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Effective Equations of Motion for Quantum Systems

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Abstract

In many situations, one can approximate the behavior of a quantum system, i.e. a wave function subject to a partial differential equation, by effective classical equations which are ordinary differential equations. A general method and geometrical picture is developed and shown to agree with effective action results, commonly derived through path integration, for perturbations around a harmonic oscillator ground state. The same methods are used to describe dynamical coherent states, which in turn provide means to compute quantum corrections to the symplectic structure of an effective system.

1 Introduction

Many applications of quantum systems are placed in a realm close to classical behavior, where nevertheless quantum properties need to be taken into account. In view of the more complicated structure of quantum systems, both of a conceptual as well as technical nature, it is then often helpful to work with equations of classical type, i.e. systems of ordinary differential equations for mechanical systems, which are amended by correction terms resulting from quantum theory. From a mathematical point of view, the question arises how well the behavior of a (wave) function subject to a partial differential equation can be approximated by finitely many variables subject to a system of coupled but ordinary differential equations.

One very powerful method is that of effective actions [1, 2] which have been developed and are widely used for quantum field theories. The effective action of a free field theory is identical to the classical action, while interacting theories receive quantum corrections "from integrating out irrelevant degrees of freedom." The language is suggestive for the physical intuition behind the formalism, but the technical details and the mathematical relation between classical and quantum theories remain less clear.

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In this article we develop, building on earlier work [3, 4, 5], a geometrical picture of effective equations of motion for a quantum mechanical system with a clear-cut relation between the classical and quantum system: the classical phase space can literally be embedded into the quantum system.[1] We discuss several examples and show that, in the regime where effective action techniques can be used, they coincide with our method.

2 Effective Actions

For any system with classical action $S[q]$ as a functional of the classical coordinates q, thus satisfying

$$\frac{\delta S}{\delta q} = -J \tag{1}$$

in the presence of an external source J, one can formally define the effective action $\Gamma[q]$ satisfying the same relation

$$\frac{\delta \Gamma}{\delta q} = -J \tag{2}$$

but containing $\hbar$-dependent quantum corrections. If the generating functional $Z[J]$ of Greens functions is known, Γ is obtained as the Legendre transform of $-i\hbar \log Z[J]$ [6].

This procedure is well-motivated from particle physics where additional contributions to Γ can be understood as resulting from perturbative quantum interactions ("exchange of virtual particles"). Indeed, effective actions are mostly used in perturbative settings where the generating functional Z can be computed by perturbing around free theories, using e.g. Gaussian path integrations.

For other systems, or quantum mechanical applications, Eq. (2) can, however, be seen at best as a formal justification. The effective action can rarely be derived in general, but its properties can make an interpretation very complicated. First, Γ is in general complex and so are the effective equations (2) as well as their solutions. In fact, q in (2) is not the classical q and not even the expectation value of $\hat{q}$ in a suitable state of the quantum system. Instead, in general it is related to non-diagonal matrix elements of $\hat{q}$ [2]. Secondly, Γ is in general a non-local functional of q which cannot be written as the time integral of a function of q and its derivatives. In most applications, one employs a derivative expansion assuming that higher derivatives of q are small. In this case, each new derivative order introduces additional degrees of freedom into the effective action which are not classical, but whose relation to quantum properties of, e.g., the wave function is not clear either. Indeed, in this perturbative scheme, not all solutions of the higher derivative effective action are consistent perturbatively as many depend non-analytically on the perturbation parameter $\hbar$ [7]. For those solutions, it is then not guaranteed that they capture the correct perturbative behavior considering that next order corrections,

[1] Using the geometrical picture of [3] for this purpose and the idea of horizontality as well as the appearance of additional quantum degrees of freedom in this context were suggested to us by Abhay Ashtekar.

non-analytical in the perturbation parameter, can dominate the leading order. Such non-analytical solutions have to be excluded in a perturbative treatment, which usually brings down the number of solutions to the classical value even if perturbative corrections are of higher derivative form [7]. The description, even in a local approximation, can thus be quite complicated, given by higher derivative equations with many general solutions subject to the additional condition that only solutions analytic in the perturbation parameter are to be retained. The formulation is thus very redundant if higher derivative terms are used. Moreover, where there seem to be additional (quantum) degrees of freedom associated with higher derivative corrections, their role remains dubious given that many solutions have to be excluded.

There are other technical difficulties if one tries to generalize beyond the usual realm of perturbing around the ground state of a free field theory or, in quantum mechanics, the ground state of a harmonic oscillator. In the latter case, for a system with classical action

$$S[q(t)] = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - U(q) \tag{3}$$

one can derive the effective action [2]

$$\Gamma_{\rm eff}[q(t)] = \int {\rm d}t \left[\frac{1}{2}\left(m + \frac{\hbar U'''(q)^2}{2^5 m^2 \left(\omega^2 + m^{-1}U''(q)\right)^{\frac{5}{2}}}\right)\dot{q}^2 - U(q) - \frac{\hbar\omega}{2}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{\frac{1}{2}}\right] \tag{4}$$

to first order in $\hbar$ and in the derivative expansion, using path integral techniques. The quantum system is here described effectively in an expansion around the ground state of the Harmonic oscillator. On the other hand, a quantum system allows more freedom and one could, e.g., want to find an effective formulation for a quantum system which is prepared to be initially close to a squeezed state, or a state of non-minimal uncertainty. This freedom is not allowed by the usual definition of an effective action.

Other problems include the presence of "infrared problems": In the free particle limit, corresponding to a massless field theory, one has $U(q) = -\frac{1}{2}m\omega^2 q^2$ for which (4) becomes meaningless. Still, at least for some time the free particle should be possible to be described in an effective classical manner. Other generalizations, such as for systems to be perturbed around a Hamiltonian non-quadratic in momenta as they occur, e.g., in quantum cosmology, look even more complicated since one could not rely on Gaussian path integrations.

For all these reasons it is of interest to develop a scheme for deriving effective equations of a quantum system based on a geometrical formulation of quantum mechanics. This has been used already in the context of quantum cosmology [4, 5] where usual techniques fail. As we show here, it reduces to the effective action result (4) in the common range of applicability, but is much more general. Moreover, it provides a clear, geometrical picture for the relation between the classical and quantum systems, the role of quantum degrees of freedom and the effective approximation.

3 A Geometrical Formulation of Quantum Mechanics

The formalism of quantum theory has been studied for almost a century already and a prominent understanding of its structure, based mainly on functional analysis, has been achieved. From this perspective, quantum mechanics appears very different from classical mechanics not only conceptually but also mathematically. While in classical physics the viewpoint is geometrical, employing symplectic or Poisson structures on a phase space, quantum theory is analytical and based on Hilbert space structures and operator algebras. There are, however, some contributions which develop and pursue a purely geometrical picture of quantum mechanics, in which the process of quantization and kinematical as well as dynamical considerations are generalizations of classical structures. The process of quantization is described in a geometrical manner in geometric quantization [8], employing line bundles with connections, but the picture of the resulting theory remains analytical based on function spaces and operators thereon. Independently, a geometrical formulation of quantum mechanics has been developed which, irrespective of the quantization procedure, provides a geometrical viewpoint for all the ingredients necessary for the basic formulation of quantum physics [3]. It is the latter which will be crucial for our purposes of developing a geometrical theory of effective equations of motion and the classical limit.

Let us assume that we are given a quantum system, specified by a Hilbert space $\mathcal{H} = (\mathcal{V}, \langle\cdot,\cdot\rangle)$ with underlying vector space $\mathcal{V}$ equipped with inner product $\langle\cdot,\cdot\rangle$, together with an algebra of basic operators and a Hamiltonian $\hat{H}$. The Hamiltonian defines a flow on $\mathcal{H}$ by $\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -i\hbar^{-1}\hat{H}\Psi$.

Lemma 1 ([3]) *Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. The inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$ defines a Kähler structure on $\mathcal{V}$.*

Proof: To start with we note, following [3], that the inner product can be decomposed as

$$\langle\Phi,\Psi\rangle = \frac{1}{2\hbar}G(\Phi,\Psi) + \frac{i}{2\hbar}\Omega(\Phi,\Psi) \tag{5}$$

where $G(\Phi,\Psi)$ and $\Omega(\Phi,\Psi)$ denote the real and complex parts of $2\hbar\langle\Phi,\Psi\rangle$, respectively. It follows from the properties of an inner product that G is a metric and Ω a symplectic structure on the vector space $\mathcal{V}$, identified with its tangent space in any of its points. Also by definition, the metric and symplectic structure are related to each other by

$$G(\Phi,\Psi) = \Omega(\Phi, i\Psi)\,.$$

With the obvious complex structure, $(\mathcal{V}, G, \Omega)$ is thus Kähler. □

As used in the proof, points and tangent vectors of the Kähler manifold $\mathcal{K} = (\mathcal{V}, G, \Omega)$ correspond to states in the Hilbert space, and functions densely defined on $\mathcal{V}$ can be associated to mean values of operators acting on $\mathcal{H}$: Any operator $\hat{F}$ on $\mathcal{H}$ defines a function $F := \langle\hat{F}\rangle$ on $\mathcal{K}$ taking values $F(\Psi) = \langle\Psi, \hat{F}\Psi\rangle$ in points Ψ of its domain of definition.

Any state $\eta \in \mathcal{H}$ defines a constant vector field on $\mathcal{K}$, which can be used to compute the Lie derivative

$$\pounds_\eta F(\Psi) := \frac{\mathrm{d}}{\mathrm{d}t} F(\Psi + t\eta)|_{t=0} \,. \tag{6}$$

This allows us to show

Lemma 2 *Let $F = \langle \hat{F} \rangle$ be a function on $\mathcal{K}$ associated with a self-adjoint operator $\hat{F}$ on $\mathcal{H}$. Its Hamiltonian vector field is given by*

$$X_F(\Psi) := \frac{1}{i\hbar} \hat{F}\Psi \,.$$

Proof: Using the definition of a Lie derivative and self-adjointness of $\hat{F}$ we have

$$\begin{aligned} \pounds_\eta F(\Psi) &= \frac{\mathrm{d}}{\mathrm{d}t} \langle \Psi + t\eta, \hat{F}(\Psi + t\eta) \rangle|_{t=0} = \langle \eta, \hat{F}\Psi \rangle + \langle \Psi, \hat{F}\eta \rangle \\ &= -i\hbar \left(\langle -i\hbar^{-1} \hat{F}\Psi, \eta \rangle - \langle \eta, -i\hbar^{-1} \hat{F}\Psi \rangle \right) = \Omega \left(-i\hbar^{-1} \hat{F}\Psi, \eta \right) \end{aligned} \tag{7}$$

for any vector η, from which X_F can immediately be read off. □

Remark: Such vector fields are also known as Schrödinger vector fields, as their flow is generated on $\mathcal{H}$ by a Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t} |\Psi\rangle = \frac{1}{i\hbar} \hat{F} |\Psi\rangle \,. \tag{8}$$

The flow is a family of unitary transformations, i.e. automorphisms of the Hilbert space which preserve the Hilbert space structure. Therefore, the flow preserves not only the symplectic structure of $\mathcal{K}$, as any Hamiltonian vector field does, but also the metric. Hamiltonian vector fields thus are Killing vector fields, and since each tangent space has a basis of Killing vectors the Kähler space is maximally symmetric.

For two functions $F = \langle \hat{F} \rangle$ and $K = \langle \hat{K} \rangle$ the symplectic structure defines the Poisson bracket

$$\{F, K\} := \Omega(X_F, X_G) = \frac{1}{i\hbar} \langle [\hat{F}, \hat{K}] \rangle \,. \tag{9}$$

For, e.g., $q := \langle \hat{q} \rangle$ and $p := \langle \hat{p} \rangle$ we have $\{q, p\} = 1$ from $[\hat{q}, \hat{p}] = i\hbar$.

Of physical significance in quantum theory are only vectors of the Hilbert space up to multiplication with a non-zero complex number. Physical information is then not contained in the vector space $\mathcal{V}$ but in the projective space $\mathcal{V}/\mathbb{C}^*$. From now on we will take this into account by working only with norm one states.

4 Classical and Quantum Variables

For any quantum system, the algebra of basic operators, which is a representation of the classical Poisson algebra of basic phase space variables, plays an important role. We will assume mainly, for simplicity, that this basic algebra is given by a set of position and momentum operators, $\hat{q}^i$ and $\hat{p}_i$ for $1 \leq i \leq N$, with canonical commutation relations. This distinguished set of operators leads to further structure on $\mathcal{K}$:

Definition 1 *The set of fundamental operators $(\hat{q}^i, \hat{p}_i)$ on $\mathcal{H}$ defines a fiber bundle structure on $\mathcal{V}$ where the bundle projection identifies all points Φ, Ψ for which $\langle\Psi, \hat{q}^i\Psi\rangle = \langle\Phi, \hat{q}^i\Phi\rangle$ and $\langle\Psi, \hat{p}_i\Psi\rangle = \langle\Phi, \hat{p}_i\Phi\rangle$ for all i. The base manifold can be identified with the classical phase space as a manifold.*

Remark: The Hilbert space used for the quantization of a classical system is always infinite dimensional, which implies that the fibers of the bundle are infinite dimensional. For instance, for an analytic wave function one can consider the collection of numbers associated to the mean values of products of the fundamental operators, $a_n = \langle\Psi, \hat{q}^n\Psi\rangle$ and $b_n = \langle\Psi, \hat{q}^n\hat{p}\Psi\rangle$ for all $n \geq 0$. Usually denominated by the name of Hamburger momenta [9], the (a_n, b_n) are a complete set in the sense that they uniquely determine the wave function. Indeed, from linear combinations c_n of the Hamburger momenta with coefficients corresponding to some orthogonal polynomials, taking Hermite polynomials $\{H_n(q) = \sum_l h_{n,l}q^l\}$ for definiteness, we have

$$c_n = \sum_l h_{n,l}a_l = \int dq|\Psi(q)|^2 H_n(q) \tag{10}$$

giving the absolute value of the wave function as

$$|\Psi(q)|^2 = e^{-q^2} \sum_n \frac{c_n H_n(q)}{2^n \pi n!} \,. \tag{11}$$

The b_n, on the other hand, provide information about the phase $\alpha(q)$ of the wave function up to a constant:

$$b_n = -\int dq\Psi(q)^* q^n i\partial_q \Psi(q) = -\int \mathrm{d}q|\Psi(q)|q^n i\partial_q|\Psi(q)| - \int \mathrm{d}q|\Psi(q)|^2 q^n i\partial_q \alpha(q) \tag{12}$$

from which $\partial_q\alpha(q)$ is determined as before, using the already known norm of Ψ.

One could thus use Hamburger momenta as coordinates on the fiber bundle, but for practical purposes it is more helpful to choose coordinates which are not only adapted to the bundle structure but also to the symplectic structure. We thus require that, in addition to the classical variables q^i and p_i, coordinates of the fibers generate Hamiltonian vector fields symplectically orthogonal to $\partial/\partial q^i$ and $\partial/\partial p_i$.

Definition 2 *The* quantum variables *of a Hilbert space $\mathcal{H}$ are defined as*

$$G^{i_1\ldots i_n} := \langle(\hat{x}^{(i_1} - x^{(i_1}) \cdots (\hat{x}^{i_n)} - x^{i_n)})\rangle = \sum_{k=0}^{n} (-)^k \tbinom{n}{k} x^{(i_1} \cdots x^{i_k} \langle \hat{x}^{i_{k+1}} \cdots \hat{x}^{i_n)}\rangle \tag{13}$$

with respect to fundamental operators $\{\hat{x}^i\}_{1\leq i\leq 2N} := \{\hat{q}^k, \hat{p}_k\}_{1\leq i\leq N}$.

Variables of this type have been considered in quantum field theories; see, e.g., [10].

Lemma 3 *The fiber coordinates $G^{i_1\cdots i_n}$ on $\mathcal{K}$ are symplectically orthogonal to the classical coordinates x^i.*

Proof: We compute the Poisson bracket with x^j to obtain

$$\begin{aligned}
\left\{x^j, G^{i_1\cdots i_n}\right\} &= \sum_{k=0}^{n}(-)^k\tbinom{n}{k}\left[\left\{x^j, x^{(i_1}\cdots x^{i_k}\right\}\langle\hat{x}^{i_{k+1}}\cdots\hat{x}^{i_n)}\rangle + x^{(i_1}\cdots x^{i_k}\left\{x^j,\langle\hat{x}^{i_{k+1}}\cdots\hat{x}^{i_n)}\rangle\right\}\right]\\
&= \sum_{k=0}^{n}(-)^k\tbinom{n}{k}\left[k\epsilon^{j(i_n}x^{i_1}\cdots x^{i_{k-1}}\langle\hat{x}^{i_k}\cdots\hat{x}^{i_{n-1})}\rangle + (n-k)\epsilon^{j(i_n}x^{i_1}\cdots x^{i_k}\langle\hat{x}^{i_{k+1}}\cdots\hat{x}^{i_{n-1})}\rangle\right]\\
&= \sum_{l=0}^{n-1}(-)^{(l+1)}\tbinom{n}{l}(n-l)\epsilon^{j(i_n}x^{i_1}\cdots x^{i_l}\langle\hat{x}^{i_{l+1}}\cdots\hat{x}^{i_{n-1})}\rangle\\
&\quad+\sum_{k=0}^{n}(-)^k\tbinom{n}{k}(n-k)\epsilon^{j(i_n}x^{i_1}\cdots x^{i_k}\langle\hat{x}^{i_{k+1}}\cdots\hat{x}^{i_{n-1})}\rangle = 0
\end{aligned}\tag{14}$$

where we used repeatedly the Leibnitz rule and introduced $\epsilon^{ij} = \{x^i, x^j\}$. □

Remark: An alternative proof proceeds by computing the Poisson bracket between the function $\langle e^{\alpha_i(\hat{x}^i - x^i)}\rangle$ and x^j, restricting to the dense subspace in which such functions are analytic in $\{\alpha_i\}$, and expanding.

Since the fibers are symplectic, Ω defines a natural decomposition of tangent spaces of $\mathcal{K}$ as a direct sum of a vertical space tangent to the fibers and a horizontal space $\mathrm{Hor}^\Omega\mathcal{K}$ as the symplectic complement:

Corollary 1 $(\mathcal{K}, \pi, \mathcal{B})$ *is a fiber bundle with connection over the classical phase space $\mathcal{B}$ as base manifold.*

We now know the Poisson relation between the classical variables x^i and between x^i and the $G^{j_1,\ldots,j_m}$. In order to compute the remaining Poisson brackets $\{G^{i_1,\ldots,i_n}, G^{j_1,\ldots,j_m}\}$ for N canonical degrees of freedom we introduce a new notation

$$G^{a_{k_1},\ldots,a_{k_N}}_{b_{k_1},\ldots,b_{k_N}} = \langle(\hat{q}^{k_1} - q^{k_1})^{a_{k_1}}\cdots(\hat{q}^{k_N} - q^{k_N})^{a_{k_N}}(\hat{p}_{k_1} - p_{k_1})^{b_{k_1}}\cdots(\hat{p}_{k_N} - p_{k_N})^{b_{k_N}}\rangle_{\rm Weyl}$$

the label "Weyl" meaning that the product of operators is Weyl or fully symmetric ordered. The notation allows us to drop indices whose values are zero so whenever we are dealing with a single pair of degrees of freedom we use the notation where $G^{a,n} := G^{n-a}_a$.

Lemma 4 *The Poisson brackets for the variables above are*

$$\begin{aligned}
\left\{G^{a_{k_1},\ldots,a_{k_N}}_{b_{k_1},\ldots,b_{k_N}}, G^{c_{k_1},\ldots,c_{k_N}}_{d_{k_1},\ldots,d_{k_N}}\right\} &= -\sum_{r,s,e_1,\ldots,e_N}(-)^{r+s}(\tfrac{1}{2}\hbar)^{2r}\delta_{e_1+\cdots+e_N,2r+1}K^{\{a\}\{b\}\{c\}\{d\}}_{r,s,\{e\}}G^{a_{k_1}+c_{k_1}-e_1,\ldots,a_{k_N}+c_{k_N}-e_N}_{b_{k_1}+d_{k_1}-e_1,\ldots,b_{k_N}+d_{k_N}-e_N}\\
&\quad-\sum_{f=1}^{N}\left(a_{k_f}d_{k_f}G^{a_{k_1},\ldots,a_{k_f}-1,\ldots,a_{k_N}}_{b_{k_1},\ldots,b_{k_N}}G^{c_{k_1},\ldots,c_{k_N}}_{d_{k_1},\ldots,d_{k_f}-1,\ldots,d_{k_N}} - b_{k_f}c_{k_f}G^{a_{k_1},\ldots,a_{k_N}}_{b_{k_1},\ldots,b_{k_f}-1,\ldots,b_{k_N}}G^{c_{k_1},\ldots,c_{k_f}-1,\ldots,c_{k_N}}_{d_{k_1},\ldots,d_{k_N}}\right)
\end{aligned}$$

with indices running as

$$1 \le 2r+1 \le \textstyle\sum_{f=1}^{N} \left(\min(a_f,d_f)+\min(b_f,c_f)\right),$$
$$0 \le s \le \min\left(r, \textstyle\sum_{f=1}^{N}\min(b_f,c_f)\right),$$
$$0 \le e_f \le \min(a_f,d_f,s)+\min(b_f,c_f,2r+1-s).$$

and coefficients given by

$$K^{\{a\}\{b\}\{c\}\{d\}}_{r,s,\{e\}} = \sum_{g_1,\dots,g_n} \frac{\delta_{g_1+\cdots+g_N,2r+1-s}}{s!(2r+1-s)!} \prod_f \frac{\binom{a_f}{e_f-g_f}\binom{b_f}{g_f}\binom{c_f}{g_f}\binom{d_f}{e_f-g_f}}{\binom{2r+1-s}{g_f}\binom{s}{e_f-g_f}} \tag{15}$$

where

$$\max(e_f-s, e_f-a_f, e_f-d_f, 0) \le g_f \le \min(b_f, c_f, 2r+1-s, e_f)\,.$$

Proof: Consider first the Poisson bracket between functions of the form $D(\alpha)=\langle e^{\alpha_i(\hat{x}^i-x^i)}\rangle$. For analytical wave functions in the mean values, $D(\alpha)$ is an analytical function and so is the Poisson bracket between two such functions $D(\alpha)$ and $D(\beta)$. We can therefore take the coefficients in a Taylor expansion for all orders in α_i and β_j. Using the relation $[e^{\alpha_i\hat{x}^i}, e^{\beta_j\hat{x}^j}] = 2i\sin(\frac{\hbar}{2}\alpha_j\beta_k\epsilon^{jk})e^{(\alpha+\beta)_i\hat{x}^i}$, which follows from the Baker–Campbell–Hausdorff formula and the commutator $[\alpha_i\hat{x}^i, \beta_j\hat{x}^j] = i\hbar\epsilon^{ij}\alpha_i\beta_j$, we find that

$$\{D(\alpha), D(\beta)\} = \frac{2}{\hbar}\sin(\tfrac{1}{2}\hbar\alpha_j\beta_k\epsilon^{jk})D(\alpha+\beta) - \alpha_j\beta_k\epsilon^{jk}D(\alpha)D(\beta)\,. \tag{16}$$

Now, we use $D(\alpha)=\langle e^{\alpha_i(\hat{x}^i-x^i)}\rangle = \sum_{\{a\},\{b\}} G^{a_1\cdots a_N}_{b_1\cdots b_N}\prod_{i=1}^{N}\alpha_{q^i}^{a_i}\alpha_{p_i}^{b_i}(a_i!b_i!)^{-1}$, and substitute

$$\frac{2}{\hbar}\sin(\tfrac{1}{2}\hbar\alpha_j\beta_k\epsilon^{jk})D(\alpha+\beta) = -\sum(-)^{r+s}(\tfrac{1}{2}\hbar)^{2r}G^{a_1+c_1,\cdots,a_N+c_N}_{b_1+d_1,\cdots,b_N+d_N} \times\prod_{f=1}^{N}\frac{\alpha_{q^f}^{a_f+g_f}\alpha_{p_f}^{b_f+e_f}\beta_{q^f}^{c_f+e_f}\beta_{p_f}^{d_f+g_f}}{a_f!b_f!c_f!d_f!e_f!g_f!(2r+1-s-e_f)!(s-g_f)!} \tag{17}$$

where we sum over all collections of numbers a_f, b_f, c_f, d_f, e_f, g_f, r and s such that $\sum_f g_f = s$, $\sum_f e_f = 2r-s$ and $s \le 2r+1$. Since the equality (16) holds for any α and β, coefficients in the expansion have to fulfill the equality. □

5 Uncertainty Principle

The fibers of $\mathcal{K}$ as a fiber bundle over the classical phase space are not vector spaces, and the quantum variables $G^{i_1,\dots,i_n}$ are not allowed to take arbitrary values. Similarly, not any collection of numbers is a collection of Hamburger momenta. With $\mathcal{K}$ being a Kähler space, the fibers are bounded by relations following from Schwarz inequalities. A special case of this fact is well-known and commonly written as the uncertainty relation

$$(\triangle q)^2(\triangle p)^2 \geq \frac{\hbar^2}{4} + \langle(\hat{q}\hat{p}+\hat{p}\hat{q})/2 - qp\rangle^2 \geq \frac{\hbar^2}{4} \tag{18}$$

where $(\triangle a)^2 = \langle(\hat{a}-a)^2\rangle$, or in our notation

$$G^{0,2}G^{2,2} \geq \frac{\hbar^2}{4} + (G^{1,2})^2\,. \tag{19}$$

More generally, the Schwarz inequality for a Kähler manifold with metric g and symplectic structure ω is

$$g(u,u)g(v,v) \geq |g(u,v)|^2 + |\omega(u,v)|^2 \tag{20}$$

for all tangent vectors u and v. This results in bounds to be imposed on the quantum variables.

Lemma 5 *The function $D(\alpha) = \langle e^{\alpha_i(\hat{x}^i - x^i)}\rangle$ is subject to*

$$\left(D(2\alpha)-D(\alpha)^2\right)\left(D(2\beta)-D(\beta)^2\right) \geq D(\alpha+\beta)^2-2\cos(\tfrac{1}{2}\hbar\alpha\times\beta)D(\alpha+\beta)D(\alpha)D(\beta)+D(\alpha)^2D(\beta)^2\,. \tag{21}$$

Proof: For the Schwarz inequality we need to know the metric and pre-symplectic structure on the space of states of unit norm, which we compute by evaluating them on vector fields that generate transformations only along the submanifold of unit vectors in the Hilbert space. To an arbitrary vector $X_F = \frac{1}{i\hbar}\hat{F}\Psi$ we associate the vector given by $\tilde{X}_F = (1-|\Psi\rangle\langle\Psi|)\,X_F = \frac{1}{i\hbar}(\hat{F}-F)\Psi$. This ensures that the transformation generated by $\tilde{X}_F$ maps normalized states to normalized states, which is most easily seen infinitesimally using $|(1+\epsilon\tilde{X}_F)\Psi|^2 = |\Psi|^2 - 2i\hbar^{-1}\epsilon\langle\Psi,(\hat{F}-F)\Psi\rangle + O(\epsilon^2) = |\Psi|^2 + O(\epsilon^2)$. The metric on the space of physical states evaluated in Hamiltonian vector fields induces a symmetric bracket

$$(F,K) = g(X_F,X_K) = G((1-|\Psi\rangle\langle\Psi|)\,X_F,(1-|\Psi\rangle\langle\Psi|)\,X_K)\,. \tag{22}$$

The symplectic structure is as before, $\omega(X_F,X_K) = \Omega(X_F,X_K)$. For the corresponding operators, g and ω result in the anticommutator $[\cdot,\cdot]_+$ and commutator $[\cdot,\cdot]$, respectively.

For functions $\langle e^{\alpha.\hat{x}}\rangle$ and $\langle e^{\beta.\hat{x}}\rangle$ (parameterized by α_i and β_i) the Schwarz inequality implies

$$(\langle e^{2\alpha.\hat{x}}\rangle - \langle e^{\alpha.\hat{x}}\rangle^2)(\langle e^{2\beta.\hat{x}}\rangle - \langle e^{\beta.\hat{x}}\rangle^2) \geq \left|\frac{1}{2}\langle[e^{\alpha.\hat{x}},e^{\beta.\hat{x}}]_+\rangle - \langle e^{\alpha.\hat{x}}\rangle\langle e^{\beta.\hat{x}}\rangle\right|^2 + \frac{1}{4}\left|\langle[e^{\alpha.\hat{x}},e^{\beta.\hat{x}}]\rangle\right|^2 \tag{23}$$

which upon using, as before, the Baker–Campbell–Hausdorff formula for the commutator and anticommutator and multiplying both sides with $e^{-2(\alpha+\beta).x}$ proves the lemma. □

This gives us a large class of inequalities thus specifying bounds on the variables $G^{i_1,\ldots,i_n}$. The boundary, obtained through saturation of the inequalities, is characterized by relations which result from the lemma order by order in α and β.

6 Quantum Evolution

The dynamical flow of the quantum system is given as the unitary Schrödinger flow on $\mathcal{H}$ of a self-adjoint Hamiltonian operator $\hat{H}$. As before, this flow is also Hamiltonian when viewed on the Kähler space $\mathcal{K}$. It is generated by the Hamiltonian function obtained as the mean value of the Hamiltonian operator. In terms of coordinates on the manifold the Hamiltonian function is obtained by Taylor expanding the mean value of the Hamiltonian operator which in our convention is taken to be Weyl ordered:

Definition 3 *The* quantum Hamiltonian *on $\mathcal{K}$ is the function*

$$H_Q := \langle H(\hat{x}^i)\rangle_{\mathrm{Weyl}} = \langle H(x^i + (\hat{x}^i - x^i))\rangle = \sum_{n=0}^{\infty}\sum_{a=0}^{n} \frac{1}{n!}\binom{n}{a}\frac{\partial^n H(q,p)}{\partial p^a \partial q^{n-a}} G^{a,n} \tag{24}$$

generating Hamiltonian equations of motion

$$\begin{aligned} \dot{x}^i &= \{x^i, H_Q\} \\ \dot{G}^{a,n} &= \{G^{a,n}, H_Q\}\,. \end{aligned} \tag{25}$$

This Hamiltonian flow is equivalent to the Schrödinger equation of the Hamiltonian operator. As such, it is an equivalent description of the quantum dynamics and only superficially takes a classical form, albeit for infinitely many variables, in its mathematical structure. Nevertheless, the reformulation makes it possible to analyze the classical limit in a direct manner, and to derive effective equations in appropriate regimes. Classical dynamics is to arise in the limit of "small" quantum fluctuations which, when the fluctuations are completely ignored or switched off by $\hbar \to 0$, should give rise to classical equations of motion. In practice, this limit is not easy to define, and the most direct way is to derive first effective equations of motion, which still contain $\hbar$, and then take the limit $\hbar \to 0$.

In this procedure, the main problem is to reduce the infinite set of coupled quantum equations of motion to a set of differential equations for only a finite set of variables. Additional degrees of freedom without classical analogs carry information about, e.g., the spreading of the wave function around the peak, which itself is captured by expectation values. For a formulation of classical type, taking into account only a finite number of degrees of freedom, a system has to allow a *finite-dimensional* submanifold of the quantum space $\mathcal{K}$ which is preserved by the quantum flow. We start by generalizing the situation encountered in [3]:

Definition 4 *A* strong effective classical system $(\mathcal{P}, H_{\mathrm{eff}})$ *for a quantum system* $(\mathcal{H}, \hat{H})$ *is given by a finite dimensional pre-symplectic subspace $\mathcal{P}$ of the Kähler space $\mathcal{K}$ associated with $\mathcal{H}$ satisfying the following two conditions:*

1. *For each $p \in \mathcal{P} \subset \mathcal{K}$ the tangent space $T_p\mathcal{P}$ contains the horizontal subspace* $\mathrm{Hor}^{\Omega}_p\mathcal{K}$ *of p in $\mathcal{K}$ defined by the symplectic structure:* $\mathrm{Hor}^{\Omega}_p\mathcal{K} \subset T_p\mathcal{P}$ *for all $p \in \mathcal{P}$* (base horizontality).

2. *$\mathcal{P}$ is fixed under the Schrödinger flow of $\hat{H}$ and, if $\mathcal{P}$ is symplectic, the restriction of the flow to $\mathcal{P}$ agrees with the Hamiltonian flow generated by the* effective Hamiltonian $H_{\rm eff}$.

Remark: A strong effective classical system agrees with the quantum system both at the kinematical and quantum level since its symplectic structure as well as the Hamiltonian flow are induced by the embedding. As such, the conditions are very strong since they require a quantum system to be described *exactly* in terms of a *finite dimensional* system $\mathcal{P}$. In addition to agreement between the strong effective and the quantum dynamics, the first condition ensures that the classical variables are contained in $\mathcal{P}$ and fulfill the classical Poisson relations.

In the simplest case we require the effective system to have the same dimension as the classical system, such that potentially only correction terms will appear in $H_{\rm eff}$ (to be discussed further in Theorem 1 below) but no additional degrees of freedom. Quantum variables, in general, cannot simply be ignored since they evolve and back react on the classical variables. Sometimes one may be forced to keep an odd number of quantum variables, such as the three $G^{a,2}$, in the system which we allow by requiring the effective phase space $\mathcal{P}$ to be only pre-symplectic. For a strong effective system of the classical dimension, however, the dynamics of the quantum variables in the embedding space occurs only as a functional dependence through the classical coordinates:

$$\dot{G}^{a,n} = \dot{x}^i \partial_{x^i} G^{a,n}(x^j)\,. \tag{26}$$

The effective equations of motion, generated by $H_{\rm eff}$ are then obtained by inserting solutions $G^{a,n}(x)$ in the equations for x^i:

$$\dot{x}^i = \{x^i, H_Q\}|_{G^{a,n}(x)} = \sum_{n=0}^{\infty} \frac{1}{n!}\{x^i, H(x^i)_{,i_1\ldots i_n}\}G^{i_1,\ldots,i_n}(x)\,. \tag{27}$$

7 Examples

We now demonstrate the applicability of the general procedure by presenting examples, which will then lead the way to a weakened definition and, in the following section, a proof that the results coincide with standard effective action techniques when both can be applied.

Example 1: Harmonic Oscillator
The quantum Hamiltonian (24) for a harmonic oscillator is

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + \frac{1}{2}m\omega^2 G^{0,2} + \frac{1}{2m}G^{2,2} \tag{28}$$

giving equations of motion

$$\begin{aligned}\dot{p} = &\quad \{p, H_Q\} &= -m\omega^2 q \\ \dot{q} = &\quad \{q, H_Q\} &= \frac{1}{m}p \\ \dot{G}^{a,n} = &\quad \{G^{a,n}, H_Q\} &= \frac{1}{m}(n-a)G^{a+1,n} - m\omega^2 a G^{a-1,n}\,.\end{aligned} \tag{29}$$

In this case, the set of infinitely many coupled equations splits into an infinite number of sets, for each n as well as the classical variables, each having a finite number of coupled equations. Independently of the solutions for the $G^{a,n}$ we obtain the same set of effective equations for q and p agreeing with the classical ones. Therefore the effective Hamiltonian for a system of the classical dimension is here identical to the classical one (up to a constant which can be added freely). We can also define higher dimensional (but non-symplectic) systems by including the variables $G^{a,n}$ for a finite set of values for n.

Along the classical evolution, the evolution of the additional parameters is then given by linear differential equations which we write down in a dimensionless form, defining

$$\tilde{G}^{a,n} = \hbar^{-n/2}(m\omega)^{n/2-a}G^{a,n}\,. \tag{30}$$

The requirement that dynamics be restricted to the classical subspace parameterized by q and p implies

$$\frac{1}{\omega}\left(\frac{1}{m}p\partial_q - m\omega^2 q\partial_p\right)\tilde{G}^{a,n} = (n-a)\tilde{G}^{a+1,n} - a\tilde{G}^{a-1,n} =: {}^{(n)}M^a{}_b\tilde{G}^{b,n} \tag{31}$$

whose solution is

$$\tilde{G}^{a,n}(r,\theta) = (\exp\theta^{(n)}M)^a{}_b A^b(r) \tag{32}$$

where $r = \sqrt{\frac{1}{m}p^2 + m\omega^2 q^2}$, $\tan(\theta) = m\omega q/p$ and $A^{a,n}(r)$ are $n+1$ arbitrary functions of r. For, e.g., $n=2$ we have

$$\tilde{G}^{0,2}(r,\theta) = A^{0,2}(r) - e^{2i\theta}A^{2,2}(r) - e^{-2i\theta}A^{-2,2}(r) \tag{33}$$

$$\tilde{G}^{1,2}(r,\theta) = -ie^{2i\theta}A^{2,2}(r) + ie^{-2i\theta}A^{-2,2}(r) \tag{34}$$

$$\tilde{G}^{2,2}(r,\theta) = A^{0,2}(r) + e^{2i\theta}A^{2,2}(r) + e^{-2i\theta}A^{-2,2}(r) \tag{35}$$

In terms of the constants $A^{a,n}$, the uncertainty relation (19) reads:

$$(A^{0,2}(r))^2 - 4A^{2,2}(r)A^{-2,2}(r) \geq \frac{1}{4} \tag{36}$$

We are thus allowed to choose $A^{2,2} = 0 = A^{-2,2}$ and $A^{0,2} = \frac{1}{2}$ which saturates the uncertainty bound and makes the $G^{a,2}$ constant. In fact, these values arise from quantum evolution given by coherent states $|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}|0\rangle$ which corresponds to trajectories of *constant* quantum variables

$$\tilde{G}^{a,n} = \frac{1}{2^n}\frac{a!}{(a/2)!}\frac{(n-a)!}{((n-a)/2)!} \tag{37}$$

for even a and n, and $\tilde{G}^{a,n} = 0$ otherwise. This implies that any truncation of the system by including only a finite set of values for n, which as already seen is consistent with the dynamical equations, and choosing initial conditions to be that of a coherent state gives a base horizontal subspace as required by Def. 4. In other words, the harmonic oscillator allows an infinite set of strong effective classical systems, including one of the classical dimension. The last case is symplectic, with effective Hamiltonian $H_{\rm eff} = H + {\rm const}$.

In particular, for $n = 2$ we see that the uncertainty relations are saturated. For other states, the quantum variables will in general vary during evolution, which means that the spreading of states changes in time. Nevertheless, the variables remain bounded and the system will stay in a semiclassical regime of small uncertainties if it starts there. With varying G, we will not obtain a strong effective system as horizontality will be violated. Nevertheless, such states are often of interest and suitable for an effective description, which we will provide in a weakened form later on.

Example 2: Linear systems

The harmonic oscillator is a special case of systems, where a complete set of functions on the classical phase space exists such that they form a Lie algebra with the Hamiltonian. For such systems, which we call linear, semiclassical aspects can be analyzed in an elegant manner using *generalized coherent states*: a family of states — of the dimension of the algebra minus the dimension of its subalgebra that generates the stability subgroup of a given, so-called extremal state — with respect to which the mean values of operators can be approximated very well by their classical expressions [11].

In this example we assume that basic variables of the quantum system are not necessarily canonical but given by the Lie algebra elements $\hat{L}^i$ of a linear quantum system. Thus, our classical variables are mean values $L^i := \langle \hat{L}^i \rangle$, and quantum variables are

$$G_L^{i_1,\ldots,i_n} = \langle (\hat{L}^{(i_1} - L^{i_1}) \cdots (\hat{L}^{i_n} - L^{i_n)}) \rangle \,.$$

Poisson brackets between these functions on the infinite dimensional Kähler manifold $\mathcal{K}$ can easily be found to be

$$\{L^i, L^j\} = f^{ij}{}_k L^k$$

and

$$\{L^i, G_L^{i_1,\ldots,i_n}\} = \sum_{r,j} f^{ii_r}{}_j G_L^{i_1,\ldots,i_{r-1} j i_{r+1},\ldots,i_n} \,.$$

It is then immediately seen that the Hamiltonian dynamics of all degrees of freedom is linear, the L^i decouple from the quantum variables, and that the dynamics of any $G_L^{i_1,\ldots,i_n}$ depends only on other $G_L^{j_1,\ldots,j_n}$ with the same n. As in the harmonic oscillator case, the dynamics of infinitely many degrees of freedom thus decouples into infinitely many sectors containing only finitely many variables. This shows

Corollary 2 *Any linear quantum system admits a class of finite dimensional subspaces preserved by the quantum flow, including one of the classical dimension.*

This is not sufficient for the existence of a strong effective system, for which we also have to discuss base horizontality. As in the harmonic oscillator example, one can try to use coherent states which have been widely analyzed in this context. Nevertheless, the issue of base horizontality, i.e. finding coherent states for which all G are constant, in general is more complicated.

A special family of states is generated by acting with the Lie algebra on an extremal state, i.e. a lowest weight of a module representation, which can thus be seen to be in one-to-one correspondence with the factor space of the Lie algebra by the stabilizer of the state. More explicitly those states are of the form

$$|\eta\rangle_{\Lambda,\Omega} = e^{\sum_\alpha \eta_\alpha E_\alpha - H.c.}|\mathrm{ext}\rangle = N(\tau(\eta),\tau(\eta)^*)^{-1} e^{\sum_\alpha \tau_\alpha(\eta)E_\alpha}|\mathrm{ext}\rangle\,,$$

where Λ is a representation of the Lie algebra, Ω is the quotient of the Group manifold by its stabilizer, $|\mathrm{ext}\rangle$ is an extremal state, $E_{-\alpha}|\mathrm{ext}\rangle = 0$ for all positive roots α and η_α or τ_α are coordinate charts of the homogeneous space. Since the flow is generated by an element of the Lie algebra, generalized coherent states define a preserved manifold according to the Baker–Campbell–Hausdorff formula.

In this situation one can compute the mean values of elements L^i of the Lie algebra and the quantum variables $G_L^{i_1,\ldots,i_n}$ as functions over the classical phase space. With this construction of coherent states, the semiclassical phase space associated to the Lie algebra and the dimension of the classical theories would differ depending on the choice of the extremal state and each of these would provide us with diffeomorphisms from the set of L^i to the τ_α, these last ones being the only dynamical variables of this subspace (when all conditions are satisfied, we have by definition *dynamical coherent states*).

We can notice as well that a natural emergence of a Kähler structure for this submanifold of the space of states, as observed within the context of the geometrical formulation of quantum mechanics, is also justified in Gilmore's construction.

We are not aware of general expressions for the G or special choices of constant values as they exist for the harmonic oscillator. It is, however, clear that such constant choices are not possible in general for a linear system as the counter-example of the free particle demonstrates.

Example 3: Free Particle

The free particle is an example for a linear system and can be obtained as the limit of a harmonic oscillator for $\omega \to 0$. However, the limit is non-trivial and the semiclassical behavior changes significantly. If we re-instate units into the uncertainty formulas of the harmonic oscillator, we obtain in the case of constant $G^{a,2}$:

$$G^{0,2} = \frac{\hbar}{2m\omega} \quad , \quad G^{1,2} = 0 \quad , \quad G^{2,2} = \frac{\hbar m\omega}{2}\,.$$

The fixed point of the evolution of quantum variables which exists for the harmonic oscillator thus moves out to infinity in the free particle limit and disappears. Moreover, the closed classical orbits break open and become unbounded. Even non-constant bounded solutions for the G then cease to exist, a fact well-known from quantum mechanics where the wave function of a free particle has a strictly growing spread, while harmonic oscillator states always have bounded spread as follows from (33), (34) and (35). For a free particle, one can thus not expect to have a valid semiclassical approximation for all times.

One can see this explicitly by computing eigenvalues of the matrices ${}^{(n)}M$ in (31) for arbitrary n which in the limit of vanishing frequency become degenerate. More precisely, the solutions of

$$\frac{p}{m}\partial_q G^{a,n} = \frac{n-a}{m} G^{a+1,n} \tag{38}$$

are given by

$$G^{a,n}(q,p) = p^a \sum_{i=0}^{n-a} \frac{c_{i,n}(n-a)!}{(n-a-i)!} q^{n-a-i} \tag{39}$$

with integration constants $c_{i,n}$, $i = 0, \ldots, n$. Minimal uncertainty requires for $n = 2$ that $2c_0c_2 - c_1^2 = \frac{\hbar^2}{4p^2}$. Initial conditions could be chosen by requiring the initial state to be a Harmonic oscillator coherent state at the point (q_0, p_0). Since, due to the degeneracy of eigenvalues, solutions for the G are now polynomials in q and the classical trajectories are unbounded, the spread is unbounded when the whole evolution is considered. In particular, no constant choice and so no strong effective system exists. With unbounded quantum variables, the system cannot be considered semiclassical for all times, but for limited amounts of time this can be reasonable. If this is done, the equations of motion for the classical variables q and p are unmodified such that there is no need for introducing an effective Hamiltonian different from the classical one if one is interested only in an effective system of the classical dimension.

Example 4: Quantum Cosmology

So far we have mainly reproduced known results in a different language. To illustrate the generality of the procedure we now compute effective equations for an unbounded Hamiltonian, using the example of isotropic quantum cosmology coupled to matter in the form of dust (constant matter energy E) in Ashtekar variables [12] where $H = -3\gamma^{-2}\kappa^{-1}c^2\sqrt{p} + E$. (This is formally similar to a system with varying mass as discussed in [13].) For details of the variables (c, p) used we refer to [14, 15]. For our purposes here it suffices to know that the canonical variables are (c, p) with $\{c, p\} = \frac{1}{3}\gamma\kappa$ where γ is a real constant, the so-called Barbero–Immirzi parameter [16, 17], and $\kappa = 8\pi G$ the gravitational constant. The geometrical meaning can be seen from $|p| = a^2$ and $c = \frac{1}{2}\gamma\dot{a}$ in terms of the scale factor a of a Friedmann–Robertson–Walker metric. For a semiclassical universe, we thus have $c \ll 1$ and $p \gg \ell_{\rm P}^2 = \hbar\kappa$ In contrast to a, p can also be negative in general with the sign corresponding to spatial orientation, but we will assume $p > 0$ in this example. The Hamiltonian H is actually a constraint in this case, but we will not discuss aspects of constrained systems in the geometric formulation here.

To simplify calculations we already weaken the notion of a strong effective system and require agreement between quantum and effective dynamics only up to corrections of the order $\hbar$. Performing the $\hbar$ expansion of the mean value of the Hamiltonian we obtain

$$H_Q = H + \frac{1}{2}\hbar\kappa H_{,ij}\,\tilde{G}^{ij} + O(\hbar^{\frac{3}{2}}) = H - \frac{3\hbar}{\gamma^2}\left(\sqrt{p}\tilde{G}^{0,2} + \frac{c}{\sqrt{p}}\tilde{G}^{1,2} - \frac{c^2}{8\sqrt{p^3}}\tilde{G}^{2,2}\right) + O(\hbar^{\frac{3}{2}})\,. \quad (40)$$

in terms of $\tilde{G}^{a,n} = \ell_{\rm P}^{-n}G^{a,n}$. These variables are motivated by the uncertainty relations, with for the symplectic structure in this example read $G^{0,2}G^{2,2} - (G^{1,2})^2 \geq \frac{1}{36}\gamma^2\ell_{\rm P}^4$. Thus, one can expect that for minimal uncertainty the $\tilde{G}$ (which are not dimensionless) do not contribute further factors of $\hbar$. We will now perform a more detailed analysis.

From the commutation relation $[c,p] = \frac{1}{3}i\gamma\ell_{\rm P}^2$ we obtain

$$\dot{G}^{a,n} = (\dot{c}\partial_c + \dot{p}\partial_p)G^{a,n} = -\frac{1}{\gamma\sqrt{p^3}}\left(-2ap^2G^{a-1,n} + (n-2a)cpG^{a,n} - \frac{(n-a)c^2}{4}G^{a+1,n}\right)\,.$$

At this point it is useful to define $G^{a,n} =: c^{n-a}p^a g^{a,n}$ with dimensionless g, leading to

$$(\tfrac{1}{2}c\partial_c - 2p\partial_p)g^{a,n} = -ag^{a-1,n} + \frac{1}{4}(n+a)g^{a,n} - \frac{1}{8}(n-a)g^{a+1,n}\,.$$

This system of partial differential equations can be simplified by introducing coordinates (x,y) by $e^{2x} = \ell c^2/\sqrt{p}$ and $y := c^2\sqrt{p}/\ell$ with a constant ℓ of dimension length, e.g. $\ell = \kappa E$ as the only classically available length scale independent of the canonical variables, such that $\frac{1}{2}c\partial_c - 2p\partial_p = \partial_x$ and $(\frac{1}{2}c\partial_c - 2p\partial_p)f(y) = 0$ for any function f independent of x.

The general solution for $n=2$ then is

$$\begin{aligned}
g^{0,2} &= g_0(y) + g_{\frac{3}{2}}(y)e^{\frac{3}{2}x} + g_3(y)e^{3x}\\
g^{1,2} &= 2g_0(y) - g_{\frac{3}{2}}(y)e^{\frac{3}{2}x} - 4g_3(y)e^{3x}\\
g^{2,2} &= 4g_0(y) - 8g_{\frac{3}{2}}(y)e^{\frac{3}{2}x} + 16g_3(y)e^{3x}
\end{aligned}$$

subject to the uncertainty relation

$$4g_0g_3 - g_{\frac{3}{2}}^2 \geq \frac{\gamma^2\ell_{\rm P}^4}{2^2 3^4 \ell^{\frac{3}{2}}(c^2\sqrt{p})^{\frac{5}{2}}}\,. \quad (41)$$

Since H is a constraint, y will be constant physically such that we can also consider g_0, $g_{\frac{3}{2}}$ and g_3 as constants. On the constraint surface, the right hand side of the uncertainty relation is then of the order $(\ell_{\rm P}/\kappa E)^4$ for the above choice of ℓ and thus very small.

Note first that, unlike the free particle and the harmonic oscillator examples, solutions for the $G^{a,n}$ do not leave unaffected the effective system. In this example, provided that it allows an effective Hamiltonian description, we would thus encounter an effective Hamiltonian different from the classical one. Spreading back-reacts on the dynamics according to the effective equations

$$\gamma\dot{c} = -c^2p^{-\frac{1}{2}}\left(1 + \frac{1}{2}g_0 - g_{\frac{3}{2}}(\ell c^2p^{-\frac{1}{2}})^{3/4} + 11g_3(\ell c^2p^{-\frac{1}{2}})^{3/2} + \cdots\right) \quad (42)$$

$$\gamma\dot{p} = c\sqrt{p}\left(4 + 2g_0 + 2g_{\frac{3}{2}}(\ell c^2p^{-\frac{1}{2}})^{3/4} - 16g_3(\ell c^2p^{-\frac{1}{2}})^{3/2} + \cdots\right) \quad (43)$$

There is no explicit $\hbar$ in the correction terms because we use dimensionless variables, but the uncertainty relation shows that for constants close to minimal uncertainty the corrections are of higher order in the Planck length.

Moreover, as in the free particle case no constant solutions for the $G^{a,n}$ exist. We thus have to weaken not only the condition of a preserved embedding, but also its horizontality. Since we are interested in effective equations only up to a certain order in $\hbar$, which we already used in the dynamics of this example, it is reasonable to require constant G also only up to terms of some order in $\hbar$. This means that the quantum variables do not need to be strictly constant, but change only slowly. In this example, we have

$$\begin{aligned}
\dot{G}^{0,2} &= -\gamma^{-1}c^3p^{-1/2}(g_0 + \tfrac{5}{2}g_{\frac{3}{2}}e^{\frac{3}{2}x} + 4g_3e^{3x}) \\
\dot{G}^{1,2} &= 3\gamma^{-1}c^2p^{1/2}(g_0 + 2g_3e^{3x}) \\
\dot{G}^{2,2} &= 4\gamma^{-1}cp^{3/2}(2g_0 + 5g_{\frac{3}{2}}e^{\frac{3}{2}x} + 2g_3e^{3x})
\end{aligned}$$

where e^x is small for a large, semiclassical universe and the dominant terms are given by g_0. For large p, $\dot{G}^{2,2}$ grows most strongly, but we can ensure that it is small by using small g_0. It is easy to see that the uncertainty relation allows g_0 to be small enough such that the $\dot{G}^{a,2}$ are small and at most of the order $\hbar$. For instance, $g_{\frac{3}{2}} = 0$, $g_3 = 1$ and $g_0 \sim \ell_{\rm P}^4\ell^{-3/2}(c^2\sqrt{p})^{-5/2}$ is a suitable choice where correction terms to the classical equations are small and the strongest growth of the second order quantum variables, given by $\dot{G}^{2,2} \sim \ell_{\rm P}^4\ell^{-3/2}c^{-4}p^{1/4}$ is small on the constraint surface and using $\ell \sim \kappa E$: $\dot{G}^{2,2} \sim \ell_{\rm P}^4(\kappa E)^{-7/2}p^{5/4}$. To the $\hbar$-order of the equations derived here the system is thus almost preserved, and quantum variables do not grow strongly for some time of the evolution provided that the integration constants g_a are chosen appropriately. (Similar results, without using explicit quantum variables G, have been obtained in [5, 4].)

In the following section we will formalize the weakened conditions on an effective system and show that this allows one to reproduce standard effective action results.

8 An-Harmonic oscillator

We now come to the main part of this paper. As motivated by the preceding examples, we first weaken the effective equation scheme developed so far and then show that it reproduces the standard effective action results when quantum dynamics is expanded around the ground state of a harmonic oscillator. From what we discussed so far, one can already see that basic properties are the same: First, the harmonic oscillator ground state (or any coherent state) gives a quantum dynamics with constant quantum variables such that the quantum Hamiltonian differs from the classical one only by a constant. Effective equations of motion are then identical to the classical ones, which agrees with the usual result. If there is an anharmonic contribution to the potential, however, the evolution of classical variables depends on the quantum variables, and moreover there is no finite set of decoupled quantum variables. Thus, for an exact solution all infinitely many quantum variables have to be taken into account, and in general *no strong effective system exists*. This is the analog of the non-locality of the standard effective action which in general cannot be written as a time integral of a functional of the q^i and finitely many of their time derivatives. In standard effective actions, a derivative expansion is an important approximation, and similarly we have to weaken our definition of effective systems by introducing approximate notions.

The classical Hamiltonian is now given by $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q)$, and the quantum Hamiltonian in terms of dimensionless quantum variables (30), dropping the tilde from now on, is

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q) + \frac{\hbar\omega}{2}(G^{0,2} + G^{2,2}) + \sum_n \frac{1}{n!}(\hbar/m\omega)^{n/2} U^{(n)}(q) G^{0,n} \,. \quad (44)$$

This generates equations of motion

$$\begin{aligned}
\dot{q} &= m^{-1}p \\
\dot{p} &= -m\omega^2 q - U'(q) - \sum_n \frac{1}{n!}(m^{-1}\omega^{-1}\hbar)^{n/2} U^{(n+1)}(q) G^{0,n} \qquad (45) \\
\dot{G}^{a,n} &= -a\omega G^{a-1,n} + (n-a)\omega G^{a+1,n} - \frac{aU''}{m\omega} G^{a-1,n} \\
&\quad + \frac{\sqrt{\hbar} a U'''(q)}{2(m\omega)^{\frac{3}{2}}} G^{a-1,n-1} G^{0,2} + \frac{\hbar a U''''(q)}{3!(m\omega)^2} G^{a-1,n-1} G^{0,3} \\
&\quad - \frac{a}{2}\left(\frac{\sqrt{\hbar} U'''(q)}{(m\omega)^{\frac{3}{2}}} G^{a-1,n+1} + \frac{\hbar U''''(q)}{3(m\omega)^2} G^{a-1,n+2} \right) \\
&\quad + \frac{a(a-1)(a-2)}{3 \cdot 2^3} \left(\frac{\sqrt{\hbar} U'''(q)}{(m\omega)^{\frac{3}{2}}} G^{a-3,n-3} + \frac{\hbar U''''(q)}{(m\omega)^2} G^{a-3,n-2} \right) + \cdots
\end{aligned}$$

showing explicitly that a potential of order higher than two makes the equations of motion for the $G^{a,n}$ involve $G^{a,n+1}$, $G^{a,n+2}$ and so on, therefore requiring one to solve an infinite set of coupled non-linear equations. However, for semiclassical dynamics the $G^{a,n}$ should be small as they are related to the spreading of the wave function. This allows the implementation of a perturbative expansion in $\hbar^{1/2}$ powers to solve the equations for G, where the number of degrees of freedom involved to calculate the equations of motion for the classical variables up to a given order is finite.

We emphasize that corrections appear at half-integer powers in $\hbar$, except for the linear order. This is in contrast to what is often intuitively expected for quantum theories, where only corrections in powers of $\hbar$ are supposed to appear. (Correction terms of half-integer order do not appear only if the classical Hamiltonian is even in all canonical variables.) However, this is much more natural from a quantum gravity point of view where not $\hbar$ but the Planck length $\ell_{\rm P} = \sqrt{\kappa\hbar}$ is the basic parameter, which is a fractional power of $\hbar$ (see the quantum cosmology example).

To solve the equations, we expand $G^{a,n} = \sum_e G^{a,n}_e \hbar^{e/2}$. If we want to find a solution up to kth order we have to calculate the solutions to (45) for $G^{0,2}$ up to the order $k-2$

and $G^{0,3}$ to the order $k-3$. At the same time, these will be functions of the $G^{a,n}$ to all orders up to $G_l^{a,3+2(k-3)-l}$ for all positive integer $l \leq 2k-3$.
Example: For $U(q) = \frac{\delta}{4!}q^4$ we have equations of motion

$$
\begin{aligned}
\dot{G}_0^{a,n} &= -a\omega G_0^{a-1,n} + (n-a)\omega G_0^{a+1,n} - \frac{\delta q^2 a}{2m\omega} G_0^{a-1,n} \\
\dot{G}_1^{a,n} &= -a\omega G_1^{a-1,n} + (n-a)\omega G_1^{a+1,n} - \frac{\delta q^2 a}{2m\omega} G_1^{a-1,n} + \frac{\delta a q}{2(m\omega)^{\frac{3}{2}}} G_0^{0,2} G_0^{a-1,n-1} \\
&\quad - \frac{\delta a q}{2(m\omega)^{\frac{3}{2}}} \left(G_0^{a-1,n+1} - \frac{(a-1)(a-2)}{12} G_0^{a-3,n-3} \right) \\
\dot{G}_2^{a,n} &= -a\omega G_2^{a-1,n} + (n-a)\omega G_2^{a+1,n} - \frac{\delta q^2 a}{2m\omega} G_2^{a-1,n} + \frac{\delta a q}{2(m\omega)^{\frac{3}{2}}} (G_1^{0,2} G_0^{a-1,n-1} + G_0^{0,2} G_1^{a-1,n-1}) \\
&\quad - \frac{\delta a q}{2(m\omega)^{\frac{3}{2}}} \left(G_1^{a-1,n+1} - \frac{(a-1)(a-2)}{12} G_1^{a-3,n-3} \right) \\
&\quad + \frac{\delta a}{3!(m\omega)^2} G_0^{0,3} G_0^{a-1,n-1} - \frac{\delta a q}{6(m\omega)^2} \left(G_0^{a-1,n+2} - \frac{(a-1)(a-2)}{4(m\omega)^2} G_0^{a-3,n-2} \right)
\end{aligned}
$$

up to second order.

Now, in order to construct a strong effective theory of the system we would again have to find a submanifold which is invariant under the action of the Hamiltonian. The only dynamics contained in our quantum degrees of freedom then comes via the submanifold: $\dot{G}^{a,n} = \dot{x}^i \partial_i G^{a,n}$, e.g. for a potential $U(q) = \frac{\delta}{4!}q^4$

$$
\dot{G}^{a,n} = \left(\frac{1}{m} p \partial_q - \left(m\omega^2 q + \frac{\delta}{3!} q^3 + \frac{\hbar \delta q}{2m\omega} G^{0,2} + \frac{\hbar^{3/2}\delta}{3!(m\omega)^{\frac{3}{2}}} G^{0,3} \right) \partial_p \right) G^{a,n} . \qquad (46)
$$

It seems convenient to perform an expansion in δ in addition to $\hbar$ in order to solve the system of equations. However, solutions of these equations, perturbative or exact, are in general not single valued functions of the classical variables and therefore an exactly preserved semiclassical submanifold does not exist. In fact, we have

Lemma 6 *Let $(\mathcal{H}, \hat{H})$ be a quantum mechanical system such that $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{q})$. If $(\mathcal{H}, \hat{H})$ admits a strong effective system of the classical dimension then $(\mathcal{H}, \hat{H})$ is linear.*

Proof: By assumption, we have an embedding of the classical phase space into the quantum phase space such that the quantum flow is everywhere tangential to the embedding and the classical symplectic structure is induced. We can thus take the quantum Hamiltonian vector field and choose additional horizontal vector fields generated by functions L^i on $\mathcal{K}$ such that they span the tangent space to $\mathcal{P}$ in each point $p \in \mathcal{P}$. Since, by construction, the collection of all those vector fields can be integrated to a manifold, they are in involution. Vector fields on the bundle, finally, correspond to linear operators on the Hilbert space having the same commutation relations as the Poisson relations of the generating functions. There is thus a complete set of operators of the quantum system which includes the Hamiltonian and is in involution. □

The notion of a strong effective system then does not allow enough freedom to include many physically interesting systems. Indeed, the dynamics of a strong effective system does not significantly differ from the classical one:

Theorem 1 *For any strong effective system of classical dimension,* $H_{\rm eff} = H + {\rm const}$ *differs from the classical Hamiltonian only by a constant of order* $\hbar$*.*

Proof: From the preceding lemma it follows that a strong effective system can exist only when the Hamiltonian is at most quadratic in the complete classical phase space functions L^i. In an expansion as in (44) we then have only the linear order in $\hbar$ containing $G^{a,2}$. Since by assumption the strong effective system is of the classical dimension, horizontality implies that the $G^{a,2}$ are constant. Thus, $H_Q - H = \hbar c$ with a constant c, and H_Q directly gives the effective Hamiltonian. □

If quantum degrees of freedom are included in a strong effective system of dimension higher than the classical one, they are then only added onto the classical system without interactions, which is not of much interest. On the other hand, for effective equations one is not necessarily interested in precisely describing whole orbits of the system, for which single valued solutions $G(q,p)$ would be required, but foremost in understanding the local behavior compared to the classical one, i.e. modifications of time derivatives of the classical variables. The conditions for a strong effective system, however, are requirements on the whole set of orbits of the system. Thus, as noted before, we have to weaken our definition of effective systems. We first do so in a manner which focuses on the finite dimensionality of classical systems but ignores more refined notions of semiclassicality:

Definition 5 *An* effective system of order k *for a quantum system* $(\mathcal{H},\hat{H})$ *is a dynamical system* $(\mathcal{M},X_{\rm eff})$*, i.e. a finite-dimensional manifold* $\mathcal{M}$ *together with an effective flow defined by the vector field* $X_{\rm eff}$*, which can locally be embedded in the Kähler manifold* $\mathcal{K}$ *associated with* $\mathcal{H}$ *such that it is* almost preserved*: for any* $p\in\mathcal{M}$ *there is an embedding* ι_p *of a neighborhood of* p *in* $\mathcal{K}$ *such that* $X_H(p)-\iota_{p*}X_{\rm eff}(p)$ *is of the order* $\hbar^{k+1}$ *with the vector field* X_H *generated by the quantum Hamiltonian.*

An effective system in this sense allows one to describe a quantum system by a set of finitely many equations of motion, as we encountered it before in the examples. The only concept of classicality is the finite dimensionality, while otherwise the quantum variables included in the effective system can change rapidly and grow large even if an initial state has small fluctuations. Moreover, the finite dimensional space of an effective system is not required to be of even dimension or, even if it is of even dimension, to be a symplectic space. In general, it is only equipped locally with a pre-symplectic form through the pull-back of Ω on $\mathcal{K}$. A stronger notion, taking these issues into account, is

Definition 6 *A* Hamiltonian effective system $(\mathcal{P}, H_{\text{eff}})$ of order k *for a quantum system* $(\mathcal{H}, \hat{H})$ *is a finite-dimensional subspace* $\mathcal{P}$ *of the Kähler manifold* $\mathcal{K}$ *associated with* $\mathcal{H}$ *which is*

1. symplectic*, i.e. equipped with a symplectic structure* $\Omega_{\mathcal{P}} = \iota^*\Omega_{\mathcal{K}} + O(\hbar^{k+1})$ *agreeing up to order* $\hbar^{k+1}$ *with the pull back of the full symplectic structure, and*

2. almost preserved and Hamiltonian*, i.e. there is a Hamiltonian vector field* X_{eff} *generated by the* effective Hamiltonian H_{eff} *on* $\mathcal{P}$ *such that for any* $p \in \mathcal{P}$ *the vector* $X_H(p) - X_{\text{eff}}(p)$ *is of the order* $\hbar^{k+1}$ *with the vector field* X_H *generated by the quantum Hamiltonian.*

By using a symplectic subspace we ensure that the commutator algebra of the quantum system, which determines the symplectic structure on $\mathcal{K}$, is reflected in the symplectic structure of the effective system. Moreover, as in the previous definition the dynamics of the effective system is close to the quantum dynamics. Still, the effective Hamiltonian is not directly related to the quantum Hamiltonian: one generally expands the quantum Hamiltonian in powers of $\hbar$, solves some of the equations of motion for $G^{a,n}$ and reinserts solutions into the expansion. Nevertheless, to low orders in $\hbar$ most fluctuations can be ignored and it is often possible to work directly with the quantum Hamiltonian as the expectation value in suitably peaked states. This is the case for effective equations of quantum cosmology [18, 4, 5] where this procedure has been suggested first.

In this definition, we still do not include any reference to the corresponding classical system. In general, its dynamics will not be close to the effective dynamics, but there are usually regimes where this can be ensured for at least some time starting with appropriate initial states. Also the symplectic structure $\Omega_{\mathcal{P}}$ can differ from the classical one. This is realized also for effective actions such as (4), where the symplectic structure also receives correction terms of the same order in $\hbar$ as the Hamiltonian. The effective and classical symplectic structures are close if the embedding of $\mathcal{P}$ in $\mathcal{K}$ is "almost horizontal" which can be formalized by requiring that for any $p \in \mathcal{P}$ and $v \in \text{Hor}_p^{\Omega}\mathcal{K}$ there is a $w \in T_p\mathcal{P}$ such that $w - v \in T_p\mathcal{K}$ is of some appropriate order in $\hbar$.

We do not make this definition of almost horizontality more precise since it turns out not to be needed to reproduce usual effective action results. Moreover, its practical implementation can be rather complicated: The quantum cosmology example showed that the order to which one can ensure almost horizontality is not directly related to the order in $\hbar$ to which equations of motion are expanded. If one has an almost horizontal embedding, ignored quantum degrees of freedom remain almost constant such that they do not much influence the evolution for an appropriately prepared initial state. But not any system can be approximated in this manner, and so the condition of almost horizontality implies that for some systems only higher dimensional Hamiltonian effective systems exist. In such a case there are some quantum degrees of freedom which can by no means be ignored for the effective dynamics. On the other hand, in such a case it may be difficult to guarantee the existence of a symplectic structure. This happens, for instance, if the $G^{a,2}$ change too rapidly, but not higher G. One can then use a 5-dimensional effective system with variables $(q, p, G^{0,2}, G^{1,2}, G^{2,2})$ which can only be pre-symplectic and thus not Hamiltonian. Alternatively, one can drop the condition of almost horizontality, but then has to accept a new (pre-)symplectic structure which is not necessarily related to the classical one by only correction terms. These constraints show that a discussion of quantum variables in higher-dimensional effective systems can be complicated if one insists on the presence of a canonical structure. Moreover, computing the symplectic structure on the Kähler space and its pull-back to the effective manifold in an explicit manner is usually complicated (see, however, Sec. 9 for a brief discussion).

We thus present a final definition which does not require an explicit form of the quantum symplectic structure but is sufficient for the usual setting of effective actions:

Definition 7 *An* adiabatic effective system of order (e, k) *for a quantum system* $(\mathcal{H}, \hat{H})$ *is an effective system* $(\mathcal{M}, X_{\text{eff}})$ *of order k in the sense of Def. 5 such that the local embeddings are given by solutions up to order e in an adiabatic expansion of those quantum variables not included as variables of the effective system.*

Here, adiabaticity intuitively captures the physical property of a weak influence of quantum degrees of freedom on the classical ones: in the adiabatic approximation they change only slowly compared to the classical variables. Provided that a semiclassical initial state is chosen it is then guaranteed that the system remains semiclassical for some time.

This viewpoint is still much more general than the usual definition of an effective action, and it allows much more freedom by choosing different finite-dimensional subspaces. For an explicit derivation of effective equations, of course, one has to find solutions $G^{a,n}(x^i)$ as they appear in the quantum Hamiltonian, which requires one to solve an infinite set of coupled differential equations for infinitely many variables. Only in exceptional cases, such as integrable systems, can this be done without approximations. Moreover, general solutions for $G^{a,n}(x^i)$ contain infinitely many constants of integration which then also appear in the effective equations after inserting the $G^{a,n}(x^i)$. On the one hand, this allows much more freedom in choosing the states, such as squeezed or of non-minimal uncertainty, to perturb around. But it also means that one needs criteria to fix the integration constants in situations of interest. One such situation is that of

Theorem 2 *A system with classical Hamiltonian* $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q)$ *admits an adiabatic effective system of order* $(2, 1)$ *whose dynamics is governed by the effective action (4).*

Proof: In order to find the subspace $\mathcal{P}$ and the dynamics on it we expand the quantum Hamiltonian in powers of $\hbar$ and solve the equations of motion for $G^{a,n}$ in an adiabatic approximation.

The adiabatic approximation of slowly varying fields in the equations of motion is an expansion in a parameter λ introduced for the sake of the calculation, but in the end set to $\lambda = 1$. Derivatives with respect to time are scaled as $\frac{\mathrm{d}}{\mathrm{d}t} \to \lambda \frac{\mathrm{d}}{\mathrm{d}t}$ and, expanding $G^{a,n} = \sum_e G_e^{a,n} \lambda^e$, the equations of motion

$$\dot{x}^i \partial_i G^{a,n} = \{G^{a,n}, H_Q\}_Q$$

imply

$$\dot{x}^i \partial_i G^{a,n}_{e-1} = \{G^{a,n}_e, H_Q\}_Q \,.$$

In addition to the adiabatic approximation we also perform a semiclassical expansion in powers of $\hbar$. In what follows, we will calculate the first order in $\hbar$ and go to second order in λ for $G^{a,2}$.

To zeroth order in λ the equations to solve are

$$0 = \{G^{a,n}_0, H_Q\}_Q = \omega\left((n-a)G^{a+1,n}_0 - a\left(1+\frac{U''}{m\omega^2}\right)G^{a-1,n}_0\right)$$

with general solution

$$G^{a,n}_0 = \tbinom{n/2}{a/2}\tbinom{n}{a}^{-1}\left(1+\frac{U''}{m\omega^2}\right)^{a/2} G^{0,n}_0$$

for even a and n, and $G^{a,n}_0 = 0$ whenever a or n are odd. This still leaves the value of $G^{0,n}_0$ free, which will be fixed shortly. To first order in λ,

$$(n-a)G^{a+1,n}_1 - a\left(1+\frac{U''}{m\omega^2}\right)G^{a-1,n}_1 = \frac{1}{\omega}\dot{G}^{a,n}_0$$

implies

Lemma 7

$$\sum_{a\text{ even}} \tbinom{n/2}{a/2}\left(1+\frac{U''}{m\omega^2}\right)^{\frac{n-a}{2}} \dot{G}^{a,n}_0 = 0$$

Proof: From the equation above

$$\sum_a \tbinom{n/2}{a/2}\left(1+\frac{U''}{m\omega^2}\right)^{\frac{n-a}{2}} \dot{G}^{a,n}_0 = \sum_a \tbinom{n/2}{a/2}\left(1+\frac{U''}{m\omega^2}\right)^{\frac{n-a}{2}}\left((n-a)G^{a+1,n}_1 - a\left(1+\frac{U''}{m\omega^2}\right)G^{a-1,n}_1\right)$$

manipulating the first term of the right hand side expression we shift $a \to a-2$ leaving the limits for a unaffected in the summation to obtain

$$\sum_a \frac{(n/2)!\left(1+\frac{U''}{m\omega^2}\right)^{\frac{n-a+2}{2}}}{((a-2)/2)!((n-a+2)/2)!}(n-a+2)G^{a-1,n}_1 = \sum_a \tbinom{n/2}{a/2} a\left(1+\frac{U''}{m\omega^2}\right)^{\frac{n-a+2}{2}} G^{a-1,n}_1$$

which cancels then the second term to finish the proof. □

This imposes a constraint on $G^{0,n}_0$ solved by setting $G^{0,n}_0 = C_n(1+\frac{U''}{m\omega^2})^{-n/4}$. The remaining constants C_n are fixed to $C_n = \frac{n!}{2^n(n/2)!}$ by requiring that the limit $U \to 0$ reproduces the quantum variables of coherent states of the free theory (37) or equivalently by requiring the perturbative vacuum of the quantum theory to be associated to the vacuum of the effective system. Therefore,

$$G^{a,n}_0 = \frac{(n-a)!a!}{2^n((n-a)/2)!(a/2)!}\left(1+\frac{U''}{m\omega^2}\right)^{\frac{2a-n}{4}} .$$

We will need only the $n = 2$ corrections to first order in $\hbar$, and the solution to the first order equations becomes trivial: $G_1^{1,2} = \frac{1}{2\omega}\dot{G}_0^{0,2}$, the rest being zero. To second order we have

$$G_2^{2,2} - \left(1 + \frac{U''}{m\omega^2}\right) G_2^{0,2} = \frac{1}{\omega}\dot{G}_1^{1,2} = \frac{1}{2\omega^2}\ddot{G}_0^{0,2}$$

again leaving free parameters in the general solution to be fixed by the next, third order from which we obtain

$$\left(1 + \frac{U''}{m\omega^2}\right) \dot{G}_2^{0,2} + \dot{G}_2^{2,2} = 0$$

as in the Lemma before. The previous two equations can be combined to a first order differential equations for $G_2^{0,2}$ in terms of known solutions at lower orders:

$$\dot{G}_2^{0,2} - \frac{\dot{G}_0^{0,2}}{G_0^{0,2}} G_2^{0,2} + \frac{1}{\omega^2}(G_0^{0,2})^2 \dddot{G}_0^{0,2} = 0\,.$$

Its general solution is

$$G_2^{0,2} = \left(c - 2\omega^{-2}(G_0^{0,2})^{3/2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(G_0^{0,2})^{1/2}\right) G_0^{0,2}$$

where the integration constant c can be fixed to $c = 0$ by requiring the correct free limit $U = 0$ (for which the original two differential equations imply $G_2^{2,2} = -G_2^{0,2} = 0$). From this, the solution to the system is

$$G_2^{0,2} = -\frac{2}{\omega^2}(G_0^{0,2})^{\frac{5}{2}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(G_0^{0,2})^{1/2} = \frac{\left(1 + \frac{U''}{m\omega^2}\right)^{-\frac{7}{2}}}{4\omega^2}\left(\left(1 + \frac{U''}{m\omega^2}\right)\frac{U'''\ddot{q} + U''''\dot{q}^2}{4m\omega^2} - 5\left(\frac{U'''\dot{q}}{4m\omega^2}\right)^2\right)\,.$$

Finally, putting our approximate expressions for the quantum variables back into the equations of the classical variables (45), we obtain

$$\left(m + \frac{\lambda^2\hbar(U''')^2}{2^5 m^2\omega^5\left(1 + \frac{U''}{m\omega^2}\right)^{\frac{5}{2}}}\right)\ddot{q} + \frac{\lambda^2\hbar\dot{q}^2\left(4m\omega^2 U''' U''''\left(1 + \frac{U''}{m\omega^2}\right) - 5(U''')^3\right)}{2^7 m^3\omega^7\left(1 + \frac{U''}{m\omega^2}\right)^{\frac{7}{2}}}$$
$$+ m\omega^2 q + U' + \frac{\hbar U'''}{4m\omega\left(1 + \frac{U''}{m\omega^2}\right)^{\frac{1}{2}}} = 0 \qquad (47)$$

as it also follows from the effective action (4) after setting $\lambda = 1$. □

The proof demonstrates the role of the harmonic oscillator ground state and its importance for fixing constants in the effective equations. If there is no distinguished state, effective equations contain free parameters incorporating the freedom of choosing an initial state in which the system is prepared.
Remark: Since we know the effective action we can compute the effective symplectic structure which has a correction of order $\hbar$ compared to the classical one. Still, this does

not imply that the system is a Hamiltonian effective system of first order as per Def. 6 because we did not relate this symplectic structure to that following from pull-back from the quantum symplectic structure.

9 Dynamical coherent states

In addition to the effective dynamical behavior of classical and quantum degrees of freedom it is also of interest to know approximate states whose dynamics corresponds to the effective evolution. Under the name of dynamical coherent states [11], they can be obtained by collecting the information contained in the mean values of the fundamental operators and the spreading as well as higher order distortions of the state of the system. In this section, we only collect results related to the previous discussion without going into further details.

As we already stated, the task could be achieved by summing up the Hermite polynomial modes obtained through the Hamburger momenta, but a short cut to the answer is possible using Moyal's formula [19] by which four arbitrary normalizable vectors $|\Psi_1\rangle$, $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ satisfy

$$\int \frac{\mathrm{d}^2 z}{2\pi} \langle\Psi_1|e^{z\hat{a}^\dagger-\bar{z}\hat{a}}|\Psi_2\rangle\langle\Psi_3|e^{-z\hat{a}^\dagger+\bar{z}\hat{a}}|\Psi_4\rangle = \langle\Psi_1|\Psi_4\rangle\langle\Psi_3|\Psi_2\rangle \tag{48}$$

where $z = \frac{1}{\sqrt{2}}(z^q + iz^p)$ and $\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{q} + i\hat{p})$. For a bounded operator $\hat{F}$, (48) can be rewritten as

$$\int \frac{\mathrm{d}^2 z}{2\pi} \langle\Psi_1|e^{z\hat{a}^\dagger-\bar{z}\hat{a}}|\Psi_2\rangle \mathrm{Tr}\left\{\hat{F}e^{-z\hat{a}^\dagger+\bar{z}\hat{a}}\right\} = \langle\Psi_1|\hat{F}|\Psi_2\rangle\,. \tag{49}$$

For given solutions $G^{a,n}$, the reconstruction of a dynamical coherent state is completed by performing the integral with arbitrary $|\Psi_1\rangle$, $|\Psi_2\rangle$ after inserting for $\hat{F}$ the probability density operator $\hat{\rho}(q,p)$ and assuming that the state is analytical such that

$$\mathrm{Tr}\left\{\hat{\rho}(q,p)e^{-z\hat{a}^\dagger+\bar{z}\hat{a}}\right\} = e^{\frac{i}{\sqrt{\hbar}}(z^q p - z^p q)} \sum_{n=0}^{\infty}\sum_{a=0}^{n} \frac{(-)^{n-a} i^n}{n!} \tbinom{n}{a} (z^q)^a (z^p)^{n-a} G^{a,n}(q,p)$$

produce the matrix elements of $\hat{\rho}(q,p)$ in a basis of operators $e^{z\hat{a}^\dagger-\bar{z}\hat{a}}$. For the anharmonic oscillator to $0th$ order in $\hbar$ we have $G^{i_1,\ldots,i_n} = \frac{n!}{(n/2)!} G^{(i_1 i_2} \cdots G^{i_{n-1} i_n)}$ for n even, implying

$$\mathrm{Tr}\left\{\hat{\rho}_U(q,p)e^{-z\hat{a}^\dagger+\bar{z}\hat{a}}\right\} = \exp\left(\frac{i}{\sqrt{\hbar}}(z^i\epsilon_{ij}x^j) - \frac{1}{2}z^i z^j \epsilon_{ik}\epsilon_{jl} G^{kl}(q,p)\right)\,.$$

In order to perform the integral above, we choose to work with Harmonic oscillator coherent states $|\alpha\rangle = e^{\alpha\hat{a}^\dagger-\bar{\alpha}\hat{a}}|0\rangle$ for which the matrix elements of the exponential operator are $\langle\alpha|e^{z\hat{a}^\dagger-\bar{z}\hat{a}}|\alpha'\rangle = \exp(-\frac{1}{4\hbar}(\alpha'^i - \alpha^i)\delta_{ij}(\alpha'^j - \alpha^j) + \frac{i}{4\hbar}(\alpha'^i + \alpha^i)\epsilon_{ij}(\alpha'^j - \alpha^j + 2z^j))$. Finally, defining $S_i = \delta_{ij}(\alpha'^j - \alpha^j) + i\epsilon_{ij}(\alpha'^i + \alpha^i - 2x^i)$, the matrix elements of the probability density operator are

$$\langle\alpha|\hat{\rho}_{U}(q,p)|\alpha'\rangle = \frac{1}{\sqrt{\det(\frac{1}{2}\delta^{ij}+G^{ij})}}\exp\left(-\frac{1}{4\hbar}S_{i_1}\epsilon^{i_1j_1}(2G^{ij}+\delta^{ij})^{-1}_{j_1j_2}\epsilon^{j_2i_2}S_{i_2}\right) \times\exp\left(-\frac{i}{4\hbar}(\alpha'^i-\alpha^i)\epsilon_{ij}(\alpha'^j+\alpha^j)-\frac{1}{4\hbar}(\alpha'^i-\alpha^i)\delta_{ij}(\alpha'^j-\alpha^j)\right). \quad (50)$$

The trace of the operator above can now be computed to equal one whenever G^{ij} is a non-degenerate matrix. In order to be sure that ρ is a density matrix, we need to show its positivity. We do not have a complete proof for arbitrary systems, but using the fact that the assumption of the state being semi-classical requires the mean values of operators to be given by their classical expressions up to $\hbar$ corrections, a case by case study leads to the conclusion that the positive mean values above lead to positivity of the operator.

Furthermore, the state of the quantum system as given above is not in general a pure state, but if $G^{ij}=\frac{\hbar}{2}(e^{g\epsilon})^i_k(e^{g\epsilon})^j_l\delta^{kl}$, also $\hat{\rho}_U(x)^2$ has trace one and thus gives a pure state which can be realized as a squeezed coherent state labeled by the symmetric matrix g_{ij} through

$$|x,g\rangle=\exp\left(\frac{i}{2\hbar}g_{ij}(\hat{x}^i-x^i)(\hat{x}^j-x^j)\right)\exp\left(-\frac{i}{\hbar}x^i\epsilon_{ij}\hat{x}^j\right)|0\rangle. \quad (51)$$

With the help of $e^{-\frac{i}{2\hbar}g_{ij}\hat{x}^i\hat{x}^j}\hat{x}^k e^{\frac{i}{2\hbar}g_{ij}\hat{x}^i\hat{x}^j}=(e^{g\epsilon})^k_l\hat{x}^l$, the remaining fiber coordinates become

$$G^{i_1,\ldots,i_n}(g_{ij})=\frac{\hbar^{n/2}n!}{2^n(n/2)!}(e^{g\epsilon})^{i_1}_{j_1}\cdots(e^{g\epsilon})^{i_n}_{j_n}\delta^{(j_1j_2}\cdots\delta^{j_{n-1}j_n)} \quad (52)$$

Reconstructing a dynamical coherent state from the quantum variables $G^{a,n}$ also provides means to compute the symplectic structure on the effective space, as needed for a Hamiltonian effective system as per Def. 6. For the evaluation of the symplectic structure on the vector fields we obtain the pull-back $\Omega(Y,Z)=2\hbar\mathrm{Im}\langle Y,Z\rangle$ where Y and Z are tangent vectors to the embedded effective manifold. Given a dynamical coherent state $|\psi(f^i)\rangle$ as a function of classical variables f^i, we can define a basis of the tangent space spanned by $|i\rangle:=\partial|\psi\rangle/\partial f^i$. Expanding $Y=\sum_i Y_i|i\rangle$ and $Z=\sum_i Z_i|i\rangle$, we have

$$\langle Y|Z\rangle=\sum_{i,j}\bar{Y}^iZ^j\frac{\partial\langle\psi|}{\partial f^i}\frac{\partial|\psi\rangle}{\partial f^j}$$

such that we can formally write

$$\Omega=-2i\hbar\,\mathrm{d}(\langle x_1,\ldots,x_n|)\wedge\mathrm{d}(|x_1,\ldots,x_n\rangle),. \quad (53)$$

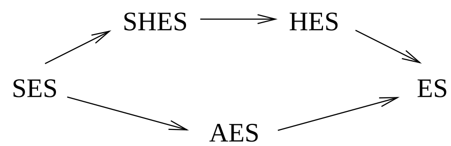
Thus, the pull-back of the symplectic structure to the subspace of squeezed states is

$$\Omega_{|x,g\rangle}=2\epsilon_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j+2^{-5}\hbar\delta^{i_1i_2}\epsilon^{i_3i_4}(\delta^{j_1}_{i_1}+(e^{g\epsilon})^{j_1}_{i_1})\cdots(\delta^{j_4}_{i_4}+(e^{g\epsilon})^{j_4}_{i_4})\mathrm{d}g_{j_1j_3}\wedge\mathrm{d}g_{j_2j_4}\,. \quad (54)$$

For an effective system of the classical dimension, corresponding to a set of solutions $g_{ij}(x^k)$, we can further pull back (54) to the classical manifold and obtain the quantum symplectic structure there. This shows that the classical symplectic structure is reproduced up to corrections of order $\hbar$ if the g do not change strongly (adiabaticity or almost horizontality), and provides means to compute those correction terms.

10 Conclusions

Comparison with common effective action techniques applicable to an-harmonic oscillators demonstrates how effective systems can be formulated more generally for any quantum system. We have extracted several definitions which have different strengths and use different mathematical structures:



Here, the strengths of each of our definitions are compared in a condensed diagram by use of implication arrows and abbreviations in which the initial S holds for strong, H for Hamiltonian, A for adiabatic and ES for effective system. The only definition not provided before is that of a strong Hamiltonian effective system which is a Hamiltonian effective system which is *exactly* preserved and whose symplectic structure is *exactly* the pull-back of the quantum symplectic structure. It is clear from the discussions before that any strong effective system is also strong Hamiltonian, and examples lead to the conjecture that the converse is also true. Still, since we are not aware of a proof, we include strong Hamiltonian effective systems in this diagram.

While the definition of Hamiltonian effective systems is most geometrical, adiabatic effective systems turn out to be more practical and are more directly related to path integral techniques. The weakest notion of an effective system can be applied to any system but does not incorporate many classical aspects except for finite dimensionality for mechanical systems. As the examples showed, in particular that of quantum cosmology, the general definitions provided here are more widely applicable and also present a more intuitive understanding of possible quantum degrees of freedom. Moreover, they are always switched on perturbatively, and no non-analyticity in perturbation parameters as with higher derivative effective actions arises.

The expansion of the quantum Hamiltonian also showed that in general half-integer powers of $\hbar$ have to be expected in correction terms and not just integer powers as often stated. The only exception is the first order in $\hbar^{1/2}$ which does not appear because the expectation value of variables G^1 would be zero by definition. Half-integer powers do not appear only if one has a system with a Hamiltonian even in all canonical variables, such as an an-harmonic oscillator with an even potential, as it often occurs in quantum field theories. These observations are relevant for quantum gravity phenomenology because an expansion in the Planck length $\ell_{\rm P} = \sqrt{\kappa\hbar}$ naturally involves half-integer powers in $\hbar$. From the perspective provided here one can expect all integer powers of the Planck length except for the linear one.

Other advantages are that the effective equations have a geometrical interpretation where only real variables, unlike $q(t)$ in the usual definition, occur. We are dealing directly with equations of motion displaying only the relevant degrees of freedom, which are automatically provided with an interpretation as properties of the wave function, and can directly deal with canonical formulations in which the scheme indeed arises most naturally. The techniques are general enough for arbitrary initial states and systems with unbounded Hamiltonians, as demonstrated by our quantum cosmology example. The infrared problem of (4) for $m \to 0$ is seen to arise only in the adiabatic approximation, but can easily be treated by using more general notions of effectivity such as by including the spreading parameters $G^{a,2}$ in a pre-symplectic effective system.

As discussed briefly in the preceding section, techniques introduced here can also be used directly at the quantum level and not just for effective semiclassical approximations. In this context, we have presented only first steps, but this already shows that the techniques can give information on dynamical coherent states. This will then also have helpful implications for the effective equation scheme itself from which such states arise, as they can give a handle on computing the pull-back of the full symplectic structure.

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