_i^m e_{\bar{j}}^n \left( \omega_{tmn} - \frac{1}{2} H_{tmn} \right), \quad (2.11c)$$

whereas the second set consists of combinations of spatial derivatives of the fields defined via

$$\mathbf{P}_{ijk} := -3! \mathbf{n} e^{-2\phi} e_i^m e_j^n e_k^p \left( \frac{1}{4} \omega_{[mnp]} + \frac{1}{24} H_{mnp} \right), \quad (2.12a)$$

$$\mathbf{P}_{\bar{i}jk} := -2 \mathbf{n} e^{-2\phi} e_{\bar{i}}^m e_j^n e_k^p \left( \frac{1}{4} \omega_{m[np]} + \frac{1}{8} H_{mnp} \right), \quad (2.12b)$$

$$\mathbf{P}_{i\bar{j}\bar{k}} := +2 \mathbf{n} e^{-2\phi} e_i^m e_{\bar{j}}^n e_{\bar{k}}^p \left( \frac{1}{4} \omega_{m[np]} - \frac{1}{8} H_{mnp} \right), \quad (2.12c)$$

$$\mathbf{P}_{\bar{i}\bar{j}\bar{k}} := +3! \mathbf{n} e^{-2\phi} e_{\bar{i}}^m e_{\bar{j}}^n e_{\bar{k}}^p \left( \frac{1}{4} \omega_{[mnp]} - \frac{1}{24} H_{mnp} \right), \quad (2.12d)$$

where  $\omega$  is the spin connection with respect to the rescaled vielbein  $e_m^a$ . Its definition is analogous to (2.2), implying e.g.  $\omega_{mnt} = e_{(m}^a \partial_t e_{n)a}$ . Both the indices  $i, j, \dots$  and  $\bar{i}, \bar{j}, \dots$  are frame indices taking values in the spatial directions  $1, \dots, 9$ , where we identify  $e_{\bar{i}}^m$  and  $e_i^m$ , so that for example  $\partial_{\bar{i}} = \partial_i$ . However, they will have different transformation properties under an  $SO(9) \times SO(9)$  group we now introduce. To be more precise, the unbarred indices are  $SO(9)$  vector indices of the first factor, whereas the barred indices are vector indices of the second factor. The spatial  $SO(9)$  Lorentz group is the diagonal subgroup of  $SO(9) \times SO(9)$  (see also [17]). The fields  $\mathbf{Q}_{ij}$ ,  $\mathbf{Q}_{i\bar{j}}$ ,  $\mathbf{P}_{ijk}$  and  $\mathbf{P}_{\bar{i}\bar{j}\bar{k}}$  are totally antisymmetric, whereas the mixed  $\mathbf{P}_{i\bar{j}}$ ,  $\mathbf{P}_{\bar{i}jk}$  and  $\mathbf{P}_{i\bar{j}\bar{k}}$  are only antisymmetric in indices belonging to the same  $SO(9)$  factor of  $SO(9) \times SO(9)$ . Repeated indices on the same level are summed over with  $\delta^{ij}$  or  $\delta^{\bar{i}\bar{j}}$ .

We note that the total number of components in  $P_{ijk}$ ,  $P_{\bar{i}jk}$ ,  $P_{i\bar{j}k}$  and  $P_{i\bar{j}\bar{k}}$  is 816 whereas the number of independent components of the supergravity variables  $\omega_{mnp}$  and  $H_{mnp}$  involved in the redefinition is only 408 so that the redefinition is not one-to-one. Hence, there are equivalent ways of expressing a supergravity expression in these new variables. Our choice is such that it connects well to the  $DE_{10}$  analysis.

### 2.3 Supergravity dynamics

We now take certain combinations of the equations of motion (2.5) after separating the time index 0 from the spatial indices  $a$ . The independent components of the equations of motion then are

$$\begin{aligned}\hat{K}_{ab} &= 0, & \hat{K}_{00} &= 0, & \hat{K}_{a0} &= 0, \\ \hat{M}_{ab} &= 0, & \hat{M}_{a0} &= 0, & \hat{S} &= 0.\end{aligned}\quad (2.13)$$

We combine the symmetric Einstein equation and the antisymmetric two-form equation into a single tensor equation with no definite symmetry

$$N^2 \left( \hat{K}_{ab} - \frac{1}{4} \delta_{ab} \hat{S} + \frac{1}{2} e^{\frac{1}{2}\Phi} \hat{M}_{ab} \right) = 0. \quad (2.14)$$

In the new variables (2.11) and (2.12), the equation (2.14) takes the form

$$\begin{aligned}\mathbf{n} D_t (\mathbf{n}^{-1} P_{i\bar{j}}) &- e^{2\phi} (P_{ikl} P_{\bar{j}kl} + 2P_{\bar{k}il} P_{l\bar{k}\bar{j}} + P_{i\bar{k}l} P_{\bar{j}\bar{k}l}) \\ &= \mathbf{n} \partial_p [e_k{}^p P_{\bar{j}ki} - e_{\bar{k}}{}^p P_{i\bar{k}\bar{j}}] - e^{2\phi} (P_{ikl} P_{\bar{j}kl} + P_{i\bar{k}l} P_{\bar{j}\bar{k}l} - 2P_{\bar{k}il} P_{l\bar{k}\bar{j}}) \\ &\quad - \mathbf{n}^2 e^{-2\phi} \partial_{(i} v_{j)} + \mathbf{n} e^{-\phi} \partial_{(i} [\mathbf{n} e^{-\phi} (\mathbf{n}^{-1} \partial_{j)} \mathbf{n} - \partial_{j)} \phi]\end{aligned}\quad (2.15)$$

with the  $SO(9) \times SO(9)$  covariant derivative  $D_t$  acting on  $P_{i\bar{j}}$  via

$$D_t P_{i\bar{j}} := \partial_t P_{i\bar{j}} + Q_{ik} P_{k\bar{j}} + Q_{\bar{j}\bar{k}} P_{i\bar{k}}. \quad (2.16)$$

The dilaton equation of motion (2.5c) can be combined with the spatial trace of the Einstein equation by

$$N^2 \left( \frac{1}{4} \hat{S} - \delta^{ab} \hat{K}_{ab} \right) = 0 \quad (2.17)$$

to give the first scalar equation of motion

$$\begin{aligned}\mathbf{n} \partial_t (\mathbf{n}^{-1} \partial_t \phi) &+ \frac{1}{6} e^{2\phi} (P_{ijk} P_{ijk} + 3P_{\bar{i}jk} P_{\bar{i}jk} + 3P_{i\bar{j}\bar{k}} P_{i\bar{j}\bar{k}} + P_{i\bar{j}\bar{k}} P_{i\bar{j}\bar{k}}) \\ &= \mathbf{n}^2 e^{-2\phi} \left[ 2\partial_a v_a - \frac{1}{2} \Omega^{abc} \Omega_{acb} - v_a v_a \right] \\ &\quad - \mathbf{n} e^{-\phi} \partial_a \left( \mathbf{n} e^{-\phi} [\mathbf{n}^{-1} \partial_a \mathbf{n} - \partial_a \phi] \right) + \mathbf{n}^2 e^{-2\phi} v_a [\mathbf{n}^{-1} \partial_a \mathbf{n} - \partial_a \phi].\end{aligned}\quad (2.18)$$

Furthermore, we have an independent second scalar equation

$$-N^2(\hat{K}_{00} + \delta^{ab}\hat{K}_{ab}) = 0, \quad (2.19)$$

which is proportional to the Hamiltonian constraint. In the new variables it reads

$$\begin{aligned} & -(\partial_t\phi)^2 + \mathbf{P}_{i\bar{j}}\mathbf{P}_{i\bar{j}} + \frac{1}{6}e^{2\phi}(\mathbf{P}_{ijk}\mathbf{P}_{ijk} + 3\mathbf{P}_{i\bar{j}k}\mathbf{P}_{i\bar{j}k} + 3\mathbf{P}_{i\bar{j}\bar{k}}\mathbf{P}_{i\bar{j}\bar{k}} + \mathbf{P}_{i\bar{j}\bar{k}}\mathbf{P}_{i\bar{j}\bar{k}}) \\ & = \mathbf{n}^2e^{-2\phi}\left[2\partial_a\mathbf{v}_a - \frac{1}{2}\Omega^{abc}\Omega_{acb} - \mathbf{v}_a\mathbf{v}_a\right]. \end{aligned} \quad (2.20)$$

Finally, we have two vector equations stemming from the Gauss constraint on the two-form field,  $\hat{M}_{a0} = 0$ , and the diffeomorphism constraint,  $\hat{K}_{a0} = 0$ . We combine them by

$$Ne^{-\frac{1}{4}\Phi}(\hat{K}_{a0} \pm \frac{1}{2}e^{\frac{1}{2}\Phi}\hat{M}_{a0}) = 0 \quad (2.21)$$

and get the two vector constraint equations

$$0 = \mathbf{n}\partial_m[e_j^m\mathbf{n}^{-1}\mathbf{P}_{i\bar{j}}] + 2\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{jki}\mathbf{P}_{k\bar{j}} + \mathbf{n}e^{-\phi}\partial_i(\mathbf{n}^{-1}e^\phi\partial_t\phi), \quad (2.22a)$$

$$0 = \mathbf{n}\partial_m[e_j^m\mathbf{n}^{-1}\mathbf{P}_{j\bar{i}}] - 2\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{j\bar{k}i}\mathbf{P}_{j\bar{k}} + \mathbf{n}e^{-\phi}\partial_{\bar{i}}(\mathbf{n}^{-1}e^\phi\partial_t\phi). \quad (2.22b)$$

The equations (2.15)–(2.22) are completely equivalent to the set of bosonic equations of motion (2.5) upon substitution of the definitions (2.11) and (2.12).

We conclude this section by giving the Bianchi identities (2.6a) and (2.6b) in an appropriate form. Starting from the equations

$$\mathbf{n}e^{-2\phi}D_t(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{ijk}) + 3\mathbf{P}_{[i\bar{l}]\bar{l}][j\bar{k}]} = -\frac{3}{2}\partial_{[i}\mathbf{Q}_{jk]}, \quad (2.23a)$$

$$\mathbf{n}e^{-2\phi}D_t(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{i\bar{j}k}) + \mathbf{P}_{\bar{l}i}\mathbf{P}_{ljk} + 2\mathbf{P}_{[j\bar{l}]\bar{l}][k\bar{i}]} = -\frac{1}{2}\partial_{\bar{i}}\mathbf{Q}_{jk} + \partial_{[j}\mathbf{P}_{k]\bar{i}}, \quad (2.23b)$$

$$\mathbf{n}e^{-2\phi}D_t(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{i\bar{j}\bar{k}}) + \mathbf{P}_{i\bar{l}}\mathbf{P}_{\bar{l}\bar{j}\bar{k}} + 2\mathbf{P}_{l[j}\mathbf{P}_{\bar{k}]i\bar{l}} = \frac{1}{2}\partial_i\mathbf{Q}_{\bar{j}\bar{k}} - \partial_{[\bar{j}}\mathbf{P}_{i|\bar{k}]}, \quad (2.23c)$$

$$\mathbf{n}e^{-2\phi}D_t(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{i\bar{j}\bar{k}}) + 3\mathbf{P}_{l[\bar{i}]\bar{l}][j\bar{k}]} = \frac{3}{2}\partial_{\bar{i}}\mathbf{Q}_{\bar{j}\bar{k}}, \quad (2.23d)$$

one recovers the Bianchi identities by taking suitable combinations. For the Bianchi identity  $\hat{D}_{[0}\hat{H}_{abc]} = 0$  one has to sum (2.23a) and (2.23d), whereas  $\hat{R}_{[0bcd]} = 0$  corresponds to the difference between (2.23a) and (2.23d). The difference between (2.23b) and (2.23c) gives the identity  $R_{[0ab]c} = 0$ . The Bianchi identities with purely spatial indices will not be discussed here but they can also be rewritten in the new variables of (2.11) and (2.12).

### 3 The geodesic $DE_{10}/K(DE_{10})$ coset model

#### 3.1 Abstract derivation of the equations of motion

The abstract  $D = 1$   $\sigma$ -model on any group coset  $G/H$  is given in terms of a representative  $\mathcal{V}(t) \in G/H$ , where  $t$  is the parameter along the world-line. The velocity along this worldline pulled backed to the identity is the  $\text{Lie}(G)$  valued expression  $\partial_t \mathcal{V} \mathcal{V}^{-1}$  that can be decomposed into generators along  $\text{Lie}(H)$  and  $\text{Lie}(G/H)$  as

$$\partial_t \mathcal{V} \mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P}, \quad (3.1)$$

where  $\mathcal{Q} \in \text{Lie}(H)$  are the unbroken gauge connections in the language of non-linear realisations, and  $\mathcal{P} \in \text{Lie}(G/H)$  correspond to the velocity components in the direction of the ‘broken’ generators. Using the invariant symmetric form ( $\equiv$  Cartan–Killing form)<sup>7</sup>  $\langle \cdot | \cdot \rangle$  on  $\text{Lie}(G)$ , we define a Lagrange function that determines the dynamics of the bosonic  $D = 1$   $\sigma$ -model

$$L = \frac{1}{4} n^{-1} \langle \mathcal{P} | \mathcal{P} \rangle. \quad (3.2)$$

The Lagrange function is invariant under the standard non-linear transformation  $\mathcal{V}(t) \rightarrow h(t) \mathcal{V}(t) g^{-1}$  for local  $h(t) \in H$  and global  $g \in G$ . The factor  $n(t)$  ensures reparametrisation invariance along the world-line and, since we have no mass term in  $L$ , the massless particle will move on a null trajectory.<sup>8</sup>

In order to derive the equations of motion from this Lagrange function we consider variations of the field  $\mathcal{V}$  associated with a derivation  $\delta$  which is assumed to commute with time derivative  $\partial_t$ . Under this variation we get a similar decomposition  $\delta \mathcal{V} \mathcal{V}^{-1} = \Sigma + \Lambda$  for  $\Sigma \in \text{Lie}(H)$  and  $\Lambda \in \text{Lie}(G/H)$ . Substituting this variation into the Lagrange function (3.2) leads to the  $H$  covariant  $\sigma$ -model equations of motion

$$n \partial_t (n^{-1} \mathcal{P}) - [\mathcal{Q}, \mathcal{P}] = 0 \quad (3.3)$$

and the null constraint<sup>9</sup>

$$\langle \mathcal{P} | \mathcal{P} \rangle = 0. \quad (3.4)$$

Using the  $H$  covariant derivative

$$\mathcal{D} = \partial_t - \mathcal{Q}, \quad (3.5)$$

---

<sup>7</sup>This is the invariant trace in the adjoint representation in the finite-dimensional case.

<sup>8</sup>This is only possible if the invariant form  $\langle \cdot | \cdot \rangle$  is *indefinite* as in our case.

<sup>9</sup>We assume here that  $\mathcal{V}$  and  $n$  are independent.

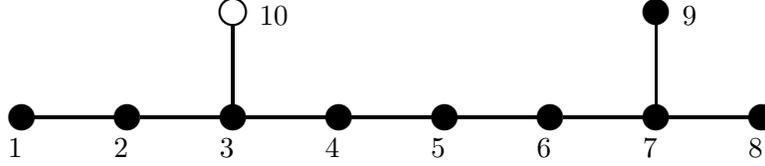


Figure 1: Dynkin diagram of  $\text{Lie}(DE_{10})$  with numbering of nodes. The solid nodes mark the  $\mathfrak{so}(9,9) \equiv \text{Lie}(D_9)$  subalgebra.

where  $\mathcal{Q}$  acts on a  $H$  representation, here the algebraic coset  $\text{Lie}(G/H)$ , eq. (3.3) can also be written as

$$\mathcal{D}(n^{-1}\mathcal{P}) = 0. \quad (3.6)$$

For any Kac-Moody algebra [27], there is a generalized transposition map  $-\omega$  mapping the Chevalley generators  $e_i, f_i$  and  $h_i$  with  $i = 1, \dots, \text{rk}(\text{Lie}(G))$  to themselves by

$$-\omega(e_i) = f_i \quad , \quad -\omega(f_i) = e_i \quad , \quad -\omega(h_i) = h_i. \quad (3.7)$$

As  $\omega^2 = \mathbf{1}_{\text{Lie}(G)}$ , we can decompose any  $\text{Lie}(G)$ -valued object into eigenspaces  $\mathcal{Q} \in \text{Lie}(H)$  and  $\mathcal{P} \in \text{Lie}(G/H)$  via

$$-\omega(\mathcal{P}) = \mathcal{P} \quad , \quad -\omega(\mathcal{Q}) = -\mathcal{Q}. \quad (3.8)$$

$H$  is then referred to as the maximal compact<sup>10</sup> subgroup of  $G$ , which we denote by  $K(G)$ . Now we study this general set-up for the case of  $G = DE_{10}$  and  $H = K(DE_{10})$ .

### 3.2 The $D_9$ level decomposition of $\text{Lie}(DE_{10})$

The Lie algebra  $\text{Lie}(DE_{10})$  is an infinite-dimensional hyperbolic Kac-Moody algebra [27] with Dynkin diagram given in figure 1 and we consider it in split real form. In order to analyse the dynamical equation (3.3), we need to know the structure constants of  $\text{Lie}(DE_{10})$ . However, a closed representation of  $\text{Lie}(DE_{10})$  is not known. The only known presentation of  $\text{Lie}(DE_{10})$  is in terms of simple generators  $e_i, f_i$  and  $h_i$  (for  $i = 1, \dots, 10$ ) and defining relations among them [27]. These simple generators and their independent multiple commutators form a basis of the vector spaces  $\mathfrak{n}_+, \mathfrak{n}_-$  and the

<sup>10</sup>In the case of a finite-dimensional Lie group  $G$ , this is the standard notion of a compact manifold.

Cartan subalgebra  $\mathfrak{h}$  respectively. Thus, one obtains the following decomposition of  $\text{Lie}(DE_{10})$ :

$$\text{Lie}(DE_{10}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \quad (3.9)$$

Next, we define the level  $\ell$  of a homogeneous element of  $\mathfrak{n}_+$  with respect to  $D_9 \equiv \text{Lie}(SO(9,9))$  to be the number of times  $e_{10}$  appears in the corresponding multiple commutator. This definition can be extended to the entire Kac–Moody algebra by counting  $f_{10}$  negatively and setting the degree of  $\mathfrak{h}$  to zero. Thus, we have constructed an integer grading of  $\text{Lie}(DE_{10})$ , given by the level decomposition into subspaces labelled by levels  $\ell \in \mathbb{Z}$ . The level  $\ell$  piece in this decomposition is finite-dimensional and is mapped to itself under the adjoint action of the  $\ell = 0$  piece. Therefore any fixed level  $\ell$  is a sum of irreducible representations of the  $\ell = 0$  subalgebra of  $\text{Lie}(DE_{10})$  and we first study  $\ell = 0$ .<sup>11</sup>

Following from this definition, the subspace with level  $\ell = 0$  consists of all multiple generators of  $e_i, f_i$  for  $i = 1, \dots, 9$  and all ten Cartan subalgebra elements. Leaving out all  $e_{10}$  and  $f_{10}$  generators in commutators one arrives at a  $\text{Lie}(D_9)$  subalgebra of  $\text{Lie}(DE_{10})$  as also evident from figure 1. A certain linear combination of the ten Cartan elements  $h_i$  is orthogonal to this  $\mathfrak{so}(9,9)$  and therefore the resulting  $\ell = 0$  subalgebra of  $\text{Lie}(DE_{10})$  is the direct sum  $\mathfrak{so}(9,9) \oplus \mathfrak{gl}(1)$ .

We denote the  $SO(9,9)$  generators by  $M^{IJ} = -M^{JI}$  and take their commutation relation to be

$$[M^{IJ}, M^{KL}] = \eta^{KI} M^{JL} - \eta^{KJ} M^{IL} - \eta^{LI} M^{JK} + \eta^{LJ} M^{IK} \quad (3.10)$$

with

$$\eta^{IJ} = \text{diag}(\mathbf{1}_9, -\mathbf{1}_9) \iff \eta^{ij} = \delta^{ij} = -\eta^{\bar{i}\bar{j}}; \eta^{i\bar{j}} = \eta^{\bar{j}i} = 0, \quad (3.11)$$

where we made use of the  $SO(9) \times SO(9)$ -indices  $I \equiv (i, \bar{i})$ .<sup>12</sup> We use  $\eta_{IJ}$  to raise and lower  $SO(9,9)$  indices in the standard fashion. With the Cartan–Killing form

$$\langle M^{IJ} | M^{KL} \rangle = \eta^{KJ} \eta^{IL} - \eta^{KI} \eta^{JL}, \quad (3.12)$$

we can split the generators into compact and non-compact ones

$$J^{ij} := M^{ij} \quad , \quad J^{\bar{i}\bar{j}} := -M^{\bar{i}\bar{j}} \quad ; \quad S^{i\bar{j}} := M^{i\bar{j}}, \quad (3.13)$$

<sup>11</sup>Further details on this decomposition can be found in [17]; the general technique of level decompositions is explained for example in [20, 28, 29, 30].

<sup>12</sup>The explicit mapping between the  $D_9$  Chevalley operators and the  $M^{IJ}$  can be found in [17].

where  $J^{ij}$  and  $J^{\bar{i}\bar{j}}$  generate the two  $SO(9)$ -groups of  $SO(9) \times SO(9)$ , whereas the symmetric  $S^{i\bar{j}}$  is a coset generator. The generator of  $GL(1)$  will be denoted  $T$  and satisfies

$$\begin{aligned} [T, M^{IJ}] &= 0, \\ \langle T|T \rangle &= -1. \end{aligned} \quad (3.14)$$

Restricting  $\text{Lie}(DE_{10})$  to level  $\ell = 1$  constitutes an irreducible representation of  $\text{Lie}(D_9 \times GL(1))$  which is in an antisymmetric three-tensor representation of  $SO(9, 9)$ , denoted by  $E^{IJK}$ , and carries  $GL(1)$  charge  $+1$ :

$$[T, E^{IJK}] = E^{IJK}, \quad (3.15a)$$

$$[M^{IJ}, E^{KLM}] = 3\eta^{I[K} E^{LM]J} - 3\eta^{J[K} E^{LM]I}. \quad (3.15b)$$

The commutation relations for the elements  $F_{IJK} := -\omega(E^{IJK})$  on level  $\ell = -1$  are obtained by using the generalized transposition  $-\omega$  on (3.15a) and (3.15b) to give

$$[T, F^{IJK}] = -F^{IJK}, \quad (3.16a)$$

$$[M^{IJ}, F^{KLM}] = 3\eta^{I[K} F^{LM]J} - 3\eta^{J[K} F^{LM]I}. \quad (3.16b)$$

Here, indices have been raised with  $\eta^{IJ}$ . If we fix the normalization by<sup>13</sup>

$$\langle E^{IJK} | F_{LMN} \rangle := 12\delta_{LMN}^{IJK}, \quad (3.17)$$

the invariance of the Cartan–Killing form yields the  $\text{Lie}(DE_{10})$  commutation relation

$$[E^{IJK}, F_{LMN}] = -12\delta_{LMN}^{IJK}T - 36\delta_{[LM}^{[IJ} \eta_{N]P} M^{K]P}. \quad (3.18)$$

In summary, we have the following set of generators of  $DE_{10}$  on levels  $|\ell| \leq 1$  in a decomposition with respect to  $D_9$ :

	$\ell = -1$	$\ell = 0$	$\ell = 1$
Generator	$F_{IJK}$	$M^{IJ}, T$	$E^{IJK}$
Dimension	816	153+1	816

As we will not use any further generators in this paper, we will not discuss the representation theory of the higher levels. More details and extensive tables up to  $\ell = 5$  can be found in [17].

<sup>13</sup>We use  $\delta_{L_1 \dots L_l}^{I_1 \dots I_l} := \delta_{L_1}^{[I_1} \dots \delta_{L_l}^{I_l]}$  and antisymmetrisations of strength one.

### 3.3 $\sigma$ -model equation truncated at $D_9$ level 1

Given the knowledge of the generators up to level  $\ell = 1$  we can parametrise the  $DE_{10}/K(DE_{10})$  coset element  $\mathcal{V}$  by

$$\mathcal{V}(t) = e^{\varphi(t)T} e^{\frac{1}{2}v_{IJ}(t)M^{IJ}} e^{\frac{1}{3!}A_{IJK}(t)E^{IJK}} \dots \quad (3.19)$$

in a Borel gauge consisting of only generators of levels  $\ell \geq 0$ . Singling out the  $GL(1)$ -generator  $T$  will introduce a factor  $e^{\ell\varphi}$  for the level  $\ell$  term when evaluating the velocity (3.1), which we truncate for all  $\ell \geq 2$ .<sup>14</sup> Hence, we get the parametrisation  $\partial_t \mathcal{V} \mathcal{V}^{-1} = \mathcal{P} + \mathcal{Q}$  with

$$\begin{aligned} \mathcal{P} &= \partial_t \varphi T + P_{i\bar{j}} S^{i\bar{j}} + \frac{1}{3!} e^\varphi P_{IJK} S^{IJK}, \\ \mathcal{Q} &= \frac{1}{2} Q_{ij} J^{ij} + \frac{1}{2} Q_{i\bar{j}} J^{i\bar{j}} + \frac{1}{3!} e^\varphi P_{IJK} J^{IJK}, \end{aligned} \quad (3.20)$$

where we have introduced a new notation for the coset generator  $S^{IJK}$  and the  $K(DE_{10})$ -generator  $J^{IJK}$

$$S^{IJK} := \frac{1}{2} (E^{IJK} - \omega(E^{IJK})), \quad (3.21a)$$

$$J^{IJK} := \frac{1}{2} (E^{IJK} + \omega(E^{IJK})). \quad (3.21b)$$

The occurrence of the same coefficient  $P_{IJK}$  in  $\mathcal{P}$  and  $\mathcal{Q}$  in (3.20) is due to our Borel gauge condition. Although we could work out  $P_{IJK}$  explicitly in terms of the coset coordinate fields  $v_{IJ}$  and  $A_{IJK}$  we will leave it in this compact form since this will be sufficient for the comparison with the  $\sigma$ -model equations of motion (3.3).

In section 3.1, we mentioned that the  $\sigma$ -model equations of motion (3.3) are only  $H = K(DE_{10})$ -covariant, whereas the  $G = DE_{10}$ -covariance is broken. However,  $SO(9, 9)$  is not a subgroup of  $K(DE_{10})$ , only its maximal compact subgroup  $SO(9) \times SO(9)$  is. Hence for our truncation chosen, we can only expect to get  $SO(9) \times SO(9)$  covariant equations of motion. This is already obvious from the definitions of the generators (3.21a) and (3.21b):  $S^{IJK}$  and  $J^{IJK}$  do not transform as  $SO(9, 9)$  tensors, but as  $SO(9) \times SO(9)$  tensors. Therefore, we also decompose the  $SO(9, 9)$  tensor  $P_{IJK}$  into its irreducible  $SO(9) \times SO(9)$  components  $P_{ijk}$ ,  $P_{\bar{i}\bar{j}\bar{k}}$ ,  $P_{i\bar{j}\bar{k}}$  and  $P_{\bar{i}j\bar{k}}$  and write the  $\sigma$ -model equations of motion (3.3) for the level  $\ell = 0$ , *i.e.* projected on

<sup>14</sup>In principle, we should add all higher level  $\ell$   $D_9$ -representations  $E^{(\ell)}$  with appropriate coefficients  $P^{(\ell)}$  to the parametrization. However, in the present discussion we want to restrict the levels to  $\ell \leq 1$ , which is consistently achieved by setting all coefficients  $P^{(\ell)} = 0$  initially for all  $\ell \geq 2$  [25].

the coset generators  $T$  and  $S^{i\bar{j}}$ , as

$$0 = n\partial_t(n^{-1}\partial_t\varphi) + \frac{1}{6}e^{2\varphi}(P_{ijk}P_{ijk} + 3P_{\bar{i}jk}P_{\bar{i}jk} + 3P_{i\bar{j}\bar{k}}P_{i\bar{j}\bar{k}} + P_{\bar{i}\bar{j}\bar{k}}P_{\bar{i}\bar{j}\bar{k}}), \quad (3.22a)$$

$$0 = nD_t(n^{-1}P_{i\bar{j}}) - e^{2\varphi}(P_{ikl}P_{\bar{j}kl} + 2P_{k\bar{i}l}P_{l\bar{k}\bar{j}} + P_{i\bar{k}l}P_{\bar{j}k\bar{l}}). \quad (3.22b)$$

Here,  $D_t$  is the  $SO(9) \times SO(9)$  covariant derivative of (2.16) with the connection  $(Q_{ij}, Q_{\bar{i}\bar{j}})$ . The  $\sigma$ -model equations of motion (3.3) for the level  $\ell = 1$ , *i.e.* projected on the generators  $S^{ijk}$ ,  $S^{\bar{i}\bar{j}\bar{k}}$ ,  $S^{i\bar{j}\bar{k}}$  and  $S^{\bar{i}j\bar{k}}$ , are

$$ne^{-2\varphi}D_t(n^{-1}e^{2\varphi}P_{ijk}) + 3P_{[i|\bar{l}}P_{l|jk]} = 0, \quad (3.23a)$$

$$ne^{-2\varphi}D_t(n^{-1}e^{2\varphi}P_{\bar{i}jk}) + P_{\bar{l}i}P_{ljk} + 2P_{[j|\bar{l}}P_{k]i\bar{l}} = 0, \quad (3.23b)$$

$$ne^{-2\varphi}D_t(n^{-1}e^{2\varphi}P_{i\bar{j}\bar{k}}) + P_{i\bar{l}}P_{l\bar{j}\bar{k}} + 2P_{l[\bar{j}}P_{k]i\bar{l}} = 0, \quad (3.23c)$$

$$ne^{-2\varphi}D_t(n^{-1}e^{2\varphi}P_{\bar{i}j\bar{k}}) + 3P_{l[\bar{i}}P_{l]j\bar{k}} = 0. \quad (3.23d)$$

As we set  $P^{(\ell)} = 0$  for  $\ell \geq 2$  initially, the equation (3.3), describing its time evolution, preserves this setting. However, written in terms of the field  $\mathcal{V}$  parametrising the coset  $DE_{10}/K(DE_{10})$  this implies a non-trivial time evolution of the higher level fields. We stress that the terms extending the  $SO(9) \times SO(9)$  covariant derivative  $D_t$  in (3.23) are the next terms in the full  $K(DE_{10})$  covariant derivative  $\mathcal{D}_t$  of (3.5).

We conclude this section with the null constraint  $\langle \mathcal{P}|\mathcal{P} \rangle = 0$ , cf. (3.4), in this parametrisation

$$0 = -(\partial_t\varphi)^2 + P_{i\bar{j}}P_{i\bar{j}} + \frac{1}{6}e^{2\varphi}(P_{ijk}P_{ijk} + 3P_{\bar{i}jk}P_{\bar{i}jk} + 3P_{i\bar{j}\bar{k}}P_{i\bar{j}\bar{k}} + P_{\bar{i}\bar{j}\bar{k}}P_{\bar{i}\bar{j}\bar{k}}). \quad (3.24)$$

### 3.4 Comparison of the $\sigma$ -model with supergravity

Now we turn to the comparison of the level  $\ell = 0$   $\sigma$ -model equations of motion (3.22) with the rewritten dynamical supergravity equations (2.15) and (2.18) and of the  $\ell = 1$  equations (3.23) with the Bianchi constraints (2.23). We will also compare the Hamiltonian constraint (2.20) with the null constraint (3.24). All these equations contain the following objects:

$D = 10$ pure type I supergravity	$DE_{10}/K(DE_{10})$ $\sigma$ -model
$P_{i\bar{j}}(t, \mathbf{x}), Q_{ij}(t, \mathbf{x}), Q_{\bar{i}\bar{j}}(t, \mathbf{x})$	$P_{i\bar{j}}(t), Q_{ij}(t), Q_{\bar{i}\bar{j}}(t)$
$P_{ijk}(t, \mathbf{x}), P_{\bar{i}jk}(t, \mathbf{x}), P_{i\bar{j}\bar{k}}(t, \mathbf{x}), P_{\bar{i}j\bar{k}}(t, \mathbf{x})$	$P_{ijk}(t), P_{\bar{i}jk}(t), P_{i\bar{j}\bar{k}}(t), P_{\bar{i}j\bar{k}}(t)$
$\phi(t, \mathbf{x})$	$\varphi(t)$
$\mathbf{n}(t, \mathbf{x})$	$n(t)$

Here, we have explicitly re-instated the dependence on the coordinates. Working locally in one coordinate chart  $(t, \mathbf{x})$  and keeping the spatial point  $\mathbf{x}$  fixed, the time coordinate  $t$  of supergravity can be identified with the parameter along the world-line of the coset model, as already anticipated in the table above.

By comparing the supergravity equations and the  $\sigma$ -model equations, we see that we can match large parts of the equations by demanding that the supergravity quantities evaluated at the fixed spatial point  $\mathbf{x}$  correspond to the  $\sigma$ -model quantities. In other words, the *dynamical dictionary* which maps (parts of) the supergravity equations to the  $\sigma$ -model equations consists of letting

$$\mathbf{P}_{i\bar{j}}(t, \mathbf{x}) \leftrightarrow P_{i\bar{j}}(t) \quad \text{for all } t \text{ (and fixed } \mathbf{x}) \quad (3.25)$$

and similarly for the other objects in the table. To be more precise the  $\sigma$ -model equations (3.22)–(3.24) coincide with the left hand sides of the equations of motion (2.18), (2.15) and (2.20) and the Bianchi identity (2.23) of  $D = 10$  pure supergravity. However, the terms on the right hand sides do not match in this correspondence which we now discuss in more detail, together with the vector constraint equations (2.22a) and (2.22b) which do not have corresponding  $\sigma$ -model equations.

We begin with the tensor equation (2.15) and the two vector constraints (2.22a) and (2.22b), where we want to show that, in some sense, we have only neglected spatial derivatives. Our identification fixed an arbitrary spatial position  $\mathbf{x}$  in a coordinate chart and considered the evolution of the fields  $\mathbf{P}$  in time only. However, direct spatial derivatives  $\partial_q$  of  $\mathbf{P}$  (and  $\mathbf{Q}$ ) are not expected to be represented in this truncated correspondence. They are thought to be represented by higher level fields  $P^{(\ell > 1)}$  [20] which we ignored in the  $\sigma$ -model and therefore this disagreement is not surprising. In the tensor equation (2.15), we have a term in the second line which seems not to be directly connected to a spatial derivative. However, if we assume that the  $SO(9) \times SO(9)$  symmetry can be gauged and if we introduce an  $SO(9) \times SO(9)$  valued zehnbein  $e_K{}^p := (e_k{}^p, e_{\bar{k}}{}^p)$  analogously to the  $SU(8)$  valued elfbein in [5], we can write the second line of the tensor equation (2.15) as follows<sup>15</sup>

$$\delta_i^I \delta_j^J \left( \mathbf{n} \partial_p [e_K{}^p \delta^{KL} \mathbf{P}_{ILJ}] + e^{2\phi} \mathbf{P}_I{}^{KL} \mathbf{P}_{JKL} \right), \quad (3.26)$$

---

<sup>15</sup> $\delta^{KL}$  denotes the second invariant tensor of  $SO(9) \times SO(9)$  which is similarly defined as the first one,  $\eta^{KL}$ , but with  $\delta^{\bar{k}\bar{l}} = +\delta^{kl}$ .

where indices have been raised with  $\eta^{KL}$ . Furthermore, the two vector equations combine to a single one

$$0 = -\mathbf{n}\partial_m [e_K{}^m\eta^{KL}\mathbf{n}^{-1}\mathbf{P}_{IL}] + e^{2\phi}\mathbf{P}_I{}^{KL}\mathbf{n}^{-1}\mathbf{P}_{KL} + \mathbf{n}e^{-\phi}\partial_I (\mathbf{n}^{-1}e^\phi\partial_t\phi), \quad (3.27)$$

if we set  $P_{ij} = P_{\bar{i}\bar{j}} = 0$ . This looks like an  $SO(9) \times SO(9)$ -covariant derivative  $D_m$  with respect to *space* whereas before we only considered  $D_t$ . This would imply that the term<sup>16</sup>

$$e^{2\phi}\mathbf{P}_i{}^{KL}\mathbf{P}_{\bar{j}KL} = e^{2\phi} (\mathbf{P}_{ikl}\mathbf{P}_{\bar{j}kl} + \mathbf{P}_{i\bar{k}\bar{l}}\mathbf{P}_{\bar{j}\bar{k}\bar{l}} - 2\mathbf{P}_{\bar{k}il}\mathbf{P}_{l\bar{k}\bar{j}}) = -\frac{1}{2}\mathbf{n}^2 e^{-2\phi}\Omega_{(i|kl}\Omega_{\bar{j})lk}$$

should not be separated from the discussion of spatial derivatives. The third line in the tensor equation (2.15) is an explicit spatial derivative, which concludes the discussion of this equation.

Turning to the scalar equations we see that there are two different mismatches. The first one is the common term in the second lines of (2.18) and (2.20) whose value depends on the choice of coordinate system and local Lorentz gauge. These freedoms could be used, e.g., to let this term vanish or to fix  $v_a(t, \mathbf{x}) = 0$ , as was suggested in [17].<sup>17</sup>

The second mismatch in the scalar equations concerns the final line in (2.18). However, in order to account for this term we also have the densitised lapse  $\mathbf{n}$  at our disposal which we can choose as convenient as long as it does not vanish. Evidently, choosing  $\mathbf{n}$  suitably we can cause this line to vanish identically at the fixed spatial point  $\mathbf{x}$ .

A fascinating possibility for taking the terms containing spatial gradients into account in the correspondence was proposed in [20], where it was suggested that they are related to some higher level fields of the  $\sigma$ -model which have been truncated in our analysis. The proper interpretation of these terms is still an open problem.

## 4 Fermions and supersymmetry

In this section, we extend our analysis to take into account the fermionic degrees of freedom. We will in particular check that the supersymmetry transformations (2.4) can be stated in an  $SO(9) \times SO(9)$  covariant form, which is necessary for the  $K(DE_{10})$  covariance that is conjectured to hold if all levels  $\ell$  are fixed appropriately. We start with the discussion of the fermionic variations, before we move on to the bosonic fields.

<sup>16</sup>The following equality shows that this term is precisely the term which is also problematic in the Einstein equation of the maximal theory [25].

<sup>17</sup>The quantity  $v_a$  exactly is the spatial trace of the  $D = 11$  spin connection used in [17].

#### 4.1 $K(DE_{10})$ covariance of the fermionic transformations

The supersymmetry transformation of the fermions  $\lambda$  and  $\psi_M$  have been stated in equations (2.4d) and (2.4e). In order to uncover the  $SO(9) \times SO(9)$  covariance, we have to reparametrise the fermions as we have done with the bosons in (2.8) and (2.9), where we explicitly break the  $SO(1,9)$  covariance again by treating the time component in a special way

$$\begin{aligned}
\varepsilon &:= (\det(\hat{E}_m{}^a))^{-\frac{1}{2}} \epsilon, \\
\chi_t &:= (\det(\hat{E}_m{}^a))^{-\frac{1}{2}} \left( \psi_t - N \Gamma_0 \Gamma^a \hat{\psi}_a \right), \\
\chi_{\bar{i}} &:= (\det(\hat{E}_m{}^a))^{\frac{1}{2}} \left( \frac{1}{4} \Gamma_{\bar{i}} \lambda - \hat{\psi}_{\bar{i}} \right), \\
\chi &:= (\det(\hat{E}_m{}^a))^{\frac{1}{2}} \left( \Gamma^a \hat{\psi}_a - \frac{1}{4} \lambda \right). \tag{4.1}
\end{aligned}$$

The hats denote, as in section 2, the projection onto the orthonormal frame  $\hat{\psi}_a \equiv \hat{E}_a{}^m \psi_m$ . The fermions in (4.1) can be assigned  $SO(9) \times SO(9)$  transformation properties as follows: All spinors transform as 16-component Majorana spinors of the first  $SO(9)$  factor and trivially under the second  $SO(9)$  except for  $\chi_{\bar{i}}$  which transforms as a vector. This is consistent with the different  $SO(1,9)$  chiralities of the type I fermions since single  $(32 \times 32)$   $\Gamma$ -matrices, defined in appendix A, intertwine between these two chiralities. The  $SO(9) \times SO(9)$  representations considered here are the chiral half of the representations of [17, 36].

Using the redefinitions (4.1), (2.8) and (2.9) in the supersymmetry variations (2.4d) and (2.4e), we arrive at the following results in leading fermion order

$$\begin{aligned}
\delta_\varepsilon \chi_t &= \partial_t \varepsilon + \frac{1}{4} \Gamma^{ij} \varepsilon \mathbb{Q}_{ij} + \frac{1}{3!} e^\phi \Gamma^{ijk} \Gamma^0 \varepsilon \mathbb{P}_{ijk} \\
&\quad + \mathbf{n} e^{-\phi} \Gamma^a \Gamma^0 \left\{ -\partial_a \varepsilon + \frac{1}{2} \varepsilon \partial_a \phi + \frac{1}{2} \varepsilon \mathbf{v}_a \right\} \\
&\quad + \frac{1}{2} \mathbf{n} e^{-\phi} \Gamma^a \Gamma^0 \varepsilon \left[ \mathbf{n}^{-1} \partial_a \mathbf{n} - \partial_a \phi \right], \tag{4.2a}
\end{aligned}$$

$$\begin{aligned}
\delta_\varepsilon \chi_{\bar{i}} &= -\frac{1}{2} \mathbf{n}^{-1} \Gamma^j \Gamma^0 \varepsilon \mathbb{P}_{j\bar{i}} + \frac{1}{2} \mathbf{n}^{-1} e^\phi \Gamma^{jk} \varepsilon \mathbb{P}_{ijk} \\
&\quad + e^{-\phi} \left\{ -\partial_{\bar{i}} \varepsilon + \frac{1}{2} \varepsilon \partial_{\bar{i}} \phi \right\}, \tag{4.2b}
\end{aligned}$$

$$\begin{aligned}
\delta_\varepsilon \chi &= -\frac{1}{2} \mathbf{n}^{-1} \Gamma^0 \varepsilon \partial_t \phi - \frac{1}{3!} \mathbf{n}^{-1} e^\phi \Gamma^{ijk} \varepsilon \mathbb{P}_{ijk} \\
&\quad - e^{-\phi} \Gamma^a \Gamma^0 \left\{ -\partial_a \varepsilon + \frac{1}{2} \varepsilon \partial_a \phi + \frac{1}{2} \varepsilon \mathbf{v}_a \right\}, \tag{4.2c}
\end{aligned}$$

where we used the abbreviations defined in (2.11) and (2.12). We observe again that the  $SO(9) \times SO(9)$  structure is preserved. From (4.2a) one can also read off the beginning of an extension of the  $SO(9) \times SO(9)$  covariance to  $K(DE_{10})$  covariance along the lines of [17, 31, 32, 33, 34] as we now discuss.

The key to unravelling the  $K(DE_{10})$  structure is to assume that the bosonic  $\sigma$ -model can be extended to a  $K(DE_{10})$  gauge invariant and locally *supersymmetric*  $D = 1$  coset model. This requires the introduction of fermionic fields transforming in  $K(DE_{10})$  representations. In such a model there will be a superpartner  $\chi_t$  to the lapse  $n$  which acts as the one-dimensional gravitino and therefore should transform into a  $K(DE_{10})$  covariant derivative of the supersymmetry parameter

$$\begin{aligned} \delta_\varepsilon \chi_t = \mathcal{D}_t \varepsilon = & \left( \partial_t - \frac{1}{2} Q_{ij} J^{ij} - \frac{1}{2} Q_{\bar{i}\bar{j}} J^{\bar{i}\bar{j}} \right. \\ & \left. - \frac{1}{3!} P_{ijk} J^{ijk} - \frac{1}{2!} P_{\bar{i}\bar{j}\bar{k}} J^{\bar{i}\bar{j}\bar{k}} - \frac{1}{2!} P_{\bar{i}\bar{j}\bar{k}} J^{\bar{i}\bar{j}\bar{k}} - \frac{1}{3!} P_{\bar{i}\bar{j}\bar{k}} J^{\bar{i}\bar{j}\bar{k}} + \dots \right) \varepsilon \end{aligned} \quad (4.3)$$

in Borel gauge (3.20). By comparing this relation to (4.2a), we can read off the form the  $K(DE_{10})$  generators take as a matrix representation on  $\varepsilon$ . On the first two ‘levels’, the result is

$$\begin{aligned} J^{ij} \varepsilon &= -\frac{1}{2} \Gamma^{ij} \varepsilon, & J^{\bar{i}\bar{j}} \varepsilon &= 0, \\ J^{ijk} \varepsilon &= -\Gamma^{ijk} \Gamma^0 \varepsilon, & J^{\bar{i}\bar{j}\bar{k}} \varepsilon &= 0, & J^{\bar{i}\bar{j}\bar{k}} \varepsilon &= 0, & J^{\bar{i}\bar{j}\bar{k}} \varepsilon &= 0. \end{aligned} \quad (4.4)$$

This implies that only two generators are represented non-trivially. In [32, 34] it was demonstrated in the maximally supersymmetric case that such restricted transformation rules can be sufficient to prove that  $\chi_t$  is a consistent *unfaithful* representation of  $K(DE_{10})$ . It follows from (4.3) that  $\varepsilon$  has to transform in the same  $K(DE_{10})$  representation. We now give the criterion for establishing such a consistent representation and show that it is satisfied in the present situation.

As shown in [34, 35] the generators of the compact subgroup of a Kac–Moody group can be written in terms of simple generators  $x_i = e_i - f_i$  (deduced from the Chevalley generators of section 3.2) and defining relations induced by the relations satisfied by the generators  $e_i$  and  $f_i$ . Working in an  $SO(9) \times SO(9)$  covariant formalism as we are doing here, the only consistency relation to check turns out to be

$$[x_{10}, [x_{10}, x_3]] + x_3 = 0, \quad (4.5)$$

where  $x_3$  and  $x_{10}$  are given in terms of the antisymmetric generators (3.13) and (3.21b) via [17]

$$\begin{aligned} x_3 &= J^{34} + J^{\bar{3}\bar{4}}, \\ x_{10} &= \frac{1}{2} \left( J^{123} + J^{\bar{1}\bar{2}\bar{3}} + J^{1\bar{2}3} + J^{12\bar{3}} + J^{\bar{1}\bar{2}3} + J^{\bar{1}2\bar{3}} + J^{1\bar{2}\bar{3}} + J^{\bar{1}\bar{2}\bar{3}} \right). \end{aligned} \quad (4.6)$$

By substituting these generators into the consistency relation (4.5) in a specific matrix representation acting on a vector space  $V$ , one can check whether  $V$  is a representation space of  $K(DE_{10})$ .<sup>18</sup>

The specific expressions for  $J^{ij}$ ,  $J^{\bar{i}\bar{j}}$  and  $J^{IJK}$  found in (4.4) satisfy the relation (4.5) as can be checked by straight-forward  $\Gamma$ -algebra. In terms of  $(32 \times 32)$  matrices the representation matrices  $\Gamma^{ij}$  and  $\Gamma^{ijk}\Gamma^0$  are block-diagonal and so act consistently on the projected 16-dimensional (chiral) spinor  $\chi_t$ . Naturally, there is also a representation on the other 16 components whose representation matrices are given by

$$\begin{aligned} J^{ij}\eta &= 0, & J^{\bar{i}\bar{j}}\eta &= -\frac{1}{2}\Gamma^{ij}\eta, \\ J^{ijk}\eta &= 0, & J^{\bar{i}\bar{j}\bar{k}}\eta &= 0, & J^{\bar{i}j\bar{k}}\eta &= 0, & J^{\bar{i}\bar{j}k}\eta &= -\Gamma^{ijk}\Gamma^0\eta. \end{aligned} \quad (4.7)$$

Therefore, there are two inequivalent 16-dimensional unfaithful spinor representations of  $K(DE_{10})$  as already anticipated in [36].<sup>19</sup> One can write down a similar consistent unfaithful representation of  $K(DE_{10})$  on  $\chi_{\bar{i}}$  which has dimension 144; to deduce its transformation laws one needs to rewrite the fermionic equation of motion and interpret this as a  $K(DE_{10})$  covariant derivative of  $\chi_{\bar{i}}$  [32, 34].

Having discussed the first few terms in (4.2a), we now briefly explore the remaining structure of eqs. (4.2). If we compare the two scalar equations of motion (2.18) and (2.20) with the corresponding supersymmetry transformations (4.2a) and (4.2c), the similarities are striking: Apart from the global factor of  $-\mathbf{n}$ , the second lines of (4.2a) and (4.2c) completely agree as in (2.18) and (2.20) and the third line of (4.2a) has the same structure as the one in (2.18). The final interpretation of these lines is still an open problem. However, the proposals which we discussed in section 3.4 can be equally applied here.

<sup>18</sup>We reiterate that we assume  $SO(9) \times SO(9)$  covariance of all generators — if this is not guaranteed there are additional relations which need to be verified.

<sup>19</sup>The 32-dimensional unfaithful spinor representation of  $K(E_{10})$  [31, 32] decomposes into the sum of these two inequivalent 16-dimensional spinor representations of the  $K(DE_{10})$  subgroup of  $K(E_{10})$ .

## 4.2 Supersymmetry transformation of the bosons

The redefined variables  $\mathbf{P}$  and  $\mathbf{Q}$  defined in (2.11) and (2.12) have been assigned  $SO(9) \times SO(9)$  transformation properties. Given the  $SO(9) \times SO(9) \subset K(DE_{10})$  fermions of the preceding section it is natural to study the  $SO(9) \times SO(9)$  and  $K(DE_{10})$  transformation properties of  $\mathbf{P}$  and  $\mathbf{Q}$  after a supersymmetry transformation  $\delta_\varepsilon$ .

We recall the definitions of  $\mathbf{P}_{i\bar{j}}$ ,  $\mathbf{Q}_{ij}$  and  $\mathbf{Q}_{\bar{i}\bar{j}}$  from (2.11), where the latter two play the role of  $SO(9) \times SO(9)$  gauge connections in the equations of motion (2.15) and (2.23). It will prove useful to define analogous quantities by replacing the time derivative  $\partial_t$  by the supersymmetry variation  $\delta_\varepsilon$  in (2.11) and hence get<sup>20</sup>

$$\Lambda_{i\bar{j}} := e_{(i}{}^m \delta_\varepsilon e_{m|\bar{j})} - \frac{1}{2} e_i{}^m e_{\bar{j}}{}^n \delta_\varepsilon B_{mn}, \quad (4.8a)$$

$$\Sigma_{ij} := e_{[i}{}^m \delta_\varepsilon e_{m|j]} + \frac{1}{2} e_i{}^m e_j{}^n \delta_\varepsilon B_{mn}, \quad (4.8b)$$

$$\Sigma_{\bar{i}\bar{j}} := e_{[\bar{i}}{}^m \delta_\varepsilon e_{m|\bar{j}]} - \frac{1}{2} e_{\bar{i}}{}^m e_{\bar{j}}{}^n \delta_\varepsilon B_{mn}. \quad (4.8c)$$

Substituting in the supersymmetry variations (2.4) as well as the redefinitions of the bosons (2.8) and the fermions (4.2), we find explicitly

$$\Lambda_{i\bar{j}} = -i\varepsilon \Gamma_i \chi_{\bar{j}}. \quad (4.9)$$

Furthermore, in analogy to the  $SO(9) \times SO(9)$  covariant derivative  $D_t$  defined in (2.16), we define an  $SO(9) \times SO(9)$  covariant supersymmetry transformation  $\underline{\delta}_\varepsilon$  by adding a local (in time) and field dependent  $SO(9) \times SO(9)$  gauge transformation  $\delta_\Sigma$  to the supersymmetry variation  $\delta_\varepsilon$

$$\underline{\delta}_\varepsilon = \delta_\varepsilon + \delta_\Sigma, \quad (4.10)$$

as was done in the  $SU(8)$  case in [5].<sup>21</sup> With the definitions (2.2) and (2.11), a short calculation yields the identities

$$\underline{\delta}_\varepsilon \mathbf{P}_{i\bar{j}} = \delta_\varepsilon \mathbf{P}_{i\bar{j}} + \Sigma_{ik} P_{k\bar{j}} + \Sigma_{\bar{j}\bar{k}} P_{i\bar{k}} = D_t \Lambda_{i\bar{j}}, \quad (4.11a)$$

$$\underline{\delta}_\varepsilon \mathbf{Q}_{ij} = \delta_\varepsilon \mathbf{Q}_{ij} + 2\Sigma_{[i|l} \mathbf{Q}_{l|j]} - \partial_t \Sigma_{ij} = 2\mathbf{P}_{[i|\bar{l}} \Lambda_{j] \bar{l}}, \quad (4.11b)$$

$$\underline{\delta}_\varepsilon \mathbf{Q}_{\bar{i}\bar{j}} = \delta_\varepsilon \mathbf{Q}_{\bar{i}\bar{j}} + 2\Sigma_{[\bar{i}|\bar{l}} \mathbf{Q}_{\bar{l}|\bar{j}]} - \partial_t \Sigma_{\bar{i}\bar{j}} = 2\mathbf{P}_{\bar{l}[\bar{i}} \Lambda_{\bar{l}]\bar{j}}, \quad (4.11c)$$

where the gauge fields  $\mathbf{Q}_{ij}$  and  $\mathbf{Q}_{\bar{i}\bar{j}}$  transform with explicit time derivatives of the gauge transformation parameters  $\Sigma_{ij}$  and  $\Sigma_{\bar{i}\bar{j}}$  as usual.

<sup>20</sup>As in section 2.2, there is no distinction between barred and unbarred frame indices at this point.

<sup>21</sup>As  $\Sigma = (\Sigma_{ij}, \Sigma_{\bar{i}\bar{j}})$  is of order two in fermions and as we have neglected higher order fermion contributions throughout the paper, the introduction of the covariant supersymmetry transformation  $\underline{\delta}_\varepsilon$  does not affect the discussion of the fermions in section 4.1.

For the fields defined in (2.12), we get

$$\mathbf{n}e^{-2\phi}\underline{\delta}_\varepsilon\left(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{ijk}\right) = -3\Lambda_{[i|\bar{l}]\bar{l}|j]k} - \frac{3}{2}\partial_{[i}\Sigma_{jk]}, \quad (4.12a)$$

$$\begin{aligned} \mathbf{n}e^{-2\phi}\underline{\delta}_\varepsilon\left(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{ijk}\right) &= -\Lambda_{\bar{l}\bar{i}}\mathbf{P}_{ljk} - 2\Lambda_{[j|\bar{l}]\bar{k}]\bar{i}} \\ &\quad - \frac{1}{2}\partial_{\bar{i}}\Sigma_{jk} + \partial_{[j}\Lambda_{k]\bar{i}}, \end{aligned} \quad (4.12b)$$

$$\begin{aligned} \mathbf{n}e^{-2\phi}\underline{\delta}_\varepsilon\left(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{i\bar{j}\bar{k}}\right) &= -\Lambda_{\bar{i}\bar{l}}\mathbf{P}_{\bar{l}\bar{j}\bar{k}} - 2\Lambda_{l[j\bar{k}]\bar{i}l} \\ &\quad + \frac{1}{2}\partial_{\bar{i}}\Sigma_{\bar{j}\bar{k}} - \partial_{[j}\Lambda_{|\bar{k}]|\bar{i}}, \end{aligned} \quad (4.12c)$$

$$\mathbf{n}e^{-2\phi}\underline{\delta}_\varepsilon\left(\mathbf{n}^{-1}e^{2\phi}\mathbf{P}_{i\bar{j}\bar{k}}\right) = -3\Lambda_{l[\bar{i}]\bar{l}|\bar{j}\bar{k}] + \frac{3}{2}\partial_{\bar{i}}\Sigma_{\bar{j}\bar{k}}. \quad (4.12d)$$

Again, we observe the  $SO(9) \times SO(9)$  covariance. The appearance of spatial derivatives of the  $SO(9) \times SO(9)$  transformation parameter  $\Sigma$  indicates that we should gauge the symmetry group  $SO(9) \times SO(9)$  with respect to space-time in fact by introducing also an  $SO(9) \times SO(9)$  derivative  $D_m$  as discussed in 3.4. However, this is not the way we want to pursue; instead we now study supersymmetry transformations in the  $DE_{10}/K(DE_{10})$  coset structure.

In the derivation of the coset equations of motion in section 3, we started with an element  $\mathcal{V}$  in the group coset  $DE_{10}/K(DE_{10})$  and considered variations of  $\mathcal{V}$  under a general variation  $\delta$ . As the supersymmetry operator is also realized as a derivative operator  $\delta_\varepsilon$ , which commutes with the time derivative  $\partial_t$ , we can use the same chain of arguments of section 3 to derive the supersymmetry variation of  $\mathcal{P}$  and  $\mathcal{Q}$ . This means we first decompose the  $\text{Lie}(DE_{10})$  valued expression

$$\delta_\varepsilon\mathcal{V}\mathcal{V}^{-1} = \tilde{\Lambda} + \tilde{\Sigma} \quad (4.13)$$

into generators  $\tilde{\Sigma} \in \text{Lie}(K(DE_{10}))$  and  $\tilde{\Lambda} \in \text{Lie}(DE_{10}/K(DE_{10}))$ . Then, we parametrise  $\tilde{\Sigma}$  and  $\tilde{\Lambda}$  similarly to (3.20) in a level decomposition truncated for  $\ell \geq 2$ , *i.e.*

$$\begin{aligned} \tilde{\Lambda} &= \delta_\varepsilon\varphi T + \tilde{\Lambda}_{i\bar{j}}S^{i\bar{j}} + \frac{1}{3!}e^\varphi\tilde{\Lambda}_{IJK}S^{IJK}, \\ \tilde{\Sigma} &= \frac{1}{2}\tilde{\Sigma}_{ij}J^{ij} + \frac{1}{2}\tilde{\Sigma}_{\bar{i}\bar{j}}J^{\bar{i}\bar{j}} + \frac{1}{3!}e^\varphi\tilde{\Lambda}_{IJK}J^{IJK}, \end{aligned} \quad (4.14)$$

where we again work in Borel gauge. Finally, from the fact that both derivative operators commute it follows that we get the variations after projecting

on the  $\ell = 0$  generators  $S^{i\bar{j}}$ ,  $T$  and the gauge orbit generators  $J^{ij}$  and  $J^{\bar{i}\bar{j}}$

$$\underline{\delta}_\varepsilon(\partial_t\varphi) = \delta_\varepsilon(\partial_t\varphi) = \partial_t(\delta_\varepsilon\varphi), \quad (4.15a)$$

$$\underline{\delta}_\varepsilon P_{i\bar{j}} = \delta_\varepsilon P_{i\bar{j}} + \tilde{\Sigma}_{ik} P_{k\bar{j}} + \tilde{\Sigma}_{\bar{j}\bar{k}} P_{i\bar{k}} = D_t \tilde{\Lambda}_{i\bar{j}}, \quad (4.15b)$$

$$\underline{\delta}_\varepsilon Q_{ij} = \delta Q_{ij} + 2\tilde{\Sigma}_{[i|k} Q_{k|j]} - \partial_t \tilde{\Sigma}_{ij} = 2P_{[i|\bar{l}} \tilde{\Lambda}_{j]\bar{l}}, \quad (4.15c)$$

$$\underline{\delta}_\varepsilon Q_{\bar{i}\bar{j}} = \delta Q_{\bar{i}\bar{j}} + 2\tilde{\Sigma}_{[\bar{i}|\bar{k}} Q_{\bar{k}|\bar{j}]} - \partial_t \tilde{\Sigma}_{\bar{i}\bar{j}} = 2P_{k[\bar{l}} \tilde{\Lambda}_{k|\bar{j}]}, \quad (4.15d)$$

where we have used the covariant derivative  $D_t$  as in (3.22b). With the covariant supersymmetry transformation  $\underline{\delta}_\varepsilon$  we can write the equations resulting from the projection onto the  $\ell = 1$  generators  $S^{ijk}$ ,  $S^{\bar{i}\bar{j}\bar{k}}$ ,  $S^{i\bar{j}\bar{k}}$  and  $S^{\bar{i}\bar{j}k}$  as

$$\underline{\delta}_\varepsilon P_{ijk} - 3\tilde{\Lambda}_{[i|\bar{l}} P_{l|jk]} = D_t \tilde{\Lambda}_{ijk} - 3P_{[i|\bar{l}} \tilde{\Lambda}_{l|jk]}, \quad (4.16a)$$

$$\underline{\delta}_\varepsilon P_{\bar{i}\bar{j}k} - \tilde{\Lambda}_{\bar{l}i} P_{ljk} + 2\tilde{\Lambda}_{[j|\bar{l}} P_{k]\bar{l}} = D_t \tilde{\Lambda}_{\bar{i}\bar{j}k} - P_{\bar{l}i} \tilde{\Lambda}_{ljk} + 2P_{[j|\bar{l}} \tilde{\Lambda}_{k]\bar{l}}, \quad (4.16b)$$

$$\underline{\delta}_\varepsilon P_{i\bar{j}\bar{k}} - 2\tilde{\Lambda}_{m[\bar{j}} P_{\bar{k}]im} + \tilde{\Lambda}_{i\bar{l}} P_{l\bar{j}\bar{k}} = D_t \tilde{\Lambda}_{i\bar{j}\bar{k}} - 2P_{m[\bar{j}} \tilde{\Lambda}_{\bar{k}]im} + P_{i\bar{l}} \tilde{\Lambda}_{l\bar{j}\bar{k}}, \quad (4.16c)$$

$$\underline{\delta}_\varepsilon P_{\bar{i}\bar{j}k} - 3\tilde{\Lambda}_{\bar{l}i} P_{l\bar{j}k} = D_t \tilde{\Lambda}_{\bar{i}\bar{j}k} - 3P_{\bar{l}i} \tilde{\Lambda}_{l\bar{j}k}. \quad (4.16d)$$

It should be noted that an inclusion of higher level terms  $\ell \geq 2$  in the expansions (3.20) and (4.14) above does not alter the equations (4.15) and (4.16) in contradistinction to the equations of motion (3.22) and (3.23). This is a general property of the Borel gauge [34].

As we have identified the supergravity variables  $\mathbf{P}$  and  $\mathbf{Q}$  at a fixed spatial point  $\mathbf{x}$  with the coset variables  $P$  and  $Q$  in section 3.4, a comparison of the equations (4.11) and (4.15) forces us to identify  $\Lambda_{i\bar{j}}$  with  $\tilde{\Lambda}_{i\bar{j}}$  and  $(\Sigma_{ij}, \Sigma_{\bar{i}\bar{j}})$  with  $(\tilde{\Sigma}_{ij}, \tilde{\Sigma}_{\bar{i}\bar{j}})$ , respectively. Using the explicit form of  $\Lambda_{i\bar{j}}$  in terms of fermion bilinears (4.9) and the  $K(DE_{10})$  transformation rules for the fermions deduced in section 4.1, we could in principle compute  $\tilde{\Lambda}_{IJK}$  from this by comparing it with a  $K(DE_{10})$  transformation of the coset representation. We can obtain an independent answer for  $\tilde{\Lambda}_{IJK}$  by comparing with supergravity. It is not guaranteed that the two answers will agree. In [34] a similar analysis was carried out in the maximal  $D = 11$  supergravity context and some but not all expressions for the analogues of  $\tilde{\Lambda}_{IJK}$  agree. We take this as an indication that there is a disparity between the unfaithful, finite-dimensional fermionic  $K(DE_{10})$  representation and the infinite-dimensional coset representation, which is in conflict with supersymmetry on the coset side.

## 5 Discussion

In this paper we have rewritten the bosonic and fermionic fields of pure type I supergravity in terms of variables which we assigned to representations of

$SO(9) \times SO(9)$  (cf. (2.11), (2.12) and (4.1)). The relevant bosonic representations are identical to those that arise in the  $D_9$  level decomposition of  $DE_{10}$  on the levels  $\ell = 0, 1$  as shown in section 3.2. In section 4.1 we also showed that some of the relevant fermionic representations of  $SO(9) \times SO(9)$  can be consistently extended to unfaithful representations of  $K(DE_{10})$ .

At the dynamical level we demonstrated that the bosonic equations of pure type I supergravity in this parametrisation (evaluated at a fixed spatial point) coincide with those derived from a simple  $D = 1$  non-linear  $\sigma$ -model on  $DE_{10}/K(DE_{10})$  truncated consistently beyond  $\ell = 1$  up to a number of terms which can either be gauged away or can be argued to be of the form of (generalised) spatial gradients, see section 3.4.<sup>22</sup> It would be very interesting to see whether the full  $D = 10$  equations can be cast in  $SO(9) \times SO(9)$  covariant form, analogous to the treatment in [5]. Our focus was not on this question but rather if we can extend the (partial)  $SO(9) \times SO(9)$  covariance to a (partial)  $K(DE_{10})$  covariance to further test the ideas of [20]. As pointed out for example below (3.23) the truncated coset model equations of motion include terms which are part of a  $K(DE_{10})$  covariant formulation and these terms agree with identical terms in the supergravity equations (2.23). A similar phenomenon was observed for the supersymmetry variation of certain fermionic fields, see (4.3).

We consider our results as evidence that  $K(DE_{10})$  might be a dynamical symmetry of the pure type I theory. There are also a number of conundrums related to our analysis, some of which were already hinted at.

As is well known, the pure type I supergravity theory is not anomaly free. However, by adding appropriate vector multiplets [37] the anomalies can be cancelled [38, 39]. The possibilities which are realised as string theory low energy effective theories are those which have vector multiplets transforming as Yang–Mills fields of either  $SO(32)$  or  $E_8 \times E_8$ . The inclusion of these non-Abelian symmetries in the context of Kac–Moody symmetries is poorly understood. Augmenting the theory (2.1) by Abelian or non-Abelian vector fields changes the associated cosmological billiard from  $DE_{10}$  to a group called  $BE_{10}$  [18] but, at the level of coset model,  $BE_{10}$  is not appropriate for accommodating more than a single Abelian vector field. In [23] multiple Abelian vector fields were added to the pure type I theory by increasing the rank of the Kac–Moody symmetry and changing the real form.<sup>23</sup> A proper understanding of the non-Abelian symmetries from a Kac–Moody algebraic point of view is lacking at the moment.

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<sup>22</sup>Our analysis was always at the level of the equations of motion and not at the level of the action.

<sup>23</sup>See also [40] for a discussion of a non-maximal theory with vector fields obtained by orbifolding the maximal theory. In this analysis  $DE_{10}$  and further extensions of it similar to those of [23] appear.

Another interesting challenge is to extend the bosonic  $DE_{10}/K(DE_{10})$   $\sigma$ -model of (3.2) to a locally supersymmetric model in  $D = 11$ . With the fermionic representations employed in this paper, it appears impossible to construct such a model. In fact, in the present situation there is no non-vanishing combination that can be constructed from terms bilinear in the fermions (4.1) of the form  $\tilde{\Lambda}_{\bar{i}jk}$  whereas such an expression necessarily appears in the supersymmetry variation of the  $\ell = 1$  field  $P_{\bar{i}jk}$  due to (4.12) and (4.16).

Finally, it is crucial to bring the ‘gradient conjecture’ of [20] back into view. According to this conjecture the  $\sigma$ -model can capture the *full* dynamics in a neighbourhood of the fixed spatial point  $\mathbf{x}$  by translating the information about all spatial gradients of the supergravity fields into higher level degrees of freedom of the  $\sigma$ -model. As discussed in section 3.4, the concrete realisation of this translation is still an open problem.

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## A Conventions for $\Gamma$ -matrices

We use the same conventions for  $\Gamma$ -matrices as [17] which we summarise for completeness.

The  $(32 \times 32)$  real  $\Gamma$ -matrices  $\Gamma^A$  for  $A = 0, \dots, 10$  of  $SO(1, 10)$  are defined in terms of the real symmetric  $(16 \times 16)$   $\gamma$ -matrices  $\gamma^i$  ( $i = 1, \dots, 9$ ) of  $SO(9)$  by

$$\Gamma^0 = \begin{pmatrix} 0 & -\mathbf{1}_{16} \\ \mathbf{1}_{16} & 0 \end{pmatrix}, \quad \Gamma^{10} = \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}. \quad (\text{A.1})$$

The matrix  $\Gamma^0$  is the charge conjugation matrix in  $D = 11$ . After descending to  $SO(1, 9)$  the matrix  $\frac{1}{2}(\mathbf{1}_{32} \pm \Gamma^{10})$  serves as the projector on the two chiral spinors in  $D = 10$ . The type I fermions  $\psi_M$  and  $\lambda$  discussed in the paper have been projected from 32-component spinors to opposite chiralities.

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