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Multipole Radiation in a Collisionless Gas Coupled to Electromagnetism or Scalar Gravitation

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Abstract: We consider the relativistic Vlasov-Maxwell and Vlasov-Nordström systems which describe large particle ensembles interacting by either electromagnetic fields or a relativistic scalar gravity model. For both systems we derive a radiation formula analogous to the Einstein quadrupole formula in general relativity.

1. Introduction and Main Results

This paper is an investigation of the mathematical properties of certain models for the interaction of matter, described by a kinetic equation, with radiation, described by hyperbolic equations. The first model, the relativistic Vlasov-Maxwell system, plays an important role in plasma physics. The motivation for studying the second model, the Vlasov-Nordström system, comes from the theory of gravitation. On a mathematical level the Vlasov-Maxwell system can also give insights into gravity.

The most precise existing theory of gravitation, general relativity, predicts that certain astrophysical systems, such as colliding black holes or neutron stars, will give rise to gravitational radiation. There is a major international effort under way to detect these gravitational waves [5]. In order to relate the general theory to predictions of what the detectors will see it is necessary to use approximation methods - the exact theory is too complicated. The mathematical status of these approximations remains unclear although partial results exist. This paper is intended as a contribution to understanding the mathematical structures involved.

Since the solutions of the equations of general relativity are so difficult to analyze rigorously it is useful to start with model problems. One possibility is the scalar theory of gravitation considered here, the Vlasov-Nordström theory [6]. It has already been used as a model problem for numerical relativity in [21].

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Among the approximation methods used to study gravitational radiation those which are most accessible mathematically are the post-Newtonian approximations. Some information on these has been obtained in [17] and [18]. Results which are analogous to these but go much further have been obtained for the Vlasov-Maxwell and Vlasov-Nordström systems in [4] and [2] respectively. None of these results include radiation explicitly. Here we take a first step in doing so. On the other hand, for the case of finite particle systems interacting with their self-induced fields there are several rigorous results concerning radiation; see [22] for an up-to-date review.

Our main results (Theorem 1.4 and Theorem 1.9 below) are relations between the motion of matter and the radiation flux at infinity for the Vlasov-Maxwell and Vlasov-Nordström systems respectively. They are analogues of the Einstein quadrupole formula [23, (4.5.13)] which is a basic tool in computing the flux of gravitational waves from a given source. In the case of the Einstein and Maxwell equations a spherically symmetric system does not radiate. For the Vlasov-Nordström system a spherical system can radiate and the specialization of the general formula to that case is computed. In [21] a difference between the spherically symmetric and the general case was claimed but we have not succeeded in connecting this to our results. The main theorems are obtained under plausible assumptions on the behavior of global solutions of the relevant system (Assumption 1.1 and Assumption 1.6 below). The former can be proved to hold in the case of small data.

For the systems we are going to consider the (scalar) energy density e and the (vector) momentum density $\mathcal{P}$ are related by the conservation law

$$\partial_t e + \nabla \cdot \mathcal{P} = 0.$$

Defining the local energy in the ball of radius $r > 0$ as

$$\mathcal{E}_r(t) = \int_{|x|\le r} e(t,x)\,dx,$$

this conservation law and the divergence theorem imply that

$$\begin{aligned}\frac{d}{dt}\,\mathcal{E}_r(t) &= \int_{|x|\le r} \partial_t e(t,x)\,dx = -\int_{|x|\le r} \nabla\cdot\mathcal{P}(t,x)\,dx\\ &= -\int_{|x|=r} \bar{x}\cdot\mathcal{P}(t,x)\,d\sigma(x),\end{aligned} \tag{1.1}$$

where $\bar{x} = \frac{x}{|x|}$ denotes the outer unit normal. More specifically, for the relativistic Vlasov-Maxwell system with two particle species,

$$\begin{aligned}e_{\mathrm{RVM}}(t,x) &= c^2 \int \sqrt{1+c^{-2}p^2}\,(f^+ + f^-)(t,x,p)\,dp\\ &\quad + \frac{1}{8\pi}\left(|E(t,x)|^2 + |B(t,x)|^2\right),\end{aligned} \tag{1.2}$$

$$\mathcal{P}_{\mathrm{RVM}}(t,x) = c^2 \int p(f^+ + f^-)(t,x,p)\,dp + \frac{c}{4\pi}\,E(t,x)\times B(t,x), \tag{1.3}$$

whereas for the Vlasov-Nordström system,

$$
\begin{aligned}
e_{\mathrm{VN}}(t,x) &= c^2 \int \sqrt{1+c^{-2}p^2}\, f(t,x,p)\,dp \\
&\quad + \frac{c^2}{8\pi}\left((\partial_t\phi(t,x))^2 + c^2|\nabla\phi(t,x)|^2\right), \qquad (1.4)\\
\mathcal{P}_{\mathrm{VN}}(t,x) &= c^2 \int p f(t,x,p)\,dp - \frac{c^4}{4\pi}\,\partial_t\phi(t,x)\,\nabla\phi(t,x).
\end{aligned}
$$

Our assumptions on the support of the distribution function will be such that the contributions of $\int p(f^+ + f^-)\,dp$ to $\mathcal{P}_{\mathrm{RVM}}$ and $\int pf\,dp$ to $\mathcal{P}_{\mathrm{VN}}$ vanish for $|x| = r$ large. Hence we arrive at

$$
\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{RVM}}(t) = \frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t,x)\,d\sigma(x)
$$

for the relativistic Vlasov-Maxwell system, and

$$
\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{VN}}(t) = \frac{c^4}{4\pi}\int_{|x|=r} \bar{x}\cdot(\partial_t\phi\nabla\phi)(t,x)\,d\sigma(x)
$$

for the Vlasov-Nordström system.

The main results of this paper are concerned with the expansion of these energy fluxes for $r, c \to \infty$ and $|t - c^{-1}r| \leq$ const. Under suitable assumptions we will prove that, to leading order,

$$
\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{RVM}}(t) \sim -\frac{2}{3c^3}\,|\partial_t^2\mathcal{D}(u)|^2,
$$

where $u = t - c^{-1}r$ denotes the retarded time and $\mathcal{D}(u) = \int x\,\rho_0(u,x)\,dx$ is the dipole moment associated to the Newtonian limit of the relativistic Vlasov-Maxwell system. Similarly,

$$
\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{VN}}(t) \sim -\frac{1}{4\pi c^5}\int_{|\omega|=1} (\partial_t\mathcal{R}(\omega,u))^2\,d\sigma(\omega),
$$

with a more complicated radiation term $\mathcal{R}$ associated to the Newtonian limit of the Vlasov-Nordström system. In the spherically symmetric case, $\partial_t\mathcal{R}(\omega,u)$ is found to be proportional to $\partial_t\mathcal{E}_{\mathrm{kin}}(u)$, the change of kinetic energy of the Newtonian system. The exact statements are contained in Theorems 1.4 and 1.9 below.

1.1. Dipole radiation in the relativistic Vlasov-Maxwell system. The relativistic Vlasov-Maxwell system describes a large ensemble of particles which move at possibly relativistic speeds and interact only by the electromagnetic fields which the ensemble creates collectively. Collisions among the particles are assumed to be sufficiently rare to be neglected [13]. In order to see effects due to radiation damping it is necessary that there are at least two species of particles with different charge-to-mass ratios. For the sake of simplicity we assume that there are exactly two species with their masses normalized to unity and their charges normalized to plus and minus unity, respectively. The density of the positively and negatively charged particles in phase space is given by the non-negative distribution functions $f^{\pm} = f^{\pm}(t, x, p)$, depending on time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$, and momentum $p \in \mathbb{R}^3$. Their dynamics is governed by the relativistic Vlasov-Maxwell system

$$\left.\begin{array}{c} \partial_t f^{\pm} + \hat{p} \cdot \nabla_x f^{\pm} \pm (E + c^{-1}\hat{p} \times B) \cdot \nabla_p f^{\pm} = 0, \\ c\,\nabla \times E = -\partial_t B, \quad c\,\nabla \times B = \partial_t E + 4\pi j, \\ \nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0, \\ \rho = \int (f^+ - f^-)\,dp, \quad j = \int \hat{p}\,(f^+ - f^-)\,dp, \end{array}\right\} \quad \text{(RVMc)}$$

where

$$\hat{p} = \gamma p, \quad \gamma = (1 + c^{-2}p^2)^{-1/2}, \quad p^2 = |p|^2, \quad \text{and} \quad \int = \int_{\mathbb{R}^3}. \tag{1.5}$$

The electric field $E = E(t, x) \in \mathbb{R}^3$ and the magnetic field $B = B(t, x) \in \mathbb{R}^3$ satisfy the wave equations

$$(-\partial_t^2 + c^2\Delta)E = 4\pi(c^2\nabla\rho + \partial_t j) \quad \text{and} \quad (-\partial_t^2 + c^2\Delta)B = -4\pi c\,\nabla \times j. \tag{1.6}$$

In order to determine the radiation of the system at infinity, we have to consider solutions that are isolated from incoming radiation. For the wave equations in (1.6), this means that we need to restrict ourselves to the retarded part of the solutions. Accordingly, (RVMc) is replaced by

$$\left.\begin{array}{c} \partial_t f^{\pm} + \hat{p} \cdot \nabla_x f^{\pm} \pm (E + c^{-1}\hat{p} \times B) \cdot \nabla_p f^{\pm} = 0, \\ E(t, x) = -\int (\nabla\rho + c^{-2}\partial_t j)(t - c^{-1}|y - x|, y)\,\dfrac{dy}{|y - x|}, \\ B(t, x) = c^{-1}\int \nabla \times j(t - c^{-1}|y - x|, y)\,\dfrac{dy}{|y - x|}, \\ \rho = \int (f^+ - f^-)\,dp, \quad j = \int \hat{p}\,(f^+ - f^-)\,dp, \end{array}\right\} \quad \text{(retRVMc)}$$

which we call the retarded relativistic Vlasov-Maxwell system. The motivation for considering this system is as follows. In physics situations are often important where radiation impinging on the matter from far away has a negligible effect on the dynamics and it is therefore an appropriate idealization to use the retarded solution of the field equations. When this is done the specification of a solution requires only data for the matter, in contrast to what happens in the usual initial value problem for the corresponding system of equations. We prescribe initial data

$$f^{\pm}(0, x, p) = f^{\pm,\circ}(x, p), \quad x, p \in \mathbb{R}^3, \tag{1.7}$$

for the densities at $t = 0$; these data do not depend on c. However, the corresponding solution (f^+, f^-, E, B) does depend on c, but we do not make explicit this dependence through our notation. We refer to Remark 1.5(c) below for the case of initial data varying with c. Our standing assumption is that the initial data are non-negative, smooth, and compactly supported,

$$f^{\pm,\circ} \in C_0^{\infty}(\mathbb{R}^3 \times \mathbb{R}^3), \quad f^{\pm,\circ} \geq 0, \tag{1.8}$$

and we fix positive constants R_0, P_0, S_0 such that

$$f^{\pm,\circ}(x, p) = 0 \quad \text{for} \quad |x| \geq R_0 \quad \text{or} \quad |p| \geq P_0, \quad \text{and} \quad \|f^{\pm,\circ}\|_{W^{3,\infty}} \leq S_0. \tag{1.9}$$

Every solution of (retRVMc) satisfies the identity

$$f^{\pm}(t,x,p) = f^{\pm,\circ}(X^{\pm}(0,t,x,p), P^{\pm}(0,t,x,p)), \tag{1.10}$$

where $s \mapsto (X^{\pm}(s,t,x,p), P^{\pm}(s,t,x,p))$ solves the characteristic system

$$\dot{x} = \hat{p}, \quad \dot{p} = \pm(E + c^{-1}\hat{p} \times B), \tag{1.11}$$

with data $X^{\pm}(t,t,x,p) = x$ and $P^{\pm}(t,t,x,p) = p$. Hence $0 \le f^{\pm}(t,x,p) \le \|f^{\pm,\circ}\|_{\infty}$. In order to derive our results on radiation, we have to assume certain a priori bounds on the corresponding solutions of (retRVMc). In particular, the latter have to exist globally in time.

Assumption 1.1. *(a) For each* $c \ge 1$ *the system (retRVMc) has a unique solution* $f^{\pm} \in C^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$, $E \in C^2(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$, $B \in C^2(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$*, satisfying the initial condition (1.7).*

(b) There exists $P_1 > 0$ *such that* $f^{\pm}(t,x,p) = 0$ *for* $|p| \ge P_1$ *and all* $c \ge 1$*. In particular,* $f^{\pm}(t,x,p) = 0$ *for* $|x| \ge R_0 + P_1|t|$ *by (1.11).*

(c) For every $T > 0$, $R > 0$*, and* $P > 0$ *there exists a constant* $M_1(T,R,P) > 0$ *such that*

$$|\partial_t^{\alpha+1} f^{\pm}(t,x,p)| + |\partial_t^{\alpha}\nabla_x f^{\pm}(t,x,p)| \le M_1(T,R,P)$$

for $|t| \le T$, $|x| \le R$, $|p| \le P$*, and* $\alpha = 0, 1$*, uniformly in* $c \ge 1$.

Note that none of the constants in Assumption 1.1 may depend on c. The constants

$$R_0, P_0, S_0, P_1, M_1$$

from (1.9) and Assumption 1.1 are considered to be the "basic" ones. Any other constant which appears in an estimate is only allowed to depend on these. Checking the arguments from [7, 8], it can be shown that Assumption 1.1 holds at least for sufficiently "small" initial data $f^{\pm,\circ}$. A more precise investigation of the set of initial data leading to solutions which satisfy Assumption 1.1 is not part of this paper. The main point we want to make here is that whenever Assumption 1.1 is verified, then the technique described below can be employed.

We will need estimates relating the solutions of (retRVMc) to the corresponding Newtonian problem obtained in the limit $c \to \infty$. This sort of information usually goes under the name of post-Newtonian approximation; see [19, 4]. For this, one formally expands the solutions in powers of c^{-1} as

$$\begin{aligned} f^{\pm} &= f_0^{\pm} + c^{-1} f_1^{\pm} + c^{-2} f_2^{\pm} + \cdots, \\ E &= E_0 + c^{-1} E_1 + c^{-2} E_2 + \cdots, \\ B &= B_0 + c^{-1} B_1 + c^{-2} B_2 + \cdots, \end{aligned}$$

with coefficient functions $f_j^{\pm}$, E_j, and B_j independent of c. Moreover, by (1.5),

$$\hat{p} = p - (c^{-2}/2)p^2 p + \cdots, \quad \gamma = 1 - (c^{-2}/2)p^2 + \cdots.$$

These expansions can be substituted into (retRVMc), and comparing coefficients at every order gives a sequence of equations for the coefficients. The Newtonian limit of (retRVMc) is given by the plasma physics case of the Vlasov-Poisson system:

$$\left.\begin{aligned} \partial_t f_0^\pm + p\cdot\nabla_x f_0^\pm \pm E_0\cdot\nabla_p f_0^\pm &= 0,\\ E_0(t,x) &= \int \frac{x-y}{|x-y|^3}\,\rho_0(t,y)\,dy,\\ \rho_0 &= \int (f_0^+ - f_0^-)\,dp,\\ f_0^\pm(0,x,p) &= f^{\pm,\circ}(x,p). \end{aligned}\right\} \tag{VPpl}$$

The following proposition addresses the well-known solvability properties of (VPpl). Clearly, (f_0^+, f_0^-, E_0) is independent of c, and we refer to e.g. [20, 16] for the regularity of the solution.

Proposition 1.2. *There are constants $R_2, P_2 > 0$, and for every $T > 0$, $R > 0$, and $P > 0$, there is a constant $M_2(T,R,P) > 0$, with the following properties. For initial data $f^{\pm,\circ}$ as above, there exists a unique global solution $(f_0^\pm, E_0)$ of (VPpl) so that*

(a) $f_0^\pm \in C^\infty(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ and $E_0 \in C^\infty(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$,
(b) if $|t|\le 1$, then $f_0^\pm(t,x,p) = 0$ for $|x|\ge R_2$ or $|p|\ge P_2$,
(c) if $|t|\le T$, $|x|\le R$, $|p|\le P$, and $\alpha = 0, 1$, then

$$|\partial_t^\alpha f_0^\pm(t,x,p)| + |\partial_t^\alpha E_0(t,x)| \le M_2(T,R,P).$$

For the approximation of solutions of (retRVMc) by solutions of (VPpl), we state the following result without proof; the result is derived like the analogous one for (RVMc), cf. [19, 4].

Proposition 1.3. *Under Assumption 1.1 there exist for every $T > 0$, $R > 0$, and $P > 0$ constants $M_3(T,R,P) > 0$ and $M_4(T,R) > 0$ with the following property. If $c \ge 2P_1$ and if $(f^\pm, E, B)$ and $(f_0^\pm, E_0)$ denote the global solutions of (retRVMc) and (VPpl) provided by Assumption 1.1 and Proposition 1.2, respectively, with initial data as above, then*

(a) $|f^\pm(t,x,p) - f_0^\pm(t,x,p)| \le M_3(T,R,P)\,c^{-2}$ for $|t|\le T$, $|x|\le R$, and $|p|\le P$,
(b) $|E(t,x) - E_0(t,x)| \le M_4(T,R)c^{-2}$ for $|t|\le T$ and $|x|\le R$,
(c) $|B(t,x)| \le M_4(T,R)\,c^{-1}$ for $|t|\le T$ and $|x|\le R$.

It is important to note that all the "derived" constants R_2, P_2, M_2, M_3, M_4 appearing above do only depend on the basic constants R_0, P_0, S_0, P_1, M_1. In order to determine the constants $M_3(T,R,P)$ and $M_4(T,R)$ for given parameters $T > 0$, $R > 0$, and $P > 0$ the constant $M_1(T', R', P')$ from Assumption 1.1 is needed for certain parameters $T' > T$, $R' > R$, $P' > P$. We are now ready to state our first main result.

Theorem 1.4 (Radiation for (retRVMc)). *Put $r_* = \max\{2(R_0 + P_1), R_2\}$ and*

$$\mathcal{M}_{\mathrm{RVM}} = \{(t,r,c) : r \ge 2r_*,\ c \ge 2P_1,\ |t - c^{-1}r| \le 1,\ r \ge c^3\}.$$

If $(t,r,c)\in\mathcal{M}_{\mathrm{RVM}}$, then with $r = |x|$, $\bar{x} = \frac{x}{|x|}$, and $u = t - c^{-1}|x|$,

$$\left|\bar{x}\cdot(B\times E)(t,x) + c^{-4}r^{-2}\,|\bar{x}\times\partial_t^2\mathcal{D}(u)|^2\right| \le A(c^{-5}r^{-2} + c^{-2}r^{-3} + c^{-1}r^{-4}), \tag{1.12}$$

for a constant $A > 0$ depending only on R_0, P_0, S_0, P_1, M_1. In particular,

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{RVM}}(t) = \frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t,x)\,d\sigma(x)$$
$$= -\frac{2}{3c^3}\,|\partial_t^2\mathcal{D}(u)|^2 + \mathcal{O}(c^{-4} + c^{-1}r^{-1} + r^{-2}) \quad (1.13)$$

for $(t, r, c) \in \mathcal{M}_{\mathrm{RVM}}$. Here $\mathcal{E}_r^{\mathrm{RVM}}(t) = \int_{|x|\le r} e_{\mathrm{RVM}}(t,x)\,dx$, see (1.2), and

$$\mathcal{D}(u) = \int x\,\rho_0(u,x)\,dx$$

denotes the dipole moment associated to the Vlasov-Poisson system (VPpl).

Remark 1.5. (a) The condition $r \ge c^3$ in $\mathcal{M}_{\mathrm{RVM}}$ is not needed for the proof of (1.12) and (1.13). It just guarantees that $c^{-2}r^{-3} \le c^{-5}r^{-2}$ and $c^{-1}r^{-1} \le c^{-4}$.

(b) The same estimate (1.12) can be derived, possibly with a different constant A, if the condition $|u| \le 1$ is replaced by $|u| \le u_0$ for some $u_0 > 0$.

(c) As long as the constants R_0, P_0, S_0, P_1, M_1 remain independent of c, one can also allow for c-dependent initial data $f_c^{\pm,\circ}$, both for (retRVMc) and (VPpl). However, in this case the functions $(f_0^{\pm}, E_0)$ become c-dependent, too. For instance, in the particular case

$$f_c^{\pm,\circ} = f_0^{\pm,\circ} + c^{-1}f_1^{\pm,\circ} + c^{-2}f_{r,c}^{\pm,\circ},$$

with $f_0^{\pm,\circ}$, $f_1^{\pm,\circ}$, and $f_{r,c}^{\pm,\circ}$ satisfying suitable bounds (independently of c for $f_{r,c}^{\pm,\circ}$), Theorem 1.4 remains valid, if $f_0^{\pm}$ and E_0 are replaced by the approximations $\tilde{f}_0^{\pm} + c^{-1}\tilde{f}_1^{\pm}$ and $\tilde{E}_0 + c^{-1}\tilde{E}_1$, respectively. Here $(\tilde{f}_0^{\pm}, \tilde{E}_0)$ is the solution of (VPpl) for the initial data $f_0^{\pm,\circ}$, and $(\tilde{f}_1^{\pm}, \tilde{E}_1)$ solves the Vlasov-Poisson system linearized about $(\tilde{f}_0^{\pm}, \tilde{E}_0)$, under the initial condition $\tilde{f}_1^{\pm}(0) = f_1^{\pm,\circ}$.

(d) In the case of one species only, say $f^{-,\circ} = 0$, there is no dipole radiation, since then $\partial_t^2\mathcal{D} = 0$, cf. (2.13) below.

(e) For spherically symmetric solutions there is again no dipole radiation. In fact, if $\rho_0(t,-x) = \rho_0(t,x)$ for $x \in \mathbb{R}^3$, then $\mathcal{D} = 0$ by symmetry.

The proof of Theorem 1.4 is given in Sect. 2.1.

1.2. Monopole radiation in the Vlasov-Nordström system. If we set all physical constants (except the speed of light c) equal to unity, then the Vlasov-Nordström system is given by

$$\left.\begin{aligned} \partial_t f + \hat{p}\cdot\nabla_x f - \big[(S\phi)p + c^2\gamma\nabla\phi\big]\cdot\nabla_p f &= 4(S\phi)f,\\ (-\partial_t^2 + c^2\Delta)\phi &= 4\pi\mu,\\ \mu &= \int \gamma f\,dp, \end{aligned}\right\} \quad \text{(VNc)}$$

where we continue to use the notation from (1.5), and where $S = \partial_t + \hat{p}\cdot\nabla$. The matter distribution is modeled through the nonnegative density function $f = f(t,x,p)$, whereas the scalar function $\phi = \phi(t,x)$ describes the gravitational field. We refer to

[6, 11, 1, 9] for the global existence of smooth solutions to (VNc). In analogy to the passage from (RVMc) to (retRVMc), the solutions of (VNc) that are isolated from incoming radiation are the solutions of the retarded system

$$\left.\begin{aligned} \partial_t f + \hat{p}\cdot\nabla_x f - \left[(S\phi)p + c^2\gamma\nabla\phi\right]\cdot\nabla_p f &= 4(S\phi)f,\\ \phi(t,x) = -c^{-2}\int \mu(t - c^{-1}|y-x|, y)\,\frac{dy}{|y-x|}&,\\ \mu = \int \gamma f\,dp&, \end{aligned}\right\} \tag{retVNc}$$

which we call the retarded Vlasov-Nordström system. We continue to make the standing hypotheses (1.8) and (1.9) for the initial data $f(0,x,p) = f^{\circ}(x,p)$ of (retVNc). At this point it should be noted that the "physical" particle density on the mass shell in the metric $e^{2\phi}\mathrm{diag}(-1,1,1,1)$ is not f but $e^{-4\phi(t,x)}f(t,x,e^{\phi}p)$. In particular, the density f used in the formulation above is not constant along solutions of the characteristic system

$$\dot{x} = \hat{p}, \quad \dot{p} = -(S\phi)p - c^2\gamma\nabla\phi, \tag{1.14}$$

but satisfies the relation

$$f(t,x,p) = f^{\circ}(X(0,t,x,p), P(0,t,x,p))e^{4\phi(t,x)-4\phi(0,X(0,t,x,p))}, \tag{1.15}$$

where $s \mapsto (X(s,t,x,p), P(s,t,x,p))$ denotes the solution of (1.14) with $X(t,t,x,p) = x$, $P(t,t,x,p) = p$. For technical reasons we prefer to work with the above formulation of the system in terms of the 'unphysical' density f, and we make the following assumptions on the solutions of (retVNc).

Assumption 1.6. *(a) For each $c \ge 1$ the system (retVNc) has a unique solution $f \in C^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$, $\phi \in C^2(\mathbb{R}\times\mathbb{R}^3)$, satisfying the initial condition $f(0,x,p) = f^{\circ}(x,p)$.*

(b) There exists $P_1 > 0$ such that $f(t,x,p) = 0$ for $|p| \ge P_1$ and all $c \ge 1$; by (1.15), (1.14) this implies that $f(t,x,p) = 0$ for $|x| \ge R_0 + P_1|t|$.

(c) For every $T > 0$, $R > 0$, and $P > 0$ there exists a constant $M_1(T,R,P) > 0$ such that

$$|\partial_t^{\alpha} f(t,x,p)| \le M_1(T,R,P)$$

for $|t| \le T$, $|x| \le R$, $|p| \le P$, and $\alpha = 0,1,2$. In addition, for every $T > 0$ and $R > 0$ there exists a constant $M_1(T,R) > 0$ such that

$$|\phi(t,x)| + |\nabla\phi(t,x)| + |\partial_t\phi(t,x)| \le M_1(T,R)$$

for $|t| \le T$ and $|x| \le R$, uniformly in $c \ge 1$.

Again R_0, P_0, S_0, P_1, M_1 are considered to be the "basic" constants, all other constants being derived from these. We remark that for "small" initial data the existence of global-in-time solutions is shown in [12], where also bounds on the solutions are obtained. It is reasonable to expect that these solutions have the required regularity for smooth initial data, cf. [16], and that on compact time intervals estimates as in Assumption 1.6 (c) can be derived uniformly in c. The crucial assumption is the bound on the momentum support in part (b), which needs to be uniform in c as well.

The Newtonian approximation for $c \to \infty$ of (retVNc) is found by means of the formal expansion

$$f = f_0 + c^{-1} f_1 + c^{-2} f_2 + \cdots ,$$
$$\phi = \phi_0 + c^{-1}\phi_1 + c^{-2}\phi_2 + c^{-3}\phi_3 + c^{-4}\phi_4 + \cdots ,$$

see [10, 2]. Thereby it is verified that this (lowest order) Newtonian approximation of (retVNc) is given by the gravitational case of the Vlasov-Poisson system

$$\left.\begin{aligned} \partial_t f_0 + p \cdot \nabla_x f_0 - \nabla\phi_2 \cdot \nabla_p f_0 = 0, \\ \phi_2(t,x) = -\int \frac{\rho_0(t,y)}{|x-y|}\, dy, \\ \rho_0 = \int f_0\, dp, \\ f_0(0,x,p) = f^\circ(x,p). \end{aligned}\right\} \qquad \text{(VPgr)}$$

The analogue of Proposition 1.2 is valid for (VPgr). Note that (f_0, ϕ_2) is independent of c.

Proposition 1.7. *There are constants $R_2, P_2 > 0$, and for every $T > 0$, $R > 0$, and $P > 0$, there is a constant $M_2(T, R, P) > 0$, with the following properties. For initial data f° as above, there exists a unique global solution (f_0, ϕ_2) of (VPgr) so that*

(a) $f_0 \in C^\infty(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $\phi_2 \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$,
(b) if $|t| \le 1$, then $f_0(t,x,p) = 0$ for $|x| \ge R_2$ or $|p| \ge P_2$,
(c) if $|t| \le T$, $|x| \le R$, $|p| \le P$, and $\alpha = 0, 1, 2$, then

$$|\partial_t^\alpha f_0(t,x,p)| + |\partial_t^{\alpha+1}\phi_2(t,x)| + |\partial_t^\alpha \nabla\phi_2(t,x)| \le M_2(T,R,P).$$

By [3], we also have the following rigorous result concerning the Newtonian limit of (retVNc).

Proposition 1.8. *Choose the constants $P_1 > 0$ and $M_1(T,R,P) > 0$ according to Assumption 1.6. Then for every $T > 0$, $R > 0$, and $P > 0$ there are constants $M_3(T,R,P) > 0$ and $M_4(T,R) > 0$ with the following properties. If $c \ge 2P_1$, let (f, ϕ) and (f_0, ϕ_2) denote the global solutions of (retVNc) and (VPgr) provided by Assumption 1.6 and Proposition 1.7, respectively, with initial data as above. Then*

(a) $|f(t,x,p) - f_0(t,x,p)| \le M_3(T,R,P)\, c^{-2}$ for $|t| \le T$, $|x| \le R$, and $|p| \le P$,
(b) $|\nabla\phi(t,x)| \le M_4(T,R)c^{-2}$ for $|t| \le T$ and $|x| \le R$,
(c) $|\partial_t\phi(t,x) - c^{-2}\partial_t\phi_2(t,x)| + |\nabla\phi(t,x) - c^{-2}\nabla\phi_2(t,x)| \le M_4(T,R)\, c^{-4}$ for $|t| \le T$ and $|x| \le R$.

After these preparations we can state our second main result.

Theorem 1.9 (Radiation for (retVNc)). *Put $r_* = \max\{2(R_0 + P_1), R_2\}$ and*

$$\mathcal{M}_{\mathrm{VN}} = \{(t,r,c) : r \ge 2r_*,\ c \ge 2P_1,\ |t - c^{-1}r| \le 1,\ r \ge c^6\}.$$

If $(t,r,c) \in \mathcal{M}_{\mathrm{VN}}$, then with $r = |x|$, $\bar{x} = \frac{x}{|x|}$, and $u = t - c^{-1}|x|$,

$$\left|\bar{x} \cdot (\partial_t\phi\nabla\phi)(t,x) + c^{-9}r^{-2}\,(\partial_t\mathcal{R}(\bar{x},u))^2\right| \le A(c^{-10}r^{-2} + c^{-4}r^{-3}), \qquad (1.16)$$

for a constant $A > 0$ depending only on R_0, P_0, P_1, M_1, S_0. In particular,

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{VN}}(t) = \frac{c^4}{4\pi}\int_{|x|=r} \bar{x}\cdot(\partial_t\phi\nabla\phi)(t,x)\,d\sigma(x)$$
$$= -\frac{1}{4\pi c^5}\int_{|\omega|=1}(\partial_t\mathcal{R}(\omega,u))^2\;d\sigma(\omega) + \mathcal{O}(c^{-6}+r^{-1}) \qquad (1.17)$$

for $(t,r,c)\in\mathcal{M}_{\mathrm{VN}}$. Here $\mathcal{E}_r^{\mathrm{VN}}(t) = \int_{|x|\le r} e_{\mathrm{VN}}(t,x)\,dx$, see (1.4), and

$$\mathcal{R}(\bar{x},u) = -\frac{1}{4\pi}\int|\bar{x}\cdot\nabla\phi_2(u,y)|^2\,dy$$
$$-\iint(\bar{x}\cdot p)^2 f_0(u,y,p)\,dp\,dy + 4\,\mathcal{E}_{\mathrm{kin}}(u), \qquad (1.18)$$

where

$$\mathcal{E}_{\mathrm{kin}}(t) = \frac{1}{2}\iint p^2 f_0(t,x,p)\,dx\,dp \qquad (1.19)$$

denotes the kinetic energy associated to the Vlasov-Poisson system (VPgr).

Defining $\mathcal{E}_{\mathrm{pot}}(t) = -\frac{1}{8\pi}\int|\nabla\phi_2(t,x)|^2\,dx$, the total energy $\mathcal{E}(t) = \mathcal{E}_{\mathrm{kin}}(t) + \mathcal{E}_{\mathrm{pot}}(t)$ is conserved along solutions of (VPgr).

Remark 1.10. (a) Once again the condition $r \ge c^6$ in $\mathcal{M}_{\mathrm{VN}}$ is not needed for the proof of (1.16). It only has to be included in order that the second error term $\mathcal{O}(c^{-4}r^{-3})$ is at least as good as the first one, which is $\mathcal{O}(c^{-10}r^{-2})$.

(b) In the sense of Remark 1.5 (b) and (c), one could allow for $|u| \le u_0$ and/or c-dependent initial data.

For spherically symmetric solutions, Theorem 1.9 simplifies as follows.

Corollary 1.11 (Radiation for spherically symmetric solutions to (retVNc)). *Define $r_* = \max\{2(R_0+P_1), R_2\}$ and*

$$\mathcal{M}_{\mathrm{VN}} = \left\{(t,r,c) : r \ge 2r_*,\ c \ge 2P_1,\ |t-c^{-1}r| \le 1,\ r \ge c^6\right\}.$$

If $(t,r,c)\in\mathcal{M}_{\mathrm{VN}}$, then with $r=|x|$ and $u = t-c^{-1}r$,

$$\left|(\partial_t\phi\,\partial_r\phi)(t,x) + \frac{64}{9}\,c^{-9}r^{-2}\,(\partial_t\mathcal{E}_{\mathrm{kin}}(u))^2\right| \le A(c^{-10}r^{-2}+c^{-4}r^{-3}),$$

for a constant $A > 0$ depending only on R_0, P_0, S_0, P_1, M_1. In particular,

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{VN}}(t) = \frac{c^4}{4\pi}\int_{|x|=r}(\partial_t\phi\,\partial_r\phi)(t,x)\,d\sigma(x) = -\frac{64}{9c^5}\,(\partial_t\mathcal{E}_{\mathrm{kin}}(u))^2 + \mathcal{O}(c^{-6}+r^{-1})$$

for $(t,r,c)\in\mathcal{M}_{\mathrm{VN}}$.

The proofs of Theorem 1.9 and Corollary 1.11 are carried out in Sect. 2.2.

2. Proofs

2.1. Proof of Theorem 1.4. To expand $E(t,x)$ and $B(t,x)$ as given by (retRVMc), we recall from Assumption 1.1 (b) that $f^{\pm}(t,x,p)=0$ for $|x|\geq R_0+P_1|t|$. It follows that $\rho(t,x)=0$ and $j(t,x)=0$ for $|x|\geq R_0+P_1|t|$. If $(t,x,c)\in\mathcal{M}_{\mathrm{RVM}}$, then $|u|=|t-c^{-1}|x||\leq 1$ and $c\geq 2P_1$. Thus if $|y|\geq 2(R_0+P_1)$, then

$$\begin{aligned}R_0+P_1|t-c^{-1}|y-x||&=R_0+P_1|u+c^{-1}|x|-c^{-1}|y-x||\\&\leq R_0+P_1(|u|+c^{-1}|y|)\leq R_0+P_1(1+(2P_1)^{-1}|y|)\leq |y|.\end{aligned}$$

Hence $F(t-c^{-1}|y-x|,y)=0$ for both $F=-(\nabla\rho+c^{-2}\partial_t j)$ or $F=c^{-1}\nabla\times j$. Thus for the y-integrals defining E and B in (retRVMc), it is sufficient to extend these over the ball $|y|\leq\max\{2(R_0+P_1),R_2\}=r_*$.

In what follows $g=\mathcal{O}(c^{-k}r^{-l})$ denotes a function such that

$$|g(t,x)|\leq Ac^{-k}r^{-l}\quad\text{for all}\quad |x|=r\geq 2r_*,\quad c\geq 2P_1,\quad\text{and}\quad |t-c^{-1}|x||\leq 1,$$

with A only depending on the basic constants. The following lemma states a representation for E and B similar to the Friedlander radiation field; see [15, p. 91/92] and [8].

Lemma 2.1. *The fields can be written as*

$$E(t,x)=E^{\mathrm{rad}}(t,x)+\mathcal{O}(r^{-2})\quad\textit{and}\quad B(t,x)=B^{\mathrm{rad}}(t,x)+\mathcal{O}(c^{-1}r^{-2}),$$

where

$$E^{\mathrm{rad}}(t,x)=-r^{-1}\int_{|y|\leq r_*}(\nabla\rho+c^{-2}\partial_t j)(u+c^{-1}\,\bar{x}\cdot y,y)\,dy,\tag{2.1}$$

$$B^{\mathrm{rad}}(t,x)=c^{-1}r^{-1}\int_{|y|\leq r_*}\nabla\times j(u+c^{-1}\,\bar{x}\cdot y,y)\,dy.\tag{2.2}$$

Proof. Consider E first, and let $F=-(\nabla\rho+c^{-2}\partial_t j)$. According to Assumption 1.1 (c), we have $|F(\ldots)|\leq AM_1(1+r_*,r_*,p_*)=\mathcal{O}(1)$ for some constant $A>0$, where $p_*=\max\{P_1,P_2\}$ and

$$(\ldots)=(t-c^{-1}|y-x|,y)=(u+c^{-1}|x|-c^{-1}|y-x|,y).$$

If $|x|=r\geq 2r_*$ and $|y|\leq r_*$, then $\frac{|x|}{|y-x|}\leq\frac{|x|}{|x|-r_*}\leq 2$. It follows that

$$\frac{1}{|y-x|}=\frac{1}{|x|}+\frac{|x|-|y-x|}{|y-x||x|}=r^{-1}+\mathcal{O}(r^{-2})$$

for all $|y|\leq r_*$. Therefore by (retRVMc),

$$\begin{aligned}E(t,x)&=\int F(\ldots)\,\frac{dy}{|y-x|}=\int_{|y|\leq r_*}F(\ldots)\frac{dy}{|y-x|}\\&=\int_{|y|\leq r_*}F(\ldots)\left(r^{-1}+\mathcal{O}(r^{-2})\right)dy\\&=r^{-1}\int_{|y|\leq r_*}F(\ldots)\,dy+\mathcal{O}(r^{-2}).\end{aligned}$$

Next we note that for $|y| \le r_*$ and $|x| = r \ge 2r_*$,

$$\begin{aligned} |x| - |x-y| &= |x| - |x|\sqrt{1 - 2\bar{x}\cdot y/|x| + |y|^2/|x|^2} \\ &= |x| - |x|\left(1 + \frac{1}{2}\left(-2\bar{x}\cdot y/|x| + |y|^2/|x|^2\right) + \mathcal{O}(r^{-2})\right) \\ &= \bar{x}\cdot y + \mathcal{O}(r^{-1}). \end{aligned} \tag{2.3}$$

Since

$$|F(\ldots) - F(u + c^{-1}\,\bar{x}\cdot y, y)| \le \|\partial_t F\|_{L^\infty} c^{-1} ||x| - |y-x| - \bar{x}\cdot y| = \mathcal{O}(c^{-1}r^{-1})$$

by Assumption 1.1 (c) and (2.3), we get $E = E^{\mathrm{rad}} + \mathcal{O}(r^{-2})$. The proof for the magnetic field is analogous, using $F = c^{-1}\nabla \times j$. □

Now we need to investigate the relation between E^{rad} and B^{rad}. For this, we recall the continuity equation $\partial_t \rho + \nabla \cdot j = 0$ and calculate

$$\begin{aligned} \nabla\rho(*) &= \nabla_y\,[\rho(*)] + c^{-1}\bar{x}\,\nabla_y\cdot[j(*)] - c^{-2}(\bar{x}\cdot\partial_t j(*))\,\bar{x}, \\ \nabla\times j(*) &= \nabla_y\times[j(*)] - c^{-1}\bar{x}\times\partial_t j(*), \end{aligned}$$

where

$$(*) = (u + c^{-1}\,\bar{x}\cdot y, y)$$

is the argument. This follows just from evaluating the total derivatives. Since $\int_{|y|\le r_*} dy = \int dy$ in (2.1) and (2.2) by Assumption 1.1 (b), integration by parts shows that all ∇_y-terms drop out. Consequently, due to $u = t - c^{-1}r$ the relations

$$\begin{aligned} E^{\mathrm{rad}}(t,x) &= -r^{-1}\int_{|y|\le r_*}[\nabla\rho(*) + c^{-2}\partial_t j(*)]\,dy \\ &= -r^{-1}\int_{|y|\le r_*}\left[-c^{-2}(\bar{x}\cdot\partial_t j(*))\,\bar{x} + c^{-2}\partial_t j(*)\right]dy \\ &= -c^{-2}r^{-1}\,\partial_t\int_{|y|\le r_*}\left[j(u + c^{-1}\,\bar{x}\cdot y, y) - \left(\bar{x}\cdot j(u + c^{-1}\,\bar{x}\cdot y, y)\right)\bar{x}\right]dy, \qquad (2.4) \\ B^{\mathrm{rad}}(t,x) &= c^{-1}r^{-1}\int_{|y|\le r_*}\nabla\times j(*)\,dy \\ &= -c^{-2}r^{-1}\,\partial_t\int_{|y|\le r_*}\bar{x}\times j(u + c^{-1}\,\bar{x}\cdot y, y)dy \end{aligned}$$

are obtained. Note that in particular E^{rad} and B^{rad} are of the same order in c^{-1} and r^{-1}, i.e.,

$$E^{\mathrm{rad}}(t,x) = B^{\mathrm{rad}}(t,x) = \mathcal{O}(c^{-2}r^{-1}) \tag{2.5}$$

by Assumption 1.1 (c). Observing

$$\bar{x}\times\int_{|y|\le r_*}[j(*) - (\bar{x}\cdot j(*))\bar{x}]\,dy = \int_{|y|\le r_*}\bar{x}\times j(*)\,dy,$$

differentiation w.r.t. to t yields the important formula

$$\bar{x} \times E^{\mathrm{rad}}(t,x) = B^{\mathrm{rad}}(t,x). \tag{2.6}$$

Also

$$\bar{x} \cdot \int_{|y|\le r_*} [j(*) - (\bar{x}\cdot j(*))\bar{x}]\, dy = 0,$$

so that

$$\bar{x} \cdot E^{\mathrm{rad}}(t,x) = \bar{x} \cdot B^{\mathrm{rad}}(t,x) = 0. \tag{2.7}$$

Collecting the results from Lemma 2.1 and (2.5), it follows that

$$\begin{aligned} \bar{x} \cdot (B \times E) &= \bar{x} \cdot \Big([B^{\mathrm{rad}} + \mathcal{O}(c^{-1}r^{-2})] \times [E^{\mathrm{rad}} + \mathcal{O}(r^{-2})] \Big) \\ &= \bar{x} \cdot \Big(B^{\mathrm{rad}} \times E^{\mathrm{rad}} + \mathcal{O}(c^{-2}r^{-3}) + \mathcal{O}(c^{-3}r^{-3}) + \mathcal{O}(c^{-1}r^{-4}) \Big) \\ &= -|\bar{x} \times E^{\mathrm{rad}}|^2 + \mathcal{O}(c^{-2}r^{-3}) + \mathcal{O}(c^{-1}r^{-4}), \end{aligned} \tag{2.8}$$

since by (2.6) and (2.7),

$$\begin{aligned} \bar{x} \cdot (B^{\mathrm{rad}} \times E^{\mathrm{rad}}) &= \bar{x} \cdot ([\bar{x} \times E^{\mathrm{rad}}] \times E^{\mathrm{rad}}) = \bar{x} \cdot ((\bar{x}\cdot E^{\mathrm{rad}})E^{\mathrm{rad}} - |E^{\mathrm{rad}}|^2 \bar{x}) \\ &= -|E^{\mathrm{rad}}|^2 = -|\bar{x} \times E^{\mathrm{rad}}|^2. \end{aligned}$$

Equations (2.8) and (2.4) imply

$$\begin{aligned} \bar{x} \cdot (B \times E) = -c^{-4}r^{-2} & \left| \bar{x} \times \partial_t \int_{|y|\le r_*} j(u + c^{-1}\,\bar{x}\cdot y, y)\, dy \right|^2 \\ & + \mathcal{O}(c^{-2}r^{-3}) + \mathcal{O}(c^{-1}r^{-4}). \end{aligned} \tag{2.9}$$

To expand the square as $c \to \infty$, we note that $|p| \ge p_* \ge P_1$ implies $f^{\pm}(t,y,p) = 0$ for all $t \in \mathbb{R}$ and all $y \in \mathbb{R}^3$ by Assumption 1.1 (b). Therefore we can always replace the average over momentum space $\int dp$ by $\int_{|p|\le p_*} dp$. For $|p| \le p_*$,

$$\nabla_p \hat{p} = \gamma\, \mathrm{id}_{\mathbb{R}^3} - c^{-2}\gamma^3\, p \otimes p = \mathrm{id}_{\mathbb{R}^3} + \mathcal{O}(c^{-2})$$

by (1.5). Furthermore, using Assumption 1.1,

$$\begin{aligned} \nabla_x (f^+ - f^-)(*) &= \nabla_y[(f^+ - f^-)(*)] - c^{-1}\bar{x}\, \partial_t (f^+ - f^-)(*) \\ &= \nabla_y[(f^+ - f^-)(*)] + \mathbf{1}_{\{|y|\le r_*,\, |p|\le p_*\}} \mathcal{O}(c^{-1}). \end{aligned}$$

Utilizing this, (retRVMc), and Proposition 1.3, we get, writing $(*, p) = (u + c^{-1}\,\bar{x} \cdot y, y, p)$,

$$\begin{aligned}\int_{|y|\le r_*} \partial_t j(*)\,dy &= \iint \hat{p}\,\partial_t (f^+ - f^-)(*, p)\,dp\,dy\\ &= \iint \hat{p}\Big(- \hat{p}\cdot\nabla_x (f^+ - f^-)(*, p)\\ &\quad - (E + c^{-1}\hat{p}\times B)\cdot\nabla_p (f^+ + f^-)(*, p)\Big)\,dp\,dy\\ &= \mathcal{O}(c^{-1}) + \int_{|y|\le r_*}\int_{|p|\le p_*} \nabla_p\hat{p}\,(E + c^{-1}\hat{p}\times B)\\ &\qquad\times (f^+ + f^-)(*, p)\,dp\,dy\\ &= \mathcal{O}(c^{-1}) + \int_{|y|\le r_*}\int_{|p|\le p_*} \Big(\mathrm{id}_{\mathbb{R}^3} + \mathcal{O}(c^{-2})\Big)\Big(E_0 + \mathcal{O}(c^{-2})\Big)\\ &\qquad\times\Big(f_0^+ + f_0^- + \mathcal{O}(c^{-2})\Big)(*, p)\,dp\,dy\\ &= \int_{|y|\le r_*}\int_{|p|\le p_*} E_0 (f_0^+ + f_0^-)(*, p)\,dp\,dy + \mathcal{O}(c^{-1}). \end{aligned} \tag{2.10}$$

Also

$$\begin{aligned}&|E_0(f_0^+ + f_0^-)(*, p) - E_0(f_0^+ + f_0^-)(u, y, p)|\\ &\quad\le \|\partial_t\left(E_0(f_0^+ + f_0^-)\right)\|_{L^\infty} c^{-1}|\bar{x}\cdot y| = \mathcal{O}(c^{-1})\end{aligned}$$

by Proposition 1.2 (c). Thus (2.9) and (2.10) yield

$$\begin{aligned}\bar{x}\cdot(B\times E) &= -c^{-4}r^{-2}\left|\bar{x}\times\int_{|y|\le r_*}\int_{|p|\le p_*} E_0(f_0^+ + f_0^-)(u, y, p)\,dp\,dy + \mathcal{O}(c^{-1})\right|^2\\ &\quad + \mathcal{O}(c^{-2}r^{-3}) + \mathcal{O}(c^{-1}r^{-4})\\ &= -c^{-4}r^{-2}\left|\bar{x}\times\iint E_0(f_0^+ + f_0^-)(u, y, p)\,dp\,dy\right|^2\\ &\quad + \mathcal{O}(c^{-5}r^{-2}) + \mathcal{O}(c^{-2}r^{-3}) + \mathcal{O}(c^{-1}r^{-4}), \end{aligned} \tag{2.11}$$

since by Proposition 1.2 (b), $f_0^{\pm}(u, y, p) = 0$ for $|y| \ge r_* \ge R_2$ or $|p| \ge p_* \ge P_2$. Defining the dipole moment

$$\mathcal{D}(t) = \int x\,\rho_0(t, x)\,dx$$

with ρ_0 from (VPpl), we obtain by the Vlasov equation in (VPpl) that

$$\begin{aligned}\partial_t\mathcal{D} &= \iint x\,\partial_t (f_0^+ - f_0^-)\,dp\,dx\\ &= -\iint x\,\left(p\cdot\nabla_x (f_0^+ - f_0^-) + E_0\cdot\nabla_p (f_0^+ + f_0^-)\right)\,dp\,dx\\ &= \iint p\,(f_0^+ - f_0^-)\,dp\,dx\end{aligned}$$

and

$$\begin{aligned}\partial_t^2 \mathcal{D} &= \iint p\, \partial_t\, (f_0^+ - f_0^-)\, dp\, dx \\ &= -\iint p\, \left(p \cdot \nabla_x (f_0^+ - f_0^-) + E_0 \cdot \nabla_p (f_0^+ + f_0^-)\right)\, dp\, dx \\ &= \iint E_0 (f_0^+ + f_0^-)\, dp\, dx. \end{aligned} \tag{2.12}$$

Due to (2.11) it follows that

$$\bar{x} \cdot (B \times E) = -c^{-4} r^{-2}\, |\bar{x} \times \partial_t^2 \mathcal{D}(u)|^2 + \mathcal{O}(c^{-5} r^{-2}) + \mathcal{O}(c^{-2} r^{-3}) + \mathcal{O}(c^{-1} r^{-4}),$$

which completes the proof of (1.12). Concerning (1.13), we have

$$\begin{aligned}\frac{d}{dt}\, \mathcal{E}_r^{\mathrm{RVM}}(t) = \frac{c}{4\pi} \int_{|x|=r} \bar{x} \cdot (B \times E)(t, x)\, d\sigma(x) \\ -c^2 \int_{|x|=r} \int (\bar{x} \cdot p)(f^+ + f^-)(t, x, p)\, dp\, d\sigma(x)\end{aligned}$$

by (1.1) and (1.3). For $|x| = r$ and $(t, r, c) \in \mathcal{M}_{\mathrm{RVM}}$, we can estimate $R_0 + P_1 |t| = R_0 + P_1 |u + c^{-1} r| \le R_0 + P_1 + \frac{r}{2} \le r = |x|$, since $r \ge 2r_* \ge 2(R_0 + P_1)$. Therefore $f^{\pm}(t, x, p) = 0$ by Assumption 1.1 (b), and this yields

$$\frac{d}{dt}\, \mathcal{E}_r^{\mathrm{RVM}}(t) = \frac{c}{4\pi} \int_{|x|=r} \bar{x} \cdot (B \times E)(t, x)\, d\sigma(x), \quad (t, r, c) \in \mathcal{M}_{\mathrm{RVM}}.$$

Hence for (1.13) it suffices to use (1.12) and to note that

$$\begin{aligned}-\frac{c^{-3} r^{-2}}{4\pi} \int_{|x|=r} |\bar{x} \times \partial_t^2 \mathcal{D}(u)|^2\, d\sigma(x) &= -\frac{1}{4\pi c^3} \int_{|\omega|=1} |\omega \times \partial_t^2 \mathcal{D}(u)|^2\, d\sigma(\omega) \\ &= -\frac{2}{3c^3}\, |\partial_t^2 \mathcal{D}(u)|^2\end{aligned}$$

by integration. □

Proof of Remark 1.5 (d). If $f_0^-(t = 0) = f^{-,\circ} = 0$, then also $f_0^- = 0$ by (1.10). Thus defining $\phi_0(t, x) = \iint |x - y|^{-1} f_0^+(t, y, p)\, dp\, dy = \int |x - y|^{-1} \rho_0(t, y)\, dy$, we get $E_0 = -\nabla \phi_0$ and $\Delta \phi_0 = -4\pi \rho_0$. Consequently, by (2.12),

$$\partial_t^2 \mathcal{D} = \iint E_0 f_0^+\, dp\, dx = \int E_0 \rho_0\, dx = \frac{1}{4\pi} \int \nabla \phi_0\, \Delta \phi_0\, dx = 0. \tag{2.13}$$

Hence there is no dipole radiation in this case. □

2.2. Proof of Theorem 1.9. By (retVNc), $\partial_t\phi$ and $\nabla\phi$ are given by

$$\partial_t\phi(t,x) = -c^{-2}\int \partial_t\mu(t-c^{-1}|y-x|,y)\,\frac{dy}{|y-x|},$$
$$\nabla\phi(t,x) = -c^{-2}\int \nabla\mu(t-c^{-1}|y-x|,y)\,\frac{dy}{|y-x|}.$$

In full analogy to Lemma 2.1, we obtain the following representation.

Lemma 2.2. *We can write*

$$\partial_t\phi(t,x)=(\partial_t\phi)^{\rm rad}(t,x)+\mathcal{O}(c^{-2}r^{-2}) \quad and \quad \nabla\phi(t,x)=(\nabla\phi)^{\rm rad}(t,x)+\mathcal{O}(c^{-2}r^{-2}),$$

where

$$(\partial_t\phi)^{\rm rad}(t,x) = -c^{-2}r^{-1}\int_{|y|\le r_*} \partial_t\mu(u+c^{-1}\,\bar{x}\cdot y,y)\,dy, \tag{2.14}$$
$$(\nabla\phi)^{\rm rad}(t,x) = -c^{-2}r^{-1}\int_{|y|\le r_*} \nabla\mu(u+c^{-1}\,\bar{x}\cdot y,y)\,dy. \tag{2.15}$$

Let again $(*) = (u + c^{-1}\bar{x}\cdot y, y)$ denote the argument. Then $\nabla_y[\mu(*)] = c^{-1}\bar{x}\,\partial_t\mu(*) + \nabla\mu(*)$. Since $\int_{|y|\le r_*} dy = \int dy$ in (2.15), it follows that

$$\begin{aligned}\bar{x}\cdot(\nabla\phi)^{\rm rad}(t,x) &= -c^{-2}r^{-1}\,\bar{x}\cdot\int \nabla\mu(*)\,dy\\ &= c^{-3}r^{-1}\,\bar{x}\cdot\int \bar{x}\,\partial_t\mu(*)\,dy\\ &= -c^{-1}(\partial_t\phi)^{\rm rad}(t,x).\end{aligned}$$

The same argument shows that $(\nabla\phi)^{\rm rad} = \mathcal{O}(c^{-3}r^{-1})$, and also $(\partial_t\phi)^{\rm rad} = \mathcal{O}(c^{-2}r^{-1})$ by (2.14). Hence we find from Lemma 2.2 that

$$\bar{x}\cdot(\partial_t\phi\,\nabla\phi)(t,x) = -c^{-5}r^{-2}\left|\int \partial_t\mu(*)\,dy\right|^2 + \mathcal{O}(c^{-4}r^{-3}). \tag{2.16}$$

In order to expand the square we use, following [14], the differential operators

$$T = c^{-1}\bar{x}\,\partial_t + \nabla \quad\text{and}\quad S = \partial_t + \hat{p}\cdot\nabla.$$

Then

$$\partial_t = (1-c^{-1}\hat{p}\cdot\bar{x})^{-1}(S-\hat{p}\cdot T)$$

and $\nabla_y[\mu(*)] = T\mu(*)$ is a total derivative. Hence the corresponding term drops out upon integration with respect to y. Observing the relation

$$\nabla_p\cdot[(S\phi)p + c^2\gamma\nabla\phi] = 3(S\phi),$$

the Vlasov equation in (retVNc) yields

$$\begin{aligned}\int \partial_t \mu(*)\, dy &= \iint \gamma\, \partial_t f(*, p)\, dp\, dy \\ &= \iint \gamma \Big(1 - c^{-1} \hat{p}\cdot\bar{x}\Big)^{-1} \Big(\Big[(S\phi)p + c^2 \gamma \nabla\phi\Big]\cdot \nabla_p f + 4(S\phi) f\Big)(*, p) dp dy \\ &= -\iint \nabla_p \left(\gamma\left(1 - c^{-1}\hat{p}\cdot\bar{x}\right)^{-1}\right)\cdot \Big[(S\phi)p + c^2\gamma\nabla\phi\Big]\, f(*, p)\, dp\, dy \\ &\quad + \iint \gamma \left(1 - c^{-1}\hat{p}\cdot\bar{x}\right)^{-1} (S\phi)\, f(*, p)\, dp\, dy, \end{aligned} \tag{2.17}$$

where $(*, p) = (u + c^{-1}\,\bar{x}\cdot y, y, p)$. A direct calculation shows that

$$\begin{aligned}\nabla_p \left(\gamma\, (1 - c^{-1}\hat{p}\cdot\bar{x})^{-1}\right) &= \nabla_p \left(\left(\sqrt{1 + p^2/c^2} - c^{-1} p\cdot\bar{x}\right)^{-1}\right) \\ &= \gamma^2 (1 - c^{-1}\hat{p}\cdot\bar{x})^{-2} (c^{-1}\bar{x} - c^{-2}\hat{p}).\end{aligned}$$

If $|p| \geq p_* \geq P_1$, then $f(*, p) = 0$ by Assumption 1.6 (b). Furthermore, if $|u| \leq 1$ and $|x| \geq 2r_*$, then $|y| \geq r_*$ enforces $f(*, p) = 0$ as before. Therefore we can replace $\iint dp\, dy$ by $\int_{|y|\leq r_*}\int_{|p|\leq p_*} dy\, dp$ in the integrals occurring in (2.17). In other words, we may always assume that both $|y|$ and $|p|$ are bounded, with a bound depending only on the basic constants. Accordingly,

$$\gamma = 1 + \mathcal{O}(c^{-2}), \quad \gamma^2 = 1 + \mathcal{O}(c^{-2}), \quad \hat{p} = \gamma p = p + \mathcal{O}(c^{-2}), \tag{2.18}$$

$$\Big(1 - c^{-1}\hat{p}\cdot\bar{x}\Big)^{-1} = 1 + \mathcal{O}(c^{-1}), \;\; \Big(1 - c^{-1}\hat{p}\cdot\bar{x}\Big)^{-2} = 1 + 2c^{-1} p\cdot\bar{x} + \mathcal{O}(c^{-2}). \tag{2.19}$$

This results in

$$\nabla_p \left(\gamma\, (1 - c^{-1}\hat{p}\cdot\bar{x})^{-1}\right) = c^{-1}\bar{x} + c^{-2}(2(p\cdot\bar{x})\bar{x} - p) + \mathcal{O}(c^{-3}). \tag{2.20}$$

Furthermore, since $|u + c^{-1}\bar{x}\cdot y| \leq 1 + r_*$, also

$$\begin{aligned} f(*, p) &= f_0(*, p) + \mathcal{O}(c^{-2}), \\ (S\phi)(*, p) &= c^{-2}(\tilde{S}\phi_2)(*, p) + \mathcal{O}(c^{-4}) = \mathcal{O}(c^{-2}), \\ \nabla\phi(*) &= c^{-2}\nabla\phi_2(*) + \mathcal{O}(c^{-4}), \end{aligned}$$

by Proposition 1.8 and (2.18), where $\tilde{S}\phi_2 = \partial_t\phi_2 + p\cdot\nabla\phi_2$. Observe that here the constants $M_3(1 + r_*, r_*, p_*)$ and $M_4(1 + r_*, r_*)$ enter the bounds on $\mathcal{O}(c^{-2})$ and $\mathcal{O}(c^{-4})$.

Hence from (2.17), (2.18), (2.19), and (2.20) we get

$$\begin{aligned}\int \partial_t \mu(*)\, dy = &-\int_{|y|\le r_*}\int_{|p|\le p_*} \Big(c^{-1}\bar{x} + c^{-2}(2(p\cdot\bar{x})\bar{x} - p) + \mathcal{O}(c^{-3})\Big) \\ &\qquad \times \Big[\mathcal{O}(c^{-2}) + \Big(1+\mathcal{O}(c^{-2})\Big)\Big(\nabla\phi_2 + \mathcal{O}(c^{-2})\Big)\Big] \\ &\qquad \times \Big(f_0(*,p) + \mathcal{O}(c^{-2})\Big)\, dp\, dy \\ &+\int_{|y|\le r_*}\int_{|p|\le p_*} \Big(1+\mathcal{O}(c^{-2})\Big)\Big(1+\mathcal{O}(c^{-1})\Big)\Big(c^{-2}(\tilde{S}\phi_2)+\mathcal{O}(c^{-4})\Big) \\ &\qquad \times \Big(f_0(*,p) + \mathcal{O}(c^{-2})\Big)\, dp\, dy \\ = &-c^{-1}\bar{x}\cdot\int_{|y|\le r_*}\int_{|p|\le p_*} (1+2c^{-1}p\cdot\bar{x})\nabla(\phi_2 f_0)(*,p)\, dp\, dy \\ &+c^{-2}\int_{|y|\le r_*}\int_{|p|\le p_*} (\tilde{S}\phi_2 + p\cdot\nabla\phi_2) f_0(*,p)\, dp\, dy + \mathcal{O}(c^{-3}).\end{aligned} \tag{2.21}$$

Let ψ denote either $\nabla\phi_2$ or $\partial_t\phi_2$. Then by Proposition 1.7 (c),

$$\begin{aligned}(\psi f_0)(*,p) &= (\psi f_0)(u + c^{-1}\,\bar{x}\cdot y, y, p) \\ &= (\psi f_0)(u,y,p) + c^{-1}(\bar{x}\cdot y)\,\partial_t(\psi f_0)(u,y,p) + \mathcal{O}(c^{-2}) \\ &= (\psi f_0)(u,y,p) + \mathcal{O}(c^{-1}).\end{aligned}$$

Hence (2.21) yields

$$\begin{aligned}\int \partial_t\mu(*)dy = &-c^{-1}\bar{x}\cdot\int_{|y|\le r_*}\int_{|p|\le p_*}\Big((\nabla\phi_2 f_0)(u,y,p) + c^{-1}(\bar{x}\cdot y)\,\partial_t(\nabla\phi_2 f_0)(u,y,p) \\ &\qquad + \mathcal{O}(c^{-2}) + 2c^{-1}(\bar{x}\cdot p)\,\nabla\phi_2 f_0(u,y,p)\Big)\, dp\, dy \\ &+c^{-2}\int_{|y|\le r_*}\int_{|p|\le p_*}(\tilde{S}\phi_2 + p\cdot\nabla\phi_2) f_0(u,y,p)\, dp\, dy + \mathcal{O}(c^{-3}) \\ = &\,\mathcal{O}(c^{-3}) - c^{-1}\int_{|y|\le r_*}(\bar{x}\cdot\nabla\phi_2)\,\rho_0(u,y)\, dy \\ &-c^{-2}\partial_t\int_{|y|\le r_*}(\bar{x}\cdot y)\,(\bar{x}\cdot\nabla\phi_2)\,\rho_0(u,y)\, dy \\ &+c^{-2}\int_{|y|\le r_*}\int_{|p|\le p_*}\Big(\tilde{S}\phi_2 + p\cdot\nabla\phi_2 - 2(\bar{x}\cdot p)\,(\bar{x}\cdot\nabla\phi_2)\Big) \\ &\qquad \times f_0(u,y,p)\, dp\, dy,\end{aligned} \tag{2.22}$$

recalling that $u = t - c^{-1}r$. In view of Proposition 1.7 (b) we may extend all integrals over the whole space again. Now

$$\int \nabla\phi_2\rho_0(u,y)\, dy = \iint \nabla\phi_2(u,y) f_0(u,y,p)\, dp\, dy = 0$$

by Lemma 2.3 below, whence the lowest order term drops out. In addition, Lemma 2.3 also shows that

$$\iint (\tilde{S}\phi_2)(u, y)\, f_0(u, y, p)\, dp\, dy = -2\, \partial_t \mathcal{E}_{\text{kin}}(u)$$

as well as

$$\iint (p \cdot \nabla\phi_2)(u, y)\, f_0(u, y, p)\, dp\, dy = -\partial_t \mathcal{E}_{\text{kin}}(u)$$

and

$$\iint (\bar{x} \cdot p)(\bar{x} \cdot \nabla\phi_2)(u, y)\, f_0(u, y, p)\, dp\, dy = -\frac{1}{2}\, \partial_t \iint (\bar{x} \cdot p)^2 f_0(u, y, p)\, dp\, dy.$$

Finally, we can also write

$$\int (\bar{x} \cdot y)(\bar{x} \cdot \nabla\phi_2)(u, y)\rho_0(u, y)\, dy = -\mathcal{E}_{\text{pot}}(u) - \frac{1}{4\pi} \int |\bar{x} \cdot \nabla\phi_2(u, y)|^2\, dy$$

by Lemma 2.3 and since $|\bar{x}| = 1$. Using this and $\partial_t \mathcal{E}_{\text{pot}} = -\partial_t \mathcal{E}_{\text{kin}}$ (see the remarks following (1.19)) in (2.22), and collecting all the terms, it follows that

$$\int \partial_t \mu(*)\, dy = -c^{-2}\, \partial_t \mathcal{R}(\bar{x}, u) + \mathcal{O}(c^{-3}), \tag{2.23}$$

with $\mathcal{R}(\bar{x}, u)$ as in (1.18). Inserting (2.23) into (2.16), we see that

$$\begin{aligned} \bar{x} \cdot (\partial_t\phi\nabla\phi)(t, x) &= -c^{-5}r^{-2} \left| -c^{-2}\partial_t\mathcal{R}(\bar{x}, u) + \mathcal{O}(c^{-3}) \right|^2 + \mathcal{O}(c^{-4}r^{-3}) \\ &= -c^{-9}r^{-2}\, (\partial_t\mathcal{R}(\bar{x}, u))^2 + \mathcal{O}(c^{-10}r^{-2}) + \mathcal{O}(c^{-4}r^{-3}). \end{aligned}$$

Therefore (1.16) is proved. Concerning (1.17), the fact that

$$\frac{d}{dt}\, \mathcal{E}_r^{\text{VN}}(t) = \frac{c^4}{4\pi} \int_{|x|=r} \bar{x} \cdot (\partial_t\phi\nabla\phi)(t, x)\, d\sigma(x)$$

is due to $(t, r, c) \in \mathcal{M}_{\text{VN}}$, analogously to the argument in the proof of Theorem 1.4. Hence (1.17) is a direct consequence of (1.16), changing variables as $x = r\omega$. □

We still need to give the proof of

Lemma 2.3. *For the Vlasov-Poisson system (VPgr),*

$$\iint \nabla\phi_2(t, x)\, f_0(t, x, p)\, dp\, dx = 0,$$

$$\iint (\tilde{S}\phi_2)(t, x)\, f_0(t, x, p)\, dp\, dx = -2\, \partial_t \mathcal{E}_{\text{kin}}(t),$$

$$\iint p \cdot \nabla\phi_2(t, x)\, f_0(t, x, p)\, dp\, dx = -\partial_t \mathcal{E}_{\text{kin}}(t),$$

$$\begin{aligned} &\iint (\xi \cdot p)(\xi \cdot \nabla\phi_2)(t, x)\, f_0(t, x, p)\, dp\, dx \\ &\quad = -\frac{1}{2}\, \partial_t \iint (\xi \cdot p)^2 f_0(t, x, p)\, dp\, dx \quad (\xi \in \mathbb{R}^3), \end{aligned}$$

$$\begin{aligned} &\iint (\xi \cdot x)(\xi \cdot \nabla\phi_2)(t, x)\, f_0(t, x, p)\, dp\, dx \\ &\quad = -|\xi|^2\, \mathcal{E}_{\text{pot}}(t) - \frac{1}{4\pi} \int |\xi \cdot \nabla\phi_2(t, x)|^2\, dx \quad (\xi \in \mathbb{R}^3), \end{aligned}$$

where $\mathcal{E}_{\text{kin}}(t) = \frac{1}{2} \iint p^2 f_0(t, x, p)\, dp\, dx$, *see (1.19), and* $\mathcal{E}_{\text{pot}}(t) = -\frac{1}{8\pi}\int |\nabla\phi_2(t, x)|^2 dx$.

Proof. Firstly, since $\Delta\phi_2 = 4\pi\rho_0$,

$$\begin{aligned}\iint \nabla\phi_2(t,x) f_0(t,x,p)\,dp\,dx &= \int \nabla\phi_2(t,x)\,\rho_0(t,x)\,dx\\ &= \frac{1}{4\pi}\int \nabla\phi_2(t,x)\,\Delta\phi_2(t,x)\,dx\\ &= \frac{1}{8\pi}\int \nabla\cdot|\nabla\phi_2(t,x)|^2\,dx = 0.\end{aligned}$$

For the remaining assertions we define the mass current density as $j_0 = \int p f_0\,dp$. Integration of the Vlasov equation with respect to p implies the continuity equation $\partial_t\rho_0 + \nabla\cdot j_0 = 0$. Hence

$$\begin{aligned}\iint \tilde{S}\phi_2\, f_0\,dp\,dx &= \int(\partial_t\phi_2\,\rho_0 + \nabla\phi_2\cdot j_0)\,dx\\ &= \int(\partial_t\phi_2\,\rho_0 - \phi_2\,\nabla\cdot j_0)\,dx\\ &= \partial_t\int\phi_2\rho_0\,dx = 2\partial_t\mathcal{E}_{\mathrm{pot}}(t) = -2\partial_t\mathcal{E}_{\mathrm{kin}}(t)\end{aligned}$$

by conservation of energy, and

$$\begin{aligned}\iint p\cdot\nabla\phi_2\, f_0\,dp\,dx &= \int j_0\cdot\nabla\phi_2\,dx = \int \partial_t\rho_0\,\phi_2\,dx\\ &= -\iint \frac{1}{|x-y|}\,\partial_t\rho_0(t,x)\rho_0(t,y)\,dx\,dy\\ &= \partial_t\mathcal{E}_{\mathrm{pot}}(t) = -\partial_t\mathcal{E}_{\mathrm{kin}}(t).\end{aligned}$$

Furthermore, by (VPgr),

$$\begin{aligned}\partial_t\iint(\xi\cdot p)^2 f_0\,dp\,dx &= \iint(\xi\cdot p)^2\,\nabla_p\cdot(\nabla\phi_2\, f_0)\,dp\,dx\\ &= -2\iint(\xi\cdot p)(\xi\cdot\nabla\phi_2)\, f_0\,dp\,dx.\end{aligned}$$

For the last assertion, using $\Delta\phi_2 = 4\pi\rho_0$,

$$\begin{aligned}\int(\xi\cdot x)(\xi\cdot\nabla\phi_2)\rho_0\,dx &= \frac{1}{4\pi}\sum_{i,j=1}^{3}\int(\xi\cdot x)\,\xi_i\partial_i\phi_2\,\partial_j\partial_j\phi_2\,dx\\ &= -\frac{1}{4\pi}\sum_{i,j=1}^{3}\int\big((\xi\cdot x)\xi_i\,\partial_i\partial_j\phi_2 + \xi_j\xi_i\,\partial_i\phi_2\big)\,\partial_j\phi_2\,dx\\ &= -\frac{1}{8\pi}\int(\xi\cdot x)\xi\cdot\nabla|\nabla\phi_2|^2\,dx - \frac{1}{4\pi}\int|\xi\cdot\nabla\phi_2|^2\,dx\\ &= \frac{|\xi|^2}{8\pi}\int|\nabla\phi_2|^2\,dx - \frac{1}{4\pi}\int|\xi\cdot\nabla\phi_2|^2\,dx\\ &= -|\xi|^2\,\mathcal{E}_{\mathrm{pot}} - \frac{1}{4\pi}\int|\xi\cdot\nabla\phi_2|^2\,dx.\end{aligned}$$

This completes the proof of the lemma. □

2.3. Proof of Corollary 1.11. In this section we verify Corollary 1.11 by specializing Theorem 1.9 to spherically symmetric functions. We recall that initial data f° are said to be spherically symmetric, if

$$f^\circ(Ax, Ap) = f^\circ(x, p)$$

for any matrix $A \in \mathrm{SO}(3)$. Then the solution (f_0, ϕ_2) of (VPgr) provided by Proposition 1.7 remains spherically symmetric for all times. Therefore

$$f_0(t, Ax, Ap) = f_0(t, x, p), \quad \rho_0(t, Ax) = \rho_0(t, x), \quad \text{and} \quad \phi_2(t, x) = \phi_2(t, Ax) \tag{2.24}$$

holds for all $A \in \mathrm{SO}(3)$.

Firstly, this implies $\nabla\phi_2 = \bar{x}\,\partial_r\phi_2$ as well as $|\nabla\phi_2|^2 = |\partial_r\phi_2|^2$, ∂_r denoting the radial derivative. By choosing $A \in \mathrm{SO}(3)$ such that $A\bar{x} = e_j$ (the j's unit vector in $\mathbb{R}^3$), (2.24) yields

$$\begin{aligned}\frac{1}{4\pi}\int |\bar{x}\cdot\nabla\phi_2(u, y)|^2\,dy &= \frac{1}{4\pi}\int |\partial_j\phi_2(u, y)|^2\,dy = \frac{1}{4\pi}\int \left|\frac{y_j}{|y|}\,\partial_r\phi_2(u, y)\right|^2 dy\\ &= \frac{1}{12\pi}\int |\partial_r\phi_2(u, y)|^2\,dy = -\frac{2}{3}\,\mathcal{E}_{\mathrm{pot}}(u).\end{aligned}$$

Similarly, (2.24) and $|\bar{x}|^2 = 1$ implies that

$$\begin{aligned}\iint (\bar{x}\cdot p)^2 f_0(u, y, p)\,dp\,dy &= \iint p_j^2\, f_0(u, y, p)\,dp\,dy\\ &= \frac{1}{3}\iint p^2\, f_0(u, y, p)\,dp\,dy\\ &= \frac{2}{3}\,\mathcal{E}_{\mathrm{kin}}(u).\end{aligned}$$

Therefore by (1.18),

$$\begin{aligned}\mathcal{R}(\bar{x}, u) &= -\frac{1}{4\pi}\int |\bar{x}\cdot\nabla\phi_2(u, y)|^2\,dy - \iint (\bar{x}\cdot p)^2 f_0(u, y, p)\,dp\,dy + 4\,\mathcal{E}_{\mathrm{kin}}(u)\\ &= \frac{2}{3}\,\mathcal{E}_{\mathrm{pot}}(u) + \frac{10}{3}\,\mathcal{E}_{\mathrm{kin}}(u).\end{aligned}$$

Since $\partial_t\mathcal{E}_{\mathrm{pot}} = -\partial_t\mathcal{E}_{\mathrm{kin}}$ by conservation of energy, we get $\partial_t\mathcal{R}(\bar{x}, u) = \frac{8}{3}\,\partial_t\mathcal{E}_{\mathrm{kin}}(u)$. Hence Corollary 1.11 follows from (1.16) and (1.17). □

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