AEI-2005-179

E_{10} Cosmology

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Abstract: We construct simple exact solutions to the $E_{10}/K(E_{10})$ coset model by exploiting its integrability. Using the known correspondences with the bosonic sectors of maximal supergravity theories, these exact solutions translate into exact cosmological solutions. In this way, we are able to recover some recently discovered solutions of M-theory exhibiting phases of accelerated expansion, or, equivalently, S-brane solutions, and thereby accommodate such solutions within the $E_{10}/K(E_{10})$ model. We also discuss the difficulties regarding solutions with non-vanishing (constant) curvature of the internal manifold.

1 Introduction

Cosmological solutions of Einstein gravity coupled to (axionic and/or dilatonic) scalar and p-form matter have been studied for a long time. In particular, those systems which arise as the bosonic sectors of (maximal) supergravities have attracted attention due to their appearance as low energy effective actions in string and M-theory (see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11] and references therein). As was shown in [7, 8, 9] some of these solutions exhibit a phase of accelerated expansion. Although these solutions fail to generate an inflationary period on the scale required by current observational data [8, 9], a most intriguing feature of these results is clearly the fact that M theory (*alias* $D = 11$ supergravity [12]) admits cosmological solutions with interesting profiles for the time evolution of the cosmic scale factor, with intermittent periods of accelerated and decelerated expansion.

More recently, it has been appreciated that the dynamics of the (spatial) scale factors and dilaton fields in the vicinity of a space-like singularity can be effectively described in terms of a cosmological billiard (for a review and references to earlier work, see [13]), which in many cases of interest (and for all examples in M- or superstring theory [14, 13]) takes place in the Weyl chamber of some indefinite Kac–Moody algebra. Motivated by this unsuspected link between cosmological solutions and the theory of (indefinite) Kac–Moody algebras, a more general framework was developed in [15], which relates (a certain truncation of) the bosonic equations of motion of the Einstein-matter system to a null geodesic motion on the infinite-dimensional coset space, which is defined as the (formal) quotient of the relevant Kac–Moody group by its maximal compact subgroup. The example which has been studied in most detail is the coset space $E_{10}/K(E_{10})$, where E_{10} is a maximally extended (rank 10) hyperbolic Kac–Moody group E_{10}, and $K(E_{10})$ its maximal compact subgroup. The equations of motion of the corresponding $E_{10}/K(E_{10})$ σ-model can be expanded in terms of 'levels' defined w.r.t. certain distinguished finite-dimensional subalgebras of E_{10}. When this expansion is truncated to low levels, the resulting (truncated) equations of motion can be shown to correspond to a truncated version of the bosonic equations of motion of $D = 11$ supergravity [15, 16], massive IIA [17] and type IIB supergravity [18], respectively, where the level is defined w.r.t. the A_9, D_9 and $A_8 \times A_1$ subgroups of E_{10}, respectively. The truncation of the supergravity field equations here corresponds to a dimensional reduction to one (time) dimension, where, however, first order spatial gradients are kept. The parameter of the coset space geodesic in this context is interpreted as the time coordinate of the Einstein-matter system and all fields depend on it. According to the main conjecture in [15], some of the higher levels will then correspond to higher and higher order spatial gradients, in such a way that the dependence on the spatial coordinates (which seems to be lost in a naïve reduction to one dimension) gets 'spread over' the infinite-dimensional Kac–Moody Lie algebra. However, the vast majority of higher level representations are expected to correspond to new, M-theoretic, degrees of freedom which have no counterpart in the effective low energy supergravity theory. For some ideas on the rôle of imaginary root generators we refer to [19, 20].

The aim of this letter is to show that the hyperbolic $E_{10}/K(E_{10})$ σ-model of [15] naturally includes accelerated cosmological solutions of the type studied in [7, 8, 9]. A crucial new ingredient in our analysis is the incorporation of *Borcherds*-type subalgebras of E_{10}, which are characterized by the presence of *imaginary simple* roots.[1] Our strategy will be to exploit the integrability of the $E_{10}/K(E_{10})$ model. For clarity we restrict attention to the simplest cases as it will turn out that these already produce the solution given in [8] and outline at the end how to obtain more general solutions of cosmological type, as this generalisation is straightforward.

We note that in the context of Kac–Moody theoretic approaches to supergravity various solutions have been studied in the literature. Our approach differs from the one taken in [22, 23, 24] in that we obtain exact solutions to an abstract one-dimensional coset model and then use the known (partial) correspondences with different supergravity models to interpret these abstract solutions in cosmological terms. In [23] the equation to which the coset element is subjected is dependent on the model. The exact $E_{11}/K(E_{11})$ coset model solutions of [25] in the cosmological interpretation of [26] are similar to our approach.

This letter is structured as follows. In section 2.1 we define the $E_{10}/K(E_{10})$ coset model and demonstrate how to obtain exact solutions to this integrable system in section 2.2. Using the established Kac–Moody/supergravity dictionary, we translate one particular such solution to $D = 11$ supergravity in section 2.3. At the present stage there are difficulties with accommodating related gravity solutions with highly curved spaces as will be discussed in section 2.4. Our conclusions are presented in section 3.

2 Cosmological Solutions from E_{10}

2.1 E_{10} model

The hyperbolic Kac–Moody algebra $\mathfrak{e}_{10}$ is defined by the Dynkin diagram of fig. 1 [28]. An element of the corresponding group coset $E_{10}/K(E_{10})$ is given by an Iwasawa-like parametrisation

$$\mathcal{V}(t) = \exp\left(\sum_{\alpha\in\Delta_+} A_\alpha(t)E_\alpha\right)\exp\left(\sum_{i=1}^{10}\phi_i(t)K^i{}_i\right) \qquad (1)$$

[1]There are many Borcherds subalgebras inside $\mathfrak{e}_{10}$, with an arbitrarily large number of imaginary simple roots, see e.g. [21]. The complexifications of the rank-one Borcherds algebras considered below (for $\alpha^2 \neq 0$) are still isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, but with a compact (negative norm) Cartan generator h over the real numbers.

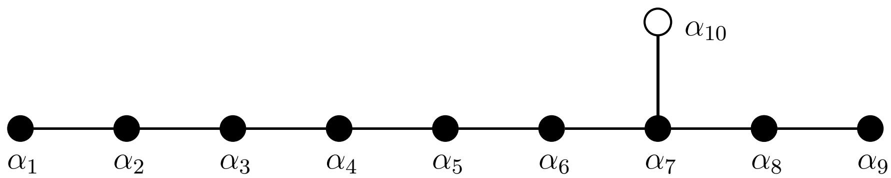

Figure 1: *Dynkin diagram of $\mathfrak{e}_{10}$. The solid nodes correspond to the regular $\mathfrak{sl}(10)$ subalgebra which can be extended to $\mathfrak{gl}(10)$.*

Here, E_α are the positive root (upper triangular) generators of $\mathfrak{e}_{10}$ (as is customary, we denote by $\Delta_\pm$ the set of positive and negative roots, respectively). Since we will be interested in the $\mathfrak{gl}(10)$ decomposition of $\mathfrak{e}_{10}$, we have already adopted a basis of ten diagonal $GL(10)$ generators $K^i{}_i$ as a basis of the Cartan subalgebra of $\mathfrak{e}_{10}$. (For details on the decomposition see [15, 27].)

The $E_{10}/K(E_{10})$ coset model is constructed from (1) in the standard fashion. Associated with $\mathcal{V}$ is the velocity

$$\mathcal{V}^{-1}\partial_t\mathcal{V} = \mathcal{P} + \mathcal{Q}, \tag{2}$$

where $\mathcal{Q}$ corresponds to the unbroken local $K(E_{10})$ invariance of the model, and is therefore fixed by the Chevalley involution $\varnothing$ defining the invariant subalgebra Lie$(K(E_{10}))$. The remaining components $\mathcal{P}$ are along the coset directions in E_{10}. They contain the physical fields of the model. The time reparametrisation invariant Lagrangian is given by

$$\mathcal{L} = \frac{1}{2n}\langle\mathcal{P}|\mathcal{P}\rangle, \tag{3}$$

with the standard invariant bilinear form $\langle\cdot|\cdot\rangle$ [28]. The equations of motion implied by this Lagrangian are

$$n\partial_t(n^{-1}\mathcal{P}) = [\mathcal{P},\mathcal{Q}], \qquad \mathcal{H} \equiv \langle\mathcal{P}|\mathcal{P}\rangle = 0. \tag{4}$$

the latter being the Hamiltonian constraint associated with the invariance under time reparametrisations. The system is formally integrable since the $\mathfrak{e}_{10}$-valued Noether current

$$\mathcal{J} = n^{-1}\mathcal{V}\mathcal{P}\mathcal{V}^{-1} \tag{5}$$

contains infinitely many conserved charges. The formal solution of (4) is

$$\mathcal{V} = \exp(\nu(t)\mathcal{J})\,\mathcal{V}_0\,k(t) \qquad \text{with} \quad \nu(t) := \int_{t_0}^{t} n(t')dt' \tag{6}$$

for any choices of charges $\mathcal{J}$ and initial configuration $\mathcal{V}_0$; the compensating $K(E_{10})$ rotation $k(t)$ is determined so as to bring $\mathcal{V}$ into the triangular gauge (1), and depends on both $\mathcal{J}$ and $\mathcal{V}_0$.

2.2 Construction of $E_{10}/K(E_{10})$ Solutions

Even though the general solution to the $E_{10}/K(E_{10})$ model is known, the usefulness of (6) is limited by (at least) two facts. First, the structure of E_{10} is not known in closed form which complicates the evaluation of (6), in particular the determination of $k(t)$ — even disregarding ambiguities in defining the Kac–Moody groups. Secondly, the relation of the $E_{10}/K(E_{10})$ model to $D = 11$ supergravity is presently only understood for a truncation of the system (4) to low levels [15, 16]. Hence, an interpretation of the general solution (6) in terms of supergravity or M-theory variables is not amenable. For these two reasons we here take a conservative approach, by restricting ourselves to solutions of the equations (4) for *finite-dimensional* subspaces of $E_{10}/K(E_{10})$ only, for which the 'dictionary' between the E_{10} and the supergravity variables is known. We will keep the discussion as simple as possible by only exciting one positive root, although, of course, the analysis can be straightforwardly extended to more general (but still finite-dimensional) subspaces.

To study the geodesic motion on finite-dimensional subspaces of the $E_{10}/K(E_{10})$ coset model, we consider a generator E_α of E_{10} in the root space of a positive root $\alpha = \sum_{i=1}^{10} m_i\alpha_i$ of the algebra. Our labelling of the simple roots of E_{10} can be read off from fig. 1. Let $E_{-\alpha} = -\o(E_\alpha)$ where $\o$ is the Chevalley involution [28]. Then we have the following relations

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{10} m_i h_i \equiv h, \qquad [h, E_{\pm\alpha}] = \pm\alpha^2 E_{\pm\alpha}. \tag{7}$$

where h lies in the Cartan subalgebra (CSA) of E_{10}. The generators are normalised as follows

$$\langle E_\alpha | E_{-\alpha}\rangle = 1, \qquad \langle h|h\rangle = \alpha^2. \tag{8}$$

Since $\mathfrak{e}_{10}$ is a hyperbolic Kac–Moody algebra with symmetric generalized Cartan matrix of signature $(-+\cdots+)$, α^2 can take the values $\alpha^2 = 2, 0, -2, \ldots$. For $\alpha^2 = 2$ we have a standard $\mathfrak{sl}(2)$ algebra (in split real form). For $\alpha^2 \leq 0$

the simple root is imaginary, and consequently we are dealing with a rank-one *Borcherds algebra* with generalized Cartan matrix (α^2). Details on Borcherds algebras can be found in [29] or §11.13 of [28]. For imaginary roots α, the E_{10} root multiplicities $\text{mult}_{E_{10}}(\alpha)$ increase exponentially with $-\alpha^2$, and we will thus have a large set of Lie algebra elements $\{E_{\alpha,s}|s = 1, \dots, \text{mult}_{E_{10}}(\alpha)\}$ to choose from. Below, however, we will need only one particular element from this root space, and will therefore omit the label s.

We will thus consider a special two-dimensional subspace of the (infinite-dimensional) coset $E_{10}/K(E_{10})$ coset, which is parametrized as follows in triangular gauge

$$\mathcal{V}(t) = e^{A(t)\cdot E_\alpha}\, e^{\phi(t)\cdot h} \quad \Longrightarrow \quad \mathcal{V}^{-1}\partial_t\mathcal{V} = \underbrace{e^{-\alpha^2\phi}\partial_t A}_{P_\alpha}\cdot E_\alpha + \partial_t\phi\cdot h. \tag{9}$$

The Lagrangian (3) evaluated on these fields is

$$\mathcal{L} = \frac{\alpha^2}{2}(\partial_t\phi)^2 + \frac{1}{2}P_\alpha^2, \tag{10}$$

where we have adopted the affine gauge $n = 1$. The equations of motions (4) now read

$$\begin{aligned} e^{-2\alpha^2\phi}\partial_t A &= a \qquad \Longleftrightarrow \qquad P_\alpha = a e^{\alpha^2\phi}, \\ \partial_t^2\phi &= -\frac{1}{2}a^2 e^{2\alpha^2\phi}. \end{aligned} \tag{11}$$

The A equation has already been integrated once using the cyclicity of A, with a the real constant of integration. If $\alpha^2 \neq 0$ we define

$$\varphi(t) := \alpha^2\,\phi(t) \tag{12}$$

which obeys the Liouville equation

$$\partial_t^2\varphi = -Ce^{2\varphi}, \qquad C = \frac{1}{2}a^2\alpha^2. \tag{13}$$

Obviously, the sign of C is positive if α is real and negative if α is (pure) imaginary, *i.e.* time-like. Eq. (13) can be integrated once, with one integration constant E

$$(\partial_t\varphi)^2 + Ce^{2\varphi} = E\ , \tag{14}$$

which is the energy of the effective one-dimensional system obtained by integrating out $A(t)$. For $C \neq 0$, Eq. (14) can be integrated explicitly with the result (see e.g. [10])

$$\begin{aligned}
\varphi(t) &= -\ln\left[\sqrt{\frac{C}{E}}\cosh(\sqrt{E}t)\right] && \text{for } C>0\,,\ E>0\ \ , \\
\varphi(t) &= -\ln\left[\sqrt{-\frac{C}{E}}\sinh(\sqrt{E}t)\right] && \text{for } C<0\,,\ E>0\ \ , \\
\varphi(t) &= -\ln\left[\sqrt{-C}t\right] && \text{for } C<0\,,\ E=0\ \ , \\
\varphi(t) &= -\ln\left[\sqrt{\frac{C}{E}}\cos(\sqrt{-E}t)\right] && \text{for } C<0\,,\ E<0. \qquad (15)
\end{aligned}$$

(for vanishing time shifts). The time dependence of A is then easily deduced by integrating the equation $\partial_t A = ae^{2\varphi}$. The trajectories $\big(\varphi(t), A(t)\big)$ are geodesics on the non-compact coset spaces $SL(2)/SO(2)$ (for $C > 0 \Longleftrightarrow \alpha^2 > 0$) and $SL(2)/SO(1,1)$ (for $C < 0 \Longleftrightarrow \alpha^2 < 0$).

If $C = 0$, we must distinguish two cases depending on the value α^2. For $\alpha^2 \neq 0$ (hence $a = 0$), the solution is a straight motion $\varphi(t) = \pm\sqrt{E}t$, which can be transformed back into a straight motion in the original ϕ space by eq. (12).[2] If $\alpha^2 = 0$ (imaginary lightlike simple root), φ is not a valid coordinate on the coset space and the solution for ϕ has to be computed from eq. (11), yielding $\phi(t) = -\frac{1}{2}a^2t^2 + \sqrt{E}t$.

The above equations solve the σ-model equations of motion, but in addition we have to satisfy the Hamiltonian constraint $\mathcal{H} = \langle\mathcal{P}|\mathcal{P}\rangle = 0$ of the σ-model, according to eq. (4). We can compute the contribution $\mathcal{H}_\alpha$ of the above sector to $\mathcal{H}$ for $\alpha^2 \neq 0$ easily from (11):

$$\mathcal{H}_\alpha = \frac{1}{2}a^2e^{2\alpha^2\phi} + \alpha^2(\partial_t\phi)^2 \equiv \frac{1}{2}a^2e^{2\varphi} + \frac{1}{\alpha^2}(\partial_t\varphi)^2 = \frac{E}{\alpha^2}. \qquad (16)$$

Evidently, unless $E = 0$, more fields are required to satisfy the full Hamiltonian constraint of the $E_{10}/K(E_{10})$ σ-model in order for the resulting trajectory to be a *null* geodesic on the $E_{10}/K(E_{10})$ manifold. There are many ways to achieve this; in all cases, we need at least one timelike imaginary root (or at least the associated Cartan element) for this. For the present

[2]This is just the Kasner solution.

purposes it is, however, sufficient to superimpose two commuting rank-one algebras with different signs for E. Commuting subalgebras are convenient because the equations of motion decouple[3] with the following result for the Hamiltonian

$$\mathcal{H} = \sum_i \mathcal{H}_{\alpha_i}, \tag{17}$$

if α_i labels the different commuting subalgebras. A necessary condition for two rank-one subalgebras to commute is $\langle \alpha_i | \alpha_j \rangle = 0$. For real roots α_i and α_j this is also a sufficient condition.

2.3 Example: Accelerated Cosmology

We now illustrate this by exhibiting a solution with two commuting subalgebras, one of which is $\mathfrak{sl}(2)$ and the other one is Borcherds (although we only excite the $\mathfrak{u}(1)$ CSA part of it). We will put tildes on all quantities belonging to the Borcherds subalgebra. Concretely we take the two roots (written in a 'root basis')

$$\begin{aligned} \alpha &= [0,0,0,0,0,0,0,0,0,1] \equiv \alpha_{10} \\ \tilde{\alpha} &= [6,12,18,24,30,36,42,28,14,21] \end{aligned}$$

so that

$$\alpha^2 = \langle \alpha | \alpha \rangle = 2, \qquad \tilde{\alpha}^2 = \langle \tilde{\alpha} | \tilde{\alpha} \rangle = -42 \qquad \langle \alpha | \tilde{\alpha} \rangle = 0 \,. \tag{18}$$

The element $\tilde{\alpha}$ is the fundamental weight [28] Λ_7 corresponding to node 7, and its multiplicity is $\text{mult}_{E_{10}}(\tilde{\alpha}) = 4\,348\,985\,101$; its A_9 level is equal to the last entry, whence $\ell = 21$. Correspondingly, the level of α is $\ell = 1$ and we identify the generator E_α with the component $E^{8\,9\,10}$ of the rank three antisymmetric tensor representation occurring at $\ell = 1$ [15, 16]. The commutator $[E_\alpha, E_{\tilde{\alpha}}]$ will belong to the root space of $\alpha + \tilde{\alpha}$, whose multiplicity is $\text{mult}_{E_{10}}(\alpha + \tilde{\alpha}) = 2\,221\,026\,189$.[4] By choosing $E_{\tilde{\alpha}}$ appropriately we can therefore arrange that this commutator vanishes, so that the two algebras

[3] In the general case one would have to deal with a Toda-type system (if the relevant algebra is still finite-dimensional).

[4] For a list of E_{10} multiplicities see e.g. [30].

commute. The explicit expressions for the Cartan generators corresponding to (18) in the $\mathfrak{gl}_{10}$ basis are (cf. eqs. (2.14) and (2.15) of [16])

$$
\begin{aligned}
h &= -\frac{1}{3}(K^1{}_1 + \cdots + K^7{}_7) + \frac{2}{3}(K^8{}_8 + K^9{}_9 + K^{10}{}_{10}) \equiv h_{10} \\
\tilde{h} &= K^1{}_1 + \cdots + K^7{}_7
\end{aligned}
\tag{19}
$$

The identification of the E_{10} 'matrix' $\mathcal{V}$ with the supergravity degrees of freedom is straightforward only for the diagonal components of the metric (at a fixed spatial point). Abstractly, the dictionary identifies the diagonal spatial metric as [15]

$$
g_{mn}(t) = e^{2\phi(t)\cdot h + 2\tilde{\phi}(t)\cdot\tilde{h}} \ . \tag{20}
$$

In components, for the fields generators h and $\tilde{h}$ from eq. (19) above this leads to

$$
g_{\tilde{1}\tilde{1}} = \cdots = g_{\tilde{7}\tilde{7}} = e^{2\tilde{\phi} - 2\phi/3} \quad , \qquad g_{\tilde{8}\tilde{8}} = g_{\tilde{9}\tilde{9}} = g_{\tilde{10}\tilde{10}} = e^{4\phi/3}, \tag{21}
$$

where tildes indicate that the corresponding indices are curved. The (tt)-component of the metric in the σ-model correspondence (in $n = 1$ gauge) is given as the square of the lapse function $N = \det(e_m{}^a)$ [16], yielding

$$
g_{tt} = -e^{14\tilde{\phi} - 2\phi/3}. \tag{22}
$$

For the off-diagonal $GL(10)$ generators and E_{10} generators outside $GL(10)$, on the other hand, the precise correspondence is only known for low levels. Instead of presenting here all known correspondences [15, 17, 16, 18, 33], we present only an identification we will need for A_9 level $\ell = 1$

$$
P^{(1)}_{abc} = F_{t\,abc} \ , \tag{23}
$$

where $F_{t\,abc}$ is the 4-form field strength of $D = 11$ supergravity, with curved time index t and *flat* spatial indices a, b, c. The quantity on the l.h.s. multiplies the level-one generators E^{abc} in the expansion of the Cartan form (2).

We first consider the case $a \neq 0$ and $\tilde{a} = 0$, hence $C > 0$ and $\tilde{C} = 0$. Therefore the two energies of the solutions satisfy $E > 0$ and $\tilde{E} \geq 0$, with the full Hamiltonian of the E_{10} model now evaluated to be

$$
\mathcal{H} \equiv \mathcal{H}_\alpha + \mathcal{H}_{\tilde{\alpha}} = \frac{1}{2}E - \frac{1}{42}\tilde{E} = 0 \quad \Longrightarrow \quad \tilde{E} = 21E. \tag{24}
$$

We summarise the translation of the exact E_{10} solution (15) with non-trivial fields ϕ, $P_{8\,9\,10}$ and $\tilde{\phi}$ to an exact supergravity solution as

$$\begin{aligned} ds^2 &= -e^{14\tilde{\phi}-2\phi/3}dt^2 + e^{4\phi/3}(dx^2+dy^2+dz^2) + e^{2\tilde{\phi}-2\phi/3}d\Sigma^2_{7,0}, \\ F_{txyz} &= \frac{E}{a\cosh^2(\sqrt{E}t)} = a\,e^{4\phi}, \end{aligned} \tag{25}$$

with

$$\begin{aligned} \phi(t) &= -\frac{1}{2}\ln\left[\frac{a}{\sqrt{E}}\cosh(\sqrt{E}(t-t_0))\right], \\ \tilde{\phi}(t) &= \frac{1}{42}\sqrt{\tilde{E}}(t-t_1). \end{aligned} \tag{26}$$

Here we have used eq. (23). Note also that the flat spatial indices $8, 9, 10$ on the 4-form field strength have been converted to *curved* indices x, y, z in (25). The spatial coordinates of a four-dimensional space-time are now designated as $x \equiv x_8, y \equiv x_9$ and $z \equiv x_{10}$, and the $d\Sigma^2_{7,k}$ represents the metric of constant curvature k on a (compact) seven-dimensional manifold; here, however, we restrict attention to the flat metric with $k = 0$. This solution is identical to the cosmological solution of [8].[5] The solution (25) exhibits a period of accelerated expansion for the four-dimensional universe with coordinates (t, x, y, z). This was shown in [8] by a careful analysis of the relevant 'frames'. More precisely, in order to ascertain whether or not the solution exhibits accelerated expansion, one must first bring the metric of four-dimensional space-time into Einstein conformal frame [7] (*i.e.* no dependence on scalar fields in front of the four-dimensional Einstein-Hilbert term). To be sure, a fully realistic solution of this type remains to be found, but let us emphasize again the important fact that M theory does admit cosmological solutions with interesting profiles for the time evolution of the cosmic scale factor in a rather natural manner.

2.4 Curved internal spaces?

There exist analogues of the solution (25) for the cases when $d\Sigma^2_{7,k}$ is the metric on a compact Einstein space $\mathcal{M}_{7,k}$ with constant positive or negative

[5]The solution (25) is also equivalent to the SM2-brane solution, see [9] and references therein.

scalar curvature $R = \pm 42k^2$. These spherical resp. hyperbolic internal spaces have been studied in detail in the literature [8, 31, 9]. The solution is in form identical to (25); only the function $\tilde{\phi}$ changes its functional form from (26) to

$$\tilde{\phi}(t) = \begin{cases} -\frac{1}{6}\ln\left[\frac{42k}{\sqrt{\tilde{E}}}\cosh\left(\frac{\sqrt{\tilde{E}}}{7}(t-t_1)\right)\right] & \text{spherical space } R = 42k^2 \\ \\ -\frac{1}{6}\ln\left[\frac{42k}{\sqrt{\tilde{E}}}\sinh\left(\frac{\sqrt{\tilde{E}}}{7}(t-t_1)\right)\right] & \text{hyperbolic space } R = -42k^2 \end{cases} \tag{27}$$

We note that the second solution is very similar to the $C < 0, E > 0$ case of eq. (15). Therefore we are tempted to 'switch on' the positive step operator $E_{\tilde{\alpha}}$ belonging to the Borcherds subalgebra by allowing $\tilde{a} \neq 0$, hoping that this $\ell = 21$ root would mimic the effect of the hyperbolic internal space. However, the coefficient of the logarithm is different in (27) and (15). For the E_{10} solution it is $\frac{1}{42}$ (after dividing (15) by $\tilde{\alpha}^2$), as compared to $-\frac{1}{6}$ in (27). This discrepancy also manifests itself in the comparison of the Hamiltonian constraint for the E_{10} model and the Hamiltonian constraint following from the $D = 11$ supergravity equations of motion, expressed in terms of the fields prior to redefinition (12)

$$\begin{aligned} 2(\partial_t\phi)^2 + \frac{1}{2}a^2 e^{4\phi} - 42(\partial_t\tilde{\phi})^2 + \frac{1}{2}\tilde{a}^2 e^{-84\tilde{\phi}} &= 0 \qquad (E_{10}) \\ 2(\partial_t\phi)^2 + \frac{1}{2}a^2 e^{4\phi} - 42(\partial_t\tilde{\phi})^2 + 42k^2 e^{12\tilde{\phi}} &= 0 \qquad (D = 11) \end{aligned} \tag{28}$$

Therefore we are led to conclude that if the $E_{10}/K(E_{10})$ model incorporates the solution with hyperbolic internal space, it must use more positive step generators than simply $E_{\tilde{\alpha}}$. This is not unexpected since the required dependence on the coordinates of the compact space has to be rather complicated in order to allow for the constant negative curvature.

However, the spherical solution in (27) is more puzzling. Looking at the set of solutions (15) to the Liouville equation, we see that the hyperbolic cosine only appears for real roots. The split of coordinates in (25) however suggests taking the CSA elements h and $\tilde{h}$ of eq. (19) with h real and $\tilde{h}$ imaginary. This is related to the question raised in [13] concerning the sign of contributions to the Hamiltonian constraint in supergravity and in E_{10}. There are cosmological solutions to gravity which contribute negatively to the Hamiltonian constraint. The spherical solutions with positive constant curvature above is an example of such a cosmology. In E_{10}, on the other hand, all generators except for one direction in the CSA have a positive contribution to the Hamiltonian constraint. We have not been able to remedy this mismatch.

Let us at this point make one remark concerning the limit $k \to 0$ of the solutions (27). In [8] it was noted that the limit $k \to 0$ is not continuous and does not give the correct flat space solution (26). This can be remedied by noting that in the construction of the solution (27) the time shift appears as an integration constant which is related to the curvature scale by $\ln k$. Therefore, contrary to previous appearances, the limit $k \to 0$ *can* smoothly reproduce the flat space solution, via

$$\lim_{k\to 0} \left[k \sinh(t - \ln k)\right] = \frac{1}{2} e^t, \tag{29}$$

leading to the functional behaviour of the flat case solution.

3 Conclusion

We have seen that the E_{10} equations can be solved rather easily for dimension two subspaces, and that the solution to the resulting Liouville equation translates into a (known) interesting cosmological solution to the bosonic $D = 11$ theory. The generalisation to subspaces of dimension greater than two is straightforward, since the resulting equations will be of Toda-type and can be integrated in closed form (see for example [32]). We stress again that this abstract solution of geodesic motion on a group coset can be interpreted in many different ways as solutions to the maximal supergravity theories or their reductions by employing different correspondences to type IIA, IIB, pure type I and pure Einstein gravity which were detailed in [17, 18, 33]. In this way one finds from (15) all extremal S-branes. Their intersections rules are also implemented by an orthogonality of two dimension two subspaces, similar to the results of [34, 35, 36]. We note that our approach does not require the introduction of additional phantom fields unlike, for instance, Ref. [10] since there is a negative norm field contained in both the $E_{10}/K(E_{10})$ model and the reduction of $D = 11$. The known dictionaries also cover one light-like (isotropic) root of $\mathfrak{e}_{10}$, belonging to a Heisenberg subalgebra of $\mathfrak{e}_{10}$. One can easily construct a solution using this Heisenberg subcoset of $E_{10}/K(E_{10})$. This solution is a purely gravitational solution.

On the basis of the present results, it appears that the $E_{10}/K(E_{10})$ model disfavors static solutions, such as $AdS_4 \times S^7$. The reason is that negative contributions to the Hamiltonian constraint always come from the Cartan subalgebra, and the latter will only contribute if the diagonal metric degrees of freedom depend on time. The situation is somewhat reminiscent of 'Einstein's dilemma', namely the impossibility of finding static solutions for Einstein's gravitational field equations, that led him to introduce a cosmological constant. However, it appears that, for the E_{10} σ-model, such a remedy is not at hand.[6]

The action of the E_{10} Weyl group $\mathcal{W}$ on these solutions should contain interesting features. The orbit of a real root (like α in the example of section 2.3) under $\mathcal{W}$ covers *all* positive roots and therefore relates solutions of different type: The SM2 and SM5 solution are mapped into one another but also into pure gravitational waves or monopole solutions. This is well-known from U-duality [37, 38] but for E_{10} the action of $\mathcal{W}$ also relates these solutions to as yet poorly understood scenarios involving higher level generators.

Acknowledgements
The authors thank M. Henneaux and M. Wohlfarth for clarifying correspondence. AK would like to thank the organisers of the Young researchers workshop in Crete for the opportunity to present the results of this paper. This work was supported in part by the EU network grant MRTN-CT-2004 – 512194.

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