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A family of asymptotically hyperbolic manifolds with arbitrary energy-momentum vectors

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A family of non-radial solutions to the Yamabe equation, modeled on the hyperbolic space, is constructed using power series. As a result, we obtain a family of asymptotically hyperbolic metrics, with spherical conformal infinity, with scalar curvature greater than or equal to $-n(n-1)$, but which are *a priori* not complete. Moreover, any vector of $\mathbb{R}^{n+1}$ is performed by an energy-momentun vector of one suitable metric of this family. They can in particular provide counter-examples to the *positive energy-momentum theorem* when one removes the completeness assumption. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4759581]

I. INTRODUCTION

For the last decade, the study of asymptotically hyperbolic manifolds has enjoyed a significant effort from the mathematical General Relativity community as well as from a pure geometric point of view. It is generally motivated by the same questions that arise in the study of asymptotically flat manifolds (or asymptotically Minkowskian initial data). Indeed, hyperbolic manifolds, or anti-de Sitter initial data appear as ground states in general relativity with negative cosmological constant, and one can define mass-type geometric invariants for initial data that approach these models at infinity, and conjecture some positivity results in the spirit of the *positive mass theorem*.

Significant results with the hyperbolic background have been obtained by Wang[19] and by Chruściel and Herzlich.[8] They both show that a set of global invariants can be defined for asymptotically hyperbolic manifolds (A.H. manifolds for short), forming the *energy-momentum vector*, although this has been done in a more general way in Ref. 8. They further establish a *positive energy-momentum theorem*, which, like the asymptotically flat positive mass theorem, ensures that a positivity property of these invariants is satisfied provided the manifold is complete and fulfills the *dominant energy condition*. This latter condition translates here as a lower bound on the scalar curvature, which is precisely the value of the scalar curvature for the model considered in each case, 0 for $\mathbb{R}^n$ and $-n(n-1)$ for $\mathbb{H}^n$.

The aim of the present work is to construct examples of asymptotically hyperbolic metrics which satisfy the above assumptions except the completeness, and which violate the positivity result of the positive energy-momentum theorem.

Such results aim at giving a better knowledge of the behaviour of these invariants and the role played by each assumption of the positive energy-momentum theorem. Indeed, this theorem has not yet been proven in the most general and satisfying case. The results obtained by Wang and Chruściel-Herzlich indeed require a further assumption of topological nature (namely the existence of a spin structure), whereas, in an attempt to remove this further assumption, Andersson, Cai, and Galloway in Ref. 1 needed another extra condition on the structure at infinity of the manifold.

In the asymptotically flat case, it is straightforward to exhibit (non-complete) Riemannian manifolds of non-negative scalar curvature and of negative mass, by considering the exterior of a Schwarzschild manifold of mass $m < 0$. In fact, the mass invariant of an asymptotically flat manifold

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(see Ref. 3) is a number, coincides with the parameter m for Schwarzschild manifolds, and thus can take any real value.

However, when the model at infinity is the hyperbolic space $\mathbb{H}^n$ (with conformal infinity $\mathbb{S}^{n-1}$), the set of mass-type invariants is in fact more complicated, since the natural mass-type object to consider for an asymptotically hyperbolic manifold is rather the "energy-momentum" vector, made of $n + 1$ components.

The *positive energy-momentum conjecture* states that this vector has to be timelike and future-directed if the manifold is moreover complete and has scalar curvature greater than $-n(n - 1)$ (which is the scalar curvature of $\mathbb{H}^n$).

As for the asymptotically flat case, one can exhibit metrics that satisfy these assumptions except the completeness and that violate the energy-momentum conjecture. Well-known examples of such metrics are the Kottler-Schwarzschild-anti de Sitter metrics with negative mass parameter m, given by the expression in local coordinates

$$b_m = \frac{dr^2}{1 - \frac{2m}{r^{n-2}} + r^2} + r^2\sigma_{n-1} \ ,$$

where σ_{n-1} is the standard metric on the unit sphere $\mathbb{S}^{n-1}$. Note that we obtain here for $m = 0$ the expression of the hyperbolic metric. Their energy-momentum vector are then timelike, past-directed and we can moreover obtain any timelike, future or past-directed energy-momentum vector by considering the family of *boosted Kottler metrics*, see e.g., Ref. 7

But no example of such a metric, which satisfies the assumptions of the positive energy-momentum theorem except the completeness, and which has a spacelike or null energy-momentum vector, has been exhibited so far. A work[17] from Shi and Tam shows the existence, in the three-dimensional case, of asymptotically hyperbolic metrics (A.H. metrics for short) of constant scalar curvature -6 on the unit ball of $\mathbb{R}^3$, with energy-momentum vectors whose Minkowskian norm is positive, prescribed up to a small error, and with further properties on the existence of horizons. But this result holds only for timelike, future-directed energy-momentum vectors.

The aim of the present work is to fill this gap by proving the main result of this paper:

Theorem 1.1: *Let* $\mathbf{p}$ *be a vector of the Minkowski space* $\mathbb{R}^{1,n}$*. Then, there exists a compact set* $K \subset \mathbb{R}^n$ *and a Riemannian, asymptotically hyperbolic metric g defined on* $\mathbb{R}^n \setminus K$*, with scalar curvature* $R_g \geq -n(n - 1)$*, such that the energy-momentum vector of g is well defined and coincides with* $\mathbf{p}$.

In fact, to prove this theorem, we construct an explicit family of metrics which have other remarkable properties such as being conformally flat and having a constant scalar curvature $-n(n - 1)$. Such a family may then be useful when one wishes to approximate any asymptotically hyperbolic metric of general energy-momentum vector by one from this family. This approximation "near infinity" is made possible from the gluing method introduced by Corvino and Schoen,[6,9,10] and it may be used as a first step in a future attempt to prove the positive energy-momentum theorem, simplifying the structure at infinity. In the same spirit, for the asymptotically flat case, one can start the Schoen-Yau's proof of the positive mass theorem by gluing an asymptotically flat manifold of negative mass with a Schwarzschild manifold of (arbitrarily close) negative mass parameter, for example using Corvino's gluing result,[9] or from a direct calculation such as in Ref. 18. The result of this is that one can consider only metrics having nicely behaved asymptotics without loss of generality.

The present paper is organized as follows:

In Sec. II, we give the specific material to work with asymptotically hyperbolic manifolds, in particular their various definitions, the definition of the energy-momentum vector, and a quick retrospective of the current knowledge on the positive energy-momentum theorem.

In Sec. III, we establish the Theorem 1.1, which is the main of this work. We actually look for a family of conformally hyperbolic metrics, whose conformal factor satisfy the Yamabe equation such that the scalar curvature is constant and equal to $-n(n - 1)$. We seek an expression for these conformal factors as power series in $1/r$ –where r is the standard radial coordinate – whose coefficients are non-constant functions defined on the $(n-1)$-unit sphere.

In Sec. IV at last, we derive some applications, one concerning a discussion around a "positive energy-momentum theorem with boundary" proved by Chruściel and Herzlich in Ref. 8, and the other one showing that the family of metrics constructed in Sec. III may well be used as a family of models at infinity in the process of gluing, similarly to what is done by Chruściel and Delay in Ref. 7. One can therefore approximate an asymptotically hyperbolic metric g of constant scalar curvature by one with much nicer properties at infinity, and whose energy-momentum vector is arbitrarily close to the one of g.

II. ASYMPTOTICALLY HYPERBOLIC MANIFOLDS AND ENERGY-MOMENTUM

A. Asymptotically hyperbolic metrics

For $n \geq 3$, let (M, g) be a smooth n-dimensional Riemannian manifold with boundary ∂M. We will consider such manifolds (M, g) which are furthermore non-compact and which have an end in which g tends to the hyperbolic metric b.

These manifolds appear naturally in general relativity as initial data of asymptotically anti-de Sitter spacetimes.

In the sequel, we give accurate definitions, following standard notations used, e.g., in Ref. 11:

Definition 2.1: A manifold (M, g) is said to be conformally compact if there exists:

- *a smooth compact manifold $\bar{M}$, with interior $\mathring{M}$ and with boundary $\partial\bar{M}$ such that $\partial\bar{M}$ is the union of ∂M and $\partial_\infty M$, with $M = \mathring{M} \cup \partial M$,*
- *a defining function, i.e., a smooth function $\rho : \bar{M} \to \mathbb{R}_+$ such that $\rho^{-1}(0) = \partial_\infty M$, the 1-form $d\rho$ does not vanish on $\partial_\infty \bar{M}$, and the metric tensor $\bar{g} := \rho^2 g$ of M extends to a smooth metric on $\bar{M}$.*

The boundary component $\partial_\infty M$ will often be referred to as the *boundary at infinity*, or as the *conformal boundary*, whereas ∂M will be referred to as the *inner boundary*.

Definition 2.2: A conformally compact manifold (M, g) is asymptotically hyperbolic *(A.H.) if one has the further condition $|d\rho|_{\bar{g}} = 1$ on $\partial_\infty M$.*

Remark 2.3:

- *With respect to the above notations, one has in particular $\sec_g \to -1$ as $\rho \to 0$: the sectional curvature converges to -1 near the boundary at infinity. This motivates the "asymptotically hyperbolic" terminology.*
- *Under these assumptions, there exists a unique defining function ρ and a neighborhood U of $\partial_\infty M$ of the form $(0, \varepsilon] \times \partial_\infty M$ such that, under a diffeomorphism Φ between U and the complement M_{ext} of a compact region of M, the metric takes the form*

$$\Phi^* g = \frac{1}{\sinh^2 \rho}\left(d\rho^2 + h_\rho\right)$$

on U, where (h_ρ) is a family of metrics on a $(n-1)$-dimensional compact boundaryless manifold N, smooth with respect to ρ, such that the limit as $\rho \to 0$ is a metric h_0 on N with constant scalar curvature $R_h = k(n-1)(n-2)$, $k \in \{-1, 0, 1\}$.
- *It is common and convenient to express these definitions using a "radial" coordinate r such that*

$$\frac{1}{r} = \sinh \rho \ .$$

In these coordinates, the hyperbolic metric b takes the form

$$b = \frac{dr^2}{1 + r^2} + r^2 \sigma_{n-1} \ , \tag{1}$$

where σ_{n-1} is the standard metric on the unit $(n-1)$-sphere, whereas the definition for an asymptotically hyperbolic metric would write

$$\Phi^* g = \frac{dr^2}{r^2+k} + r^2 h_r \ , \tag{2}$$

where Φ: $[R, +\infty) \times N \to M_{ext}$ is a diffeomorphism and where $(h_r)_r$ is a family of metrics on $\partial_\infty M$, smooth with respect to r, such that the limit as $r \to +\infty$ is a metric h_0 on *$\partial_\infty M$ with constant scalar curvature $R_h = k(n-1)(n-2)$, $k \in \{-1, 0, 1\}$.*

There exists a more restrictive notion of asymptotic hyperbolicity, such as presented in Refs. 1 and 19:

Definition 2.4: An asymptotically hyperbolic manifold (M, g) is strongly asymptotically hyperbolic *(S.A.H. for short) if:*

- *the conformal infinity $\partial_\infty M$ is the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$, equipped with its standard metric h_0,*
- *if ρ is a defining function of the conformal infinity such that one can write $g = \sinh^{-2}\rho(d\rho^2 + h_\rho)$, one then has the asymptotic expansion as $\rho \to 0$:*

$$h_\rho = h_0 + \frac{\rho^n}{n} h + O(\rho^{n+1}) \ ,$$

where the terms of the expansion can be differentiated twice.

The tensor h which appears in the above definition is a rank 2 symmetric tensor on $\mathbb{S}^{n-1}$, called the *mass-aspect tensor*.

B. Mass and energy-momentum of asymptotically hyperbolic manifolds

We recall here how the mass and the energy-momentum invariants for an asymptotically hyperbolic (A.H.) manifold arise. As for the mass in an asymptotically flat Riemannian manifold, these invariants appear from an Hamiltonian formulation of general relativity, as described in Refs. 2 and 4, in particular they are related to symmetries of the background (M_0, b) which is considered (either the Euclidean or the hyperbolic space). Indeed, let $\mathcal{K}_0$ be the space of the *static Killing initial data* (or *static KIDs*) as defined by Chruściel and Beig in Ref. 5, and more precisely for our particular case at the beginning of Ref. 8. This space of static KIDs here consists of the set of functions $f \in C^\infty(M_0)$ such that the spacetime metrics $-f^2 dt^2 + b$ are *static* solutions to the vacuum Einstein equations. The word "static" means that the orbits of the Killing vector ∂_t are timelike and orthogonal to totally geodesic hypersurfaces (here the level sets of the t-coordinate function).

Note that in such a spacetime, the vector ∂_t takes the form $\partial_t = f\nu$, where ν is the unit (timelike future-directed) vector normal to M_0 in the spacetime. Then, an important result from Moncrief[16] relates the space of static KIDs to the cokernel of the linearized scalar curvature operator evaluated at the background metric b (or for general KIDs to the cokernel of the linearized constraint operator linearized at the background initial data considered). To apply this here, we start writing the linearized scalar curvature operator at b, tested against a (0, 2)-symmetric tensor h:

$$L_b h := DR(b)h$$

and its formal L^2-adjoint, against a function $f \in C^\infty(M_0)$:

$$L_b^* f = -(\Delta_b f) b + \mathrm{Hess}_b f - f \mathrm{Ric}_b \ ,$$

where Δ_b, Hess_b, and Ric_b are respectively the Laplace-Beltrami operator, the Hessian and the Ricci curvature tensor of b. Following the notations of Refs. 8 and 15, let g an asymptotically hyperbolic metric in the sense defined above, and a diffeomorphism at infinity Φ such that $\Phi^* g$ is asymptote to

b (no matter of the topology of the conformal boundary). From Moncrief's result, one then has

$$\mathcal{K}_0 = \ker L_b^* \,. \tag{3}$$

The mass integrals (see Refs. 8 and 15) then appear for each $V \in \mathcal{K}_0$, as limits of flux integrals

$$H_\Phi(V) = \lim_{R\to+\infty} \int_{r=R} \mathbb{U}(V, \Phi^*g - b)(\nu_r)dS_r \,, \tag{4}$$

where ν_r is the unit normal vector to the hypersurface $\{r = R\}$, where dS_r is the induced measure on this hypersurface from b and where the integrand term reads, for $e = \Phi^*g - b$, as (see Ref. 15)

$$\mathbb{U}(V, e) = V\,(\mathrm{div}\; e - d(\mathrm{tr}\; e)) - \iota_{\nabla V} e + (\mathrm{tr}\; e)dV \,, \tag{5}$$

where the divergence and the trace are computed relatively to b.

The integrals (4) depend in general on the chosen chart at infinity Φ. However, in Refs. 8 and 15, it is proven that, for $k = 0$ or -1, the space $\mathcal{K}_0$ is one-dimensional and that given a non-trivial element $V_{(0)}$ of $\mathcal{K}_0$, the quantity $m = H_\Phi(V_{(0)})$ does not depend on Φ provided suitable conditions on the decay of $\Phi^*g - b$ at infinity hold (and provided a volume normalization for $k = 0$). The quantity m is referred to as the *mass* of an A.H. manifold with conformal boundary of negative ($k = -1$) or zero ($k = 0$) scalar curvature.

The case $k = 1$ (conformal boundary of positive Yamabe type) is slightly more complicated. We will only consider the particular case where $\partial_\infty M$ is the $(n - 1)$-sphere $\mathbb{S}^{n-1}$, with $h_0 = \sigma_{n-1}$. For quotients of $\mathbb{S}^{n-1}$ by a discrete subgroup Γ of its group of isometries, see the discussion in Ref. 8.

In this case of a spherical conformal boundary, $\mathcal{K}_0$ is $(n + 1)$-dimensional and it is proven in Refs. 8 and 15 that the Minkowskian norm of the linear form H_Φ does not depend on Φ, again provided suitable conditions on the decay of $\Phi^*g - b$ at infinity hold. Namely, if one imposes a basis $\left(V_{(\mu)}\right)_{\mu=0\cdots n}$ of $\mathcal{K}_0$ and defines $p_{(\mu)} = H_\Phi(V_{(\mu)})$, then the quantity $\eta(p_{(\mu)}, p_{(\mu)}) = p_{(0)}^2 - \sum_{i=1}^n p_{(i)}^2$ is independent of Φ. We can now define an important notion. We first observe that, if the hyperbolic metric is written as in (1), then the functions

$$V_{(0)} = \sqrt{1 + r^2}\,,\;\; V_{(i)} = x^i$$

form a basis of $\mathcal{K}_0$, where the x^i are the Cartesian coordinates of $\mathbb{R}^n$.

Definition 2.5: Let (M, g) *an asymptotically hyperbolic manifold, with a spherical conformal boundary. Let* Φ *be a diffeomorphism* $[R, \infty) \times N \longrightarrow M_{ext}$*, with a radial coordinate r. The* energy-momentum vector *of g (or of (M, g)) is the vector* $\mathbf{p_g}$ *with components* $p_{(0)} = H_\Phi(\sqrt{1 + r^2})$*,* $p_{(i)} = H_\Phi(x^i)$*. If* $\mathbf{p_g}$ *is timelike or null and future-directed, its mass is the number* m_g *such that* $m_g = \sqrt{\eta(\mathbf{p_g}, \mathbf{p_g})}$*.*

Using the Minkowskian form η of signature $(+, -, \ldots, -)$, we can say that $\mathbf{p_g}$ is *spacelike* (resp. *null*, *timelike*) if $\eta(\mathbf{p_g}, \mathbf{p_g}) < 0$ (resp. $= 0$, >0). Moreover, for a timelike energy-momentum vector, we say that $\mathbf{p_g}$ is *future-directed* (resp. *past-directed*) if the first coordinate $p_{(0)}$ is positive (resp. negative). It is important to note that from the invariance property stated above, the η-norm of an energy-momentum $\mathbf{p_g}$ does not depend on the diffeomorphism Φ, provided the suitable decay properties of $\Phi^*g - b$ stated in Refs. 8 and 15 are satisfied, although the vector $\mathbf{p_g}$ itself transforms under isometries of the hyperbolic space.

Note that when one considers the initial data (g, k) of an asymptotically anti-de Sitter space-time, one can similarly define the corresponding energy-momentum vector, see, e.g., the work of Maerten[13,14] for more details.

A rather more straightforward way exists to define the notions of mass and of energy-momentum for a S.A.H. manifold (M, g).

Under the Definition (2.4) above, one introduces the notion of *mass-aspect function* of g, defined on the conformal infinity of (M, g) (here $\mathbb{S}^{n-1}$) as the trace with respect to h_0 of the mass-aspect tensor h:

$$\mu_{h_0} := \mathrm{tr}_{h_0}(h) = h_0^{AB} h_{AB} \ ,$$

where the summation is made on indices A and B which refer to coordinates y^A on $\mathbb{S}^{n-1}$. One can now define the energy-momentum vector (see also Ref. 19) $\mathbf{p_g}$ as the vector of $\mathbb{R}^{1,n}$:

$$\mathbf{p_g} := \left(\int_{\mathbb{S}^{n-1}} \mu_{h_0}(x)\, d\sigma_{h_0}, \int_{\mathbb{S}^{n-1}} \mu_{h_0}(x)\, x^1\, d\sigma_{h_0}, \dots, \int_{\mathbb{S}^{n-1}} \mu_{h_0}(x)\, x^n\, d\sigma_{h_0} \right) \ ,$$

where $d\sigma_{h_0}$ is the volume associated to the round metric h_0 of $\mathbb{S}^{n-1}$. This definition coincides with the previous definition of Ref. 8, up to a constant positive factor, which does not affect the causal character of the vector in the Minkowski spacetime.

Remark 2.6: This terminology of "mass" and "energy-momentum" is meaningful in mathematical general relativity (see Ref. 12 for the mass). Whereas the term "energy-momentum" is not introduced in Ref. 8, several authors use it, such as in Refs. 7 and *19. This is moreover justified since it has the same nature as the energy-momentum vector for asymptotically flat initial data in the Minkowski space. Indeed, in this case, the space of KIDs to be considered is the space of sections $V = (f, Y)$ of $\mathbb{R} \times T\mathbb{R}^{1,n}$ such that the vector $X = f\partial_t + Y$ are Killing vectors of the Minkowski spacetime. This space of KIDs is spanned by* $(1, 0)$, $(x^i, 0)$, $(0, \partial_{x^i})$, and $(0, x^j \partial_{x^i} - x^i \partial_{x^j})$ *for all $i, j \in \{1, \dots, n\}$. The energy-momentum vector of some asymptotically flat initial data set (g, k) is then obtained using mass integrals computed against the $(n + 1)$ elements $(f, 0)$ with $f = 1, x^1, \dots, x^n$ (see Ref. 10).*

One of the main concerns of mathematical relativists is to prove that these quantities are in fact "positive" or "well-oriented" under local geometric assumptions corresponding to the positivity of the density of energy.[2] In the present context of A.H. manifolds, the statement is:

Conjecture 2.7: Let (M, g) be a complete, n-dimensional Riemannian, asymptotically hyperbolic manifold whose conformal infinity is the $(n - 1)$-unit sphere. Assume that the scalar curvature R_g of g satisfies $R_g \geq -n(n - 1)$. Then the energy-momentum $\mathbf{p_g}$ is timelike future-directed, unless (M, g) is isometric to the hyperbolic space.

This general result is yet an open question. However, the last decade has seen significant progress towards the proof of it. Indeed, Wang (Ref. 19) for S.A.H. manifolds, then Chruściel and Herzlich (Ref. 8) for general asymptotically hyperbolic manifolds, have independently proved the following version of the "positive energy-momentum theorem":

Theorem 2.8: *Let (M, g) be a complete Riemannian manifold, asymptotically hyperbolic and spin, with dimension $n \geq 3$, and whose scalar curvature satisfies the inequality $R_g \geq -n(n - 1)$. Then, the energy-momentum vector $\mathbf{p_g}$, if it exists and is non-zero, is timelike future-directed. It is zero if and only if (M, g) is isometric to the hyperbolic space $\mathbb{H}^n$.*

Without the spin assumption, only partial results are known; one of them follows from Andersson, Cai, and Galloway, as in Ref. 1:

Theorem 2.9: *Let (M, g) be a complete Riemannian manifold, S.A.H., with dimension n with $3 \leq n \leq 7$, whose scalar curvature satisfies $R_g \geq -n(n - 1)$. If moreover the mass-aspect function does not change of sign (strictly speaking) on the conformal boundary, then this sign is positive, or zero if and only if the manifold (M, g) is isometric to the hyperbolic space $\mathbb{H}^n$.*

This statement is not yet completely satisfying, especially when one compares the knowledge upon the equivalent statement in the asymptotically flat case. We can however state the following remark:

Remark 2.10: The assumption on the constant sign of the mass-aspect function in Theorem 2.9 implies that the energy-momentum vector is causal.

Indeed, for a mass-aspect function u_0 of (strict) constant sign, and for all i, one has:

$$\left(\int x_i \frac{|u_0| d\sigma_{h_0}}{\int |u_0| d\sigma_{h_0}}\right)^2 \leq \int x_i^2 \frac{|u_0| d\sigma_{h_0}}{\int |u_0| d\sigma_{h_0}} ,$$

from the Jensen inequality applied to the measure $|u_0| d\sigma_{h_0}$ and to the convex function $X \mapsto X^2$. The integrals are here computed on the conformal (spherical) infinity $\{x_1^2 + \cdots + x_n^2 = 1\}$, which yields

$$\sum_{i=1}^{n} \left(\int x_i u_0(x) d\sigma_{h_0}\right)^2 \leq \left(\int u_0(x) d\sigma_{h_0}\right)^2 .$$

This result encourages us to seek functions u_0, non-radial (i.e., which do not only depend on the defining function of the conformal infinity), to play the role of mass-aspect functions of A.H. metrics with arbitrary energy-momentum vectors. This is the object of Sec. III.

III. EXISTENCE THEOREM

This section is devoted to establish the following result:

Theorem 3.1: *Let* **p** *be a vector of the Minkowski spacetime $\mathbb{R}^{1,n}$. Then there exists a compact set $K \subset \mathbb{R}^n$ and a Riemannian metric g, asymptotically hyperbolic, conformally flat defined on $\mathbb{R}^n \backslash K$, with constant scalar curvature $R_g = -n(n-1)$, such that the energy-momentum vector of* g exists, and coincides with **p**.

Proof: We first notice that if g is such a metric, then it takes the form $g = u^{\frac{4}{n-2}} b$ in a neighborhood of the boundary at infinity, where b is the hyperbolic metric. Hence u solves the Yamabe equation

$$-4 \frac{n-1}{n-2} \Delta_b u + R_b u = R_g u^{\frac{n+2}{n-2}} ,$$

where Δ_b is the Laplace-Beltrami operator of b. Given the above conditions on the scalar curvature, u has to solve the equation:

$$\Delta_b u = \frac{n(n-2)}{4} \left(u^{\frac{n+2}{n-2}} - u\right) . \tag{6}$$

We introduce now coordinates r (radial coordinate) and θ, an angular coordinate on the $(n-1)$-sphere, such that the hyperbolic metric reads

$$b = \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, \sigma_{n-2}) ,$$

where σ_{n-2} is the standard metric of the $(n-2)$-dimensional unit sphere. The coordinate r is related to the defining function ρ by $r^{-1} = \sinh \rho$. We seek u, solution to (6), as a power series in $\frac{1}{r}$:

$$u = 1 + \sum_{k=0}^{\infty} \frac{u_k}{r^{n+k}} .$$

Our motivation to do so is that if the power series converges near the conformal boundary, then the metric g is S.A.H., with a mass-aspect function that coincides with u_0 up to a positive constant factor.

We start by looking for coefficients u_k defined on the sphere $\mathbb{S}^{n-1}$ that depend only on the angular coordinate θ. In this case, the energy-momentum vector of g takes the form:

$$\mathbf{p_g} = \lambda \left(\int_{\theta=0}^{\pi} u_0(\theta) \sin\theta d\theta \,, 0 \,, \cdots, 0 \,, \int_{\theta=0}^{\pi} u_0(\theta) \sin\theta \cos\theta d\theta\right) , \tag{7}$$

where λ is a positive constant. To see how this expression arises, one can write $\mu_{h_0}(x) = au_0(\theta)$, where $a > 0$, and $\int_{\mathbb{S}^{n-1}} \mu_{h_0}(x) d\sigma_{h_0} = a\omega_{n-2} \int_{\theta=0}^{\pi} u_0(\theta) \sin\theta d\theta$, where ω_{n-2} is the volume of the unit $(n-2)$-sphere. Then, the coordinate θ is chosen so that, for $i \in \{1, \ldots, n-1\}$, one has, on the sphere $\mathbb{S}^{n-1}$, $x^i = \sin\theta y^i$, where y^i are coordinates on $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, while $x^n = \cos\theta$. In particular, the integrals $\int_{\mathbb{S}^{n-1}} \mu_{h_0}(x) x^i d\sigma_{h_0}$ vanish for $i \in \{1, \ldots, n-1\}$ since there is a factor which is the integral of y^i on the interval $[-1, 1]$.

From this, since u depends only on the coordinates r and θ, one has:

$$\Delta_b u = (1 + r^2)\frac{\partial^2 u}{\partial r^2} + \left(\frac{n - 1 + nr^2}{r}\right)\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + (n-2)\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \theta}\right) ,$$

which yields, after a term-by-term differentiation in the series:

$$\Delta_b u = \sum_{k=0}^{\infty} w_k(\theta) r^{-(n+k)} ,$$

where

$$w_k = (k+1)(k+n)u_k + k(k+n-2)u_{k-2} + u''_{k-2} + (n-2)\frac{\cos\theta}{\sin\theta}u'_{k-2} , \tag{8}$$

for all $k \geq 0$, with the convention $u_{-1} = u_{-2} = 0$, and where the symbols $'$ and $''$ indicate first and second derivatives with respect to the variable θ.

In order to handle the right-hand-side term of (6), we write $u^{\frac{n+2}{n-2}}$ as

$$u^{\frac{n+2}{n-2}} = 1 + \sum_{p=1}^{\infty} \binom{\frac{n+2}{n-2}}{p} \left(\sum_{k=0}^{\infty} \frac{u_k}{r^{n+k}}\right)^p ,$$

where $\binom{\frac{n+2}{n-2}}{p}$ is the combinatorial coefficient

$$\binom{\frac{n+2}{n-2}}{p} = \frac{\frac{n+2}{n-2} \cdots \left(\frac{n+2}{n-2} - 1\right)\left(\frac{n+2}{n-2} - p + 1\right)}{p!} .$$

One can also write

$$u^{\frac{n+2}{n-2}} = 1 + \sum_{p=1}^{+\infty} \binom{\frac{n+2}{n-2}}{p} \sum_{k=0}^{\infty} \left(\sum_{k_1 + \cdots + k_p = k} u_{k_1} \cdots u_{k_p}\right) \frac{1}{r^{pn+k}} .$$

We then identify the coefficients of the series $u^{\frac{n+2}{n-2}} = 1 + \sum_{l=0}^{\infty} \frac{v_l}{r^{n+l}}$. If we denote by $\mathcal{E}_{n,k}$ the set of elements (p, l) such that $p \geq 2$, $l \geq 0$, $pn + l = n + k$, we have

$$v_k = \frac{n+2}{n-2}u_k + \sum_{(p,l)\in\mathcal{E}_{n,k}} \binom{\frac{n+2}{n-2}}{p} \sum_{l_1+\cdots+l_p=l} u_{l_1} \cdots u_{l_p}$$

and Eq. (6) together with the expression of w_k yields, for all $k \geq 0$:

$$(k+1)(k+n)u_k + k(k+n-2)u_{k-2} + (n-2)\frac{\cos\theta}{\sin\theta}u'_{k-2} + u''_{k-2} = \frac{n(n-2)}{4}(v_k - u_k) . \tag{9}$$

Note that when considering this equation for $k = 0$, one gets that the coefficient $u_0(\theta)$ can be chosen freely, whereas for $k = 1$, the equation above forces $u_1(\theta)$ to be 0. Then, from Eq. (9) and the formula for v_k, all the coefficients u_k are completely determined for $k \geq 2$ by the coefficients of lower rank.

More precisely, one can write

$$\left((k+n+1)k+\frac{n(n-2)}{4}\right)u_k+k(k+n-2)u_{k-2}+(n-2)\frac{\cos\theta}{\sin\theta}u'_{k-2}+u''_{k-2} = \frac{n(n-2)}{4}P_k(u_0,\cdots,u_{k-1})\ , \tag{10}$$

where

$$P_k=\sum_{p\geq 2\ ,\ (p-1)n\leq k}\binom{\frac{n+2}{n-2}}{p}\sum_{l_1+\cdots+l_p=k-(p-1)n}u_{l_1}\cdots u_{l_p}\ .$$

In the sequel, we choose

$$u_0(\theta)=\beta+\cos\theta\ ,$$

where $\beta\in\mathbb{R}$. When doing this choice, we will obtain energy-momentum vectors given by the formula (7) which are either timelike, or null, or even spacelike, depending on the values of β. We use the subsequent fact, valid for all β:

Lemma 3.2: With this choice of u_0, for all $k\geq 1$, $u_k(\theta)$ is a polynomial in the variable $\cos\theta$, *with degree at most $k-1$.*

Proof: This property is trivial for $k=1$; let $k\geq 2$, such that $\deg(u_l)|_{\cos\theta}\leq l-1$, $\forall l\in\{1,\ldots,k-1\}$. Then $\left(u''_{k-2}+(n-2)\frac{\cos\theta}{\sin\theta}u'_{k-2}+k(k+n-2)u_{k-2}\right)$ is a polynomial in $\cos\theta$, with degree $\leq k-1$ by assumption, and since $\deg(u_0)|_{\cos\theta}=1$.

Concerning the part $P_k=P_k(u_0,\ldots,u_{k-n})$, each term $\sum_{l_1+\cdots+l_p=l}u_{l_1}\cdots u_{l_p}$ (with $p\geq 2$) is made of products $u_{l_1}\cdots u_{l_p}$, which have, by assumption, a degree $\leq(l_1+1)+\cdots+(l_p+1)=l+p=k-(p-1)n+p=k-(n-1)(p-1)+1$ which is less than $k-1$ as desired since $n\geq 3$ and $p\geq 2$. The formula (10) enables us to conclude the proof of the lemma. □

In the whole sequel, if $v=\sum_i v_i(\cos\theta)^i$ is a polynomial in $\cos\theta$, we introduce the notation $|v|_1=\sum_i|v_i|$.

Lemma 3.3: There exists a number $\alpha>0$ such that, for all positive integer k, one has

$$|u_k|_1\leq\frac{\alpha^k}{(k+1)^2}\ . \tag{11}$$

Proof: If $v=\sum_i v_i(\cos\theta)^i$ is a polynomial in $\cos\theta$, one has $\frac{\cos\theta}{\sin\theta}v'=\sum_i -i v_i(\cos\theta)^i$ and $v''=\sum_i v_i(-i(\cos\theta)^i+i(i-1)\sin^2\theta(\cos\theta)^{i-2})$.

Thus one obtains the following bounds on the norms $|.|_1$ of $\frac{\cos\theta}{\sin\theta}v'$ and of v'':

$$\left|\frac{\cos\theta}{\sin\theta}v'\right|_1\leq d|v|_1$$

and

$$|v''|_1\leq d(d-1)|v|_1\ ,$$

where d is the degree of v as a polynomial function of $\cos\theta$.

We prove the lemma by induction: assume that we have found a suitable α, such that the statement of the lemma holds at the order $l\in\{0,\ldots,k-1\}$ (it is obviously the case for $k=1$). We infer from the result obtained above on u_l that

$$\left|u''_{k-2}+(n-2)\frac{\cos\theta}{\sin\theta}u'_{k-2}+k(k+n-2)u_{k-2}\right|_1 \leq\frac{\alpha^{k-2}}{(k-1)^2}((k-1)(k-2)+(n-2)(k-1)+k(k+n-2))\ . \tag{12}$$

On the other hand, from our assumptions, one has

$$|P_k|_1 \leq \sum_{(p,l)\in\mathcal{E}_{n,k}} \left|\binom{\frac{n+2}{n-2}}{p}\right| \sum_{l_1+\cdots+l_p=l} |u_{l_1}|_1 \cdots |u_{l_p}|_1 \tag{13}$$

$$\leq \sum_{(p,l)\in\mathcal{E}_{n,k}} \left|\binom{\frac{n+2}{n-2}}{p}\right| \sum_{l_1+\cdots+l_p=l} \frac{\alpha^l}{(l_1+1)^2\cdots(l_p+1)^2} \ ,$$

since for all $(p, l) \in \mathcal{E}_{n,k}$, one has $l = k - (p - 1)n \leq k - 1$. We now wish to evaluate the sums which appear above, for $(p, l) \in \mathcal{E}_{n,k}$, as

$$\mathcal{S}_p(l) := \sum_{l_1+\cdots+l_p=l} \frac{1}{(l_1+1)^2}\cdots\frac{1}{(l_p+1)^2} \ .$$

We first need the following general result, simply obtained by a decomposition into simple elements of a rational function (see in the Appendix):

$$\sum_{r=0}^{q} \frac{1}{(r+1)^2(q-r+1)^2} \leq \frac{\pi^2}{(q+2)^2} \ . \tag{14}$$

From this, there is no difficulty to find an upper bound for $\mathcal{S}_p(l)$ as:

$$\mathcal{S}_p(l) \leq \frac{\pi^{2(p-1)}}{(l+p)^2} \ .$$

Hence, one can write

$$|P_k|_1 \leq \sum_{p\geq 2\ ,\ (p-1)n\leq k} \left|\binom{\frac{n+2}{n-2}}{p}\right| \frac{\pi^{2(p-1)}\alpha^{k-(p-1)n}}{(k-(p-1)n+p)^2} \ ,$$

so we need to estimate the combinatorial term $\left|\binom{\frac{n+2}{n-2}}{p}\right|$. We can in fact show (see the Appendix) that it is bounded by Ce^p, where C is a positive constant.

On the other hand, we wish to find an upper bound for the ratio $\left(\frac{k+1}{k-(p-1)n+p}\right)^2$. We find (see the Appendix) that n^2 is a valid upper bound of this ratio for all $p \geq 2$ such that $(p - 1)n \leq k$.

Combining these results, we can write

$$|P_k|_1 \leq \frac{\alpha^k}{(k+1)^2} \sum_{p\geq 2\ ,\ (p-1)n\leq k} C_n \left(\frac{e\pi^2}{\alpha^n}\right)^{p-1} \ ,$$

where C_n is a number that depends only on n. Thus, for all α such that $e\pi^2 < \alpha^n$, one concludes that

$$|P_k|_1 \leq \left(\frac{C_n}{1-\frac{e\pi^2}{\alpha^n}} \frac{e\pi^2}{\alpha^n}\right) \frac{\alpha^k}{(k+1)^2} \ , \tag{15}$$

and the term inside the bracket can be made as small as desired for α large enough, for any given n.

Putting together (10), (12), and (15), we obtain

$$|u_k|_1 \leq \left[\frac{(k-1)(k+n-4)+k(k+n-2)}{k^2+(n+1)k+\frac{n(n-2)}{4}} \times \frac{1}{\alpha^2} \times \left(\frac{k+1}{k-1}\right)^2 \right. \tag{16}$$

$$\left. + \frac{\frac{n(n-2)}{4}}{k^2+(n+1)k+\frac{n(n-2)}{4}} \times \frac{C_n}{1-\frac{e\pi^2}{\alpha^n}}\frac{e\pi^2}{\alpha^n}\right] \frac{\alpha^k}{(k+1)^2} \ .$$

In particular, the term inside the bracket admits an upper bound independent of k and can be made as small as desired when choosing a sufficiently large value of α. Therefore, if we choose α large enough (independently of k) we have

$$|u_k|_1 \leq \frac{\alpha^k}{(k+1)^2} \ , \tag{17}$$

so that we can conclude by induction on k the proof of the lemma. □

We infer from this result that $|u_k|_\infty := \sup_\theta |u_k(\theta)| \leq \frac{\alpha^k}{(k+1)^2}$ for all k, since the u_k are polynomials in $\cos\theta$. We have therefore obtained that $u(r, \theta)$, solution to (6), as a power series in $1/r$, with a radius of convergence less than α^{-1}; in other words, we have constructed a metric $g = u^{\frac{4}{n-2}} b$, defined on a neighborhood of the conformal (spherical) boundary, of the form

$$M_{ext} = (\alpha, +\infty) \times \mathbb{S}^{n-1} \ ,$$

with the desired properties, and non-radial.

For $\beta = 0$, from the formula (7), the energy-momentum vector $\mathbf{p_g}$ is spacelike, whereas it is timelike for large enough values of $|\beta|$. Moreover, these computations can be conducted also for functions u_0 of the form $u_0(\theta) = \beta + \gamma\cos\theta$, with $\gamma \in \mathbb{R}$, leading to a similar conclusion. Hence, when β and γ vary in $\mathbb{R}$, the energy-momentum $\mathbf{p_g}$ of the resulting metrics describes the two-dimensional subspace $\mathbb{R} \times \{0\}^{n-1} \times \mathbb{R}$ of the Minkowski space $\mathbb{R}^{1,n}$. One can now reach any vector $\mathbf{p}$ of $\mathbb{R}^{1,n}$ when using furthermore the action of isometries "at infinity" of the hyperbolic space, namely using the group $O(1, n)$. □

IV. APPLICATIONS

A. Optimality of a positive energy-momentum theorem with boundary

In this section, we explain why the result obtained in Sec. III illustrates that the Theorem 4.7 in Ref. 8 is somehow optimal.

Let us first recall the statement of this theorem:

Theorem 4.1 (Chruściel-Herzlich 03): *Let (M, g) be a n-dimensional Riemannian manifold, complete, spin, where g is a $\mathcal{C}^2$, and M has a (inner) compact, non-empty boundary of mean curvature*

$$\Theta \leq n - 1 \ ,$$

and where the scalar curvature of metric g satisfies

$$R_g \geq -n(n-1) \ .$$

Under suitable assumptions on the asymptotic behaviour and if the conformal infinity is the $(n-1)$-unit sphere, then the energy-momentum vector $\mathbf{p_g}$ is timelike future-directed.

As we will see, this result is no longer valid if the assumption on the mean curvature Θ of the inner boundary is removed.

The first obvious counter-example is given by the hyperbolic space $\mathbb{H}^n$ itself, as the energy-momentum vector is trivial in this case, while the scalar curvature is $R_b = -n(n-1)$. Indeed for the hyperbolic metric, one can check that there is no compact hypersurface in $\mathbb{H}^n$ with mean curvature $\Theta \leq n-1$. Indeed, let Σ be a closed hypersurface in $\mathbb{H}^n$. Consider any $(n-1)$-sphere that contains Σ in its interior and tangent to Σ at at least one point p. Then, the value of the mean curvature of Σ at p is at least the value of the mean curvature of this $(n-1)$-sphere, which is precisely $(n-1)\coth r$ for a sphere of radius r, hence strictly larger than $n-1$.

More generally speaking, let us consider a S.A.H. n-dimensional manifold (M, g), endowed with a defining function ρ. Outside some compact subset of M, the metric takes the expression, in the limit $\rho \to 0$:

$$g = \sinh^{-2}\rho \left(d\rho^2 + h_0 + \frac{\rho^n}{n} h + o(\rho^n) \right) \ ,$$

where h_0 is the standard metric on the $(n - 1)$-unit sphere. Let us now compute the mean curvature of "large spheres" in M, whose center lies at the origin of the coordinate system of the chart at infinity considered here.

Defining the coordinate s by $\sinh s = \sinh^{-1}\rho$, the S.A.H. metric takes the following form as $s \to \infty$:

$$g = ds^2 + \sinh^2 s\; h_s \ ,$$

where $h_s = h_0 + \frac{2^n}{n} e^{-ns} h + o(e^{-ns})$. The second fundamental form of the sphere S_s of Euclidean radius s centered at the origin of this coordinate system reads:

$$II_{S_s} = \frac{1}{2}\frac{\partial}{\partial s}\left(\sinh^2 s\; h_s\right) = 2\sinh s \cosh s\; h_s + \sinh^2 s \frac{\partial h_s}{\partial s} \ ,$$

and the mean curvature is therefore

$$\Theta_{S_s} = \left(\sinh^2 s\; h_s\right)^{-1} II_{S_s} = (n-1)\frac{\cosh s}{\sinh s} + \frac{1}{2} h_s^{-1}\frac{\partial h_s}{\partial s} \ .$$

Then, one has $\partial_s h_s = -2^n e^{-ns} h + o(e^{-ns})$ as $s \to +\infty$. Thus, one has

$$\Theta_{S_s} = (n-1)\frac{\cosh s}{\sinh s} - 2^{n-1} e^{-ns}\mu_{h_0} + o(e^{-ns}) \ ,$$

and we recover the mass-aspect function μ_{h_0} which does not depend on s. Hence, for $n \geq 3$, one obtains the asymptotic behaviour of the mean curvature of large $(n - 1)$-spheres:

$$\Theta_{S_s} = (n-1)\left(1 + 2e^{-2s}\right) + O(e^{-3s}) \ .$$

This last quantity is strictly larger than $n - 1$ for all s large enough. One notices that the above expansion is valid for all S.A.H. metrics, without any restriction on the scalar curvature or on the mass-aspect function μ_{h_0}.

Hence, if one replaces the condition of the mean curvature of the inner boundary in Chruściel-Herzlich's Theorem by a condition

$$\Theta \leq \alpha(n-1)$$

for $\alpha > 1$, then the condition is completely general: given such an number α, every S.A.H. metric has an inner compact, boundaryless hypersurface whose mean curvature is less than $\alpha(n - 1)$, in particular this is the case for metrics constructed in the previous section, thus with arbitrary energy-momentum vector.

B. A gluing result for asymptotically hyperbolic metrics

We show here that the gluing results of Chruściel and Delay in Ref. 7 apply when one, instead of using the family of *boosted* Kottler (Schwarzschild-anti de Sitter) metrics, uses the family constructed in Sec. III as models for the asymptotic region.

Let us give a quick overview of the Corvino-Schoen's gluing principle, at least for *static* initial data. We start by considering a non-compact Riemannian manifold (M, g) which asymptotes to a reference metric b in the asymptotic region. (We will assume for simplicity that (M, g) has only one end). The metric g is furthermore assumed to have constant scalar curvature $R_g = R_b$. The gluing result consists in finding compact regions $M_1 \subset\subset M_2 \subset M$, a family of "model" metrics $(\mathring{g}_Q)_{Q\in\mathcal{F}}$ in the asymptotic region of M and a new metric $\tilde{g}$ on M satisfying the following properties:

- $\tilde{g}$ coincides with g on M_1,
- $\tilde{g}$ coincides with $\mathring{g}_Q$ on $M\backslash M_2$ for some parameter Q in $\mathcal{F}$,
- the scalar curvature $R_{\tilde{g}}$ of $\tilde{g}$ is constant on M and equal to R_b.

The proof of such results consists in interpolating g and $\mathring{g}_Q$ by defining $g_Q = (1-\chi)g + \chi\mathring{g}_Q$, with a cut-off function $0 \leq \chi \leq 1$ which is equal to 0 in M_1 and to 1 in $M\backslash M_2$, and then to find a compactly supported (in $M_2\backslash M_1$) perturbation δg such that we recover $R(g_Q + \delta g) = R_b$. The kernel $\mathcal{K}_0$ of the adjoint of the linearized scalar curvature operator plays here an essential obstruction role, and only a big enough space $\mathcal{F}$ of parameters for a suitable family $(\mathring{g}_Q)$ of models at infinity ensures the existence of an appropriate $Q \in \mathcal{F}$ which solves the perturbation problem. In fact (see Refs. 6 and 10), the dimension of $\mathcal{F}$ has to be at least equal to the dimension of $\mathcal{K}_0$.

Going back to the situation of this paper, the metrics exhibited in Sec. III form an "admissible" family (in the sense of Ref. 10) of models for the infinity, in the sense that they satisfy the asymptotic decay requirements imposed in Ref. 7, and their global charges (in fact their energy-momentum vectors) describe a non-trivial $(n+1)$-dimensional open set in the $(n+1)$-dimensional Minkowski space of all the possible global charges. In fact, from what we saw in Sec. III, the energy-momentum vectors of our family reach the whole Minkowski space, which means that the space of parameters $\mathcal{F}$ coincides with $\mathbb{R}^{1,n}$.

We start by introducing some notations and definitions that appear in Ref. 7. If ρ is a defining function of the (spherical) conformal infinity, we define

$$M_\varepsilon := \{x \in M\ ,\ \rho(x) < \varepsilon\}$$

and the annulus

$$A_{\delta,\varepsilon} := M_\varepsilon \backslash M_\delta$$

for $0 < \delta < \varepsilon$. We also denote by ∇ the Levi-Cività connection of the (hyperbolic) metric b.

For every vector $p_{(\mu)}$ in the Minkowski space $\mathbb{R}^{1,n}$, we define $\mathring{g}_{p_{(\mu)}}$ to be the metric that we have constructed in Sec. III with an energy-momentum vector $p_{(\mu)}$. For all $n \geq 5$, we define

$$\alpha_n := \max\left(8, \frac{8+n}{2}\right) .$$

We can now state the result, directly adapted from Theorem 1.2 of Ref. 7:

Theorem 4.2: *Let $n \geq 5$, $l \in \mathbb{N}$ with $l > \lfloor \frac{n}{2} \rfloor + 4$, $\lambda \in (0, 1)$ and let $\alpha > \alpha_n$. Let g be a $C^{l,\lambda}$-asymptotically hyperbolic metric on M with constant scalar curvature $-n(n-1)$ and an energy-momentum vector $p^0_{(\mu)}$ such that the conditions hold:*

$$|g - \mathring{g}_{p^0_{(\mu)}}|_b + \cdots + |\nabla^{l-2}(g - \mathring{g}_{p^0_{(\mu)}})|_b = O(\rho^\alpha) ,$$

and

$$|g - \mathring{g}_{p^0_{(\mu)}}|_b + \cdots + |\nabla^{l}(g - \mathring{g}_{p^0_{(\mu)}})|_b = O(1) .$$

Then, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$, there exists a metric $\tilde{g}$ on M satisfying the following conditions:

- *$\tilde{g}$ has constant scalar curvature $-n(n-1)$,*
- *$\tilde{g}$ coincides with g on $M \backslash M_{4\delta}$,*
- *$\tilde{g}$ coincides with the metric $\mathring{g}_{p_{(\mu)}}$ on M_δ.*

Moreover, $\tilde{g}$ is smooth if g is.

Proof: Most of the details of the proof can be found in Ref. 7. In particular, the results of Sec. III in that paper hold. For Sec. IV, we just replace the family of boosted Kottler metrics by the family of metrics constructed here in Sec. III, and g is here allowed to have any energy-momentum vector (not necessarily timelike). □

Remark 4.3 A few comments can be done:

- For $n \geq 3$, one may also find an adapted version of Theorem 1.1 of Ref. 7, replacing Kottler metrics by metrics $\mathring{g}_{p_{(\mu)}}$ constructed above.
- As in Ref. 7 with the boosted Kottler metrics, one can obtain a metric $\mathring{g}_{p_{(\mu)}}$ with energy-momentum $p_{(\mu)}$ as close as desired to the energy-momentum $p^0_{(\mu)}$ of g, provided the number δ_0 is small enough. This means that one can deform any asymptotically hyperbolic metric g with $R_g = -n(n-1)$ into a metric $\tilde{g}$, which is also asymptotically hyperbolic (and even S.A.H.),

with $R_{\tilde{g}} = -n(n-1)$, which is conformally flat in a neighborhood of the conformal infinity, and whose energy-momentum vector is as close as desired from the energy-momentum vector of g.

- As pointed out in Ref. 7, the asymptotic decay rate of g towards $\mathring{g}_{p^0_{(\mu)}}$ required for the theorem to hold ($\alpha > \alpha_n$) is undesirably high, especially in low dimensions, compared to the decay rate needed to define the energy-momentum vector (where $\alpha > n/2$ suffices).

Note that the construction performed in this work may well be extended to mass-aspect functions which take the form of polynomials in $\cos\theta$ and $\sin\theta$ or even more generally to analytic functions with conditions on the coefficient growth, although the simpler construction presently considered already suffices to achieve our goals concerning the energy-momentum vector. It would however be of interest to study this further and find whether there are conditions for a function to be realized as a mass-aspect function of some S.A.H. metric, satisfying the scalar curvature requirement $R_g = -n(n-1)$.

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APPENDIX: WE ESTABLISH HERE THE INTERMEDIATE RESULTS NEEDED TO COMPLETE THE PROOF OF THEOREM 3.1

Lemma A.1: For all $k \geq 0$, $n \geq 3$, *and* $p \geq 2$ *such that* $(p-1)n \leq k$, *the following inequality holds:*

$$\left(\frac{k+1}{k-(p-1)n+p}\right)^2 \leq n^2 .$$

Proof: The quantity on the left-hand side is positive and also takes the form $\left(1-\frac{(p-1)(n-1)}{k+1}\right)^{-2}$, and by assumption, $(p-1)(n-1) \leq k\frac{n-1}{n}$, hence we have

$$\left(\frac{k+1}{k-(p-1)n+p}\right)^2 \leq \left(1-\frac{k}{k+1}\frac{n-1}{n}\right)^{-2} = \left(\frac{n+k}{(k+1)n}\right)^{-2} ,$$

and the result follows. □

Lemma A.2: There exists a positive constant c such that, for all positive integers p and n, one has

$$\left|\binom{\frac{n+2}{n-2}}{p}\right| \leq ce^p .$$

Proof: We first note that $\left|\binom{\frac{n+2}{n-2}}{p}\right|$ is less than the quantity $\frac{1}{p!}\max\left\{\left(\frac{n+2}{n-2}\right)^p \ , \ p^p\right\}$. The result then comes from the Stirling formula $p! \sim p^p e^{-p}\sqrt{2\pi p}$. □

Lemma A.3: For all $q \geq 0$,

$$\sum_{r=0}^{q}\frac{1}{(r+1)^2(q-r+1)^2} \leq \frac{\pi^2}{(q+2)^2} .$$

Proof: Let us denote by $S_2(q)$ the left-hand side term. The result follows from the decomposition:

$$\frac{1}{(r+1)^2(q-r+1)^2} = \frac{1}{(q+2)^2}\left(\frac{1}{(r+1)^2}+\frac{1}{(q-r+1)^2}\right)+\frac{2}{(q+2)^3}\left(\frac{1}{r+1}+\frac{1}{q-r+1}\right). \tag{A1}$$

Indeed, one can write

$$S_2(q) = \frac{2}{(q+2)^2}\sum_{r=0}^{q}\left(\frac{1}{(r+1)^2}+\frac{2}{(q+2)(r+1)}\right) \leq \frac{6}{(q+2)^2}\sum_{r=0}^{q}\frac{1}{(r+1)^2},$$

and the last term above is bounded by $\frac{6\zeta(2)}{(q+2)^2} = \frac{\pi^2}{(q+2)^2}$. □

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