Consistency Check on Volume and Triad Operator Quantisation in Loop Quantum Gravity I

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Preprint AEI-2005-100

Abstract

The volume operator plays a pivotal role for the quantum dynamics of Loop Quantum Gravity (LQG). It is essential in order to construct Triad operators that enter the Hamiltonian constraint and which become densely defined operators on the full Hilbert space even though in the classical theory the triad becomes singular when classical GR breaks down.

The expression for the volume and triad operators derives from the quantisation of the fundamental electric flux operator of LQG by a complicated regularisation procedure. In fact, there are two inequivalent volume operators available in the literature and, moreover, both operators are unique only up to a finite, multiplicative constant which should be viewed as a regularisation ambiguity.

Now on the one hand, classical volumes and triads can be expressed directly in terms of fluxes and this fact was used to construct the corresponding volume and triad operators. On the other hand, fluxes can be expressed in terms of triads and therefore one can also view the volume operator as fundamental and consider the flux operator as a derived operator.

In this paper we mathematically implement this second point of view and thus can examine whether the volume, triad and flux quantisations are consistent with each other. The results of this consistency analysis are rather surprising. Among other findings we show: 1. The regularisation constant can be uniquely fixed. 2. One of the volume operators can be ruled out as inconsistent. 3. Factor ordering ambiguities in the definition of triad operators are immaterial for the classical limit of the derived flux operator.

The results of this paper show that within full LQG triad operators are consistently quantized.
In this paper we merely present ideas and results of the consistency check. In a companion paper we supply detailed proofs.

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1 Introduction

The major unresolved problem in Loop Quantum Gravity (LQG) (see [1] for books and [2] for reviews) is the satisfactory implementation of the quantum dynamics which in turn is governed by the Wheeler – DeWitt quantum constraint [3, 4], also called the Hamiltonian constraint. The volume operator [5, 6] plays a pivotal role for the very definition of the Hamiltonian constraint because with its help one can quantize triad functions which enter the classical expression for the Hamiltonian constraint. In particular, one takes advantage of a Poisson bracket identity between the triads and the Poisson bracket among the Ashtekar connection and the classical scalar volume function. Namely, one uses the methods of canonical quantisation, that is, the axioms of quantum mechanics, according to which Poisson brackets between classical functions are turned into commutators between the corresponding operators divided by $i\hbar$, at least to lowest order in $\hbar$. Quite surprisingly, the Hamiltonian constraint operator and similarly also length operators [7] are then densely defined on the full kinematical Hilbert space of LQG although the classical triad becomes singular in physically relevant situations such as black holes or the big bang.
While playing such a distinguished role for the most important open problem of LQG, the volume operator and thus the derived triad operators have never been critically examined concerning their physical correctness and mathematical consistency. By the first we mean that it has never been shown within full LQG that the volume operator has the correct classical limit with respect to suitably chosen kinematical semiclassical states, for example those constructed in [8][1].
By the second we mean the following: The fundamental kinematical algebra $\mathfrak{A}$ on which LQG is based is the Holonomy – Flux algebra [10] and its representation theory together with background independence leads to a unique kinematical Hilbert space [11]. Now classically the volume and triad can be written as limits of functions of the flux. To implement them at the quantum level, one has to go through a complicated regularisation procedure and to take the limit. It is surprising that the resulting operators are densely defined at all and in fact have a discrete spectrum because they are highly non – polynomial expressions as functions of fluxes. This is the payoff for background independence, since in background dependent formulations, such as the standard Fock representations these operators are too singular. On the other hand, since there is little experience with non Fock representations, it is not at all clear whether the corresponding operators have anything to do with their classical counterpart. In fact, there are at least two ambiguities already at the level of the volume operator: First of all, there are in fact two unitarily inequivalent volume operators [5, 6] which come from two, a priori equally justified background independent regularisation techniques. We will denote them by Rovelli – Smolin (RS) and Ashtekar – Lewandowski (AL) volume respectively for the rest of this paper. Secondly, both volume operators are anyway only determined up to a multiplicative regularisation constant C_{reg} [12] which remains undetermined when taking the limit, quite similar to finite regularisation constants that appear in counterterms of standard renormaisation of ordinary QFT. The ambiguity is further enhanced by factor ordering ambiguities once we consider triad operators. These ambiguities are parameterized by a spin quantum number $\ell = 1/2, 1, 3/2, ...$
In this paper we will be able to remove all those ambiguities by the following consistency check: As we mentioned above, the volume and triad can be considered as functions of the fluxes. But the converse is also true: The fluxes can be written in terms of triads and thus the volume. Is it then true that there exists a regularisation constant for the volume operator and a factor ordering of the flux operator considered as a function of the triad operator or volume operator such that the corresponding alternative flux operator agrees (at least in the correspondence limit of large eigenvalues of the volume operator) with the fundamental flux operator, independent of the choice of ℓ? This better be possible as otherwise the inescapable conclusion would be that the volume operator is inconsistently quantised[2].

[1]This has been achieved, so far, only within a certain approximation [9] which essentially consists in replacing $SU(2)$ by $U(1)^3$. The necessary calculations in the Non Abelean case are much more complictaed due to two reasons: 1. The spectrum of the volume operator is not available in analytical form and 2. the semiclassical analysis is calculationally more difficult. However, work is now in progress in order to fill this gap.

[2]In contrast, the triad operator follows from the volume operator by the axioms of quantum mechanics namely that Poisson brackets be replaced by commutators divided by $i\hbar$ and therefore it is not possible that the source of a possible problem is in

We will be able to precisely answer this question affirmatively. In more detail we will show:

1. The RS volume operator is inconsistent with the flux operator, the AL volume operator is consistent.
2. $C_{reg} = 1/48$ can be uniquely fixed, there is no other choice which is semiclassically acceptable. Remarkably, this is precisely the value that was obtained in [6] by a completely different argument.
3. The choice of ℓ plays no role semiclassically, in fact it drops out of the final expression for the alternative operator altogether. Therefore the afore mentioned factor ordering ambiguity is absent as far as the flux operator is concerned.
4. There is yet one more ambiguity in LQG which is already present classically: Classically it is possible to consider the electric field either as a two form or as a pseudo two form. The corresponding sign of the determinant of the triad is then either encoded in the electric field or in the conjugate connection. One can take either point of view without affecting the symplectic structure of the theory. We will be able to show that one *must* consider the electric field as a pseudo two form, otherwise the alternative flux operator becomes the zero operator!
5. As expected, the alternative and fundamental flux operator agree for all values of the Immirzi parameter [14], hence it cannot be fixed by our analysis which is good because it has been fixed already by arguments coming from quantum black hole physics [15].
6. The calculations in this paper make extensive use of certain advances in technology concerning the matrix elements of the volume operator [16]. Thus, our calculations provide an independent check of [16] as well.
7. The factor ordering of the alternative flux operator is unique if one insists on the principle of minimality[3].

These results show that instead of taking holonomies and fluxes as fundamental operators one could instead use holonomies and volumes as fundamental operators. It also confirms that the method to quantise the triad developed in [3] is mathematically consistent.

This paper is organised as follows:

In section two we review the regularisation and definition of the fundamental flux operator for the benefit of the reader and in order to make the comparison with the alternative quantisation easier.
In section three we eplain the idea on which the construction of the usual flux operator is based on.
In section four we derive the classical expression for the alternative flux operator.
In section five and six we discuss our results for the alternative flux operator and compare them with the corresponding results of the usual flux operator.
Finally in section seven we summarise and conclude.

All the technical details and tools that are needed to perform this consistency check are providid in our companion paper [19].

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the quantisation of the triad operator.

[3]By this we mean that given a function f with self adjoint quantisation $\hat{f}$ and a multiplication operator g for which g^{-1} is defined everywhere on the Hilbert space, we may always consider instead $\hat{f}' = (g\hat{f}g^{-1} + \overline{g^{-1}\hat{f}\overline{g}})/2$ as the operator corresponding to f if it has selfadjoint extensions as well. By minimalistic we mean the choice $g = 1$. This issue is always present even in ordinary quantum mechanics however it is usually not mentioned because one usually deals with polynomials and $g \neq 1$ would destroy polynomiality. In GR the expressions are generically non – polynomial from the outset and thus polynomiality is not available as a simplistic criterion. However, we may still insist on a minimal number of such kind of factor ordering ambiguities.

2 Review of the usual Flux Operator

In LQG the classical electric flux $E_k(S)$ through a surface S is the integral of the densitised triad E^a_k over a two surface S

$$E_k(S) = \int\limits_S E^a_k\, n^S_a, \tag{2.1}$$

where n^S_a is the conormal vector with respect to the surface S. In order to define a coresponding flux operator in the quantum theory, we have to regularise the classical flux and then define the action of the operator on an arbitary spin network funtion (SNF) $T_{\gamma,\vec{j},\vec{m},\vec{n}} : G^{|E(\gamma)|} \to \mathbb{C}$,where G is the corresponding gauge group, namely $SU(2)$ in our case, as the action of the regularised expression, denoted by $E^\epsilon_k(S)$ in the limit where the regularisation parameter ϵ is removed

$$\widehat{E}_k(S)T_{\gamma,\vec{m},\vec{n}} := i\hbar \lim_{\epsilon\to 0} \left\{E^\epsilon_k(S), T_{\gamma,\vec{m},\vec{n}}\right\}. \tag{2.2}$$

Here the limit is to be understood in the following way: The Poisson bracket on the right hand side of eqn (2.2) is calculated by viewing $T_{\gamma,\vec{j},\vec{m},\vec{n}}$ as functions of smooth connections. After taking $\epsilon \to 0$ one extends the result to functions of distributional connections and thus ends up with an operator defined on the Hilbert space of LQG.
Classically, we have

$$\Big\{E^\epsilon_k(S), T_{\gamma,\vec{j},\vec{m},\vec{n}}(\{h_e(A)\}_{e\in E(\gamma)})\Big\} = \sum_{e\in E(\gamma)} \left\{E^{a,\epsilon}_k, (h_e)_{AB}\right\} \frac{\partial T_{\gamma,\vec{j},\vec{m},\vec{n}}}{\partial (h_e)_{AB}}. \tag{2.3}$$

Here $(h_e)_{AB}$ denotes the $SU(2)$-holonomy. The regularisation can be implemented by smearing the two surface S into the third dimension, shown in figure 1, so that we get an array of surfaces S_t. The surface associated with $t = 0$ is our original surface S

$$E^\epsilon_k(S) := \frac{1}{2\epsilon} \int\limits_{-\epsilon}^{+\epsilon} dt\, E_k(S_t). \tag{2.4}$$

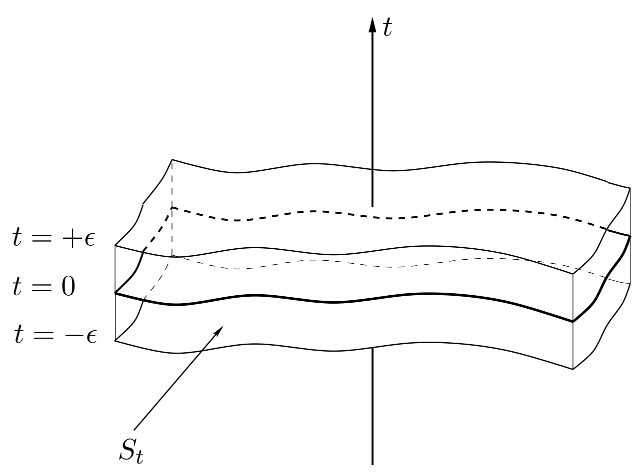

Figure 1: Smearing of the surface S into the third dimension. We obtain an array of surfaces S_t labelled by the parameter t with $t \in \{-\epsilon, +\epsilon\}$. The original surface S is associated with $t = 0$.

In order to derive the action of the flux operator on an arbitrary SNF, we would have to analyse the Poisson bracket among the flux and every possible SNF. Fortunately, it turnes out that each edge belonging to the

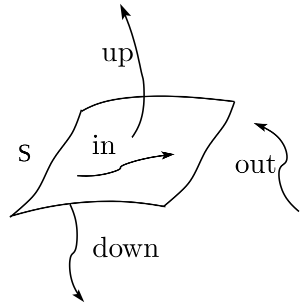

Figure 2: Edges of type up, down, in and out with respect to the surface S.

associated graph γ of $T_{\gamma,\vec{j},\vec{m},\vec{n}}$ can be classified as (i) up, (ii) down, (iii) in and (iv) out with respect to the surface S.(See figure 2 for a graphical illustration.) An arbitrary $T_{\gamma,\vec{j},\vec{m},\vec{n}}$ contains then a particular amount of edeges of each type. Accordingly, if we have the knowledge of the Poisson bracket among the flux and any of these types of edges, we will be able to derive the Poisson bracket among E_k^ϵ and any arbitrary $T_{\gamma,\vec{j},\vec{m},\vec{n}}$. The calculation of the regularised Poisson bracket can be found e.g. in the second reference of [1]. After having removed the regulator we end up with the following action of the flux operator on a SNF $T_{\gamma,\vec{j},\vec{m},\vec{n}}$

$$\widehat{E}_k(S)T_{\gamma,\vec{m},\vec{n}} = \frac{i}{2}\,\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e,S)\left[\frac{\tau_k}{2}\right]_{AB} \frac{\partial T_{\gamma,\vec{j},\vec{m},\vec{n}}(h_{e'})_{e'\in E(\gamma)}}{\partial(h_e)_{AB}}, \tag{2.5}$$

where τ_k is related to the Pauli matrices by $\tau_k := -i\sigma_k$. The sum is taken over all edges of the graph γ associated with $T_{\gamma,\vec{j},\vec{m},\vec{n}}$. The function $\epsilon(e,S)$ can take the values $\{-1,0,+1\}$ depending on the type of edge that is considered. It is +1 for edges of type up, -1 one for down and 0 for edges of type in or out.
By introducing right invariant vector fields X_k^e, defined by $(X_k^e f)(h) := \frac{d}{dt}f(e^{t\tau_k}h)\Big|_{t=0}$, we can rewrite the action of the flux operator as

$$\widehat{E}_k(S)T_{\gamma,\vec{j},\vec{m},\vec{n}} = \frac{i}{4}\,\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e,S)X_k^e\,T_{\gamma,\vec{j},\vec{m},\vec{n}}. \tag{2.6}$$

Note, the right invariant vector fields fulfill the following commutator relations $[X_e^r, X_e^s] = -2\epsilon_{rst}X_e^t$. By means of introducing the self-adjoint right invariant vector field $Y_e^k := -\frac{i}{2}X_e^k$, we achieve commutator relations for Y_e^k that are similar to the one of the angular momentum operators in quantum mechanics $[Y_e^r, Y_e^s] = i\epsilon_{rst}Y_e^t$. Therefore, we also can describe the action of $\widehat{E}_k(S)$ by the action of the self-adjoint right invariant vector field Y_e^k on $T_{\gamma,\vec{j},\vec{m},\vec{n}}$

$$\widehat{E}_k(S)T_{\gamma,\vec{j},\vec{m},\vec{n}} = -\frac{1}{2}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e,S)Y_e^k\,T_{\gamma,\vec{j},\vec{m},\vec{n}}. \tag{2.7}$$

3 Idea of the Alternative Flux Operator

Our starting point will be the Poisson bracket of the Ashtekar-connection A_a^j and the densitised triad E_k^b given by

$$\left\{A_a^j(x), E_k^b(y)\right\} = \delta^3(x,y)\delta_b^a\delta_j^k \tag{3.1}$$

which we take as fundamental. In order to go from the ADM-formalism to the formulation in terms of Ashtekar varibales, one uses a canonical transformation. There exist two possibilities of choosing such a canonical transformation that both lead to the Poisson bracket above. These two possiblities are

$$\begin{array}{lll} \text{I} & A_a^j = \Gamma_a^j + \gamma\,\mathrm{sgn}(\det(e))K_a^j & , \; E_j^a = \frac{1}{2}\epsilon_{krs}\epsilon^{abc}e_b^r\,e_c^s \\ \text{II} & A_a^j = \Gamma_a^j + \gamma K_a^j & , \; E_j^a = \frac{1}{2}\epsilon_{krs}\epsilon^{abc}e_b^r\,e_c^s\,\mathrm{sgn}(\det(e)) \end{array} \tag{3.2}$$

Here, Γ_a^j is the SU(2)-spin connection, $K_{ab} = K_a^j e_b^j$ the extrinsic curvature (when the Gauss constraint holds) and γ the Imirzi-parameter. Recall again the definition of the regularised classical flux $E_k^\epsilon(S)$ in eqn (2.4). Now the idea of defining an alternative regularised flux

$$\widetilde{E}_k^\epsilon(S) := \frac{1}{2\epsilon}\int_{-\epsilon}^{+\epsilon} dt\, \widetilde{E}_k(S_t)\,; \qquad \widetilde{E}_k(S_t) = \int_{S_t} E_k^a\, n_a^{S_t} \tag{3.3}$$

is to express the densitised triad E_k^a in terms of the triads as above. Due to the two possibile canonical transformations, we have also two possibilities in defining an alternative densitised triad

$$E_k^a = \left\{ \begin{array}{ll} \det(e) e_k^a & = \frac{1}{2}\epsilon_{krs}\epsilon^{abc} e_b^r\, e_c^s \\ \sqrt{\det(q)} e_k^a & = \frac{1}{2}\epsilon_{krs}\epsilon^{abc} \underbrace{\mathrm{sgn}(\det(e))}_{=:\,\mathcal{S}}\, e_b^r\, e_c^s \end{array} \right\} =: \left\{ \begin{array}{l} E_k^{a,I} \\ E_k^{a,II} \end{array} \right\}, \tag{3.4}$$

where e_a^j is the cotriad related to the intrinsic metric as $q_{ab} = e_a^j e_b^j$. From now on we will use $E_k^{a,I}$ and $E_k^{a,II}$, respectively for the two cases.
So, instead of quantising the densitised triad directly, we could use the above classical identities, quantize them via the Poisson bracket identity in eqn (4.1) and check whether both quantisation procedures are consistent.
The main difference between these two definitions is basically a signum factor which we will denote by $\mathcal{S}$. From the mathematical point of view both definitions in eqn (3.3) are equally viable, thus we will keep both possibilities and emphasise the differences that occur when we choose one or the other definition. Notice however that $\det(E^I) = \det(e)^2 \geq 0$ gives an anholonomic constraint which appears to be inconsistent with the definition of $\widehat{E}_k(S)$ as a derivative operator as in eqn (2.5), as for instance pointed out in [13]. We will see that this is indeed the case.
Inserting the alternative expression for E_k^a into eqn (2.4), we obtain

$$\widetilde{E}_k(S_t) = \left\{ \begin{array}{ll} \int_{S_t} \frac{1}{2}\epsilon_{krs}\epsilon^{abc} e_b^r\, e_c^s n_a^{S_t} & , E_k^{a,I} = \det(e) e_k^a \\ \int_{S_t} \frac{1}{2}\epsilon_{krs}\epsilon^{abc} e_b^r\, \mathcal{S}\, e_c^s n_a^{S_t} & , E_k^{a,II} = \sqrt{\det(q)} e_k^a \end{array} \right\}. \tag{3.5}$$

4 Construction of the Alternative Flux Operator

The strategy for quantising the alternative Flux Operator will be as follows. As a first step we will apply the Poisson bracket identity in order to replace the triads e_b^r by its associated Poisson bracket among the connection A_b^r and the scalar volume funtion $V(R) = \int_R d^3x \sqrt{\det(q)} = \int_R d^3x \sqrt{|\det(E^I)|} = \int_R d^3x \sqrt{|\det(E^{II})|}$. The Poisson bracket identity for the two cases is shown below

$$\{A_b^s, V(R)\} = \left\{ \begin{array}{ll} -\frac{\kappa}{2}\mathcal{S} e_b^s\ , & E_k^{a,I} = \frac{1}{2}\epsilon_{kst}\epsilon^{abc} e_b^s e_c^t \\ -\frac{\kappa}{2} e_b^s\ , & E_k^{a,II} = \frac{1}{2}\epsilon_{kst}\epsilon^{abc}\mathcal{S} e_b^s e_c^t \end{array} \right\}. \tag{4.1}$$

This relation is different for the two cases, because in deriving this relation we have to use the definition of the densitised triad E_k^a in terms of the triads e_b^s which is different for case I and case II. The difference between $E_k^{a,I}$ and $E_k^{a,II}$ is again a signum factor $\mathcal{S}$. Going back to eqn (3.5) we note that there is no $\mathcal{S}$ in $E_k^{a,I}$. Since we have to replace two triads e_b^r, e_c^s by Poisson brackets, we get two factors of $\mathcal{S}$ in the case of $E_k^{a,I}$. As $\mathcal{S}^2$ is always one classically (since q_{ab} is non-degenerate), it drops out in this case. In contrast, for $E_k^{a,II}$ we have a signum factor $\mathcal{S}$ occuring in the alternative flux in eqn (3.5), but no $\mathcal{S}$ in the Poisson bracket identity. Accordingly, we get only one $\mathcal{S}$ here, that does not dissapear classically, because it can take the (constant) values ± 1. Thus

$$\begin{aligned}\widetilde{E}^{I}_{k}(S_t) &= \frac{2}{\kappa^2}\int\limits_{S_t} \epsilon_{krs}\epsilon^{abc}\left\{A^r_b, V(R)\right\}\left\{A^s_c, V(R)\right\} n^{S_t}_a \\ \widetilde{E}^{II}_{k}(S_t) &= \frac{2}{\kappa^2}\int\limits_{S_t} \epsilon_{krs}\epsilon^{abc}\left\{A^r_b, V(R)\right\}\mathcal{S}\left\{A^s_c, V(R)\right\} n^{S_t}_a \end{aligned} \tag{4.2}$$

When later on we replace the classical expression by their corresponding operators, the main difference between $\widetilde{E}^{I}_{k}(S_t)$ and $\widetilde{E}^{II}_{k}(S_t)$ will be a so called signum operator $\widehat{\mathcal{S}}$. Before, we have to replace the connections A^r_b by holonomies since A^r_b cannot be promoted to well defined operators in the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL}$. Hence, we choose a partition $\mathcal{P}_t$ of each surface S_t into small squares of area ϵ'^2. In the limit where ϵ' is small enough we are allowed to replace the connection A^r_b along the edge e_I by its associated holonomy $h(e_I)$. The partition is shown in figure (3). Usually this is done for holonomies in the fundamental

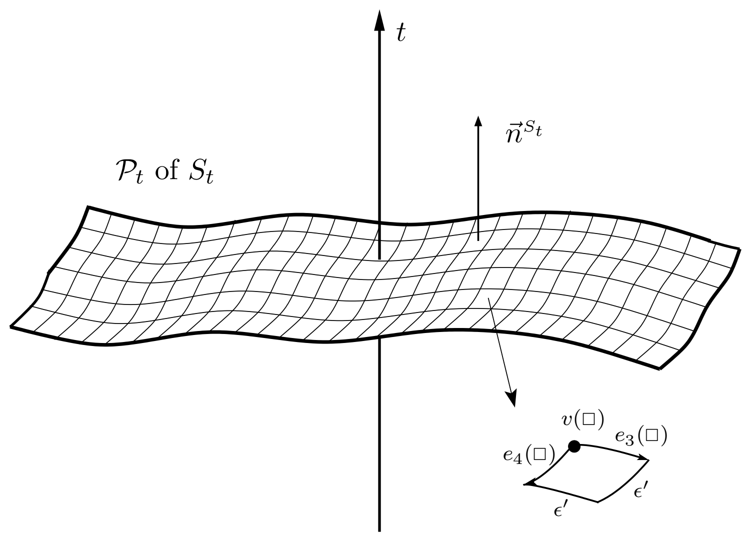

Figure 3: Partition $\mathcal{P}_t$ of the surface S_t into small squares with an parameter edge length ϵ'.

representation of 1/2. But, as we want to keep our construction of the alternative flux operator as general as possible and to study the effect of factor ordering ambiguities we will consider holonomies with arbitrary representation weights ℓ. The corresponding relation between the connection integrated along the edge e_I, denoted by A^r_I from now on, and the associated holonomy is given by

$$\left\{A^r_I(\Box), V(R_{v(\Box)})\right\}\frac{1}{2}\pi_\ell(\tau_r) + o(\epsilon'^2) = +\pi_\ell(h_{e_I})\left\{\pi_\ell(h^{-1}_{e_I}), V(R_{v(\Box)})\right\}, \tag{4.3}$$

whereby we indicate a representation with weight ℓ by π_ℓ.
Considering eqn (4.3), we end up with the following classical identity for ${}^{(\ell)}\widetilde{E}^{I}_{k}(S_t)$

$$\begin{aligned}{}^{(\ell)}\widetilde{E}^{I}_{k}(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\epsilon_{krs}\frac{4}{\kappa^2}\left\{A^r_3(\Box), V(R_{v(\Box)})\right\}\left\{A^s_4(\Box), V(R_{v(\Box)})\right\} \\ &= \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \\ &\qquad \mathrm{Tr}\left(\pi_\ell(h_{e_3(\Box)})\left\{\pi_\ell(h^{-1}_{e_3(\Box)}), V(R_{v(\Box)})\right\}\pi_\ell(\tau_k)\pi_\ell(h_{e_4(\Box)})\left\{\pi_\ell(h^{-1}_{e_4(\Box)}), V(R_{v(\Box)})\right\}\right)\end{aligned} \tag{4.4}$$

and in the case of ${}^{(\ell)}\widetilde{E}_k^{II}(S_t)$ with

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}_k^{II}(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{krs}\frac{4}{\kappa^2}\left\{A_3^r(\square),V(R_{v(\square)})\right\}\mathcal{S}\left\{A_4^s(\square),V(R_{v(\square)})\right\} \\
&= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \\
&\qquad \mathrm{Tr}\left(\pi_\ell(h_{e_3(\square)})\left\{\pi_\ell(h^{-1}_{e_3(\square)}),V(R_{v(\square)})\right\}\pi_\ell(\tau_k)\mathcal{S}\pi_\ell(h_{e_4(\square)})\left\{\pi_\ell(h^{-1}_{e_4(\square)}),V(R_{v(\square)})\right\}\right)
\end{aligned}
\tag{4.5}
$$

Here we used $\mathrm{Tr}\left(\pi_\ell(\tau_r\pi_\ell(\tau_k))\pi_\ell(\tau_s)\right) = -\frac{4}{3}\ell(\ell+1)(2\ell+1)$ that is derived in appendix A of [19].
Now, in LQG there exist two volume operators, one introduced by Rovelli and Smolin in 1994 ($\widehat{V}_{RS}$) [5] and another one published in 1995 by Ashtekar and Lewandowski ($\widehat{V}_{AL}$) [6]. Hence, we have actually in each case two different possibilities for the Poisson bracket, because we could either use $\widehat{V}_{RS}$ or $\widehat{V}_{AL}$. Thus, case I as well as case II splits into two different versions of alternative fluxes

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}_k^{I}(S_t) &\to {}^{(\ell)}\widetilde{E}_k^{I,AL}(S_t),\,{}^{(\ell)}\widetilde{E}_k^{I,RS}(S_t) \\
{}^{(\ell)}\widetilde{E}_k^{II}(S_t) &\to {}^{(\ell)}\widetilde{E}_k^{II,AL}(S_t),\,{}^{(\ell)}\widetilde{E}_k^{II,RS}(S_t)
\end{aligned}
\tag{4.6}
$$

From now on we will use the notation above for the four different cases. Before we apply canonical quantisation, we want to discuss the two volume operators and their differences a bit more in detail.

4.1 The two Volume Operators in LQG

4.1.1 The Volume Operator $\widehat{V}_{RS}$ of Rovelli and Smolin

The idea that the volume operator acts only on vertices of a given graph was first mentioned in [17]. The first version of a volume operator can be found in [5] and is given by

$$
\begin{aligned}
\widehat{V}(R)_\gamma &= \int_R d^3p\,\widehat{V}(p)_\gamma \\
\widehat{V}(p)_\gamma &= \ell_p^3\sum_{v\in V(\gamma)}\delta^{(3)}(p,v)\widehat{V}_{v,\gamma} \\
\widehat{V}_{v,\gamma}^{RS} &= \sum_{I,J,K}\sqrt{\left|\frac{i}{8}C_{reg}\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K}\right|}.
\end{aligned}
\tag{4.7}
$$

Here we sum over all triples of edges at the vertex $v\in V(\gamma)$ of a given graph γ. $\widehat{V}_{RS}$ is not sensitive to the orientation of the edges, thus also linearly dependent triples have to be considered in the sum. Moreover, we introduced a constant $C_{reg}\in\mathbb{R}^+$ that we will keep arbitrary for the moment and that is basically fixed by the particular regularisation scheme one chooses. When working with the volume operator we want to select physically relevant gauge invariant states properly. Hence, it is convenient to express our abstract angular momentum states in terms of the recoupling basis. The following identity [12] holds

$$
\frac{1}{8}\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K} = \frac{1}{4}[Y^2_{IJ},Y^2_{JK}] =: \frac{1}{4}q^{Y}_{IJK},
\tag{4.8}
$$

where $Y_{IJ} := Y^k_{e_I}+Y^k_{e_J}$ and $Y^k_{e_I}$ denotes the self-adjoint vector field $Y^k_{e_I} := -\frac{i}{2}X^k_{e_I}$.
Consequently, we get

$$
\widehat{V}(R)^{Y,RS}_\gamma|\,J\,M\,;\,M'\!> = \ell_p^3\sum_{v\in V(\gamma)\cap R}\underbrace{\sum_{I<J<K}\sqrt{\left|\frac{3!i}{4}C_{reg}\widehat{q}^{Y}_{IJK}\right|}}_{\widehat{V}^{RS}_{v,\gamma}}\;|\,J\,M\,;\,M'\!>.
\tag{4.9}
$$

The additional factor of 3! is due to the fact that we sum only over ordered triples $I < J < K$ now. The way to calculate eigenstates and eigenvalues of $\widehat{V}$ is as follows. Let us introduce the operator $\widehat{Q}^{Y,RS}_{v,IJK}$ as

$$\widehat{Q}^{Y,RS}_{v,IJK} := \ell_p^6 \frac{3!i}{4} C_{reg} \widehat{q}^{Y}_{IJK} \tag{4.10}$$

As a first step we have to calculate the eigenvalues and corresponding eigenstates for $\widehat{Q}^{Y,RS}_{v,IJK}$. If for example $|\phi>$ is an eigenstate of $\widehat{Q}^{Y,RS}_{v,IJK}$ with corresponding eigenvalue λ, then we obtain $\widehat{V}|\phi>= \sqrt{|\lambda|}|\phi>$. Consequently, we see that while $\widehat{Q}^{Y,RS}_{v,IJK}$ can have positive and negative eigenvalues, $\widehat{V}$ has only positive ones. Furthermore, if we consider the eigenvalues $\pm\lambda$ of $\widehat{Q}^{Y,RS}_{v,IJK}$ and the corresponding eigenstate $|\phi_{+\lambda}>, |\phi_{-\lambda}>$, we notice that these eigenvalues will be degenerate in the case of the operator $\widehat{V}$, as $\sqrt{|+\lambda|} = \sqrt{|-\lambda|}$.

4.1.2 The Volume Operator $\widehat{V}_{AL}$ of Ashtekar and Lewandowski

Another version of the volume operator which differs by the chosen regularisation scheme was defined in [6]

$$\widehat{V}(R)^{Y,AL}_{\gamma} |\, J\, M\, ;\, M'> = \ell_p^3 \sum_{v \in V(\gamma)\cap R} \underbrace{\sqrt{\left| \frac{3!i}{4} C_{reg} \sum_{I<J<K} \epsilon(e_I, e_J, e_K)\, \widehat{q}^{Y}_{IJK} \right|}}_{\widehat{V}^{AL}_{v,\gamma}} \;|\, J\, M\, ;\, M'> . \tag{4.11}$$

The major difference between $\widehat{V}_{AL}$ and $\widehat{V}_{RS}$ is the factor $\epsilon(e_I, e_J, e_K)$ that is sensitive to the orientation of the tangent vectors of the edges $\{e_I, e_J, e_K\}$. $\epsilon(e_I, e_J, e_K)$ is $+1$ for right handed, -1 for left handed and 0 for linearly dependent triples of edges. In the case of $\widehat{V}_{AL}$ it is convinient to introduce an operator $\widehat{Q}^{Y,AL}_{v}$ that is defined as the expression that appears inside the absolute value under the square root in $\widehat{V}^{AL}_{v,\gamma}$

$$\widehat{Q}^{Y,AL}_{v} := \ell_p^6 \frac{3!i}{4} C_{reg} \sum_{I<J<K} \epsilon(e_I, e_J, e_K)\, \widehat{q}^{Y}_{IJK} \tag{4.12}$$

By comparing eqn (4.9) with (4.11) we notice that another difference between $\widehat{V}_{RS}$ and $\widehat{V}_{AL}$ is the fact that for the first one, we have to sum over the triples of edges outside the square root, while for the latter one, we sum inside the absolute value under the square root. Apart from the difference of the sign factor, the difference in the summation will play an important role later on.

4.2 Factor Ordering Ambiguities

The discussion in this section is valid for all four different version of the alternative flux, hence we will neglect the labels I,II,RS,AL here.
The meaning of the limit when the regularisation parameter tends to zero, i.e. $\epsilon \to 0$ in combination with the limit of the partition $\mathcal{P}_t$ when the edge length parameter $\epsilon' \to 0$ is explaind in detail in [19] in section 4.3. It is the same limit as taken for the usual flux operator sketched between eqn (2.2) and eqn (2.3). We have e.g. for those S which intersects γ only in an edge of type up schematically

$$\begin{aligned} \widetilde{E}_k(S)T &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} dt \sum_I < T^I_t \,|\, \widetilde{E}_k(S_t) \,|\, T > T^I_t \\ &= \frac{1}{2} \lim_{\epsilon\to 0} \sum_I < T^I_\epsilon \,|\, \widetilde{E}_k(S_\epsilon) \,|\, T > T^I_\epsilon, \end{aligned} \tag{4.13}$$

where T^I_t are the SNW states that contribute to $\widetilde{E}_k(S_T)T$. $< T^I_\epsilon \,|\, \widetilde{E}_k(S_t) \,|\, T >$ is actually independent of ϵ and $T^I_\epsilon \to T^I_0$ as a function of smooth connections which can then be extended to distributional ones.

It turns out that the alternative flux operator can be regularised to such an extent that each plaquette of the partition $\mathcal{P}_t$ of a surface S_t has an intersection with only one single edge of a graph γ associated with an arbitrary given SNF $T_{\gamma,\vec{m},\vec{n}}$. If the considered edge lies completely inside the plaquette of S_t or completely outside (up to sets of dt measure zero if inside) it will lead to a trivial action of the alternative flux operator. This is in full agreement with the usual flux operator. The alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ attaches two additional edges, namely e_3, e_4 to the edge e. The discussion in [19] showed that these additional edges have to lie inside the surface S_t since otherwise the ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ would have only again a trivial action, see also figure 4.2. Furthermore, by attaching the additional edges e_3, e_4, ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ cre-

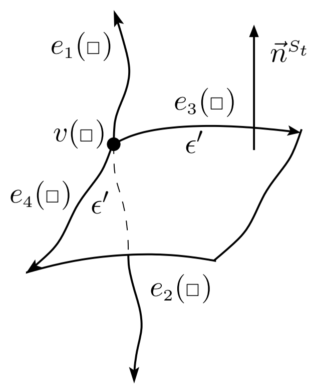

Figure 4: An non-vaishing contribution to the matrix element of ${}^{(\ell)}\widehat{\widetilde{E}}_k$ on a given arbitrary SNF T_s, i.e. $< T_{s'} \,|\, {}^{(\ell)}\widehat{\widetilde{E}}_k(\square) \,|\, T_s >$ can only be achieved if T_s contains edges of type up and/or down, respectively with respect to the surface S_t. Moreover, the edges $e_3(\square), e_4(\square)$ have to be attachted to T_s in this specific way.

ates a new vertex at a certain point of e and thus divides e into an edge e_1 of type up and an edge e_2 of type down with respect to S_t[4]. Accordinly, as for the usual flux operator, the action of the alternative flux operator is totally determined by the action on edges of type up and down with respect to the surface S_t.

Since we are familiar now with the action of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$, we are able to consider the impact of different factor orderings. Going back to the classical identity in eqn (4.4) and eqn (4.5) respectively and considering the results of the discussion above, we know that ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ adds two additional edges e_3, e_4 to a given edge e and devides this edge into an up edge e_1 and an edge of type down e_2 This is illustrated in figure 5. Let us call these SNF that involves the edges e_1, e_2 only $|\beta^{j_{12}}\, {}_{n_{12}}>$. Here we use the so called recoupling basis to express the SNF and j_{12} denotes the total angular momentum to which the two edges e_1 and e_2 couple at their single vertex v, whereas n_{12} is the associated magnetic quantum number. The SNF that results from $|\beta^{j_{12}}\, {}_{n_{12}}>$ by the action of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S_t)$ and that contains four edges $\{e_1, e_2, e_3, e_4\}$ will be denoted by $|\alpha_i^J\, {}_M>$. In this case J is the total angular momentum, M the associated magentic quantum number and i as an index is needed, because with four edges more than one state exists with the same total angular momentum, but different intermediate couplings. For more details see section 5.2 and 6 in [19]. Now, dealing still with the classical expression in eqn (4.4) and eqn (4.5) respectively, we can apply the trace and rearrange the terms in a certain manner, because classically holonomies commute in order to obtain a sensible operator ordering in the quantum theory. If we consider ${}^{(\ell)}\widetilde{E}_k^{I,AL}(S_t)$ and ${}^{(\ell)}\widetilde{E}_k^{II,AL}(S_t)$ that contain $\widehat{V}_{AL}$ we know that due to the sign factor $\epsilon(e_I, e_J, e_K)$ in $\widehat{V}_{AL}$ the action of $\widehat{V}_{AL}$ on linearly dependent triples vanishes. Therefore, it has to be ensured that the two holonomies $\widehat{\pi}_\ell(h_{e_3})_{EA}$ and $\widehat{\pi}_\ell(h_{e_4})_{IG}$ act before $\widehat{V}_{AL}$ does. This restriction reduces the number of possible factor orderings down to a single one.
The situation is different for ${}^{(\ell)}\widetilde{E}_k^{I,RS}(S_t)$ and ${}^{(\ell)}\widetilde{E}_k^{II,RS}(S_t)$, because here $\widehat{V}_{RS}$ is involved, that has an non

[4] The opposite case is also possible of course, but we will restrict our discussion to the first case and emphasize in the following disscussion where exactly the choice of e_1 as a down edge and e_2 of type up will make a difference.

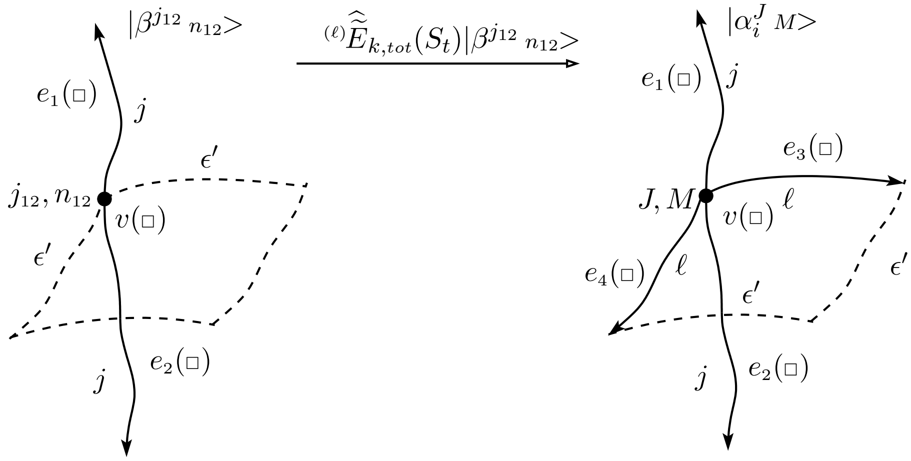

Figure 5: The SNF $|\beta^{j_{12}}{}_{n_{12}}>$ is transformed into an new SNF $|\alpha_i^J{}_M>$ by the action of $^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S_t)$.

trivial action on linearly dependent triples. Consequently, more than one possible factor ordering exists. Let us discuss the number and differences of these orderings later on and restrict ourselves for $^{(\ell)}\widetilde{E}_k^{I,RS}(S_t)$ as well as $^{(\ell)}\widetilde{E}_k^{II,RS}(S_t)$ to the single ordering that is possible for $\widehat{V}_{AL}$, first. It turnes out that for all operators with dt measure 1, the dependence of $\widetilde{E}_k(S_t)$ on t drops out and taking the average $\frac{1}{2\epsilon}\int\limits_{-\epsilon}^{+\epsilon} dt$ becomes trivial resulting in eqn (4.13). We will thus keep the label S_t in what follows, but keep in mind that the t-dependence is trivial.

4.3 Canonical Quantisation

Usually the densitised triads, appearing in the classical flux $E_k(S)$ are quantised as differential operators, while holonomies are quantised as multiplication operators. If we choose the alternative expression $\widetilde{E}_k(S)$ we will instead get the scalar volume $\widehat{V}$ and the so called signum $\widehat{\mathcal{S}}$ operator into our quantised expression. The properties of this $\widehat{\mathcal{S}}$ will be explained in more detail below. Moreover, we have to replace Poisson brackets by commutators, following the replacement rule $\{.,.\} \rightarrow (1/i\hbar)[.,.]$. The detailed derivation of the final operator can be found in section 4 of citeGT.

Clearly, we want the total operator to be self-adjoint, so we will calculate the adjoint of $^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ and define the total and final operator to be $^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S_t) = \frac{1}{2}(\widehat{\widetilde{E}}_k(S_t) + \widehat{\widetilde{E}}_k^\dagger(S_t))$ that is self-adjoint by construction. Hence, the final operator for $\widehat{V}_{RS}$ which we will use through the calculation of this paper is given by

$$
\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{I/II,RS}(S_t) = \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t} \frac{8\,\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI}\\
\Big\{ +\pi_\ell(\epsilon)_{FC}\,[\widehat{\pi}_\ell(h_{e_4})_{FG}]^\dagger \Big[[\widehat{\pi}_\ell(h_{e_3})_{BA}]^\dagger, \widehat{V}_{RS}\Big] \boxed{\widehat{\mathcal{S}}} \Big[\widehat{V}_{RS}, \widehat{\pi}_\ell(h_{e_4})_{IG}\Big]\widehat{\pi}_\ell(h_{e_3})_{EA}\\
-\pi_\ell(\epsilon)_{FB}\,[\widehat{\pi}_\ell(h_{e_3})_{IG}]^\dagger \Big[[\widehat{\pi}_\ell(h_{e_4})_{EA}]^\dagger, \widehat{V}_{RS}\Big] \boxed{\widehat{\mathcal{S}}} \Big[\widehat{V}_{RS}, \widehat{\pi}_\ell(h_{e_3})_{FG}\Big]\widehat{\pi}_\ell(h_{e_4})_{CA}\Big\}
\end{aligned}
\tag{4.14}
$$

whereby we used the identity $\widehat{\pi}_\ell(h_{e_I}^{-1})_{AB} = [\widehat{\pi}_\ell(h_{e_I})_{BA}]^\dagger$, $\pi_\ell(\epsilon)_{AB} = (-1)^{\ell-A}\delta_{A+B=0}$ and the definition of the Planck length $\ell_p^{-4} := (\hbar\kappa)^{-2}$. The box surounding the signum operator $\widehat{\mathcal{S}}$ should indicate that it is contained in the operator in case II, while it is not in case I.

Considering the operator $\widehat{V}_{AL}$, we know that for each commutator only one term will contribute, because otherwise we cannot construct linearly independet triples of edges since $\{e_1, e_2, e_{3/4}\}$ are linearly dependent.

Therefore in the case of $\widehat{V}_{AL}$ we obtain the following final expression

$$
\begin{aligned}
{}^{(\ell)}\widehat{\widehat{E}}{}^{I/II,AL}_{k,tot}(S_t) = \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{8\,\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI} \\
\Big\{ +\pi_\ell(\epsilon)_{FC}\,[\widehat{\pi}_\ell(h_{e_4})_{FG}]^\dagger\,[\widehat{\pi}_\ell(h_{e_3})_{BA}]^\dagger\,\widehat{V}_{AL}\boxed{\widehat{\mathcal{S}}}\widehat{V}_{AL}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA} \\
-\pi_\ell(\epsilon)_{FB}\,[\widehat{\pi}_\ell(h_{e_4})_{IG}]^\dagger\,[\widehat{\pi}_\ell(h_{e_3})_{EA}]^\dagger\,\widehat{V}_{AL}\boxed{\widehat{\mathcal{S}}}\widehat{V}_{AL}\widehat{\pi}_\ell(h_{e_4})_{FG}\widehat{\pi}_\ell(h_{e_3})_{CA}\Big\}. \quad (4.15)
\end{aligned}
$$

Here again for case II the signum operator is inlcuded, whereas in case I it is not.

Next, we want to calculate the matrix elements $<\beta^{j_{12}}\,\widetilde{m}_{12}\,|{}^{(\ell)}\widehat{\widehat{E}}_{k,tot}(S_t)|\,\beta^{j_{12}}\,m_{12}>$ of all four versions ${}^{(\ell)}\widehat{\widehat{E}}{}^{I,AL}_{k,tot}(S_t)$, ${}^{(\ell)}\widehat{\widehat{E}}{}^{I,RS}_{k,tot}(S_t)$, ${}^{(\ell)}\widehat{\widehat{E}}{}^{II,AL}_{k,tot}(S_t)$ and ${}^{(\ell)}\widehat{\widehat{E}}{}^{II,RS}_{k,tot}(S_t)$ of the new flux operator.
The action of the holonomy operators $\widehat{\pi}_\ell(h_{e_I})_{AB}$ on $|\beta^{j_{12}}\,m_{12}>$ can be described in the framework of angular momentum recoupling theory with the powerful tool of Clebsch-Gordan-Coefficients (CGC). The correspondence between the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL}$ and the abstract angular momentum Hilbert space is discussed in section 5.1 in [19]. Hence, the matrix element will roughly speaking have the following structure

$$
\begin{aligned}
&\sum_{\tilde{J},\widetilde{M},J,M}\sum_{\tilde{a}_3,\tilde{m}_{\tilde{a}_3},a_3,m_3} <\beta^{j_{12}}\,\widetilde{m}_{12}\,|{}^{(\ell)}\widehat{\widehat{E}}_{k,tot}(S_t)|\,\beta^{j_{12}}\,m_{12}> \\
&\propto C(j_{12},\ell;\tilde{a}_3\,m_{\tilde{a}3})C^*(\tilde{j}_{12},\ell;a_3\,m_{a_3})C^*(\tilde{a}_3,\ell;\tilde{J}\,\widetilde{M})C(a_3,\ell;J\,M)\ \ <\alpha_i^{\tilde{J}}\,\widetilde{M}\,|\,\widehat{O}\,|\,\alpha_j^J\,M> \qquad (4.16)
\end{aligned}
$$

Here $C(j_1,j_2;J\,M)$ denotes the CGC that we get if we couple the angular momenta j_1 and j_2 to a resulting angular momentum J with corresponding magnetic quantum number M and $\widehat{O}$ is an operator including the operator $\widehat{V}_{RS}$ and $\widehat{V}_{AL}$ respectively and $\widehat{\mathcal{S}}$. The details can be found in section 5.1 of [19]. Due to the symmetry properties of the alternative flux operator that are analysed in section 5.2 of [19], the only total angular momenta that will contribute to the final matrix element are $J=0,1$. (See section 5.2 of [19] for more explanations concerning this point.) Since the physically relevant states are gauge-invariant we choose $j_{12}=0$. The behaviour of ${}^{(\ell)}\widehat{\widehat{E}}_{k,tot}(S_t)$ under gauge transformations, that is discussed in section 5.3 of [19] leads to the restriction $\tilde{j}_{12}=1$. Consequently, at the end of the day we get the following form of the matrix element of ${}^{(\ell)}\widehat{\widehat{E}}_{k,tot}(S_t)$

$$
\begin{aligned}
&<\beta^1\,\widetilde{m}_{12}\,|{}^{(\ell)}\widehat{\widehat{E}}_{k,tot}(S_t)|\,\beta^0\,0> \\
&= -\lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{8\,\ell_p^{-4}(-1)^{3\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\sum_{B,C,F=-\ell}^{+\ell}\Big\{\pi_\ell(\tau_k)_{CB} \\
&\quad\Big[+(-1)^{-F}\delta_{F+C,0}\sqrt{2\ell+1}\,\delta_{\widetilde{m}_{12}+B+F,0} \\
&\qquad <1\,\widetilde{m}_{12}\,;\ell\,B\,|\,\ell\,\widetilde{m}_{12}{+}B> <\ell\,\widetilde{m}_{12}{+}B\,;\ell\,F\,|\,0\,0> \\
&\qquad <\alpha_2^0\,M=\widetilde{m}_{12}{+}B{+}F\,;\widetilde{m}'_1\,\widetilde{m}'_2\,|\,\widehat{O}_1\,|\,\alpha_1^0\,M=0\,;m'_1\,m'_2> \\
&\quad-(-1)^{-F}\delta_{F+B,0}\,\delta_{C+F,\widetilde{m}_{12}} \\
&\qquad <0\,0\,;\ell\,C|\ell\,C> <\ell\,C\,;\ell\,F|\,1\,C{+}F> \\
&\qquad\Big[+\frac{\sqrt{2\ell-1}}{\sqrt{3}}\ <\alpha_2^1\,M=\widetilde{m}_{12}\,;\widetilde{m}'_1\,\widetilde{m}'_2\,|\,\widehat{O}_2\,|\alpha_1^1\,M=C{+}F\,;m'_1\,m'_2> \\
&\qquad-\frac{\sqrt{2\ell+1}}{\sqrt{3}}\ <\alpha_3^1\,M=\widetilde{m}_{12}\,;\widetilde{m}'_1\,\widetilde{m}'_2\,|\,\widehat{O}_2\,|\alpha_1^1\,M=C{+}F\,;m'_1\,m'_2> \\
&\qquad+\frac{\sqrt{2\ell+3}}{\sqrt{3}}\ <\alpha_4^1\,M=\widetilde{m}_{12}\,;\widetilde{m}'_1\,\widetilde{m}'_2\,|\,\widehat{O}_2\,|\alpha_1^1\,M=C{+}F\,;m'_1\,m'_2>\ \Big]\Big\}, \qquad (4.17)
\end{aligned}
$$

whereby the four different cases are encoded in the operators $\widehat{O}_1$ and $\widehat{O}_2$. Explicitly, we have

$$\begin{aligned}
\widehat{O}_1^{I,AL} &= \widehat{V}_{AL}^2 \\
\widehat{O}_1^{I,RS} &= \widehat{V}_{q134}^2 + \widehat{V}_{q234}^2 + \widehat{V}_{q134}\widehat{V}_{q234} + \widehat{V}_{q234}\widehat{V}_{q134} + \widehat{V}_{q134}\widehat{V}_{q123} + \widehat{V}_{q124}\widehat{V}_{q134} + \widehat{V}_{q234}\widehat{V}_{q123} + \widehat{V}_{q124}\widehat{V}_{q234} + \widehat{V}_{q124}\widehat{V}_{q123} \\
\widehat{O}_2^{I,AL} &= \widehat{V}_{AL}^2 \\
\widehat{O}_2^{I,RS} &= \widehat{V}_{q134}^2 + \widehat{V}_{q234}^2 + \widehat{V}_{q234}\widehat{V}_{q134} + \widehat{V}_{q134}\widehat{V}_{q234} + \widehat{V}_{q123}\widehat{V}_{q134} + \widehat{V}_{q134}\widehat{V}_{q124} + \widehat{V}_{q123}\widehat{V}_{q234} + \widehat{V}_{q234}\widehat{V}_{q124} + \widehat{V}_{q123}\widehat{V}_{q124} \\
\widehat{O}_1^{II,AL} &= \widehat{V}_{AL}\widehat{\mathcal{S}}\widehat{V}_{AL} \\
\widehat{O}_1^{II,RS} &= +\widehat{V}_{q134}\widehat{\mathcal{S}}\widehat{V}_{q134} + \widehat{V}_{q234}\widehat{\mathcal{S}}\widehat{V}_{q234} + \widehat{V}_{q134}\widehat{\mathcal{S}}\widehat{V}_{q234} + \widehat{V}_{q234}\widehat{\mathcal{S}}\widehat{V}_{q134} + \widehat{V}_{q134}\widehat{\mathcal{S}}\widehat{V}_{q123} + \widehat{V}_{q124}\widehat{\mathcal{S}}\widehat{V}_{q134} + \widehat{V}_{q234}\widehat{\mathcal{S}}\widehat{V}_{q123} \\
&\quad +\widehat{V}_{q124}\widehat{\mathcal{S}}\widehat{V}_{q234} + \widehat{V}_{q124}\widehat{\mathcal{S}}\widehat{V}_{q123} \\
\widehat{O}_2^{II,AL} &= \widehat{V}_{AL}\widehat{\mathcal{S}}\widehat{V}_{AL} \\
\widehat{O}_1^{II,RS} &= +\widehat{V}_{q134}\widehat{\mathcal{S}}\widehat{V}_{q134} + \widehat{V}_{q234}\widehat{\mathcal{S}}\widehat{V}_{q234} + \widehat{V}_{q234}\widehat{\mathcal{S}}\widehat{V}_{q134} + \widehat{V}_{q134}\widehat{\mathcal{S}}\widehat{V}_{q234} + \widehat{V}_{q123}\widehat{\mathcal{S}}\widehat{V}_{q134} + \widehat{V}_{q134}\widehat{\mathcal{S}}\widehat{V}_{q124} + \widehat{V}_{q123}\widehat{\mathcal{S}}\widehat{V}_{q234} \\
&\quad +\widehat{V}_{q234}\widehat{\mathcal{S}}\widehat{V}_{q124} + \widehat{V}_{q123}\widehat{\mathcal{S}}\widehat{V}_{q124},
\end{aligned} \tag{4.18}$$

whereby we used the notation $\widehat{V}_{qIJK}$ for $\widehat{V}_{RS} = \widehat{V}_{q134} + \widehat{V}_{q234} + \widehat{V}_{q124} + \widehat{V}_{q123}$ when only the triple $\{e_I, e_J, e_K\}$ contributes to $\widehat{V}_{RS}$. The derivation of the various versions of $\widehat{O}_1, \widehat{O}_2$ can be found in [19]. These 8 versions result from taking into account a) the two volume operators, b) the signum operator $\mathcal{S}$ or not and c) the adjoint or not in eqn (4.17).

Before we will discuss our results in the next sections, we want to explain a bit more in detail what we mean by say the consistency check is affirmative or not. We managed to implement an alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S_t)$ in four different versions ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{I,AL}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{I,RS}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{II,AL}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{II,RS}(S_t)$. The usual flux operator is quantised as a differential operator in the standard way. The alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S_t)$ is quantised via the Poisson bracket identity in eqn (4.1) analogous to the quantisation of the Hamiltonian Constraint. If these two methods of quantisation are mathematically consistent with each other, the action of $\widehat{E}_k(S)$ and the one of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S_t)$ should only differ by a constant, namely

$$\widehat{E}_k(S)|\beta^0\, {}_0\!> = C(j,\ell)C_{reg}\,{}^{(\ell)}\widehat{\widetilde{E}}^{0}_{k,tot}(S)|\beta^0\, {}_0\!> = C(j,\ell)C_{reg}\sum_{\widetilde{m}_{12}} <\beta^1\, {}_{\widetilde{m}_{12}}|\,{}^{(\ell)}\widehat{\widetilde{E}}^{0}_{k,tot}(S)\,|\beta^0\, {}_0\!>\, |\beta^1\, {}_{\widetilde{m}_{12}}\!>, \tag{4.19}$$

whereby ${}^{(\ell)}\widehat{\widetilde{E}}^{0}_{k,tot}(S)$ denotes the alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S)$ where C_{reg} has been replaced by 1. The constant $C(j,\ell)$ might depend on the spin labels j,ℓ of the edges, but at least semiclassically the dependence on the spin label j and ℓ must disappear since otherwise the behaviour of $\widehat{E}_k(S)$ and ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}(S)$ in the correspondence limit of large j would disagree. Notice that also a dependence of $\lim_{j\to\infty} C(j,\ell) =: C(\ell)$ on ℓ is inacceptable, because classically the flux is independent of the factor ordering ambiguity ℓ. Moreover this constant will fix the ambiguity in the volume operator C_{reg} that is due to regularisation as we will see later. Hence, $\lim_{j\to\infty} C(j,\ell) = 1/C_{reg} = \text{const}$ as will be discussed more in detail in section 6.1.

5 Case I: Results for ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{I,RS}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{I,AL}(S_t)$

5.1 Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}_{k,tot}^{I,AL}(S_t)$

For technical reasons, we consider only a spin label $\ell = 0.5, 1$, because higher spin labels cannot computed analytically anymore. Fortunately, the main properties of this case already occurs when considering small

ℓ. The detailed calculation in section 6.2 in [19] show

$$< \alpha_2^0 \, M = \widetilde{m}_{12} \, ; \widetilde{m}_1' \, \widetilde{m}_2' \, | \, \widehat{O}_1^{I,AL} \, | \alpha_1^0 \, M = C{+}F \, ; m_1' \, m_2' > \, = 0$$
$$< \alpha_i^1 \, M = \widetilde{m}_{12} \, ; \widetilde{m}_1' \, \widetilde{m}_2' \, | \, \widehat{O}_2^{I,AL} \, | \alpha_1^1 \, M = C{+}F \, ; m_1' \, m_2' > \, = 0, \tag{5.1}$$

where $i = 2, 3, 4$. Going back to eqn (4.17), we note that the vanishing of the matrix elements above has the consequence that the whole matrix element $< \beta^{j_{12}} \, \widetilde{m}_{12} \, | {}^{(\ell)}\widehat{\widetilde{E}}{}^{I,AL}_{k,tot}(S_t) | \, \beta^{j_{12}} \, m_{12} >$ is zero. Since the action of ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,AL}_{k,tot}(S_t)$ on an arbitrary SNF can be derived from exactly this matrix element, we can conlcude that ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,AL}_{k,tot}(S_t)$ is the zero operator. Accordingly, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,AL}_{k,tot}(S_t)$ is not consistent with the usual flux operator.

5.2 Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,RS}_{k,tot}(S_t)$

Also for the operator ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,RS}_{k,tot}(S_t)$ the analogous calculations discussed in section 6.3 of [19] yields only trivial matrix elements

$$< \alpha_2^0 \, M = \widetilde{m}_{12} \, ; \widetilde{m}_1' \, \widetilde{m}_2' \, | \, \widehat{O}_1^{I,RS} \, | \alpha_1^0 \, M = C{+}F \, ; m_1' \, m_2' > \, = 0$$
$$< \alpha_i^1 \, M = \widetilde{m}_{12} \, ; \widetilde{m}_1' \, \widetilde{m}_2' \, | \, \widehat{O}_2^{I,RS} \, | \alpha_1^1 \, M = C{+}F \, ; m_1' \, m_2' > \, = 0, \tag{5.2}$$

Consequently, we can draw the same conlcusion as for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,AL}_{k,tot}(S_t)$ and state that ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,RS}_{k,tot}(S_t)$ is the zero operator and therefore inconsistent with the usual flux operator. Futhermore as either ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,AL}_{k,tot}(S_t)$ nor ${}^{(\ell)}\widehat{\widetilde{E}}{}^{I,RS}_{k,tot}(S_t)$ survive the consitency check, we can rule out, at least for the cases of $\ell = 0.5, 1$, the choice of $E^a_k = \det(e) e^a_k$ on which these operators are based on. To rule out the choice $E^a_k(S_t) = \det(e) e^a_k$ completely, we need to investigate the matrix element for arbitrary representation weights ℓ. For higher values of ℓ the calculation cannot be done analytically any more. However, the results for $\ell = 0.5, 1$ indicate that there is an abstract reason which leads to the vanishing of the matrix elements for *any* ℓ. We were not able to find such an abstract argument yet. However, even if that was not the case and there would be a range of values for ℓ for which not all of the matrix elements would vanish, it is inacceptable that the classical theory is independent of ℓ while the quantum theory strongly depends on ℓ in the correspondence limit of large j.

6 Case II: Results for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{II,RS}_{k,tot}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}{}^{II,AL}_{k,tot}(S_t)$

6.1 Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{II,AL}_{k,tot}(S_t)$

Considering the case of the operator ${}^{(\ell)}\widehat{\widetilde{E}}{}^{II,AL}_{k,tot}(S_t)$, we can read of from eqn (4.18) the expressions $\widehat{O}_1 = \widehat{V}_{AL} \widehat{\mathcal{S}} \widehat{V}_{AL} = \widehat{O}_2$. Since the signum operator $\widehat{\mathcal{S}}$ that corresponds to the classical expression $\mathcal{S} := \mathrm{sgn}(\det(e))$ does not exist in the literature so far, we will in detail explain how the operator $\widehat{\mathcal{S}}$ has to be understood.

6.2 The Signum Operator $\widehat{\mathcal{S}}$

We are dealing now with case II meaning that the densitised triad is given by $E^{a,II}_k = \mathcal{S} \det(e) e^a_k$, where $\mathcal{S} := \det(e)$. Applying the determinant onto $E^{a,II}_k$, we get

$$\det(E) = \mathrm{sgn}(\det(e)) \det(q) \quad \text{with} \quad \det(q) = [\det(e)]^2 \geq 0. \tag{6.1}$$

Therefore, we obtain

$$\mathrm{sgn}(\det(E)) = \mathrm{sgn}(\det(e)) = \mathcal{S}. \tag{6.2}$$

In the following we want to show that $\mathcal{S} = \mathrm{sgn}(\det(E))$ can be identified with the signum of the expression inside the absolute value under the square roots in the definition of the AL-volume. For this purpose let us first discuss this issue on the classical level and afterwards go back into the quantum theory and see how the corresponding operator $\widehat{\mathcal{S}}$ is connected with the operator $\widehat{Q}_v^{AL}$ in eqn (4.12).
In order to do this let us consider eqn (4.5). This equation contains the classical volume $V(R_{v(\Box)})$ where $R_{v(\Box)}$ denotes a region centred around the vertex $v(\Box)$.
The volume of such a cube is given by

$$V(R_{v(\Box)}) = \int\limits_{R_{v(\Box)}} \sqrt{\det(q)} d^3x = \int\limits_{R_{v(\Box)}} \sqrt{|\det(E)|} d^3x, \tag{6.3}$$

where we used $\det(q) = |\det(E)|$ from eqn (6.1). Introducing a parametrisation of the cube now, we end up with

$$V(R_{v(\Box)}) = \int\limits_{[-\frac{\epsilon'}{2},+\frac{\epsilon'}{2}]^3} \left|\frac{\partial X^I(u)}{\partial u_J}\right| \sqrt{|\det(E)(u)|} d^3u = \int\limits_{[-\frac{\epsilon'}{2},+\frac{\epsilon'}{2}]^3} |\det(X)| \sqrt{|\det(E)(u)|} d^3u. \tag{6.4}$$

In order to be able to carry out the integral we choose the cube $R_{v(\Box)}$ small enough and thus, the volume can be approximated by

$$V(R_{v(\Box)}) \approx \epsilon'^3 \left|\det(\frac{\partial X}{\partial u})(v)\right| \sqrt{|\det(E)(v)|}. \tag{6.5}$$

Using the definition of $\det(E) = \frac{1}{3!}\epsilon_{abc}\epsilon^{jkl}E^a_jE^b_kE^c_l$, we can rewrite eqn (6.3) as

$$V(R_{v(\Box)}) = \int\limits_{\Box} \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{jkl}E^a_jE^b_kE^c_l\right|} d^3x \tag{6.6}$$

If we again choose $R_{v(\Box)}$ small enough and define the square surfaces of the cube as S^I, we can re-express the volume integral over the densitised triads in terms of their corresponding electric fluxes through the surfaces S^I

$$V(R_{v(\Box)}) \approx \sqrt{\left|\frac{1}{3!}\epsilon_{IJK}\epsilon^{jkl}E_j(S^I)E_k(S^J)E_l(S^K)\right|}. \tag{6.7}$$

The flux through a particular surfaces S^I is defined as

$$E_j(S^I) = \int\limits_{S^I} E^a_j n^{S^I}_a \qquad n^{S^I}_a = \frac{1}{2}\epsilon^{IJK}\epsilon_{abc}X^b_{,u_J}X^c_{,u_K}\Big|_{n^I=0}. \tag{6.8}$$

Here $n^{S^I}_a$ denotes the conormal vector associated with the surface S^I. Regarding eqn (6.7) we realise that inside the absolute value in eqn (6.7) appears exactly the definition of $\det(E_j(S^I))$. Therefore we get

$$V(R_{v(\Box)}) \approx \sqrt{\left|\det(E_j(S^I))\right|}. \tag{6.9}$$

On the other hand, by taking advantage of the fact that the surfaces S^I are small enough so that the integral can be approximated by the value at the vertex times the size of the surface itself, we obtain for $\det(E_j(S^I))$

$$\begin{aligned}\det(E_j(S^I)) &\approx \det(E^a_j(v)n^{S^I}_a(v)\epsilon'^2)\\ &= \det(E^a_j(v))\det(n^{S^I}_a(v))\epsilon'^6\\ &= \det(E(v))\det(n^{S^I}_a(v))\epsilon'^6.\end{aligned} \tag{6.10}$$

If we consider the definition of the normal vector in eqn (6.8),we can show the following identity

$$\det(n_a^{S^I}) = \det(X)^3 \det(X^{-1}) = \frac{\det(X)^3}{\det(X)} = \det(X)^2. \tag{6.11}$$

Inserting eqn (6.11) back into eqn (6.10) we have

$$\det(E_j(S^I)) \approx \det(E(v))[\det(X(v))]^2 \epsilon'^6 \tag{6.12}$$

and can conclude that eqn (6.9) is consistent with the usual definition of the volume in eqn (6.5).
Since we want to identify $\mathcal{S} := \text{sgn}(\det(E))$ with the signum that appears inside the absolute value under the square root in the definition of the volume, we can read off from eqn (6.9), that we still have to show $\text{sgn}(\det(E)) = \text{sgn}(\det(E_j(S^I)))$. However, this can be done by means of eqn (6.12)

$$\begin{aligned} \text{sgn}(\det(E_j(S^I))) &\approx \text{sgn}(\det(E(v))[\det(X(v))]^2 \epsilon'^6) \\ &= \text{sgn}(\det(E(v)))\,\text{sgn}([\det(X(v))]^2)\,\text{sgn}(\epsilon'^6) \\ &= \text{sgn}(\det(E(v))). \end{aligned} \tag{6.13}$$

Consequently, we must identify $\mathcal{S}$ with the signum that appears inside the absolute value under the square root in the definition of the volume V in the classical theory, because it was precisely the expression $\det(E_j(S_I))$ that was used in the construction of the volume operator, defined as the square root of absolute value of $\det(E)$. In the quantum theory, we introduced the operator $\widehat{Q}_v^{AL}$ in eqn (4.12), which is basically the expression inside the absolute value in the definition of the volume operator. Hence, it can be seen as the squared version of the volume operator that additionally contains information about the signum of the expression inside the absolute values. Consequently, we have the operator identitity $\widehat{Q}_v^{AL} = \widehat{V}_{AL}\widehat{\mathcal{S}}\widehat{V}_{AL}$. Now we will be left with the task to calculate particular matrix elements for $\widehat{Q}_v^{AL}$ which can be done by means of the formula derived in [16].

6.3 Calculations for ${}^{(\ell)}\widehat{\tilde{E}}^{II,AL}_{k,tot}(S_t)$

One big advantage that comes along with the operator identity $\widehat{Q}_v^{AL} = \widehat{V}_{AL}\widehat{\mathcal{S}}\widehat{V}_{AL}$ is that diagonalisation of the operator $\widehat{Q}_v^{AL}$ is no longer necessary as it was in case I for $\widehat{V}_{AL}^2$, because the operator ${}^{(\ell)}\widehat{\tilde{E}}^{II,AL}_{k,tot}(S_t)$ contains only particular matrix elements of $\widehat{Q}_v^{AL}$ that can be exactly calculated, even for arbitrary ℓ, by means of the tools developed in [16]. The details of this calculation can be found in [19] as well as the corresponding matrix elements of the usual flux operator $\widehat{E}_k(S)$. If we compare the results of the usual flux operator with the one of ${}^{(\ell)}\widehat{\tilde{E}}^{II,AL}_{k,tot}(S_t)$ we can judge whether ${}^{(\ell)}\widehat{\tilde{E}}^{II,AL}_{k,tot}(S_t)$ leads to a result consistent with the usual flux operator. It transpires

$${}^{(\ell)}\widehat{\tilde{E}}^{II,AL}_{k,tot}(S)|\beta^0{}_0\rangle = 3!8C_{reg}\widehat{E}_k(S)|\beta^0{}_0\rangle \tag{6.14}$$

Therefore the two operators differ only by a positive integer constant. As there is still the regularisation constant C_{reg} in the above equation we can now fix it by requiring that both operators do exactly agree with each other. In fact there is no other choice than exact agreement because the difference would be a global constant which does not decrease as we take the corresponding limit of large quantum numbers j. Thus, we can remove the regularisation ambiguity of the volume operator in this way and choose C_{reg} to be $C_{reg} := \frac{1}{3!8} = \frac{1}{48}$.
This is exactly the value of C_{reg} that was obtained in [6] by a completely different argument. Thus the geometrical interpretation of the value we have to choose for C_{reg} is perfectly provided[5].
Note that the consistency check holds in the full theory and not only in the semiclassical sector. Consequently, the operator ${}^{(\ell)}\widehat{\tilde{E}}^{II,AL}_{k,tot}(S_t)$ is consistent with the usual flux operator.

[5] The factor $8 = 2^3$ comes from the fact that during the regularisation one integrates a product of 3 $\delta-$distributions on $\mathbb{R}$ over $\mathbb{R}^+$ only. The factor $6 = 3!$ is due to the fact that one should sum over ordered triples of edges only.

6.4 Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$

Now, considering the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$ things look differently. Here, a quantisation that is consistent with $\widehat{V}_{RS}$ of the signum operator $\widehat{\mathcal{S}}$ cannot be found due to the simple reason that $\widehat{V}_{RS}$ in contrast to $\widehat{V}_{AL}$ is a sum of single square roots. Hence, there is no origin for a global sign as it was in the case of $\widehat{V}_{AL}$. (See also section 6.6.1 in [19].) In retrospect there is a simple argument why the only possibility ${}^{(\ell)}\widehat{\widetilde{E}}^{I,RS}_{k,tot}(S_t)$ (since ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$ does not exist) is ruled out without further calculation: Namely, the lack of a factor of orientation in $\widehat{V}_{RS}$, like $\epsilon(e_I,e_J,e_K)$ in $\widehat{V}_{AL}$, leads to the following basic disagreement with the usual flux operator: Suppose we had chosen the orientation of the surface S in the opposite way. Then the type of the edge e switches between up and down and similarly for e_1,e_2. Then, the result of the usual flux operator would differ by a minus sign. In the case of $\widehat{V}_{AL}$ we would get this minus sign as well due to $\epsilon(e_I,e_J,e_K)$ contained in $Q^{Al}_v=\widehat{V}_{AL}\widehat{\mathcal{S}}\widehat{V}_{AL}$, whereas a change of the orientation of e_1,e_2 would not modify the result of the alternative flux operator if we used $\widehat{V}_{RS}$ instead, because it is not sensitive to the orientation of the edges. Accordingly, we should stop here and draw the conclusion that ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$ is not consistent with $\widehat{E}_k(S)$.

One might propose to artificially use $\widehat{V}_{RS}\widehat{\mathcal{S}}_{AL}\widehat{V}_{RS}$ for ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$. Note, that we attached the label AL to $\widehat{\mathcal{S}}$ to emphasize that its quantisation is in agreement with $\widehat{V}_{AL}$. This is artificial for the following reason. Suppose we have a classical quantity $A:=\det(E)$ and two different functions $f_1:=\sqrt{|A|}$ and $f_2:=\mathrm{sgn}(A)$. If we want to quantise the functions f_1 and f_2, we do this with the help of the corresponding selfadjoint operator $\widehat{A}$ and obtain due to the spectral theorem $\widehat{f}_1=\sqrt{|\widehat{A}|}$ and $\widehat{f}_2=\mathrm{sgn}(\widehat{A})$. The product of operators $\widehat{V}_{RS}\widehat{\mathcal{S}}_{AL}\widehat{V}_{RS}$ rather corresponds to $\widehat{g}_1=\widehat{A}'$ and $\widehat{g}_2=\mathrm{sgn}(\widehat{A})$, because $\widehat{V}_{RS}$ is quantised with a different regularisation scheme than $\widehat{S}$ is. This would only be justified if $\sqrt{|\widehat{A}|}$ and $\widehat{A}'$ would agree semiclassically. However they do not: If we compare the expressions for V_{AL} and V_{RS} then, schematically, they are related in the following way when restricted to a vertex: $\widehat{V}_{v,AL}=|\frac{3!i}{4}C_{reg}\sum\limits_{I<J<K}\epsilon(e_I,e_J,e_K)\widehat{q}_{IJK}|^{1/2}$ while $\widehat{V}_{v,RS}=\sum\limits_{I<J<K}|\frac{3!i}{4}C_{reg}\widehat{q}_{IJK}|^{1/2}$. It is clear that apart from the sign $\epsilon(e_I,e_J,e_K)$ the two operators can agree at most on states where only one of the $\widehat{q}_{IJK}$ is non vanishing (three or four valent graphs) simply because $\sqrt{|a+b|}\neq\sqrt{|a|}+\sqrt{|b|}$ for generic real numbers a,b.

Nevertheless by analysing ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$ when the artifical operator $\widehat{V}_{RS}\widehat{\mathcal{S}}_{AL}\widehat{V}_{RS}$ is involved, we obtain

$$
{}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S)|\beta^0{}_0> = C(j,\ell)C_{reg}\widehat{E}_k(S)|\beta^0{}_0>, \tag{6.15}
$$

whereby $C(j,\ell)\in\mathbb{R}$ is a constant depending non-trivially on the spin labels j,ℓ in general. One can show that $C(j,\ell)\to C(\ell)$ semiclassically, i.e. in the limit of large j, which is shown in appendix E and discussed in section 6.6.2 of of [19]. Hence ${}^{(\ell)}\widehat{\widetilde{E}}^{II,RS}_{k,tot}(S_t)$, including the artificial operator $\widehat{V}_{RS}\widehat{\mathcal{S}}_{AL}\widehat{V}_{RS}$, would be consistent with $\widehat{E}_k(S)$ within the semiclassical regime of the theory if we chose $C_{reg}=1/C(\ell)$ and if $C(\ell)$ would be a universal constant. Unfortunately, $C(\ell)$ has a non-trivial ℓ -dependence which is inacceptable because it is absent in the classical theory. Moreover, we do not see any geometrical interpretation available for the chosen value of C_{reg} for any value of ℓ in this case. One could possibly get rid of the ℓ-dependence by simply cancelling the linearly dependent triples by hand from the definition of $\widehat{V}_{RS}$. But then the so modified $\widehat{V}'_{RS}$ and $\widehat{V}_{AL}$ would practically become identical on $3-$ and $4-$valent vertices and moreover $\widehat{V}'_{RS}$ now depends on the differentiable structure of Σ. See more about this in the conclusion section.

7 Conclusion

We hope to have demonstrated in this paper that at least certain aspects of LQG are remarkably tightly defined : Certain factor ordering ambiguities turn out to be immaterial, some regularisation schemes can be ruled out as unphysical once and for all. The fact that we can exclude the RS volume from now on as far as the quantum dynamics is concerned should not be viewed as a criticism of [5] at all: The regularisation performed in [5] is manifestly background independent, natural and intuitively very reasonable. It was the first pioneering paper on quantisation of kinematical geometrical operators in LQG and had a deep impact on all papers that followed it. Most of the beautiful ideas spelled out in [5] also entered the regularisation performed in [6] and continue to be valid. One could never have guessed that the regularisation performed in [5] leads to an inconsistent result. It took a decade to develop the necessary technology in order to perform the consistency check provided in this paper. Hence, the fact that we can define the theory more uniquely now should be taken as a strength of the theory and not as a weakness of [5]. Interestingly, the very first paper on the volume operator [17] that we are aware of does look more like $\widehat{V}_{AL}$ rather than $\widehat{V}_{RS}$. We speculate that if one had completed the ideas of [17] one would have ended up with $\widehat{V}_{AL}$ rather than $\widehat{V}_{RS}$. Of course, one could take the viewpoint that the consistency check performed here is unnecessary, that one can just take some definition of the volume operator and not worry about triads. However, as triads prominently enter the dynamics of LQG such a point of view would render the quantum dynamics obsolete. In other words, the dynamics and all other operators which depend on triads such as the length operator [7] or spatially difeomorphism invariant operators forces us to use $\widehat{V}_{AL}$ rather than $\widehat{V}_{RS}$. Finally notice that one of the motivations for choosing $\widehat{V}_{RS}$ rather than $\widehat{V}_{AL}$ is that $\widehat{V}_{RS}$ does not depend on the differentiable structure of Σ. Hence one could use homeomorphism rather than diffeomorphism in order to define "diffeomorphism invariant" states as advertised in [21],[20]. This makes the Hilbert space of such states separable. However, notice that homeomorphisms are not a symmetry of the classical theory. Furthermore, there are other possibilities to arrive at a separable Hilbert space: One can decompose the Hilbert space $\mathcal{H}_{Diff}$ corresponding to the strict diffeomorphisms into an uncountably infinite direct sum of separable Hilbert spaces. In the current proposals for the quantum dynamics all of these mutually isomorphic Hilbert spaces are left invariant. If the theory are left invariant by the Dirac observables, then they would be superselected and any one of them would capture the full physics of LQG.

The technical details of the check, mainly displayed in our companion paper [19], required a substantial computational effort. There are an order of ten crucial stages in the calculation where things could have gone badly wrong. Following the details of our analysis, one sees that all the subtle issues mentioned must be properly taken care of in order to get an even qualitatively correct result. These subtleties involve, among other things:

1. The meaning of the limit as we remove the regulator and to define the alternative flux operator has to be understood in the same way as for the fundamental flux operator, otherwise the alternative flux opertor is identical to zero. This issue is discussed in full detail in section 4.3 of [19]

2. Switching from the spin network basis to the abstract angular momentum basis, discussed elaborately in section 5.1 of [19], has to take care of the precise unitary map between these two representations of the angular momentum algebra, otherwise the two operators differ drastically from each other. This unitary map is not mentioned in the literature because for gauge invariant operators it drops out of the equations. However, for the non − gauge invariant flux it has a large impact.

3. Very unexpectedly, the sign of the determinant of the triad enters the calculation in a crucial way: Classically one would expect it to be neglegible, especially in an orientable manifold. However, had we dropped it from the quantum computation then the alternative flux operator would again vanish identically. It is very pleasing to see that quantisation based on a pseudo vector density is ruled out. This is for the same reason that one cannot implement the momentum operator $i\hbar\frac{d}{dx}$ on $L_2(\mathbb{R}^+, dx)$.

4. A different ordering than the one we chose would have resulted again in the zero operator even in case II.

5. As for the fundamental flux operator one has to first smear the alternative flux operator into the direction transversal to the surface under consideration. For the fundamental flux this implies that

edges of type “in” are not acted on. Without this additional smearing the fundamental flux would be ill defined. For the alternative flux this implies furthermore that the classification of edges into the types up, down, in and out is meaningfull at all. Indeed, one actually can define the alternative flux without the additional smearing. The result is well defined. However, it would differ drastically from the fundamental flux as soon as there is a vertex of valence higher than two of the graph of the spin network state in question within the surface. The additional smearing has the effect that with dt measure one all the vertices within the sufaces S_t are bivalent.

6. Furthermore, the analyses in section 4.3 of [19] shows that without the additional smearing we would be missing the crucial factor 1/2 in eqn (4.13) and our C_{reg} would be off the value found in [6].

7. Just following the tedious calculations step by step as explicitly shown in [19] and evaluating all the CGC’s involved one sees that all the subtle signs have to be there, everything fits only when doing the calculation with 100% accuracy. The calculation is therefore a highly sensitive consistency check.

8. All the ℓ dependence disappears even at small values of j. This is especially surprising because the classical approximation of a connection by a holonomy along a given path becomes worse as we let ℓ grow. Of course in the limit we take paths of infinitesimal length, however, this is done *after* quantization and it could have happened that the quantization is affected by a non trivial ℓ dependence which however should disappear in the limit of large j.

It transpires that the reason for getting a zero operator without the signum operator unveils a so far not appreciated symmetry of the volume operator. It would be desirable to understand the symmetry from a more abstract perspective.
This paper along with our companion paper [19] is one of the first papers that tightens the mathematical structure of full LQG by using the kind of consistency argument that we used here. Many more such checks should be performed in the future to remove ambiguities of LQG and to make the theory more rigid, in particular those connected with the quantum dynamics.

Acknowledgements

It is our pleasure to thank Johannes Brunnemann for countless discussions about the volume operator. We also would like to thank Carlo Rovelli and, especially, Lee Smolin for illuminating discussions.
K.G. thanks the Heinrich-Böll-Stiftung for financial support. This research project was supported in part by a grant from NSERC of Canada to the Perimeter Institute for Theoretical Physics.

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