

Number of negative modes of the oscillating bounces

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The spectrum of small perturbations about oscillating bounce solutions recently discussed in the literature is investigated. Our study supports quite intuitive and expected result: the bounce with N nodes has exactly N homogeneous negative modes. Existence of more than one negative modes makes obscure the relation of these oscillating bounce solutions to the false vacuum decay processes.

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I. INTRODUCTION

Having in mind recent progress in string theory, predicting string landscape (many vacua) picture, it is very important to have deeper understanding of metastable (false) vacuum decay processes with gravity taken into account. Theory of false vacuum decay in flat space-time was formulated long ago [1–7]. It was shown that the metastable vacuum decay proceeds via true vacuum bubbles nucleation in the false vacuum and subsequent growth of these bubbles. Within the Euclidean approach bubble nucleation process is described by bounce [6], classical solution of the Euclidean equations of motion with certain boundary conditions. It was shown that in the WKB approximation the action of the bounce determines the tunnelling rate exponent [6] and it was rigorously proven that in flat space-time the bounce is $O(4)$ -symmetric [8]. Furthermore, the procedure for calculating the path integral in the WKB approximation around the bounce configuration was described [9] and it was shown that there is exactly one negative mode in spectrum of small perturbations about bounce solution in flat space-time. This negative mode is very essential and makes decay picture coherent. In Coleman's words: "There may exist solutions in other ways like bounces and which have more than one negative eigenvalue, but, even if they do exist, they have nothing to do with tunnelling" [10].

Bounce solution in presence of gravity was found by Coleman and De Luccia [11]. In addition some exited multi bounce solutions are known in the literature: Bousso and Linde discussed double-bubble instantons [12] and more recently the oscillating bounce solutions were studied in details by Hackworth and Weinberg [13,14]. It was suggested that in certain regimes oscillating bounces most likely will play a role.

The aim of the present letter is to investigate the number of negative modes of these oscillating bounce solutions in order to check their relevance to the tunnelling. While in flat space-time finding a negative mode about bounce is straightforward task, when gravity is taken into account it

is more involved problem [15–21]. It was shown that with the proper reduction procedure one finds a single negative mode about Coleman-De Luccia bounce [19].

The rest of the paper is organized as follows: in the next section we discuss the Euclidean equations of motion and boundary conditions for the bounce solution. In Sec. III we present Schrödinger equation for linear perturbations about the bounce and in the Sec. IV we show out numerical results for concrete choice of scalar field potentials.

II. BOUNCE SOLUTION

Let us consider the theory of a scalar field coupled to gravity which is defined by the following Euclidian action

$$S_E = \int d^4x \sqrt{g} \left(\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + V(\phi) - \frac{1}{2\kappa} R \right), \quad (1)$$

where $\kappa = 8\pi G_N$ is the reduced Newton's gravitational constant.

The most general $O(4)$ invariant metric is parameterized as

$$ds^2 = N^2(\sigma) d\sigma^2 + a^2(\sigma) d\Omega_3^2, \quad (2)$$

where $N(\sigma)$ is the Lapse function, $a(\sigma)$ is the scale factor and $d\Omega_3^2$ is metric of unit three-sphere:

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2). \quad (3)$$

For metric Eq. (2) the curvature scalar looks like

$$R = \frac{6}{a^2} - \frac{6\dot{a}^2}{a^2 N^2} - \frac{6\ddot{a}}{aN^2} + \frac{6\dot{a}\dot{N}}{aN^3}, \quad (4)$$

where $\dot{} = d/d\sigma$. Using ansatz Eq. (2) and assuming that $\phi = \phi(\sigma)$ we get the reduced action in the form

$$\begin{aligned} S_E &= S_E(\phi, N, a) \\ &= 2\pi^2 \int d\sigma \left(\frac{a^3}{2N} \dot{\phi}^2 + a^3 N V(\phi) - \frac{3aN}{\kappa} + \frac{3a\dot{a}^2}{\kappa N} \right. \\ &\quad \left. + \frac{3a^2\ddot{a}}{\kappa N} - \frac{3a^2\dot{a}\dot{N}}{\kappa N^2} \right). \end{aligned} \quad (5)$$

Corresponding field equations in the proper time gauge, $N = 1$, are

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$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{\partial V}{\partial \phi}, \quad (6)$$

$$\ddot{a} = -\frac{\kappa a}{3}(\dot{\phi}^2 + V(\phi)), \quad (7)$$

$$\dot{a}^2 = 1 + \frac{\kappa a^2}{3}\left(\frac{\dot{\phi}^2}{2} - V\right). \quad (8)$$

Now let us assume that potential $V(\phi)$ has two nondegenerate local minima at $\phi = \phi_{\text{lv}}$ and $\phi = \phi_{\text{fv}}$, with $V(\phi_{\text{fv}}) > V(\phi_{\text{lv}})$, and local maximum for some $\phi = \phi_{\text{top}}$, with $\phi_{\text{fv}} < \phi_{\text{top}} < \phi_{\text{lv}}$, Fig. 1. Euclidean solution describing vacuum decay—bounce—satisfies these equations and in case $V(\phi) > 0$ has following boundary conditions

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = 0, \quad a(0) = 0, \quad \dot{a}(0) = 1, \quad (9)$$

at $\sigma = 0$ and

$$\begin{aligned} \phi(\sigma_{\text{max}}) &= \phi_m, & \dot{\phi}(\sigma_{\text{max}}) &= 0, \\ a(\sigma_{\text{max}}) &= 0, & \dot{a}(\sigma_{\text{max}}) &= 1, \end{aligned} \quad (10)$$

at some $\sigma = \sigma_{\text{max}}$. This assumes the following Taylor series at $\sigma \rightarrow 0$

$$\begin{aligned} \phi(\sigma) &= \phi_0 + \frac{1}{8} \frac{\partial V}{\partial \phi} \Big|_{\phi=\phi_0} \sigma^2 + \frac{1}{192} \frac{\partial V}{\partial \phi} \Big|_{\phi=\phi_0} \left(\frac{\partial^2 V}{\partial \phi^2} \Big|_{\phi=\phi_0} \right. \\ &\quad \left. + \frac{2\kappa V(\phi_0)}{3} \right) \sigma^4 + O(\sigma^6), \end{aligned} \quad (11)$$

$$\begin{aligned} a(\sigma) &= \sigma - \frac{\kappa}{18} V(\phi_0) \sigma^3 - \frac{\kappa}{120} \left(\frac{3}{8} \left(\frac{\partial V}{\partial \phi} \Big|_{\phi=\phi_0} \right)^2 \right. \\ &\quad \left. - \frac{\kappa}{9} V^2(\phi_0) \right) \sigma^5 + O(\sigma^7), \end{aligned} \quad (12)$$

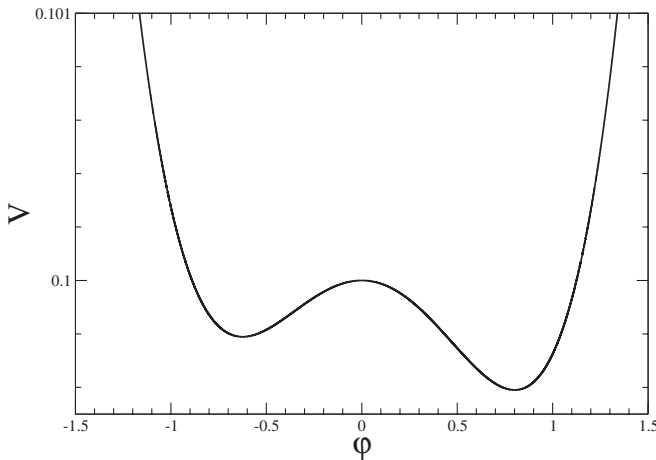


FIG. 1. Scalar field potential $V(\varphi)$.

and similar power law behavior for nonsingular bounces for $x \rightarrow 0$, where $x = \sigma_{\text{max}} - \sigma$.

Whereas bounce solution always exists in the flat space-time, when gravity is switched on, the existence of bounce depends on details of scalar field potential. For wide class of potentials the existence of bounce solution is determined by the value of parameter β [22,23],

$$\beta = |V''(\phi_{\text{top}})|/H^2, \quad (13)$$

where $H^2 = \kappa V(\phi_{\text{top}})/3$. For $\beta < 4$ no Coleman-De Luccia bounce exists in the given potential. Increasing β more and more oscillating bounce solutions appear. For broad class of potentials for a given β there are oscillating bounces with up to N nodes, where N is the largest integer such that $N(N+3) < \beta$ [13]. In addition one finds also the Hawking-Moss solution [24] which exist in any potential with positive local maximum.

III. LINEAR PERTURBATIONS

The investigation of perturbations about the bounce solution is convenient to perform in conformal frame [19,20]. Let us expand the metric and the scalar field over a $O(4)$ -symmetric background as follows

$$\begin{aligned} ds^2 &= a(\tau)^2((1 + 2A(\tau))d\tau^2 + (1 - 2\Psi(\tau))d\Omega_3^2), \\ \phi &= \varphi(\tau) + \Phi(\tau), \end{aligned} \quad (14)$$

where τ is the conformal time, a and φ are the background field values and A , Ψ and Φ are small perturbations. In what follows we will be interested in the lowest (only τ dependant, “homogeneous”) modes and consider only scalar metric perturbations, while the negative energy states are found previously exactly in this sector.

Expanding the total action, keeping terms up to the second order in perturbations and using the background equations of motion we find

$$S = S^{(0)}[a, \varphi] + S^{(2)}[A, \Psi, \Phi], \quad (15)$$

where $S^{(0)}$ is the action of the background solution and $S^{(2)}[A, \Psi, \Phi]$ is the quadratic action. The Lagrangian corresponding to this quadratic action is degenerate and describes constrained dynamical system. Applying Dirac’s formulation of generalized Hamiltonian dynamics we get unconstrained quadratic action in the form [19,20]

$$S_E^{(2)} = 2\pi^2 \int \left(\frac{1}{2} q'^2 + \frac{1}{2} W[a(\tau), \varphi(\tau)] q^2 \right) d\tau, \quad (16)$$

with the potential W whose conformal time dependence is determined by the bounce solution [20]

$$W[a(\tau), \varphi(\tau)] = \frac{a^2}{\mathcal{Q}} \frac{\delta^2 V}{\delta \varphi \delta \varphi} - \frac{10a'^2}{a^2 \mathcal{Q}} + \frac{12a'^2}{a^2 \mathcal{Q}^2} + \frac{8}{\mathcal{Q}} - 6 - 3\mathcal{Q} + \frac{\kappa a^4}{2\mathcal{Q}^2} \left(\frac{\delta V}{\delta \varphi} \right)^2 - \frac{2\kappa a a' \varphi'}{\mathcal{Q}^2} \frac{\delta V}{\delta \varphi}. \quad (17)$$

Here $q = a/\sqrt{\mathcal{Q}}\Phi$, prime denotes the derivative with respect to conformal time τ and $\mathcal{Q} = 1 - \kappa\varphi'^2/6$.

Introducing new variable $f = \sqrt{a}q$ and passing to the proper time σ quadratic action Eq. (16) can be written in the form

$$S_E^{(2)} = 2\pi^2 \int \left(\frac{1}{2} \dot{f}^2 + \frac{1}{2} U[a(\sigma), \varphi(\sigma)] f^2 \right) d\sigma, \quad (18)$$

with the potential U

$$U[a(\sigma), \varphi(\sigma)] = \frac{1}{\mathcal{Q}} \frac{\delta^2 V}{\delta \varphi \delta \varphi} - \frac{10\dot{a}^2}{a^2 \mathcal{Q}} + \frac{12\dot{a}^2}{a^2 \mathcal{Q}^2} + \frac{8}{a^2 \mathcal{Q}} - \frac{6}{a^2} - \frac{3\mathcal{Q}}{a^2} - \frac{\dot{a}^2}{4a^2} + \frac{\kappa a^2}{2\mathcal{Q}^2} \left(\frac{\delta V}{\delta \varphi} \right)^2 - \frac{2\kappa a \dot{a} \varphi}{\mathcal{Q}^2} \frac{\delta V}{\delta \varphi} - \frac{\kappa}{6} (\dot{\varphi}^2 + V), \quad (19)$$

where $\mathcal{Q} = 1 - \kappa a^2 \dot{\varphi}^2/6$. So, spectrum of small perturbations about bounce solution is determined by the following Schrödinger equation

$$-\frac{d^2}{d\sigma^2} f + U[a(\sigma), \varphi(\sigma)] f = E f, \quad (20)$$

and the number of negative modes of the bounce solution is the number of bound states of these Schrödinger equation.

IV. NUMERICAL RESULTS

Let us parameterize the general quartic scalar field potential as follows:

$$V = V_0 + H^2 \left(-\frac{\beta}{2} \varphi^2 - \frac{g}{3} \varphi^3 + \frac{\lambda}{4} \varphi^4 \right), \quad (21)$$

with $H^2 = \kappa V_0/3$.

Passing to the dimensionless variables

$$\begin{aligned} \tilde{\varphi} &= \frac{\varphi}{v}, & \tilde{a} &= av, & \tilde{\sigma} &= \sigma v, \\ \tilde{V}_0 &= \frac{V_0}{v^4}, & \tilde{H}^2 &= \frac{H^2}{v^2}, & \tilde{\kappa} &= \kappa v^2, \end{aligned} \quad (22)$$

with $v^2 = 2\beta/\lambda$ we will get the dimensionless equations of motion with the rescaled potential (comp. [13])

$$\tilde{V} = \tilde{V}_0 + \tilde{H}^2 \beta \left(-\frac{1}{2} \tilde{\varphi}^2 - \frac{\tilde{g}}{3} \tilde{\varphi}^3 + \frac{1}{2} \tilde{\varphi}^4 \right), \quad (23)$$

where $\tilde{g} = gv/\beta$. In what follows we will use dimensionless variables and omit tildes.

Potential U in the Schrödinger equation Eq. (20) close to the $\sigma = 0$ behaves as

$$U = \frac{3}{4\sigma^2} + U_0 + O(\sigma^2), \quad (24)$$

where constant U_0 depends on the initial value of scalar field and parameters of the background solution potential V . For the potential Eq. (23) it is

$$U_0 = \left(-H^2 \beta - \frac{2}{3} \kappa V_0 - 2H^2 \beta g \phi_0 + 6H^2 \beta \left(1 + \frac{\kappa}{18} \right) \phi_0^2 + \frac{2}{9} \kappa H^2 \beta g \phi_0^3 - \frac{1}{3} \kappa H^2 \beta \phi_0^4 \right). \quad (25)$$

The regular branch of the wave function f behaves as

$$f = \sigma^{3/2} \left(1 + \frac{1}{8} (U_0 - E) \sigma^2 + O(\sigma^4) \right). \quad (26)$$

Convenient way to determine the number of bound states of Schrödinger equation in a given potential is the investigation of the zero energy wave function. The number of nodes of zero energy wave function exactly counts the number of negative energy states [25].

Let us describe our results in details on concrete example. For the parameters choice

$$\begin{aligned} \kappa &= 0.001, & V_0 &= 0.1, \\ \beta &= 70.03, & g &= \frac{1}{2\sqrt{2}}, \end{aligned} \quad (27)$$

the potential Eq. (23) has local maximum at $\varphi_{\text{top}} = 0$, metastable minimum at $\varphi_{\text{fv}} = -0.6242212930$ and true vacuum at $\varphi_{\text{tv}} = 0.8009979884$, Fig. 1. There exists Coleman-De Luccia bounce solution in this potential, oscillating bounces on top of it with up to $N = 7$ nodes and, as always, the Hawking-Moss solution [24] with $\varphi \equiv \varphi_{\text{top}}$. Numerical investigation supports quite intuitive and expected result [21]: the bounce with N nodes has exactly N negative modes. Typical results are demonstrated for $N = 3$ case on Fig. 2 and 3. The zero energy wave function of

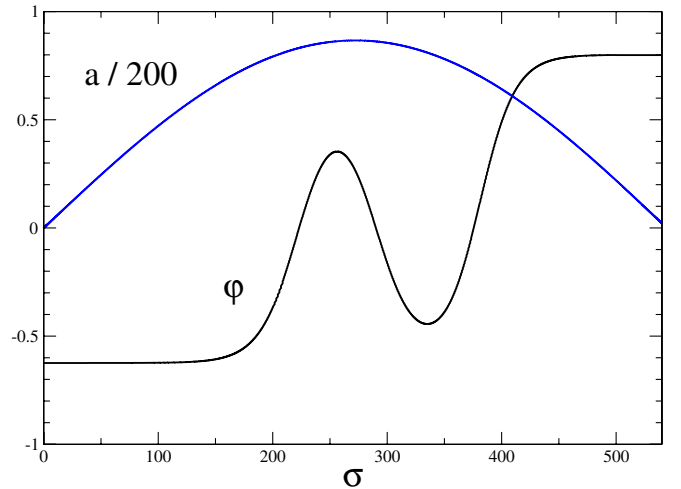


FIG. 2 (color online). Oscillating bounce solution with three nodes of φ .

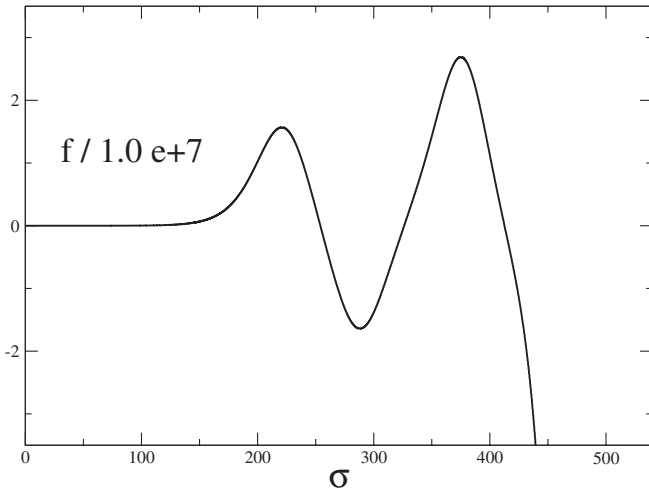


FIG. 3. Zero energy wave function f of Schrödinger equation of linear perturbations about oscillating bounce solution with three nodes of φ .

Schrödinger equation Eq. (20) has in this case three nodes, which means that there are exactly three negative

energy states for $N = 3$ oscillating bounce. We also found this states explicitly and determined their energies: $E_0 = -0.0013787$, $E_1 = -0.0004362$ and $E_2 = -0.0001207$. Corresponding Hawking-Moss solution has eight homogeneous negative modes, which is consistent with chosen value of β .

To conclude, we found that the oscillating bounces studied in [12,13] have more than one negative modes, which makes their relevance to the tunnelling processes obscure. So, question about contribution of oscillating bounces to the false vacuum decay amplitude needs further investigation.

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