

On the nature of initial singularities for solutions of the Einstein-Vlasov-scalar field system with surface symmetry

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Abstract

Global existence results in the past time direction of cosmological models with collisionless matter and a massless scalar field are presented. It is shown that the singularity is crushing and that the Kretschmann scalar diverges uniformly as the singularity is approached. In the case without Vlasov matter, the singularity is velocity dominated and the generalized Kasner exponents converge at each spatial point as the singularity is approached.

1 Introduction

In [13] the mathematical study of inhomogeneous cosmological solutions of the Einstein equations coupled to the Vlasov equation and a linear scalar field was begun. A local existence theorem and continuation criteria were proved for solutions with surface symmetry (i.e., spherical, plane or hyperbolic symmetry). In [14] this was used to prove a global existence theorem in the expanding direction. In the case of plane symmetry with only a scalar field it was shown that the solutions are future geodesically complete. In the present paper we turn to the study of the past time direction, i.e., the approach to the initial singularity.

In the case where the matter is given by the Vlasov equation alone certain results on the initial singularity have been obtained by Rein [7]. He showed that if the maximum momentum of the particles remains bounded on any interval where the solution exists then, for spherical or plane symmetry, the solutions can be extended up to $t = 0$. In the case of hyperbolic symmetry he showed

that the corresponding result holds provided the metric function μ (defined below) is initially negative. Weaver [16] showed that the solution always exists up to $t = 0$ in the case of plane symmetry. This is a special case of a result she proved for solutions of the Einstein-Vlasov system with T^2 -symmetry. Tchapanda [12] extended the result from plane symmetry to spherical symmetry and to hyperbolic symmetry under the assumption that μ is initially negative or the assumption that μ is bounded on any interval where the solution exists. His results also allow for a negative cosmological constant.

In the case where the matter is given by a scalar field alone existence up to $t = 0$ for solutions with plane symmetry follows from the work of [10]. In that case it is only necessary to solve a linear hyperbolic equation which is identical to the essential field equation in polarized Gowdy spacetimes. Using results of [5] it was shown that the initial singularity is a curvature singularity (the Kretschmann scalar blows up uniformly there) and a crushing singularity (the mean curvature of the hypersurfaces of constant t blows up uniformly as $t \rightarrow 0$). For matter described by the Vlasov equation alone the results already mentioned can be combined with theorems in [7] to show that $t = 0$ is a crushing singularity where the Kretschmann scalar blows up uniformly. For an open set of initial data belonging to a restricted class the asymptotic behaviour can be described more precisely. For convenience these will be referred to as small data. The generalized Kasner exponents converge uniformly to the values $(2/3, 2/3, -1/3)$. This means that in a certain sense the solution is approximated near the singularity by a Kasner solution with those particular values of the Kasner exponents.

In this paper some of the above results will be generalized to the case where both Vlasov matter and a scalar field are present. It is shown that the solutions exist up to $t = 0$ for spherical and plane symmetry and in the case of hyperbolic symmetry when μ satisfies the restrictions mentioned above. The Kretschmann scalar and the mean curvature of the hypersurfaces of constant t blow up uniformly as $t \rightarrow 0$. Under the assumption that the maximum momentum of any particle in the radial direction decays like a positive power of t as $t \rightarrow 0$ it is possible to obtain further interesting estimates. Unfortunately we did not succeed in generalizing the small data results for the Vlasov equation alone to the present case. For a scalar field alone these estimates allow the detailed asymptotics to be determined for all solutions. The generalized Kasner exponents converge to the values $((1 - a(r))/2, (1 - a(r))/2, a(r))$ for a continuous function $a(r)$. It may be noted that these solutions have the same type of singularity as those constructed from data on the singularity in [1].

Let us recall the formulation of the Einstein-Vlasov-scalar field system as shown in [13] and [15]. We consider a four-dimensional spacetime manifold M , with local coordinates $(x^\alpha) = (t, x^i)$ on which $x^0 = t$ denotes the time and (x^i) the space coordinates. Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On M , a Lorentzian metric g is given with signature $(-, +, +, +)$. We consider a self-gravitating collisionless gas and restrict ourselves to the case where all particles have the same rest mass, normalized to 1, and move forward in time. We denote by (p^α) the momenta of the particles. The conservation

of the quantity $g_{\alpha\beta}p^\alpha p^\beta$ requires that the phase space of the particle is the seven-dimensional submanifold

$$PM = \{g_{\alpha\beta}p^\alpha p^\beta = -1; p^0 > 0\}$$

of TM which is coordinatized by (t, x^i, p^i) . If the coordinates are such that the components g_{0i} vanish then the component p^0 is expressed by the other coordinates via

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ij}p^i p^j}$$

The distribution function of the particles is a non-negative real-valued function denoted by f , that is defined on PM . In addition we consider a massless scalar field ϕ which is a real-valued function on M . The Einstein-Vlasov-scalar field system now reads:

$$\partial_t f + \frac{p^i}{p^0} \partial_{x^i} f - \frac{1}{p^0} \Gamma_{\beta\gamma}^i p^\beta p^\gamma \partial_{p^i} f = 0$$

$$\nabla^\alpha \nabla_\alpha \phi = 0$$

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$T_{\alpha\beta} = - \int_{\mathbb{R}^3} f p_\alpha p_\beta |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p^0} + (\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\nu \phi \nabla^\nu \phi)$$

where $p_\alpha = g_{\alpha\beta}p^\beta$, $|g|$ denotes the modulus of the determinant of the metric $g_{\alpha\beta}$, $\Gamma_{\alpha\beta}^\lambda$ the Christoffel symbols, $G_{\alpha\beta}$ the Einstein tensor, and $T_{\alpha\beta}$ the energy-momentum tensor.

Note that since the contribution of f to the energy-momentum tensor is divergence-free [3], the form of the contribution of the scalar field to the energy-momentum tensor determines the field equation for ϕ .

We refer to [9] for the notion of spherical, plane and hyperbolic symmetry. We now consider a solution of the Einstein-Vlasov-scalar field system where all unknowns are invariant under one of these symmetries. We write the system in areal coordinates, i.e., coordinates are chosen such that $R = t$, where R is the area radius function on a surface of symmetry. The circumstances under which coordinates of this type exist are discussed in [2]. In such coordinates the metric g takes the form

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + t^2 (d\theta^2 + \sin_k^2 \theta d\varphi^2) \quad (1.1)$$

where

$$\sin_k \theta = \begin{cases} \sin \theta & \text{for } k = 1 \text{ (spherical symmetry);} \\ 1 & \text{for } k = 0 \text{ (plane symmetry);} \\ \sinh \theta & \text{for } k = -1 \text{ (hyperbolic symmetry)} \end{cases}$$

$t > 0$ denotes a time-like coordinate, $r \in \mathbb{R}$ and (θ, φ) range in the domains $[0, \pi] \times [0, 2\pi]$, $[0, 2\pi] \times [0, 2\pi]$, $[0, \infty[\times [0, 2\pi]$ respectively, and stand for angular

coordinates. The functions λ and μ are periodic in r with period 1. It has been shown in [7] that due to the symmetry, f can be written as a function of

$$t, r, w := e^\lambda p^1 \text{ and } F := t^4[(p^2)^2 + \sin_k^2 \theta (p^3)^2],$$

i.e., $f = f(t, r, w, F)$. In these variables, we have $p^0 = e^{-\mu} \sqrt{1 + w^2 + F/t^2}$. The scalar field is a function of t and r which is periodic in r with period 1.

We denote by a dot and by a prime the derivatives of the metric components and of the scalar field with respect to t and r respectively. Using the results of [13], the complete Einstein-Vlasov-scalar field system can be written as follows:

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1 + w^2 + F/t^2}} \partial_r f - (\dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1 + w^2 + F/t^2}) \partial_w f = 0 \quad (1.2)$$

$$e^{-2\mu}(2t\dot{\lambda} + 1) + k = 8\pi t^2 \rho \quad (1.3)$$

$$e^{-2\mu}(2t\dot{\mu} - 1) - k = 8\pi t^2 p \quad (1.4)$$

$$\mu' = -4\pi t e^{\lambda+\mu} j \quad (1.5)$$

$$e^{-2\lambda}(\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + (\dot{\lambda} + \frac{1}{t})(\dot{\lambda} - \dot{\mu})) = 4\pi q \quad (1.6)$$

$$e^{-2\lambda} \phi'' - e^{-2\mu} \ddot{\phi} - e^{-2\mu}(\dot{\lambda} - \dot{\mu} + \frac{2}{t})\dot{\phi} - e^{-2\lambda}(\lambda' - \mu')\phi' = 0 \quad (1.7)$$

where (1.7) is the wave equation in ϕ and :

$$\begin{aligned} \rho(t, r) = e^{-2\mu} T_{00}(t, r) &= \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw \\ &+ \frac{1}{2}(e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2) \end{aligned} \quad (1.8)$$

$$\begin{aligned} p(t, r) = e^{-2\lambda} T_{11}(t, r) &= \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f(t, r, w, F) dF dw \\ &+ \frac{1}{2}(e^{-2\mu} j^2 + e^{-2\lambda} \phi'^2) \end{aligned} \quad (1.9)$$

$$j(t, r) = -e^{-(\lambda+\mu)} T_{01}(t, r) = \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} w f(t, r, w, F) dF dw - e^{-(\lambda+\mu)} \dot{\phi} \phi' \quad (1.10)$$

$$\begin{aligned}
q(t, r) &= \frac{2}{t^2} T_{22}(t, r) = \frac{2}{t^2 \sin_k^2 \theta} T_{33}(t, r, \theta) \\
&= \frac{\pi}{t^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\sqrt{1+w^2+F/t^2}} f(t, r, w, F) dF dw + e^{-2\mu} \dot{\phi}^2 - e^{-2\lambda} \phi'^2
\end{aligned} \tag{1.11}$$

We prescribe initial data at time $t = 1$:

$$\begin{aligned}
f(1, r, w, F) &= \overset{\circ}{f}(r, w, F), \quad \lambda(1, r) = \overset{\circ}{\lambda}(r), \quad \mu(1, r) = \overset{\circ}{\mu}(r), \\
\phi(1, r) &= \overset{\circ}{\phi}(r), \quad \dot{\phi}(1, r) = \psi(r)
\end{aligned}$$

The choice $t = 1$ is made only for convenience. Analogous results hold in the case of prescribed data on any hypersurface $t = t_0 > 0$.

The paper is organized as follows. In section 2, we show that the solution of the Cauchy problem corresponding to system (1.2)-(1.11) exists for all $t \in]0, 1]$. In section 3, we analyze the asymptotic behaviour of solutions as $t \rightarrow 0$. The paper ends with a discussion of some interesting open problems.

2 Global existence in the past

We use the continuation criterion in the following local existence result.

Theorem 2.1 *Let $\overset{\circ}{f} \in C^1(\mathbb{R}^2 \times [0, \infty[)$ with $\overset{\circ}{f}(r+1, w, F) = \overset{\circ}{f}(r, w, F)$ for $(r, w, F) \in \mathbb{R}^2 \times [0, \infty[$, $\overset{\circ}{f} \geq 0$, and*

$$w_0 := \sup\{|w|(r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty$$

$$F_0 := \sup\{F|(r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty$$

Let $\overset{\circ}{\lambda}, \psi \in C^1(\mathbb{R})$, $\overset{\circ}{\mu}, \overset{\circ}{\phi} \in C^2(\mathbb{R})$ with $\overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1)$, $\overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1)$, $\overset{\circ}{\phi}(r) = \overset{\circ}{\phi}(r+1)$, $\psi(r) = \psi(r+1)$ and (1.5) satisfied for $t = 1$. Then there exists a unique, left maximal, regular solution (f, λ, μ, ϕ) of system (1.2)-(1.11) with $(f, \lambda, \mu, \phi)(1) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{\phi})$ and $\dot{\phi}(1) = \psi$ on a time interval $]T, 1[$ with $T \in [0, 1[$. If

1. $\sup\{|w|(t, r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty$,
2. $\sup\{(e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2)(t, r); r \in \mathbb{R}\} < \infty$,
3. μ is bounded,

then $T = 0$. If $k \geq 0$ or $\overset{\circ}{\mu} \leq 0$ then condition 3 is automatically satisfied.

This is the content of theorems 4.4 and 4.5 in [13]. For a regular solution, all derivatives which appear in the system exist and are continuous by definition (see [13]).

In order to obtain the global existence of solutions, we prove the following results :

Lemma 2.2 Let $D^+ = e^{-\mu}\partial_t + e^{-\lambda}\partial_r$; $D^- = e^{-\mu}\partial_t - e^{-\lambda}\partial_r$;
 $X = \dot{\phi}e^{-\mu} - \phi'e^{-\lambda}$; $Y = \dot{\phi}e^{-\mu} + \phi'e^{-\lambda}$;
 $a = (-\dot{\lambda} - \frac{1}{t})e^{-\mu} - \mu'e^{-\lambda}$; $b = -\frac{e^{-\mu}}{t}$; $c = (-\dot{\lambda} - \frac{1}{t})e^{-\mu} + \mu'e^{-\lambda}$
and define $X_2 = e^\mu X$, $Y_2 = e^\mu Y$. Then as a consequence of the field equations
 X and Y satisfy the system

$$D^+X = aX + bY \quad (2.1)$$

$$D^-Y = bX + cY \quad (2.2)$$

If in addition the field equations (1.3)-(1.4) are satisfied, then X_2 and Y_2 satisfy

$$D^+X_2 = e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] X_2 - \frac{e^{-\mu}}{t} Y_2 \quad (2.3)$$

$$D^-Y_2 = -\frac{e^{-\mu}}{t} X_2 + e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] Y_2 \quad (2.4)$$

Proof: This results from a straightforward calculation. \square

Lemma 2.3 Define X_2 and Y_2 as in lemma 2.2 and let

$$B(t) = \sup\{(|X_2|^2 + |Y_2|^2)^{1/2}(t, r) ; r \in \mathbb{R}\}$$

$$l(t) = \sup\left\{ \frac{1}{t} + e^{2\mu} \left[\frac{|k|}{t} + 4\pi t(\rho - p) \right] (t, r) ; r \in \mathbb{R} \right\}$$

If (X_2, Y_2) is a solution of (2.3)-(2.4), then we obtain the estimate

$$B(t)^2 \leq B(1)^2 + 2 \int_t^1 l(s)B(s)^2 ds \quad (2.5)$$

with $t \in]T, 1]$, $T > 0$.

Proof: We deduce from system (2.3)-(2.4) :

$$D^+X_2^2 = 2e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] X_2^2 - 2\frac{e^{-\mu}}{t} X_2Y_2$$

$$D^-Y_2^2 = -2\frac{e^{-\mu}}{t} X_2Y_2 + 2e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] Y_2^2$$

On the corresponding characteristic curves (t, γ_i) , $i = 1, 2$ of the wave equation,
(see [13]) D^+ or D^- is equal to $e^{-\mu} \frac{d}{dt}$ and then

$$\frac{d}{dt} X_2^2(t, \gamma_1(t)) = 2e^{2\mu} \left[\frac{k}{t} - 4\pi t(\rho - p) \right] X_2^2(t, \gamma_1(t)) - \frac{2}{t} X_2Y_2(t, \gamma_1(t))$$

$$\frac{d}{dt} Y_2^2(t, \gamma_2(t)) = -\frac{2}{t} X_2Y_2(t, \gamma_2(t)) + 2e^{2\mu} \left[\frac{k}{t} - 4\pi t(\rho - p) \right] Y_2^2(t, \gamma_2(t))$$

Integrate each of the two previous equations on $[t, 1]$ and obtain respectively :

$$\begin{aligned} X_2^2(t, \gamma_1(t)) &= X_2^2(1, \gamma_1(1)) + 2 \int_t^1 \{e^{2\mu}[-\frac{k}{s} + 4\pi s(\rho - p)]X_2^2 + \frac{1}{s}X_2Y_2\}(s, \gamma_1(s))ds \\ &\leq X_2^2(1, \gamma_1(1)) + 2 \int_t^1 \{[\frac{1}{2s} + e^{2\mu}(-\frac{k}{s} + 4\pi s(\rho - p))]X_2^2 + \frac{1}{2s}Y_2^2\}(s, \gamma_1(s))ds; \end{aligned}$$

$$Y_2^2(t, \gamma_2(t)) \leq Y_2^2(1, \gamma_2(1)) + 2 \int_t^1 \left\{ \frac{1}{2s}X_2^2 + \left[\frac{1}{2s} + e^{2\mu}(-\frac{k}{s} + 4\pi s(\rho - p)) \right] Y_2^2 \right\}(s, \gamma_2(s))ds$$

Add the two previous inequalities and take the supremum over space to obtain estimate (2.5). \square

Unless otherwise specified in what follows constants denoted by C will be positive, may depend on the initial data and may change their value from line to line.

Proposition 2.4 *Let (f, λ, μ, ϕ) be a solution of the full system (1.2)-(1.11) on a left maximal interval of existence $]T, 1]$, $T > 0$, with initial data as in theorem 2.1. If*

1. $Q(t) = \sup\{|w|(r, w, f) \in \text{supp}f(t), t \in]T, 1]\} < \infty$
2. μ is bounded.

then $T = 0$. If $k \geq 0$ or $\overset{\circ}{\mu} \leq 0$ then condition 2 is automatically satisfied.

Proof: We need to prove that $K(t) = \sup\{(|X|^2 + |Y|^2)^{1/2}(t, r) ; r \in \mathbb{R}\}$ is bounded for all $t \in]T, 1]$; where X and Y are defined in lemma 2.2. Subtract the two equations (1.8)-(1.9) to obtain :

$$\begin{aligned} \rho - p &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1 + F/t^2}{\sqrt{1 + w^2 + F/t^2}} f dF dw \\ &\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1 + F/t^2}{\sqrt{1 + F/t^2}} f dF dw \\ &\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + F/t^2} f dF dw \\ &\leq \frac{C}{t^3} ; \text{ since } Q(t) < \infty \end{aligned} \tag{2.6}$$

Using (4.15) of [13], which shows that $e^{2\mu} = O(t)$, we obtain :

$$l(t) \leq \frac{C}{t} + \frac{|k|}{C}$$

Then (2.5) implies :

$$B(t)^2 \leq B(1)^2 + 2 \int_t^1 (C(1 + \frac{1}{s}) + \frac{1}{s}) B(s)^2 ds$$

And by Gronwall's lemma,

$$B(t)^2 \leq Ct^{-C-2}$$

We have from (1.9),

$$\begin{aligned} p(s, r) &\leq \frac{\pi}{s^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{w^2}{|w|} f(s, r, w, F) dF dw + \frac{1}{4} e^{-2\mu} (X_2^2 + Y_2^2)(s, r) \\ &\leq \frac{C}{s^2} + \frac{1}{4} e^{-2\mu} B(s)^2 \\ &\leq Cs^{-2} + Cs^{-C-2} e^{-2\mu} \end{aligned}$$

Then using (1.4), we obtain the estimate :

$$\begin{aligned} e^{-2\mu(t,r)} &= \frac{e^{-2\hat{\mu}(r)} + k}{t} - k - \frac{8\pi}{t} \int_1^t s^2 p(s, r) ds \\ &\leq \frac{e^{-2\hat{\mu}(r)} + |k|}{t} + \frac{C}{t} \int_t^1 (1 + s^{-C} e^{-2\mu}) ds \\ &\leq Ct^{-1} (1 + \int_t^1 s^{-C} e^{-2\mu} ds). \end{aligned}$$

By Gronwall's lemma, we deduce that $e^{-2\mu} \leq Ct^{-1} \exp[\frac{C}{1-C}(1 - t^{1-C})]$.

Therefore,

$$(X^2 + Y^2)(t, r) = e^{-2\mu} (X_2^2 + Y_2^2)(t, r) \leq Ct^{-3-C} \exp[\frac{C}{1-C}(1 - t^{1-C})]$$

i.e $K(t)$ is bounded. We conclude by theorem 2.1 that $T = 0$. \square

In the next theorem, we prove that the solution exists on the full interval $]0, 1]$.

Theorem 2.5 *Consider a solution of the Einstein-Vlasov-scalar field system with $k \geq 0$ and initial data given for $t = 1$. Then this solution exists on the whole interval $]0, 1]$. If $k < 0$ and $\hat{\mu} \leq 0$, the same result holds.*

Proof : The strategy of the proof is the following : suppose we have a solution on an interval $]T, 1]$ with $T > 0$. We want to show that the solution can be extended to the past. By consideration of the maximal interval of existence this will prove the assertion.

Firstly let us prove that under the hypotheses of the theorem, μ is bounded above. From the field equation (1.4) and since $p(s, r) \geq 0$, we have for $k \geq 0$,

$$e^{-2\mu(t,r)} \geq \frac{e^{-2\hat{\mu}(r)} + k}{t} - k \geq e^{-2\hat{\mu}(r)}$$

For the case $k = -1$, $e^{-2\mu} \geq \frac{e^{-2\hat{\mu}} - 1}{t} + 1 \geq 1$ which gives the upper bound of μ for $\hat{\mu} \leq 0$. In either case, μ is bounded above.

Now, let us prove that w is bounded. Consider the following rescaled version of w , called u_1 , which has been inspired by the works of [16] (p. 1090) and [12] (p. 5336):

$$u_1 = \frac{e^\mu}{2t}w.$$

If we prove that μ is bounded below then the boundedness of u_1 will imply the boundedness of w . So let us show that μ is bounded below under the assumption that u_1 is bounded. We have

$$\frac{d}{dt}(te^{-2\mu}) = -k - 8\pi t^2 p. \quad (2.7)$$

Transforming the integral term defining p to u_1 as an integration variable instead of w yields

$$p = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{8\pi te^{-3\mu} u_1^2}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} f dF du_1 + \frac{1}{4}(X^2 + Y^2);$$

where X and Y are defined in lemma 2.2. The integrand in the first term in p can then be estimated by $4\pi e^{-2\mu}|u_1|$. We have from (1.8)-(1.9):

$$\begin{aligned} e^{2\mu}(\rho - p) &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{(1 + F/t^2)2te^\mu}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} f dF du_1 \\ &\leq \frac{2\pi}{t} \int_{-\infty}^{\infty} \int_0^{\infty} (1 + F/t^2) e^\mu f dF du_1 \\ &\leq Ct^{-3} \bar{u}_1 e^\mu \end{aligned}$$

where \bar{u}_1 is the maximum modulus of u_1 on the support of f at a given time. We can then estimate from (2.5) $l(s)$ by $h(s) = C \sup\{1 + s^{-1} + s^{-2} e^\mu \bar{u}_1(s, r); r \in \mathbb{R}\}$ and $B(t)^2$ by $B(1)^2 \exp(\int_t^1 h(s) ds)$. Thus

$$X^2 + Y^2 \leq e^{-2\mu} B(t)^2 \leq e^{-2\mu} B(1)^2 \exp\left(\int_t^1 h(s) ds\right).$$

Therefore, using the bound for μ and u_1 , p can be estimated by $Ce^{-2\mu}$ and so (2.7) implies that

$$\left| \frac{d}{dt}(te^{-2\mu}) \right| \leq C(1 + te^{-2\mu}),$$

Integrating this with respect to t over $[t, 1]$ and using the Gronwall inequality implies that $te^{-2\mu}$ is bounded on $]T, 1]$; that is μ is bounded below on the given time interval.

The next step is to prove that u_1 is bounded. To this end, it suffices to get a suitable integral inequality for \bar{u}_1 . Since $u_1 = u_1(t, r(t))$, we can compute \dot{u}_1 :

$$\dot{u}_1 = -\frac{e^\mu}{2t^2}w + \frac{e^\mu}{2t}w(\dot{\mu} + \dot{r}\mu') + \frac{e^\mu}{2t}\dot{w}$$

i.e.

$$\dot{u}_1 = \left(\dot{\mu} + \dot{r}\mu' - \frac{1}{t} \right) u_1 + \frac{e^\mu}{2t} \dot{w} \quad (2.8)$$

We have

$$\mu' = -4\pi t e^{\mu+\lambda} j, \quad \dot{r} = \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}}$$

and

$$\dot{w} = 4\pi t e^{2\mu} (j \sqrt{1+w^2+F/t^2} - \rho w) + \frac{1+ke^{2\mu}}{2t} w$$

so that (2.8) implies the following :

$$\dot{u}_1 |u_1| = e^{2\mu} \left[-4\pi t (\rho - p) + \frac{k}{t} \right] u_1 |u_1| + 2\pi e^{3\mu} j \frac{(1+F/t^2)|u_1|}{\sqrt{1+4t^2 e^{-2\mu} u_1^2 + F/t^2}} \quad (2.9)$$

In order to estimate the modulus of the first term on the right hand side of equation (2.9), we need an estimate of $e^{2\mu}(\rho - p)\bar{u}_1^2$. For convenience let \log_+ be defined by $\log_+(x) = \log x$ when $\log x$ is positive and $\log_+(x) = 0$, otherwise. Then estimating the integral defining $\rho - p$ shows that

$$\rho - p \leq C(1 + \log_+(\bar{w})),$$

i.e.

$$\rho - p \leq C(1 + \log_+(\bar{u}_1) - \mu).$$

The expression $-\mu$ is not under control ; however the expression we wish to estimate contains a factor $e^{2\mu}$. The function $\mu \mapsto -\mu e^{2\mu}$ has an absolute maximum at $-1/2$ which is $(1/2)e^{-1}$. Thus the first term on the right hand side of equation (2.9) can be estimated by $C\bar{u}_1^2(1 + \log_+(\bar{u}_1))$.

Next the second term on the right hand side of equation (2.9) will be estimated. By definition

$$j = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, r, w, F) dF dw - \dot{\phi} \phi' e^{-\mu-\lambda} = j_1 + j_2$$

The first term of j can be estimated by $C\bar{w}^2$, i.e.

$$|j_1| \leq C\bar{u}_1^2 e^{-2\mu}$$

and the second term $|j_2| \leq \frac{1}{2} e^{-2\mu} B(t)^2$; so that it suffices to estimate the quantity

$$\frac{(\bar{u}_1^2 + B(t)^2)(1 + F/t^2)}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} |u_1| \quad (2.10)$$

in order to estimate the second term on the right hand side of equation (2.9). But since μ and t^{-1} are bounded on the interval being considered, the quantity (2.10) can be estimated by $C(\bar{u}_1^2 + B(t)^2)$. Thus adding the estimates for the

first and second terms on the right hand side of (2.9) allows us to deduce from (2.9) that

$$|\dot{u}_1| |u_1| \leq C \bar{u}_1^2 (1 + \log_+(\bar{u}_1)) + C(\bar{u}_1^2 + B(t)^2)$$

i.e

$$\left| \frac{d}{dt} |u_1|^2 \right| \leq C(\bar{u}_1)^2 (1 + \log_+(\bar{u}_1^2)) + CB(t)^2$$

Integrating over $[t, 1]$ gives :

$$\bar{u}_1^2(t) \leq \bar{u}_1^2(1) + C \int_t^1 [\bar{u}_1^2(s)(1 + \log_+(\bar{u}_1^2(s))) + B(s)^2] ds \quad (2.11)$$

We deduce from the estimate of $\rho - p$ and from inequality (2.5), that $l(s)$ can be estimated by $C(1 + \log_+(\bar{u}_1))$. We then obtain

$$B(t)^2 \leq B(1)^2 + C \int_t^1 (1 + \log_+(\bar{u}_1^2(s))) B(s)^2 ds \quad (2.12)$$

Adding (2.11) and (2.12) gives estimate :

$$\bar{u}_1^2(t) + B(t)^2 \leq \bar{u}_1^2(1) + B(1)^2 + C \int_t^1 (1 + \bar{u}_1^2 + B(s)^2) [1 + \log_+(1 + \bar{u}_1^2(s) + B(s)^2)] ds$$

Set $v(t) = \bar{u}_1^2(t) + B(t)^2$, then the above estimate can be written

$$v(t) \leq v(1) + C \int_t^1 (1 + v(s)) [1 + \log_+(1 + v(s))] ds \quad (2.13)$$

By the comparison principle for solutions of integral equations, it is enough to show that the solution of the integral equation

$$a(t) = a(1) + C \int_t^1 (1 + a(s)) (1 + \log_+(1 + a(s))) ds$$

is bounded. The solution $a(t)$ is a non-increasing function. Thus either $a(t) \leq e$ everywhere, in which case the desired conclusion is immediate or there is some T_1 in $]T, 1]$ such that $e \leq a(t)$ on $]T, T_1]$. We take T_1 maximal with that property. Then it follows on $]T, T_1]$ that inequality

$$a(t) \leq C \left(1 + \int_t^{T_1} a(s)(1 + \log a(s)) ds \right)$$

holds for a constant C . The boundedness of $a(t)$ follows from that of the solution of the differential equation $\dot{x} = Cx(1 + \log x)$ which is $\exp(\exp(Ct) - 1)$. In either case $a(t)$ is bounded. Thus \bar{u}_1^2 and $B(t)^2$ are bounded i.e w and $K(t)$ are bounded. The proof of the theorem is complete using proposition 2.4. \square

3 On past asymptotic behaviour

In this section we examine the behaviour of solutions as $t \rightarrow 0$. Firstly we generalize the work of Ringström [11](P. S310-S311) to bound the quantity $|\phi'|e^{\mu-\lambda}$ by $C|t \log t|^{-1}$, where C is a positive constant.

Lemma 3.1 *Let $A_1 = \frac{1}{8}(-\dot{\phi} + \frac{\phi}{t \log t} + \phi' e^{\mu-\lambda})^2$ and $A_2 = \frac{1}{8}(-\dot{\phi} + \frac{\phi}{t \log t} - \phi' e^{\mu-\lambda})^2$ with $t \in]0, 1[$. If ϕ satisfies the wave equation, then*

$$\begin{aligned} (\partial_t + e^{\mu-\lambda} \partial_r) A_1 &= -\frac{1}{4t} \left(1 + \frac{1}{\log t}\right) \left[\left(-\dot{\phi} + \frac{\phi}{t \log t}\right)^2 + \phi'^2 e^{2\mu-2\lambda} \right] \\ &+ \frac{1}{2t} \left(1 + \frac{1}{\log t}\right) \phi'^2 e^{2\mu-2\lambda} + \frac{1}{4} \left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right) \left(\dot{\phi} - \phi' e^{\mu-\lambda}\right) \left(-\dot{\phi} + \frac{\phi}{t \log t} + \phi' e^{\mu-\lambda}\right) \end{aligned} \quad (3.1)$$

$$\begin{aligned} (\partial_t + e^{\mu-\lambda} \partial_r) A_2 &= -\frac{1}{4t} \left(1 + \frac{1}{\log t}\right) \left[\left(-\dot{\phi} + \frac{\phi}{t \log t}\right)^2 + \phi'^2 e^{2\mu-2\lambda} \right] \\ &+ \frac{1}{2t} \left(1 + \frac{1}{\log t}\right) \phi'^2 e^{2\mu-2\lambda} + \frac{1}{4} \left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right) \left(\dot{\phi} + \phi' e^{\mu-\lambda}\right) \left(-\dot{\phi} + \frac{\phi}{t \log t} - \phi' e^{\mu-\lambda}\right) \end{aligned} \quad (3.2)$$

Proof : This results from a straightforward calculation. \square

Proposition 3.2 *Let (f, λ, μ, ϕ) be a left maximal solution of the Einstein-Vlasov-scalar field system on the interval $]T, 1]$, $0 \leq T < e^{-1}$. Assume that*

$$Q(t) = \sup\{|w| \mid (r, w, F) \in \text{supp} f(t)\} \leq Ct^\alpha$$

for some positive constants C, α and for some $t \in]T, e^{-1}]$. Then

$$\left(-\dot{\phi} + \frac{\phi}{t \log t}\right)^2 + \phi'^2 e^{2\mu-2\lambda} \leq C(t \log t)^{-2} \quad (3.3)$$

Proof : Consider the two characteristic curves $(t, \gamma_1(t))$ and $(t, \gamma_2(t))$ of the wave operator. Since $t \in]0, e^{-1}]$, the term $(1 + \frac{1}{\log t})\phi'^2 e^{2\mu-2\lambda}$ is nonnegative and $(-\dot{\phi} + \frac{\phi}{t \log t})^2 + \phi'^2 e^{2\mu-2\lambda} = 4(A_1 + A_2)$, then from (3.1) :

$$\begin{aligned} (\partial_t + e^{\mu-\lambda} \partial_r) A_1(t, \gamma_1(t)) &\geq -\frac{1}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, \gamma_1(t)) \\ &- \frac{1}{4} \left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right) \left(-\dot{\phi} + \phi' e^{\mu-\lambda}\right) \left(-\dot{\phi} + \frac{\phi}{t \log t} + \phi' e^{\mu-\lambda}\right)(t, \gamma_1(t)) \end{aligned} \quad (3.4)$$

Similarly, we deduce from (3.2) that :

$$\begin{aligned} (\partial_t - e^{\mu-\lambda} \partial_r) A_2(t, \gamma_2(t)) &\geq -\frac{1}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, \gamma_2(t)) \\ &- \frac{1}{4} \left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right) \left(-\dot{\phi} - \phi' e^{\mu-\lambda}\right) \left(-\dot{\phi} + \frac{\phi}{t \log t} - \phi' e^{\mu-\lambda}\right)(t, \gamma_2(t)) \end{aligned} \quad (3.5)$$

Since $Q(t) \leq Ct^\alpha$, we can bound $\rho - p$ by $Ct^{-3+\alpha}$ (see (2.6)). $e^{2\mu} \leq Ct$; then subtracting (1.3) -(1.4) gives

$(\dot{\lambda} - \dot{\mu})(t) + \frac{1}{t} = -\frac{ke^{2\mu}}{t} + 4\pi te^{2\mu}(\rho - p) \leq C(1 + t^{-1+\alpha})$; and from (2.5), $l(s)$ can be bounded by $s^{-1} + C + Cs^{-1+\alpha}$. We deduce from (2.5) (consider the integral term in the interval $[t, e^{-1}]$) that

$$B(t)^2 \leq B(e^{-1})^2 \exp[2 \int_t^{e^{-1}} (s^{-1} + C + Cs^{-1+\alpha}) ds] \quad \text{i.e.,} \quad B(t)^2 \leq Ct^{-2}.$$

Therefore $|\dot{\phi}(t)|$ and $|\phi'|e^{\mu-\lambda}(t)$ are bounded each by Ct^{-1} . We can then have a lower bound of the second term of the right hand side of each inequality (3.4) and (3.5) which is $-C(t^{-2} + t^{-3+\alpha})$. Then

$$(\partial_t + e^{\mu-\lambda}\partial_r)A_1(t, \gamma_1(t)) \geq -\frac{1}{t}\left(1 + \frac{1}{\log t}\right)(A_1 + A_2)(t, \gamma_1(t)) - C(t^{-2} + t^{-3+\alpha})$$

and

$$(\partial_t - e^{\mu-\lambda}\partial_r)A_2(t, \gamma_2(t)) \geq -\frac{1}{t}\left(1 + \frac{1}{\log t}\right)(A_1 + A_2)(t, \gamma_2(t)) - C(t^{-2} + t^{-3+\alpha})$$

On the corresponding characteristic, we have $\partial_t + e^{\mu-\lambda}\partial_r$ or $\partial_t - e^{\mu-\lambda}\partial_r$ equal to $\frac{d}{dt}$. Take the supremum in the space of each of the above two inequalities and add them. Then

$$\frac{d}{dt}(A_1 + A_2)(t, r) \geq -\frac{2}{t}\left(1 + \frac{1}{\log t}\right)(A_1 + A_2)(t, r) - C(t^{-2} + t^{-3+\alpha})$$

Set $u(t) = (A_1 + A_2)(t)$ and $v(t) = (t \log t)^2 u(t)$. If $v(t)$ is bounded, then we conclude that $u(t)$ is bounded by $C(t \log t)^{-2}$. Let us prove that $v(t)$ is bounded. We have :

$$\begin{aligned} \frac{dv}{dt} &= (t \log t)^2 \left(\frac{du}{dt} + \frac{2}{t} \left(1 + \frac{1}{\log t}\right) u \right) \\ &\geq -C(\log t)^2 (1 + t^{-1+\alpha}) \end{aligned}$$

Then, $v(t) \leq v(e^{-1}) + C \int_t^{e^{-1}} (1 + s^{-1+\alpha})(\log s)^2 ds \leq v(e^{-1}) + C$. We obtain the desired conclusion of the proposition. \square

In the case $f = 0$, we obtain from the previous proposition, theorems 2.5 and 2.1, the global existence of solutions and the above estimates hold on the whole interval $]0, e^{-1}[$. In general we do not know how to use this proposition to obtain precise asymptotics. It seems that if the estimates could be improved slightly they would allow a bootstrap argument on the bound for Q similar to that used in [7]. These estimates have been included here in the hope that they might help someone else to complete the argument.

Next we prove that the curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ called the Kretschman scalar blows up as $t \rightarrow 0$. Then there is a spacetime singularity and the spacetime cannot be extended further.

Theorem 3.3 *Let (f, λ, μ, ϕ) be a regular solution of the surface-symmetric Einstein-Vlasov-scalar field system on the interval $]0, 1[$ with data given for*

$t = 1$. Then

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t, r) \geq \frac{4}{t^6} \left(\inf e^{-2\hat{\mu}} + k \right)^2, \quad (3.6)$$

with $r \in \mathbb{R}$.

Proof We can use the following expression for the Kretschman scalar from [7].

$$\begin{aligned} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= 4[e^{-2\lambda}(\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))]^2 \\ &\quad + \frac{8}{t^2}[e^{-4\mu}\dot{\lambda}^2 + e^{-4\mu}\dot{\mu}^2 - 2e^{-2(\lambda+\mu)}(\mu')^2] \\ &\quad + \frac{4}{t^4}(e^{-2\mu} + k)^2 \\ &=: K_1 + K_2 + K_3 \end{aligned}$$

The first term K_1 is nonnegative and can be dropped. Inserting the expressions

$$e^{-2\mu}\dot{\lambda} = 4\pi t\rho - \frac{k + e^{-2\mu}}{2t}; \quad e^{-2\mu}\dot{\mu} = 4\pi t p + \frac{k + e^{-2\mu}}{2t}; \quad e^{-\lambda-\mu}\mu' = -4\pi t j$$

into the formula for K_2 yields

$$K_2 = \frac{8}{t^2} \left[16\pi^2 t^2 (\rho^2 + p^2 - 2j^2) - 4\pi t (\rho - p) \frac{k + e^{-2\mu}}{t} + \frac{(k + e^{-2\mu})^2}{2t^2} \right].$$

Now

$$\begin{aligned} |j(t, r)| &\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} (1 + w^2 + F/t^2)^{1/4} f^{1/2} \frac{|w|}{(1 + w^2 + F/t^2)^{1/4}} f^{1/2} dF dw \\ &\quad + \frac{1}{2} (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) \\ &\leq \frac{\pi}{t^2} \left[\int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} f dF dw \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f dF dw \right]^{\frac{1}{2}} \\ &\quad + \frac{1}{2} (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) \text{ by the Cauchy-Schwarz inequality.} \\ &\leq \frac{1}{2} (\rho + p)(t, r). \end{aligned}$$

In fact the above inequality holds in general for all choices of matter satisfying the dominant energy condition. Therefore

$$\begin{aligned} K_2 &\geq \frac{8}{t^2} \left[8\pi t^2 (\rho - p)^2 - 4\pi t (\rho - p) \frac{k + e^{-2\mu}}{t} + \frac{(k + e^{-2\mu})^2}{2t^2} \right] \\ &\geq \frac{4}{t^2} \left[4\pi t (\rho - p) - \frac{k + e^{-2\mu}}{t} \right]^2 \geq 0. \end{aligned}$$

Recalling the expression for $e^{-2\mu}$ we get

$$\begin{aligned} e^{-2\mu} + k &= \frac{e^{-2\overset{\circ}{\mu}(r)} + k}{t} + \frac{8\pi}{t} \int_t^1 s^2 p(s, r) ds \\ &\geq \frac{e^{-2\overset{\circ}{\mu}} + k}{t} \geq \frac{\inf e^{-2\overset{\circ}{\mu}} + k}{t} \end{aligned}$$

and thus

$$K_3 = \frac{4}{t^4} (e^{-2\mu} + k)^2 \geq \frac{4}{t^6} \left(\inf e^{-2\overset{\circ}{\mu}} + k \right)^2$$

We obtain (3.6) and deduce that

$$\lim_{t \rightarrow 0} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})(t, r) = \infty,$$

uniformly in $r \in \mathbb{R}$.

Next we prove that the singularity at $t = 0$ is a crushing singularity i.e. the mean curvature of the surfaces of constant t blows up. In the case where there is only a scalar field and no Vlasov contribution this singularity is velocity dominated i.e the generalized Kasner exponents have limits as $t \rightarrow 0$.

Theorem 3.4 *Let (f, λ, μ, ϕ) be a regular solution of the surface-symmetric Einstein-Vlasov-scalar field system on the interval $]0, 1]$ with initial data given on $t = 1$. Let*

$$K(t, r) := -e^{-\mu} \left(\dot{\lambda}(t, r) + \frac{2}{t} \right)$$

denote the mean curvature of the hypersurfaces of constant t . Then

$$K(t, r) \leq -Ct^{-3/2},$$

where C is a positive constant.

Proof We use the same argument as in [7] and obtain the following :

$$K(t, r) = -\left(\dot{\lambda} + \frac{2}{t}\right)e^{-\mu}; \quad \dot{\lambda} = e^{2\mu} \left(4\pi t \rho - \frac{k + e^{-2\mu}}{2t} \right) \geq -e^{2\mu} \left(\frac{k + e^{-2\mu}}{2t} \right) \text{ and}$$

$$K(t, r) \leq \frac{k - 3e^{-2\mu}}{2t} e^{\mu}.$$

For $k = 0$ or $k = -1$,

$$K(t, r) \leq -\frac{3}{2t} e^{-\mu}.$$

and the estimate

$$e^{-2\mu} \geq \frac{e^{-2\overset{\circ}{\mu}} + k}{t}$$

implies

$$K(t, r) \leq -\frac{3}{2t} \left(\frac{\inf e^{-2\overset{\circ}{\mu}} + k}{t} \right)^{1/2} \leq -Ct^{-3/2} \text{ where } C = \frac{3}{2} (\inf e^{-2\overset{\circ}{\mu}} + k)^{1/2}.$$

For $k = 1$ we have

$$e^{-2\mu} \geq \frac{e^{-2\dot{\mu}}}{t} > 1 = k$$

thus

$$\begin{aligned} K(t, r) &\leq \left(\frac{e^{2\mu} - 3}{2}\right) \frac{e^{-\mu}}{t} \\ &\leq -\frac{e^{-\mu}}{t} \leq -\frac{\inf e^{-\dot{\mu}}}{t^{3/2}} \\ &\leq -Ct^{-3/2} \quad \text{where } C = \inf e^{-\dot{\mu}}. \end{aligned}$$

We deduce from above that

$$\lim_{t \rightarrow 0} K(t, r) = -\infty,$$

uniformly in $r \in \mathbb{R}$.

Theorem 3.5 *Let (λ, μ, ϕ) be a regular solution of the Einstein-scalar field system with spherical, plane or hyperbolic symmetry on the interval $]0, 1]$ with initial data given at $t = 1$. Then*

$$\lim_{t \rightarrow 0} \frac{K_1^1(t, r)}{K(t, r)} = a(r) ; \quad \lim_{t \rightarrow 0} \frac{K_2^2(t, r)}{K(t, r)} = \lim_{t \rightarrow 0} \frac{K_3^3(t, r)}{K(t, r)} = \frac{1}{2}(1 - a(r)),$$

uniformly in $r \in \mathbb{R}$, where

$$\frac{K_1^1(t, r)}{K(t, r)}, \quad \frac{K_2^2(t, r)}{K(t, r)}, \quad \frac{K_3^3(t, r)}{K(t, r)}$$

are the generalized Kasner exponents and $a(r)$ a continuous function of r .

Proof We have as in [7]

$$\frac{K_1^1(t, r)}{K(t, r)} = \frac{t\dot{\lambda}(t, r)}{t\dot{\lambda}(t, r) + 2}; \quad \frac{K_2^2(t, r)}{K(t, r)} = \frac{K_3^3(t, r)}{K(t, r)} = \frac{1}{t\dot{\lambda}(t, r) + 2}.$$

As we have seen previously

$$e^{2\mu(t, r)} \leq Ct$$

which implies that

$$e^{2\mu(t, r)} \rightarrow 0 \quad \text{as } t \rightarrow 0$$

Let $t_0 \in]0, e^{-1}]$ and $t \in]0, t_0]$. From (3.3),

$$\partial_t \left(\frac{\phi}{\log t} \right) = \frac{1}{\log t} \left(\dot{\phi} - \frac{\phi}{t \log t} \right) = O(t^{-1}(\log t)^{-2})$$

Since $t^{-1}(\log t)^{-2}$ is integrable on the interval $(0, t_0]$ it follows that we can define

$$A(r) = \lim_{t \rightarrow 0} \frac{\phi(t, r)}{\log t} = \frac{\phi(t_0, r)}{\log t_0} - \int_0^{t_0} (\log s)^{-1} (\dot{\phi}(s, r) - \phi(s, r)/s \log s) ds$$

Since from (3.3), $(\dot{\phi} - \frac{\phi}{t \log t}) = O((t|\log t|)^{-1})$, we have

$$t\dot{\phi} = \frac{\phi}{\log t} + O((|\log t|)^{-1})$$

so that $t\dot{\phi} \rightarrow A(r)$ as $t \rightarrow 0$. Inequality (3.3) shows also that $\phi'^2 e^{2\mu-2\lambda} = O((t|\log t|)^{-2})$. Using these limits, we have

$$t\dot{\lambda}(t, r) = 2\pi(t^2\dot{\phi}^2 + t^2\phi'^2 e^{2\mu-2\lambda}) - \frac{k}{2}e^{2\mu} - \frac{1}{2} \rightarrow 2\pi A(r)^2 - \frac{1}{2} \text{ as } t \rightarrow 0, \text{ uniformly in } r.$$

We take $a(r) = \frac{4\pi A(r)^2 - 1}{4\pi A(r)^2 + 3}$ to complete the proof. \square

4 Discussion and outlook

It is an open problem to remove the restriction on μ in Theorem 2.5, even when the scalar field vanishes. The only example where existence up to $t = 0$ is known to fail is a vacuum solution, the pseudo-Schwarzschild solution (cf. the discussion in [8], p. 115). Perhaps it can only fail in the vacuum case. As mentioned in the introduction, the result for the plane symmetric case extends to solutions of the Einstein-Vlasov system with T^2 -symmetry [16].

Once existence up to $t = 0$ is known the ideal goal is to obtain detailed information on the asymptotics. For the Einstein-Vlasov system this has been done in [7] for small data. In Proposition 3.2 an analogue of some parts of the proof of the small data theorem in [7] is obtained but this is not sufficient in order to determine the asymptotic behaviour. There is a formal similarity between the Einstein-Vlasov-scalar field system with plane symmetry and the Einstein-Vlasov system with polarized Gowdy symmetry. If the problem of asymptotics could be solved for the first problem it would probably lead to valuable insights for the second problem.

In the case of general Gowdy symmetry, results on asymptotics are available in the vacuum case but they are hard to obtain [11]. In the yet more general case of T^2 -symmetry the only thing known is a construction of a class of vacuum solutions with prescribed asymptotics [4].

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