Elliptic systems

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Summary. In this article I will review some basic results on elliptic boundary value problems with applications to General Relativity.

1 Introduction

Elliptic problems appear naturally in physics mainly in two situations: as equations which describe equilibrium (for example, stationary solutions in General Relativity) and as constraints for the evolutions equations (for example, constraint equations in Electromagnetism and General Relativity). In addition, in General Relativity they appear often as gauge conditions for the evolutions equations.

The model for all elliptic equations is the Laplace equation. Let us consider the Dirichlet boundary value problem for this equation

$$\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded, smooth, domain in $\mathbb{R}^n$ with boundary $\partial\Omega$; f, g are smooth functions and Δ is the Laplace operator in $\mathbb{R}^n$.

It is a well known result that for every source f and every boundary value g there exist a unique, smooth, solution u of (1). We would like to generalize equations (1) for more general operators and more general boundary conditions.

The first step in this generalization is given by the Neumann problem

$$\Delta u = f \text{ on } \Omega, \quad n^i\partial_i u = 0 \text{ on } \partial\Omega, \tag{2}$$

where n^i is the outward unit normal to $\partial\Omega$, the index i takes values $i = 1, \cdots, n$ and ∂_i denotes partial derivative with respect to the $\mathbb{R}^n$ coordinate x_i.

There exist two main differences between the Neumann and the Dirichlet problem: (i) The solution to the Neumann problem is not unique, for a given solution we can add a constant and obtain a new solution. Moreover, the constants are the only solutions of the homogeneous problem

$$\Delta u = 0 \text{ on } \Omega, \quad n^i \partial_i u = 0 \text{ on } \partial\Omega. \tag{3}$$

To see this, we multiply (3) by u and use the divergence theorem

$$0 = \int_\Omega u\Delta u = \int_\Omega \partial^i (u\partial_i u) - \partial^i u \partial_i u \tag{4}$$

$$= \oint_{\partial\Omega} u n^i \partial_i u - \int_\Omega \partial^i u \partial_i u \tag{5}$$

$$= -\int_\Omega \partial^i u \partial_i u. \tag{6}$$

(ii) The source f can not be arbitrary. We integrate in Ω equation (2) to obtain a necessary condition for f

$$0 = \oint_{\partial\Omega} n^i \partial_i u = \int_\Omega \Delta u = \int_\Omega f. \tag{7}$$

The following theorem says that (7) is also a sufficient condition for the existence of solution.

Theorem 1. *A solution u to the Neumann* (2) *problem exists if and only if f satisfies*

$$\int_\Omega f = 0. \tag{8}$$

Two different solutions differ by a constant.

The fact that the solution is not unique in the Neumann problem does not affect the physics of the model that is described by these equations. Take, for example, Electrostatics. The electric field E^i satisfies

$$E_i = \partial_i u, \quad \partial_i E^i = f, \tag{9}$$

where u is the electric potential and f the charge. If we prescribe $E^i n_i$ at the boundary we get a Neumann boundary problem for the potential u. The electric field E^i is invariant under the transformation $u \to u + c$, where c is a constant. We will see in section 3 that something similar happens for the constraint equations in General Relativity.

We have seen that the Neumann problem has not a unique solution. If we include lower order terms in the operator, the Dirichlet problem will not have a unique solution either. For example, for some constants $\lambda > 0$ (the eigenvalues) the following equations have a non-trivial solutions (eigenfunctions)

$$\Delta u + \lambda u = 0, \text{ on } \Omega \quad u = 0 \text{ on } \partial\Omega. \tag{10}$$

One of the main ideas in the theory of partial differential equations is that many relevant properties of the equations depends only on the principal part, that is on the terms with highest derivatives. The previous examples show that uniqueness does not depend only on the principal part. Motivated by the Neumann problem, we write the following the two main properties of elliptic equations

(i) The solutions space of the homogeneous problem (i.e., when we set the source f and the boundary values g equal to zero) is finite dimensional.
(ii) The solution will exist if and only if the sources satisfy a finite number of conditions.

We will see in the next sections that, under appropriate assumptions, (i)-(ii) depend only on the principal part of the equation and boundary conditions.

One example of a boundary condition that does not satisfy (i) is the following.

Example 1. Let Ω be the unit ball in $\mathbb{R}^3$ centered at the origin. An explicit calculation shows that the space of solutions of the homogeneous problem

$$\Delta u = 0 \text{ on } \Omega, \quad \partial_3 u = 0 \text{ on } \partial\Omega, \tag{11}$$

is *infinite dimensional* (see [27], Chapter 1, for details). Note that the vector ∂_3 is tangential to the boundary at the points $x_3 = 0$, $x_1^2 + x_2^2 = 1$.

2 Second order elliptic equations

Consider the following, second order, differential operator

$$Lu = \partial_i \left(a^{ij}(x)\partial_j u + b^i(x)u \right) + c^i(x)\partial_i u + d(x)u, \tag{12}$$

where we will assume that the coefficients are smooth functions on $\mathbb{R}^n$ and $i, j = 1, \cdots n$. We have written the operator (12) in divergence form because it will be more suitable for the following calculations; since the coefficient a^{ij} and b^i are smooth, this is equivalent to the standard formula

$$Lu = a^{ij}(x)\partial_i\partial_j u + \hat{b}^j(x)\partial_j u + \hat{d}(x)u. \tag{13}$$

where $\hat{b}^j = \partial_j a^{ij} + b^j + c^j$ and $\hat{d} = \partial_i b^i + d$.

The principal part of the operator is given by the terms which contains only second derivatives

$$l(x, \partial) = a^{ij}(x)\partial_i\partial_j. \tag{14}$$

To define the symbol of L we replace in the principal part each derivative by the component of an arbitrary constant vector in $\mathbb{R}^n$

$$l(x, \xi) = a^{ij}(x)\xi_i\xi_j, \quad \xi \in \mathbb{R}^n. \tag{15}$$

The symbol l of L is a polynomial of order 2 in the components of ξ.

We make now the crucial assumption on the symbol. We say that the operator L is *elliptic* in $\bar{\Omega}$ if

$$l(x,\xi) \neq 0 \quad \forall x \in \bar{\Omega},\, \xi \in \mathbb{R}^n, \xi \neq 0. \tag{16}$$

The next important concept is the *formal adjoint* of L. The formal adjoint L^t is defined by the relation

$$\int_\Omega vLu = \int_\Omega uL^t v \tag{17}$$

for all u, v of *compact support* in Ω. In this particular case we have

$$L^t v = \partial_j \left(a^{ij}(x)\partial_i v - c^j(x) v \right) - b^i(x)\partial_i v + d(x)v \tag{18}$$

Note that $\Delta = \Delta^t$.

We have already seen in the case of the Laplacian that the solutions of the homogeneous problem play an important role; in general L and L^t are different operator and then we have two natural null spaces defined as

$$\mathcal{N}(L) = \{u : Lu = 0 \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\} \tag{19}$$

$$\mathcal{N}(L^t) = \{u : L^t u = 0 \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}. \tag{20}$$

We can now formulate an existence result for the Dirichlet problem which will essentially ensures that properties (i)-(ii) are satisfied.

Theorem 2. *(i) Precisely one of the following statements holds:*
a) For each f there exist a unique solution of the boundary value problem

$$Lu = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{21}$$

or else
b) $\mathcal{N}(L)$ is non-trivial.
ii) Furthermore, should assertion b) hold, the dimension of $\mathcal{N}(L)$ is finite *and equals the dimension of $\mathcal{N}(L^t)$.*
iii) Finally, the boundary-value problem (21) *has a solution if and only if*

$$\int_\Omega fv = 0 \quad \text{for all } v \in \mathcal{N}(L^t). \tag{22}$$

We will consider now the analog of the Neumann problem for L. If in the integration by parts given by (17) we allow functions u and v which are not of compact support, we have to include the boundary terms; and we obtain the following relation which is called the *Green formula* for the operator L

$$\int_\Omega vL(u) - uL^t(v) = \oint_{\partial\Omega} vB(u) - uB^t(v), \tag{23}$$

where the differential boundary operators are given by

$$B(u) = n_j a^{ij}\partial_i u + b^i n_i u, \quad B^t(v) = n_j a^{ij}\partial_i v - c^i n_i v. \tag{24}$$

We want to solve the following problem

$$Lu = f \text{ on } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega. \tag{25}$$

As in the case of the Dirichlet problem, we define the null spaces

$$\mathcal{N}(L,B) = \{u : Lu = 0 \text{ on } \Omega \text{ and } B(u) = 0 \text{ on } \partial\Omega\} \tag{26}$$

$$\mathcal{N}(L^t,B^t) = \{u : L^t u = 0 \text{ on } \Omega \text{ and } B^t(u) = 0 \text{ on } \partial\Omega\}. \tag{27}$$

We have the following existence result, which looks exactly the same as the previous theorem if we replace the Dirichlet condition by the new boundary condition.

Theorem 3. *(i) Precisely one of the following statements holds:*
a) For each f there exist a unique solution of the boundary value problem

$$Lu = f \text{ on } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega, \tag{28}$$

or else
b) $\mathcal{N}(L,B)$ is non-trivial.
ii) Furthermore, should assertion b) hold, the dimension of $\mathcal{N}(L,B)$ is finite and equals the dimension of $\mathcal{N}(L^t,B^t)$.
iii) Finally, the boundary-value problem (28)–(24) has a solution if and only if

$$\int_\Omega fv = 0 \quad \text{for all } v \in \mathcal{N}(L^t,B^t). \tag{29}$$

We have written the boundary conditions in the form (24) in order to emphasize that they come naturally from the integration by parts. It is possible to write them in a perhaps more familiar form. Define the vector β^i by

$$\beta^i = n_j a^{ij}. \tag{30}$$

By the elliptic condition (16) we have $\beta^i n_i \neq 0$, that is β^i *it is never tangential to the boundary* (this excludes example 1). In the operator L only enters the symmetric part of the matrix a^{ij}, however, we have not assumed that this matrix is symmetric in the previous theorem. If we decompose $a^{ij} = a^{ij}_s + b^{ij}$ where $a^{ij}_s = a^{(ij)}_s$ and $b^{ij} = b^{[ij]}$ is an arbitrary anti symmetric matrix, then

$$\beta^i = n_j a^{ij}_s + \tau^i, \quad \tau^i = n_j b^{ij}, \tag{31}$$

where τ^i is an arbitrary tangential vector. Choosing appropriated b^i and c^i such that they do not change the operator L, we get that the function $\sigma = b^i n_i$ is also arbitrary. We conclude that the boundary condition $B(u) = 0$ is equivalent to

$$B(u) = \beta^i \partial_i u + \sigma u = 0, \tag{32}$$

where σ is an arbitrary function and β^i is an arbitrary non tangential vector field on the boundary.

Let us compare theorem 2 and 3 with the analog cases for the Laplace equation. We have now two operators L and L^t which have two different null spaces (in the case when $b^i + c^i = 0$ we have $L = L^t$ and $B = B^t$, and then only one null space). There are no statements about uniqueness or about the elements and dimension of the null spaces. We have already seen that these properties depend on the lower order terms. For the particular case of second order elliptic operators, there exist an important tool that can give uniqueness and a characterization of the null space for certain kind of lower order terms: the *maximum principle*. There exist many useful versions of the maximum principle (see for example [14]), here we mention a particular simple case, which can be generalize to other situations as we will see.

We can write the Green formula (23) in terms of a first order bilinear form $\mathbf{B}$

$$\mathbf{B}(u,v) = \oint_{\partial\Omega} vB(u) - \int_{\Omega} vL(u) = \oint_{\partial\Omega} uB^t(v) - \int_{\Omega} uL^t(v) \qquad (33)$$

where

$$\mathbf{B}(u,v) = \int_{\Omega} (a^{ij}\partial_j u + b^i u)\partial_i v - (c^i\partial_i u + du)v. \qquad (34)$$

From this equation we deduce that $u \in \mathcal{N}(L,B)$ if and only if $\mathbf{B}(u,v) = 0$ for all v. (One if is trivial, to see the other one, take test functions v which vanishes at the boundary and are arbitrary at the interior). If we assume $b^i = c^i = 0$ and $d \leq 0$, then $\mathbf{B}$ is symmetric (i.e. $\mathbf{B}(u,v) = \mathbf{B}(v,u)$) and positive

$$\mathbf{B}(u,u) \geq 0, \quad \text{for all } u. \qquad (35)$$

Moreover, $\mathbf{B}(u,u) = 0$ if and only if u is a constant and $u = 0$ if d is not identically zero. In this case we are in a similar situation as in the Neumann problem for the Laplace equation: the only elements of the null space are the constants. More general version of the maximum principle can be used to prove the followings refinements of theorems 2 and 3.

Theorem 4. *Assume $d \leq 0$. Then the Dirichlet problem*

$$Lu = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega, \qquad (36)$$

has a unique solution for every f and g.

Theorem 5. *Assume $d \leq 0$, $\sigma \geq 0$ and not both identically zero. Let β^i a vector field such that $\beta^i n_i > 0$ on $\partial\Omega$. Then the oblique derivative problem*

$$Lu = f \text{ on } \Omega, \quad \beta^i\partial_i u + \sigma u = g \text{ on } \partial\Omega, \qquad (37)$$

has a unique solution for every f and g.

In both theorems, the maximum principle can be used also to prove that the solution is positive if the sources, boundary values (and σ in theorem (5)) are positive.

Note that in theorems 2 and 3 the null spaces for the operator and the adjoint have the same dimension, we will see in the next section that this will not be the case for more general operators and boundary conditions.

We conclude this section with some examples.

Example 2. The most important second order elliptic operator is the Laplacian on a Riemannian manifold. It is given by

$$Lu = \Delta_h u = h^{ij} D_i D_j u, \tag{38}$$

where h is a Riemannian metric ($a^{ij} = h^{ij}$) and D its corresponding covariant derivative. One important example of lower order term is given by the conformal Laplacian which appears naturally in the Einstein constraint equations

$$Lu = \Delta_h u - \frac{R}{8} u, \tag{39}$$

where R is the Ricci scalar of h_{ab}.

For a Riemannian metric, the principal part of the boundary condition $B(u)$ has a geometric interpretation

$$B(u) = n^i D_i u, \tag{40}$$

where we use the standard convention $n^i = h^{ij} n_j$. That is, the vector n is now the unit normal vector with respect to the metric h_{ij}. This is sometimes denoted as *conormal boundary condition*.

An example of lower order boundary terms is the following

$$B(u) = n^i D_i u + Hu, \tag{41}$$

where H is the mean curvature of the boundary Ω with respect to the metric h_{ij}. This boundary condition appears in connection to black holes (see [21] and [8]).

3 Elliptic Systems

3.1 Definition of ellipticity

We saw in the previous section that ellipticity is a positivity condition on the symbol of the equation. In order to generalize this concept for systems of equations (this includes as particular case higher order equations) we need to define the symbol of a system. We can use the same idea as before, and define the principal part as the collection of terms which have the highest order derivatives. That is, consider the following differential operator in $\mathbb{R}^n$

$$L(u) = \sum_{|\alpha|\leq 2m} a_\alpha(x)\partial^\alpha u, \tag{42}$$

where α is a multi-index, and the coefficients a_α are $N \times N$ matrices. The principal part is defined as

$$l(x,\partial) = \sum_{|\alpha|=2m} a_\alpha(x)\partial^\alpha, \tag{43}$$

and the symbol

$$l(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha. \tag{44}$$

The operator is elliptic if $\det l(x,\xi) \neq 0$ for every $x \in \bar{\Omega}$ and $\xi \neq 0$. This is the definition that appears in most text books, we will call it *classical ellipticity* (there is no general agreement on the nomenclature, in most places these systems are called just elliptic). This definition excludes many important examples, the most remarkable is perhaps the Laplace equation as a first order system (example 3). In order to include these cases, we need to be more flexible in our definition of the principal part, in particular it is important to allow terms of different orders in it. This is particular feature of systems which does not appear in higher order equations.

It will be convenient to use a more explicit notation as the one given in (43). Let $u_1, \cdots, u_N$ be functions which depend on the coordinates $x_1, \cdots, x_n$. The operator (42) can be written as follows

$$L_{\mu\nu}(x,\partial)u^\nu(x) = f_\mu(x), \quad \nu,\mu = 1,\cdots,N; \tag{45}$$

where $L_{\mu\nu}$ are polynomials in $(\partial_1 \cdots, \partial_n)$ with coefficients depending on x. $[L_{\mu\nu}]$ is a $N \times N$ matrix, not necessarily symmetric. Note that N (dimension of the vectors u^ν) and n (dimension of $\mathbb{R}^n$) are in general different numbers.

Let $s_1, \cdots, s_N$, $t_1, \cdots, t_N$ be integers (some may be negative) such that

$$\deg(L_{\mu\nu}) \leq s_\mu + t_\nu. \tag{46}$$

where deg means the degree of the polynomial $L_{\mu\nu}$ in the derivatives ∂. The integers s_μ are attached to the equations and the t_ν to the unknowns.

We define the principal part $l_{\mu\nu}(x,\partial)$ as the terms in $L_{\mu\nu}$ which are *exactly* of order $s_\mu + t_\nu$. The symbol $l_{\mu\nu}(x,\xi)$ is obtained replacing in the principal part the derivatives by a vector ξ. We define the following polynomial in ξ

$$l(x,\xi) = \det(l_{\mu\nu}(x,\xi)). \tag{47}$$

The degree m of the systems is given by

$$m = \frac{1}{2}\deg(l(x,\xi)), \tag{48}$$

where deg means degree in ξ.

The following general definition of ellipticity was introduced in [9]

Definition 1 (Douglis-Nirenberg Ellipticity). *The system* (45) *is elliptic if there exist integer weights* s_μ *and* t_ν *which satisfy* (46) *and such* $l(x,\xi) \neq 0$ *for all real* $\xi \in \mathbb{R}^n$, $\xi \neq 0$, $x \in \bar{\Omega}$*; where* $l(x,\xi)$ *is given by* (47).

For $n = 2$ we assume in addition

Definition 2 (Supplementary condition). $l(x,\xi)$ *is of even degree* $2m$*. For every pair of linearly independent real vectors* ξ *and* ξ'*, the polynomial* $l(x,\xi + \tau\xi')$ *in the complex variable* τ *has exactly* m *roots with positive imaginary parts.*

Every elliptic systems in dimension $n \geq 3$ satisfies the supplementary condition (see [1]). This no longer true for $n = 2$, as example 5 shows. A system that is elliptic in the sense of Definition 1 and satisfies also the supplementary condition (Definition 2) will be called *properly elliptic*.

Note that the definition depends on the weights s_μ and t_ν which are not unique, a system can be elliptic for many different choices of weights. Also note that the number $2m$ is not related in general with the degree of the highest derivatives, for example for a second order system with $N = 3$ we have $m = 3$ (example 6). The degree m is important because it gives the number of boundary conditions we have to impose in order to get a well defined elliptic problem, as we will see in the next section.

There exists an important class of elliptic operators for which the Dirichlet boundary conditions will always satisfy (i)-(ii) as we will see in the next section. These systems are given by the following definition.

Definition 3 (Strong Ellipticity). *The system is called strongly elliptic if* $s_\nu = t_\nu \geq 0$ *and there exist a constant* $\epsilon > 0$ *such that*

$$Re\,(l_{\mu\nu}(x,\xi)\eta^\mu\bar{\eta}^\nu) \geq \epsilon\eta^\mu\eta_\mu\xi^i\xi_i, \tag{49}$$

for all real $\xi \in \mathbb{R}^n$ *and all complex* $\eta \in \mathbb{R}^N$.

Note that every elliptic equation (i.e., $N = 1$) is strongly elliptic. Let us discuss some examples.

Example 3 (Laplace equation as a first order system). This example was taken from [2]. Consider the Laplace equation in two dimensions

$$\partial_1^2 u + \partial_2^2 u = 0. \tag{50}$$

Every equation can be written as a first order system if we introduce the derivatives of the unknown as new variables. That is, let $u_1 = \partial_1 u$ and $u_2 = \partial_2 u$. Then we have the following system ($n = 2$ and $N = 3$)

$$\partial_1 u_1 + \partial_2 u_2 = 0, \tag{51}$$

$$\partial_1 u - u_1 = 0, \tag{52}$$

$$\partial_2 u - u_2 = 0. \tag{53}$$

In the matrix notation

$$\begin{pmatrix} 0 & \partial_1 & \partial_2 \\ \partial_1 & -1 & 0 \\ \partial_2 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ u_1 \\ u_2 \end{pmatrix} = 0. \tag{54}$$

In the classical definition, the symbol is constructed only with the terms which contains the highest order derivatives, in this case only with the terms with one derivative. Then the determinant of the symbol is

$$\begin{vmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{vmatrix} = 0, \tag{55}$$

and we conclude that the system is not classically elliptic.

Take the weights $t_1 = 2, t_2 = t_3 = 1$, for u, u_1, u_2 and $s_1 = 0, s_2 = s_3 = -1$, to the first, second and third equations, respectively. Then, we have

$$\begin{vmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \end{vmatrix} = \xi_1^2 + \xi_2^2, \tag{56}$$

and the system is elliptic with $m = 1$. Another possible choice for the weights is the following $s_i = t_i$, with $t_1 = 1$, $t_2 = t_3 = 0$.

Since $n = 2$, we have to check also that it satisfies the supplementary condition

$$0 = l(\xi + \tau\xi') = |\xi|^2 + 2\tau\xi^i\xi'_i + \tau^2|\xi'|^2 \tag{57}$$

then

$$\tau_\pm = (-\cos\theta \pm i\sin\theta)|\xi'|^{-1}|\xi| \tag{58}$$

where $|\xi|^2 = \xi^i\xi_i$ and $\xi^i\xi'_i = \cos\theta|\xi||\xi'|$. That is, we have only one root with positive imaginary part.

Example 4 (Stokes system). This example was taken from [28]. The following equations appear as the stationary linearized case of the Navier-Stokes equations (see for example [34]) for the velocity u^i and the pressure p of the fluid

$$\Delta u^i - \partial^i p = 0, \quad \partial^i u_i = 0. \tag{59}$$

The unknowns are u^i, p, that is $N = 4$, and we will assume $n = 3$. Then, in the matrix notation we have

$$\begin{pmatrix} \Delta & 0 & 0 & -\partial_1 \\ 0 & \Delta & 0 & -\partial_2 \\ 0 & 0 & \Delta & -\partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{pmatrix} = 0. \tag{60}$$

It is clear that the system is not classically elliptic. Take $t_1 = t_2 = t_3 = 2$, $t_4 = 1$ and $s_1 = s_2 = s_3 = 0$, $s_4 = -1$. Then the symbol is

$$l_{ij} = \begin{pmatrix} |\xi|^2 & 0 & 0 & -\xi_1 \\ 0 & |\xi|^2 & 0 & -\xi_2 \\ 0 & 0 & |\xi|^2 & -\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix}, \tag{61}$$

and we have

$$l = |\xi|^6, \quad m = 3. \tag{62}$$

Then, the system is elliptic. Another possible choice for the weights is the following: $s_i = t_i$, with $t_1 = t_2 = t_3 = 1$ and $t_4 = 0$.

Example 5 (Cauchy-Riemann equation). We write the Cauchy-Riemann equation $L(u) = \partial_{\bar{z}} u$, in terms of the real variables $z = x + iy$

$$L(u) = \frac{1}{2} \left(\partial_x u + i \partial_y u \right). \tag{63}$$

We have $n = 2$, $N = 1$. The symbol $l = \xi_1 + i\xi_2$, satisfies

$$l(\xi) \neq 0 \text{ for all real } \xi \neq 0, \tag{64}$$

hence the system is elliptic with $m = 1/2$. However, it does not satisfies the supplementary condition because $2m = 1$ is not an even number.

Example 6. Consider the following operator in $\mathbb{R}^3$, acting on three vectors u^i

$$L_{ij} u^j = \partial^j (\mathcal{E} u)_{ij}, \tag{65}$$

where

$$(\mathcal{E} u)_{ij} = 2\mu \partial_{(i} u_{j)} + \lambda \delta_{ij} \partial^k u_k, \tag{66}$$

and μ, λ are constants. Since in this case we have $N = n = 3$ we will use the same index notation for the index in the vectors u and in the coordinates of $\mathbb{R}^3$.

The system (65) appears in elasticity (see, for example, [20]). It also appears in General Relativity related to gauge conditions like the minimal distortion gauge (see [33]) and in the constraint equations (see [39]), usually with the choice $\mu = 1$, $\lambda = -2/3$ which makes (66) trace free.

From (65) we deduce

$$L_{ij} u^j = \left((\mu + \lambda) \partial_i \partial_j + \mu \delta_{ij} \Delta \right) u^j, \tag{67}$$

in the matrix notation we have ($\lambda' = \mu + \lambda$)

$$L_{ij} u^j \equiv \begin{pmatrix} \lambda' \partial_1^2 + \mu \Delta & \lambda' \partial_2 \partial_1 & \lambda' \partial_1 \partial_3 \\ \lambda' \partial_1 \partial_3 & \lambda' \partial_2^2 + \mu \Delta & \lambda' \partial_1 \partial_3 \\ \lambda' \partial_1 \partial_3 & \lambda' \partial_1 \partial_3 & \lambda' \partial_3^2 + \mu \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{68}$$

Take $s_i = t_i = 1$, the symbol is given by

$$l_{ij}(\xi) = \lambda' \xi_i \xi_j + \mu \delta_{ij} \xi^k \xi_k, \tag{69}$$

and

$$l = \mu^2 (2\mu + \lambda) |\xi|^6, \quad m = 3. \tag{70}$$

The operator is (classically) elliptic for $\mu > 0$, $2\mu + \lambda > 0$. It is also strongly elliptic

$$l_{ij} \eta^i \bar{\eta}^j = (\lambda + \mu)(\eta^i \xi_i)(\bar{\eta}^i \xi_i) + \mu \xi^k \xi_k \eta_i \bar{\eta}^i \geq \epsilon \xi^k \xi_k \eta_i \bar{\eta}^i, \tag{71}$$

where $\epsilon = \min\{\mu, 2\mu + \lambda\}$.

Example 7 (Einstein Constraint equations).

There exist different ways of reducing the Einstein constraint equations to an elliptic systems (see, for example, the recent review [4]). In the standard approach the principal part of the system is formed with the Laplace operator on a Riemannian manifold given in example 2 and the operator that has been discussed in example 6.

A particular interesting example is the one that has been recently used in [6] and [7] to construct new kind of solutions. This system is not elliptic in the classical sense but it satisfies definition 1 for appropriate weights (see these references for details).

Example 8 (Witten equation). The Witten equation $\partial_{AA'} u^A = 0$ (in the spinorial notation) plays an important role in the positive mass theorem of General Relativity (cf. [37]). Solutions of this equation has been analyzed in [29] and [26].

In the matrix notation ($N = 2$, and we will assume $n = 3$) this system is given by

$$\begin{pmatrix} \partial_3 & \partial_1 + i\partial_2 \\ \partial_1 - i\partial_2 & -\partial_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \tag{72}$$

The principal part, with weights $t_1 = t_2 = 1$, $s_1 = s_2 = 0$, is given by

$$l_{\nu\mu}(x, \xi) = \begin{pmatrix} \xi_3 & \xi_1 + i\xi_2 \\ \xi_1 - i\xi_2 & -\xi_3 \end{pmatrix}, \quad l = -(\xi_3^2 + \xi_2^2 + \xi_1^2), \quad m = 1. \tag{73}$$

Then, the system is elliptic.

3.2 Definition of elliptic boundary conditions

For the operator $L_{\mu\nu}$ defined in (45) we will consider boundary conditions of the following form

$$B(x, \partial)_{l\nu} u^\nu = 0, \quad l = 1, \cdots, m; \quad \nu = 1, \cdots, N; \tag{74}$$

where $B(x, \partial)_{l\nu}$ are polynomial in ∂ and m is given by (48). The order of the boundary operators, like those of the operators in (45), depends on two systems of integer weights, in this case the system t_ν already attached to the dependent variable and a new system r_l attached to each boundary condition such that

$$\deg(B_{l\nu}) \leq r_l + t_\nu. \tag{75}$$

Note that r_l can be negative and also the order of the derivatives in the boundary conditions can be higher than in the operator. The principal part $b_{l\nu}$ of the boundary operator consists of the terms in $B_{l\nu}$ which are exactly of order $r_l + t_\nu$.

For a given operator L, we would like to know for which boundary operators B the solutions of the corresponding boundary value problem will satisfies (i)-(ii). The answer to this question is given by the following definition, as we will see in the next section.

Let x_0 a point on $\partial\Omega$ and let n^i the outer normal to Ω. We consider the constant coefficient problem

$$l_{\mu\nu}(x_0, \partial)u^\nu = 0, \tag{76}$$

$$b_{l\nu}(x_0, \partial)u^\nu = 0, \tag{77}$$

on the half plane $(x^i - x^i_0) \cdot n_i < 0$ with boundary $(x^i - x^i_0)n_i = 0$.

Definition 4 (Complementing condition). *We say that the complementing condition holds at x_0 if there are no nontrivial solutions of* (76)– (77) *of the following form:*

$$u^\nu(x) = v^\nu(\eta)e^{i\xi_j(x^j - x^j_0)} \tag{78}$$

where ξ is a any nonzero, real, vector which satisfies $\xi^i n_i = 0$, $v(\eta)$ tends to zero exponentially as $\eta \to -\infty$ and the coordinate η is defined by $\eta = (x^j - x^j_0)n_j$.

In the literature, these conditions are also called *Lopatinski-Shapiro* conditions or *covering* conditions (see [2] and [38]). Let us study some examples of boundary conditions.

Example 9 (Boundary conditions for the Laplace equation.). Consider solutions of the form (78) for the Laplace equation $\Delta u = 0$. We chose coordinates in $\mathbb{R}^n$ such that $\eta = x_n$, $n^i = \delta^i_n$. Then, all the solutions of this form are given by

$$u = e^{i\xi^i x_i} e^{\pm|\xi| x_n}, \tag{79}$$

where ξ satisfies $\xi_n = 0$.

We consider different boundary conditions on the plane $x_n = 0$. For the Dirichlet condition $u(x_n = 0) = 0$ we get

$$u = e^{i\xi^i x_i} = 0, \tag{80}$$

since this is not possible there exist no solution of this form which satisfies the Dirichlet conditions. Hence, the Dirichlet boundary condition satisfies the complementing condition.

For the Neumann condition we have $\partial_{x_n} u = 0$ at $x_n = 0$, this implies $\xi = 0$ and then the solution will not decay at infinity. Hence, the Neumann conditions satisfied also the complementing conditions.

Take the oblique derivative boundary condition $\beta^i \partial_i u = 0$ at $x_n = 0$. This implies

$$i(\beta_i \xi^i) = 0, \quad \beta_n |\xi| = 0. \tag{81}$$

If $\beta_n \neq 0$ then $|\xi| = 0$, an the complementing condition is satisfied. This was the case studied in section 2. On the other hand, if $\beta_n = 0$ (like in example 1) then the complementing condition is not satisfied since we can always chose a vector ξ such that $\beta_i \xi^i = 0$ and we will get solutions of the form (78).

Consider now the following interesting example studied in [15]. At $x_n = 0$ we impose the boundary conditions

$$\delta u = 0, \tag{82}$$

where

$$\delta u = \partial_1^2 u + \cdots + \partial_{n-1}^2 u, \tag{83}$$

is the Laplacian in $n-1$ dimension. From (82) we deduce the $|\xi|^2 = 0$ and then it satisfies the complementing conditions. It is also clear that $\delta^k u = 0$ where k, is an arbitrary natural number, satisfies the complementing condition. Note that in this cases the boundary operator has derivatives of higher order than the Laplace operator. On a Riemannian manifold, these conditions can be written in geometric form where δ is the intrinsic Laplacian on the boundary. Another interesting condition which also satisfies the complementing condition is the following

$$\delta u - n^i \partial_i u = 0. \tag{84}$$

In this case, integrating by parts, it is easy to show that the only solutions of the homogeneous problem are the constants

$$0 = \int_\Omega u \Delta u = \oint_{\partial\Omega} u n^i \partial_i u - \int_\Omega \partial_i u \partial^i u \tag{85}$$

$$= \oint_{\partial\Omega} u \delta u - \int_\Omega \partial_i u \partial^i u \tag{86}$$

$$= -\oint_{\partial\Omega} |du|^2 - \int_\Omega \partial_i u \partial^i u, \tag{87}$$

where du denotes the gradient intrinsic to the boundary.

Example 10. Consider the operator discussed in example 6. Integrating by parts we get

$$\mathbf{B}(u, v) = -\int_\Omega v^i L_{ij} u^j + \oint_{\partial\Omega} (\mathcal{E}u)_{ij} n^i v^j \tag{88}$$

where

$$\mathbf{B}(u, v) = \int_\Omega (\mathcal{E}u)^{ij} \partial_i v_j. \tag{89}$$

We can write the integrand in $\mathbf{B}(u, v)$ in the following form

$$(\mathcal{E}u)^{ij}\partial_i v_j = \frac{\mu}{2}(\mathcal{L}u)_{ij}(\mathcal{L}v)^{ij} + (\lambda + \frac{2}{3}\mu)\partial_k u^k \partial_l v^l, \tag{90}$$

where $(\mathcal{L}u)_{ij}$ is the trace free part of $\partial_{(i}u_{j)}$, that is

$$(\mathcal{L}u)_{ij} = 2\partial_{(i}u_{j)} - \frac{2}{3}\delta_{ij}\partial_k u^k. \tag{91}$$

Note that $\mathbf{B}$ is symmetric $\mathbf{B}(u, v) = \mathbf{B}(v, u)$. Using this and equation (88) we get the following Green formula

$$\int_\Omega v^i L_{ij} u^j - u^i L_{ij} v^j = \oint_{\partial\Omega} (\mathcal{E}u)_{ij} n^i v^j - (\mathcal{E}v)_{ij} n^i u^j. \tag{92}$$

This is analogous to the Green formula for second order equations (23). For simplicity we have not included terms in non divergence form in the operator, that is why we have $L = L^t$ and $B = B^t$ in (92), these extra terms can be handle in the same way as in section 2.

The boundary integral in the Green formula (92) suggests that two natural boundary conditions are the Dirichlet

$$u^i = 0 \text{ on } \partial\Omega, \tag{93}$$

and the analog to the Neumann boundary condition

$$(\mathcal{E}u)_{ij} n^j = 0 \text{ on } \partial\Omega. \tag{94}$$

We want to prove that these boundary conditions satisfy the complementing conditions. We will assume that $\mu > 0$ and $2\mu + \lambda > 0$, that is, the operator is elliptic as we have seen in example 6. We will make also an extra assumption: $3\lambda + 2\mu \geq 0$; this implies that the integrand (90) is positive. Moreover, if $3\lambda + 2\mu > 0$

$$B(u, u) = 0 \iff \partial_{(i}u_{j)} = 0, \tag{95}$$

that is u is a Killing vector. If $3\lambda + 2\mu = 0$, then

$$B(u, u) = 0 \iff (\mathcal{L}u)_{ij} = 0, \tag{96}$$

then u is a conformal Killing vector. In flat space, we know explicitly all the Killing and conformal Killing vectors. Hence, we have a characterization of the null spaces for these boundary conditions. The Killing and conformal Killing are the analog of the constants for the Neumann problem for the Laplace equation.

Assume we have a solution u of the form (78). Chose Cartesian coordinates such that $\eta = x_3$. Let $L_1 = 2\pi/\xi_1$ and $L_2 = 2\pi/\xi_2$. Take as domain the infinite cubic region $x_3 \geq 0$, $0 \leq x_1 \leq L_1$, $0 \leq x_2 \leq L_2$. For this domain we use equation (88) for $u = v$. We want to prove that, on this domain, the boundary integral in (88) vanished if we impose either (93) or (94). Using these boundary conditions we get that the integrand vanishes on the face $x_3 = 0$. The integrand also vanishes on the face $x_3 = \infty$ because the solution, by hypothesis, decay at infinity. On the other faces the integrand does not vanish. However, because of the choice of L_1 and L_2, we have that the integrand of opposite faces are identical. Then, the sum of the boundary integrals vanished because the normal is always outwards. We conclude that $\mathbf{B}(u, u)$ should vanish. But there are no Killing or conformal Killing vectors which decay to zero at infinity. Hence the complementing condition is satisfied.

Example 11 (Boundary conditions for the Stoke system). If we multiply equations 59 by u^i and integrate by parts we get

$$0 = -\int_\Omega \partial_k u_i \partial^k u^i + \oint n^k (u^i \partial_k u_i - u_k p). \tag{97}$$

Using this equation and a similar argument as in the previous example it is possible to show that the boundary conditions

$$u^i = 0 \text{ on } \partial\Omega, \tag{98}$$

and p unprescribed, satisfy the complementing condition (see for example [28]).

Example 12 (Dirichlet boundary conditions for strongly elliptic systems).

Assume that the system is strongly elliptic, this implies $s_i = t_i = t'_i \geq 0$. The Dirichlet boundary conditions on $\partial\Omega$ are given by

$$(n^i \partial_i)^q u_j = 0, \quad q = 0, \cdots, t'_j - 1, \quad j = 1, \cdots, N; \tag{99}$$

when $t'_j = 0$, u_j goes unprescribed.

It can be proved that for every strongly elliptic system, the Dirichlet conditions (99) satisfy the complementing condition (see [2]).

In the case equations ($N = 1$) of order $2m$ ($t_1 = m$) these conditions reduce to

$$(n^i \partial_i)^q u = 0, \quad q = 0, \cdots, m - 1. \tag{100}$$

In particular, for second order equation ($m = 1$) we have $u = 0$ at the boundary. That is, we recover the familiar Dirichlet condition studied in sections 1 and 2. In example 6 we have $t'_i = 1$, then $q = 0$ and the Dirichlet conditions is just

$$u^j = 0 \text{ on } \partial\Omega. \tag{101}$$

As an example of a higher order equation, we have the biharmonic equation

$$\Delta\Delta u = f, \tag{102}$$

the Dirichlet conditions are given by ($N = 1$, $m = 2$)

$$u = 0, \quad n^i \partial_i u = 0 \text{ on } \partial\Omega. \tag{103}$$

Example 13. In the following example (taken from [28]), the complementing condition is *not* satisfied. Consider the following problem in $\mathbb{R}^2$, where the boundary is the line $x_2 = 0$

$$\Delta\Delta u = 0 \text{ on } \Omega \quad \Delta u = \partial_2 \Delta u = 0 \text{ on } \partial\Omega. \tag{104}$$

For every $\xi \in \mathbb{R}$ the function

$$u(x, y) = e^{i\xi x_1 - |\xi| x_2} \tag{105}$$

is a solution.

Example 14. We have seen that for strongly elliptic system the Dirichlet boundary conditions satisfy the complementing condition. This is not true for general elliptic systems. In following example (discussed in [25]) we show that there are elliptic systems for which the Dirichlet problem is not well defined.

Consider the system $(N = n = 2)$

$$\begin{pmatrix} \partial_1^2 - \partial_2^2 & -2\partial_1\partial_2 \\ 2\partial_1\partial_2 & \partial_1^2 - \partial_2^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \tag{106}$$

The symbol is given by

$$l_{ij} = \begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix}, \quad l = (\xi_1^2 + \xi_2^2)^2. \tag{107}$$

Then, the system is elliptic in the classical sense. This system can be written in the complex form, $z = x_1 + ix_2$, $w = u_1 + iu_2$, as

$$\partial_{\bar{z}}^2 w = \frac{1}{4}(\partial_1 + i\partial_2)^2 w = 0, \tag{108}$$

for which the general solution is clearly

$$w = f(z) + \bar{z} g(z), \tag{109}$$

where f and g are arbitrary functions of z. We observe that all solutions of the form

$$w = f(z)(1 - z\bar{z}) \text{ on } |z| \leq 1 \tag{110}$$

with arbitrary analytic f, vanish on the boundary of the unit disk. Thus the problem of finding a solution with Dirichlet boundary condition is not well defined.

Example 15 (Boundary conditions for the Witten equation). In the Witten equation studied in example 8 we have $m = 1$, that is, we can only impose one boundary condition. Consider the following boundary condition

$$u_1 = 0, \text{ on } \partial\Omega, \tag{111}$$

and u_2 goes unprescribed. This condition has been studied in [31] as an inner boundary condition for black holes in the positive mass theorem. We want to prove that it satisfies the complementing condition. We can explicitly calculate all the solutions of the form (78) of the equations (72)

$$u^\nu = e^{\xi^i x_i} v^\nu(x_3), \quad v^\nu = A^\nu e^{|\xi| x_3}, \tag{112}$$

where A^ν are constants such that $A_2/A_1 = (i\xi_1 + \xi_2)|\xi|^{-1}$ and we chose coordinates such that $\eta = x_3$, $\xi_3 = 0$. There is no solution of this form that satisfies $u_1(x_3 = 0) = 0$ and then the complementing condition follows.

Example 16 (Stationary solutions of Einstein equations).
In the presence of a timelike symmetry, Einstein equations can be reduced to an elliptic system. Moreover, the inner boundary conditions satisfy the complementing condition. This result was proved in [30] and it was used to prove an existence result for the non linear problem. See also [23] for a different kind of boundary conditions for the static case.

3.3 Results

In order to present a general result for properly elliptic systems with boundary conditions that satisfy the complementing condition we need to reformulate in a more precise way properties (i)-(ii). For a given operator L and boundary operator B we consider the operator A defined as $A(u) = (L(u), B(u))$. This operator will act on appropriate Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $A : \mathcal{H}_1 \to \mathcal{H}_2$. In analog way as we did in (26) we define the null space $\mathcal{N}(A)$ as

$$\mathcal{N}(A) = \{u \in \mathcal{H}_1 : A(u) = 0\}. \tag{113}$$

The range of A is defined by

$$\mathcal{R}(A) = \{w \in \mathcal{H}_2 : \exists u \in \mathcal{H}_1, A(u) = w\}, \tag{114}$$

and the complement of the range is given by

$$\mathcal{R}^\perp(A) = \{w \in \mathcal{H}_2 : (Au, w)_{\mathcal{H}_2} = 0 \text{ for all } u \in \mathcal{H}_1\}. \tag{115}$$

where $(\cdot, \cdot)$ denotes the Hilbert scalar product.

We can write now properties (i)-(ii) as follows

(i) $\mathcal{N}(A)$ has finite dimension.
(ii) $\mathcal{R}^\perp(A)$ has finite dimension.

An operator which satisfies (i)-(ii) is called a *Fredholm* operator. (we have assumed that A is bounded, otherwise in (ii) we need to impose that $\mathcal{R}(A)$ is closed, see [18] and [16]).

We have the following general result (we only sketch the statement, for details and proofs see [17],[16] and also [38])

Theorem 6. *If the system L is properly elliptic in $\bar{\Omega}$ and the boundary conditions satisfy the complementing condition for every point of $\partial\Omega$, then the operator $A(u) = (L(u), B(u))$ is Fredholm.*

We have seen that the dimension of $\mathcal{N}$ (and hence uniqueness) is not invariant if we add lower order terms to the operator. One of the consequence of theorem 6 is the existence of an invariant for elliptic problems: the Fredholm index. This number is defined as

$$I = \dim \mathcal{N}(A) - \dim \mathcal{R}^{\perp}(A). \tag{116}$$

It can be proved that the index I is stable under perturbation, in particular it does not depend on the lower order terms.

In section 2 we have used the Green formula to construct the formal adjoint operator L^t and its corresponding boundary operator B^t. In this case we can define $A^t(u) = (L^t(u), B^t(u))$, and it can be proved [1] that $N(A^t) = \mathcal{R}^{\perp}(A)$. That is, the boundary value problems considered in theorem 2 and 3 have $I = 0$. In fact these theorems also show that the index does not depend on the lower order terms in this particular case.

Boundary conditions which come from a Green formula are called *normal boundary conditions*. The advantage of them is that we have a characterization of $\mathcal{R}^{\perp}(A)$ through the formal adjoint problem, and then we can in principle compute the conditions that the sources should satisfy in order to have a solution. General results for normal boundary conditions for higher order elliptic equations can be found in [19] [32]. Since these boundary conditions come from an integration by parts, the order the boundary operators will be always less than the operator itself. We have seen that this is not necessary the case for general elliptic boundary conditions that satisfy the complementing condition. For the general case, we will not have a characterization of $\mathcal{R}^{\perp}(A)$.

We have seen that the Dirichlet boundary conditions satisfy the complementing condition for strongly elliptic systems. Using this fact, general existence results for the Dirichlet problem can be proved (see [24]). Moreover, it can be shown that the index is always zero in this case.

Finally, we want to present an existence result for the operator considered in example 6 that can be deduced from the general theorem 6 (see [36]) In this case, we have a Green formula and then we have normal boundary conditions. The following two theorems are the analogous of theorem 2 and 3.

Theorem 7. *Let L_{ij} be given by (65) with $\mu \geq 0$, $2\mu + \lambda \geq 0$, $2\mu + 3\lambda \geq 0$. Then, for every smooth f^j and g^i, there exist a unique, smooth, solution u^i of the Dirichlet problem*

$$L_{ij}u^i = f^j \text{ on } \Omega, \quad u^i = g^i \text{ on } \partial\Omega. \tag{117}$$

[1] It is important to note that for any bounded (or unbounded with dense range) operator A we can define the Hilbert adjoint A'. This is not related, in general, with the formal adjoint A^t. However, when we have a Green formula, it is possible to prove that in fact $A^t = A'$ (see theorem 8.4 of [19], [5] and also [35]

We have seen that all solution of the homogeneous problem satisfy $(\mathcal{E}v)_{ij} = 0$, that is v is a Killing or a conformal Killing vector. Uniqueness in this theorem follows because there exists no Killing or conformal Killing vector which vanishes at the boundary.

Theorem 8. *Let L_{ij} be given by* (65) *with $\mu \geq 0$, $2\mu + \lambda \geq 0$, $2\mu + 3\lambda \geq 0$. Consider the boundary value problem*

$$L_{ij}u^i = f^j \text{ on } \Omega, \quad (\mathcal{E}u)_{ij}u^i = 0 \text{ on } \partial\Omega. \tag{118}$$

This problem has a solution if and only if

$$\int_\Omega f_i v^i = 0 \quad \text{for all } v^i \text{ such that } \mathcal{E}v_{ij} = 0. \tag{119}$$

If u_1 and u_2 are two different solutions, then the difference $v = u_1 - u_2$ satisfies $(\mathcal{E}v)_{ij} = 0$.

In the case of the Einstein constraint equations, the previous theorems can be used to prove existence of solutions of the momentum constraint (see [39]). In this case the physical quantity is the second fundamental form K_{ij} which is given by

$$K_{ij} = Q_{ij} - (\mathcal{E}u)_{ij}, \tag{120}$$

where Q is an (essentially) arbitrary tensor. Then, as in the case of the Neumann problem for the Laplace equation, the lack of uniqueness in theorem 8 will not affect K_{ij}.

4 Final Comments

In order to check if a system of equations is elliptic, we should first prove that the principal part of the operator satisfy definitions 1 and 2. If the system is non linear, we should consider the corresponding linearized problem. Then we should prove that the boundary operators satisfy the complementing condition (definition 4). This can be complicated. There exist equivalent formulations of this condition (see for example [38]), some of them can be more suitable for specific problems. It is also important to know if the boundary conditions come from a Green formula (normal boundary conditions). The Green formula can be used to prove that the complementing condition holds (as we have seen in the examples). Moreover, for the case of higher order equations there exist general results that can be used (see [19]).

We have only discussed linear elliptic systems. In the non linear case, there are no general existence results like theorem 6. For non linear second order equations a good reference is [14] and for non linear systems [13] and [12]. A related issue that was not discussed here is regularity. We have assumed that the functions and the boundary are smooth. Regularity properties are crucial for non linear systems, see [14], [13] and [12] and references there.

In section 2 we have followed [11]. Good references for this section are also [10], [14] and [22]. For section 3, an introductory book is [28]; more advanced material can be found in [35] and [3]; for a complete discussion see [16], [15] and [38].

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