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Conformal entropy and stationary Killing horizons

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Abstract. Using Virasoro algebra approach, black hole entropy formula for a general class of higher curvature Lagrangians with arbitrary dependence on Riemann tensor can be obtained from properties of stationary Killing horizons. The properties used are a consequence of regularity of invariants of Riemann tensor on the horizon. As suggested by an example Lagrangian, eventual generalisation of these results to Lagrangians with derivatives of Riemann tensor, would require assuming regularity of invariants involving derivatives of Riemann tensor and that would lead to additional restrictions on metric functions near horizon.

1. Introduction

The entropy of a black hole, in Einstein gravity, is given by the well known Bekenstein–Hawking formula

$$S_{BH} = \frac{A}{4}. \tag{1}$$

For general diffeomorphism invariant Lagrangians

$$L = L(g_{ab}, R_{abcd}, \nabla R_{abcd}, \psi, \nabla\psi, ...), \tag{2}$$

the generalisation [1] of Bekenstein–Hawking formula is given by

$$S = -2\pi \int_H \hat{\boldsymbol{\epsilon}} \frac{\delta L}{\delta R_{abcd}} \eta_{ab}\eta_{cd}, \tag{3}$$

where η_{ab} is binormal to the horizon normalised so that $\eta_{ab}\eta^{ab} = -2$.

There are many independent approaches for counting the quantum microscopic states that generate the entropy of black holes. All of these approaches give results consistent with Bekenstein–Hawking formula for entropy. A possible explanation for this universality, due to Carlip [2], is that the density of microscopic states is determined by a conformal symmetry which is a consequence of Virasoro algebra of particular class of diffeomorphisms on the horizon. This approach was discussed and generalised in [3, 4, 5, 6, 7].

Here we shall describe the application [8, 9] of this approach to 4D black holes for Lagrangians of the type:

$$L = L(g_{ab}, R_{abcd}), \tag{4}$$

i.e. Lagrangians that depend arbitrarily on metric and Riemann tensors and do not contain derivatives of Riemann tensors, and Lagrangians of the type:

$$L = \frac{1}{16\pi} R + \alpha (\nabla R)^2, \tag{5}$$

i.e. Einstein gravity with additional term containing derivatives of Ricci scalar.

First we shall briefly review Carlip's conformal approach [2] (in section 2), then describe additional ingredients used in calculation (in section 3), and then describe results (in section 4).

2. Conformal approach

In approach [2] one treats horizon as boundary and fixes the following boundary conditions on the horizon

$$\frac{\chi^a \chi^b}{\chi^2} \delta g_{ab} \to 0, \quad \chi^a t^b \delta g_{ab} \to 0 \qquad \text{as } \chi^2 \to 0, \tag{6}$$

where χ^a is Killing vector that is null on the horizon (and satisfies $\nabla_a \chi^2 = -2\kappa\chi_a$ on the horizon), and ρ_a is defined with $\nabla_a \chi^2 = -2\kappa\rho_a$ (so that $\rho^a = \chi^a$ on the horizon) and t^a is any unit spacelike vector tangent to the horizon.

Diffeomorphisms compatible with boundary conditions are generated by vector fields of the form:

$$\xi^a = T\chi^a + \frac{1}{\kappa}\frac{\chi^2}{\rho^2} \nabla_\chi T \rho^a. \tag{7}$$

If $\rho^a \nabla_a T = 0$ then the Lie bracket algebra of diffeomorphisms closes:

$$\{\xi_1, \xi_2\}^a = (T_1 \nabla_\chi T_2 - T_2 \nabla_\chi T_1)\chi^a + \frac{1}{\kappa}\frac{\chi^2}{\rho^2} \nabla_\chi (T_1 \nabla_\chi T_2 - T_2 \nabla_\chi T_1)\rho^a. \tag{8}$$

For Einstein gravity, Dirac bracket algebra of Hamiltonian generators of these diffeomorphisms turns out to be (after some additional technical assumptions, for details see [2]) Virasoro algebra:

$$i\,\{J_m, J_n\}^* = (m-n)J_{m+n} + \frac{A}{8\pi} m^3 \delta_{m+n,0}, \tag{9}$$

with central charge:

$$\frac{c}{12} = \frac{A}{8\pi}. \tag{10}$$

In a conformal field theory Virasoro algebra determines asymptotic density of states and hence the entropy (Cardy formula):

$$S_{\mathrm{C}} = 2\pi \sqrt{\left(\frac{c}{6} - 4\Delta_g\right)\left(\Delta - \frac{c}{24}\right)}, \tag{11}$$

where Δ is the eigenvalue of Virasoro generator L_0 for the state we calculate the entropy and Δ_g is the smallest eigenvalue. Now, using (10), identifying Δ with $J_0 = A/8\pi$, and assuming that $\Delta_g = 0$ one obtains Bekenstein–Hawking entropy (1).

3. Near horizon metric

One would like to see what happens with algebra (9) and central charge (10) for Lagrangians of the form (4) and (5), then to use Cardy formula to obtain entropy, and then to see whether or not this conformal entropy agrees with formula (3). One way to proceed [5, 6, 7] is to use assumptions such as those of Appendix A of Ref. [2]. The alternative approach of [8, 9] is to use the general near horizon metric [10], as we shall describe below.

Consider stationary axially symmetric spacetime containing black hole, and pick coordinates t, ϕ, n, z such that t is associated with timelike stationary Killing vector, ϕ is associated with Killing vector that corresponds to axial symmetry, n is Gauss normal coordinate that corresponds to the distance from the horizon on the equal time hypersurface and the remaining coordinate z is chosen such that the metric has the form:

$$\begin{aligned} ds^2 &= -N(n,z)^2 dt^2 + g_{\phi\phi}(n,z)\,(d\phi - \omega(n,z)dt)^2 \\ &\quad + dn^2 + g_{zz}(n,z)dz^2. \end{aligned} \tag{12}$$

Absence of curvature singularities on the horizon, more precisely the finiteness of R, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ on the horizon, implies that metric coefficients have the following Taylor expansions on the horizon [10]:

$$\begin{aligned} N(n,z) &= \kappa n + \frac{1}{3!}\kappa_2(z)n^3 + O(n^4) \\ g_{\phi\phi}(n,z) &= g_{H\phi\phi}(z) + \frac{1}{2}g_{2\phi\phi}(z)n^2 + O(n^3) \\ g_{zz}(n,z) &= g_{Hzz}(z) + \frac{1}{2}g_{2zz}(z)n^2 + O(n^3) \\ \omega(n,z) &= \Omega_H + \frac{1}{2}\omega_2(z)n^2 + O(n^3). \end{aligned} \tag{13}$$

Note the absence of quadratic term in expansion of $N(n,z)$, and the absence of linear terms in other three expansions.

4. Results

It can be shown [8] that the conformal approach will reproduce the generalised Bekenstein–Hawking formula (3) if

$$\begin{aligned} &\lim_{n\to 0}\left\{\frac{\delta L}{\delta R_{abcd}}\left[\xi_1^e \eta_{ae}\nabla_d \delta_2 g_{bc} - (1\leftrightarrow 2)\right]\right\} \\ &= \lim_{n\to 0}\left\{-\frac{1}{4}\eta_{ab}\eta_{cd}\frac{\delta L}{\delta R_{abcd}}\left[\left(\frac{1}{\kappa}T_1 \dddot{T}_2 - 2\kappa T_1 \dot{T}_2\right) - (1\leftrightarrow 2)\right]\right\}, \end{aligned} \tag{14}$$

$$\lim_{n\to 0}\left\{\left[\xi_1^e \eta_{ae}\delta_2 g_{bc} - (1\leftrightarrow 2)\right]\nabla_d \frac{\delta L}{\delta R_{abcd}}\right\} = 0. \tag{15}$$

The first condition (14) can be shown to be satisfied [7] using the symmetries of $\frac{\delta L}{\delta R_{abcd}}$ (which are those of Riemann tensor).

The second (15) is more complicated due to the divergence term. For explicit form of a Lagrangian such as (4) or (5) it can be shown to be satisfied by performing explicit calculations (e.g. using *Mathematica*) as was done in [8] and [9]. For the whole class of Lagrangians of the form (4), it also can be shown by counting the powers of n in Taylor expansions of quantities that appear in (15) near horizon. For that purpose [9] it is convenient to use the basis χ^a, ρ^a, $\left(\frac{\partial}{\partial\phi}\right)^a$, $\left(\frac{\partial}{\partial z}\right)^a$. From explicit form of basis vectors and metric it is possible to obtain Taylor expansions of scalar products and derivatives of basis vectors and Taylor expansions of Riemann tensor in this basis, and from these it is possible to find out the properties of $\frac{\delta L}{\delta R_{abcd}}$ and of $\nabla_d \frac{\delta L}{\delta R_{abcd}}$ for general L, and finally to conclude that the contraction in (15) has to be zero on the horizon.

The assumptions used along the way were the regularity of scalar curvature invariants I_n (one can take e.g. $I_0 = R$, $I_1 = R_{ab}R^{ab}$, $I_2 = R_{abcd}R^{abcd}$, ...), the regularity of Lagrangian L which is a function of I_n, and also the regularity of its partial derivatives $\frac{\partial L}{\partial I_n}$ on the horizon. Note that all of the assumptions used were assumptions on scalars.

A long but straightforward calculation shows that for special case (5) usual results can be obtained provided we restrict the class of metric functions. The restrictions are

$$\omega_3 = 0 \tag{16}$$

and

$$\frac{3g_{3zz}}{g_{Hzz}} + \frac{8\kappa_3}{\kappa} + \frac{3g_{3\phi\phi}}{g_{H\phi\phi}} = 0 \tag{17}$$

where $g_{3zz}(z)$, $g_{3\phi\phi}(z)$ and $\omega_3(z)$ are coefficients of n^3 , and $\kappa_3(z)$ of n^4 in Taylor expansions (13).

These restrictions can be understood also by terms of regularity of scalar curvature invariants on horizon. Namely, if we require regularity of

$$(\nabla_a R_{bc})^2 \qquad \text{and} \qquad \nabla^2 R \tag{18}$$

we obtain relations (16) and (17).

From (16) it follows that all polynomial invariants involving Riemann tensor and its first derivatives will be regular on the horizon. This is in fact generalisation of results from [10] that regularity of invariants of Riemann tensor has implications on metric functions near horizon. Here, we see that regularity of invariants involving derivatives of Riemann tensor has additional consequences on metric functions.

5. Conclusion

We have extended Carlip's procedure for Einstein gravity to Lagrangians with arbitrary dependence on Riemann tensor (with no derivatives).

As a tool for calculation explicit form of metric near $4D$ stationary horizon has been used. That form of metric follows from regularity of curvature invariants near horizon [10] and implies restrictive power series for quantities needed to calculate central charge.

We also applied the procedure to a Lagrangian containing $(\nabla R)^2$ and in this case one needs regularity of $(\nabla_a R_{bc})^2$ and $\nabla^2 R$ on the horizon.

The compatibility of the procedure with higher order Lagrangians supports the idea that conformal field theory interpretation of entropy is a consequence only of properties of the horizon and independent of the type of interaction.

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References

[1] Wald R M 1993 *Phys. Rev.* D **48** 3427 (*Preprint* gr-qc/9307038)
Jacobson T, Kang G and Myers R C 1994 *Phys. Rev.* D **49** 6587 (*Preprint* gr-qc/9312023)
Iyer V and Wald R M 1994 *Phys. Rev.* D **50** 846 (*Preprint* gr-qc/9403028)
[2] Carlip S 1999 *Class. Quant. Grav.* **16** 3327 (*Preprint* gr-qc/9906126)
[3] Dreyer O, Ghosh A and Wisniewski J 2001 *Class. Quant. Grav.* **18** 1929 (*Preprint* hep-th/0101117)
[4] Silva S 2002 *Class. Quant. Grav.* **19** 3947 (*Preprint* hep-th/0204179)
[5] Jing J L and Yan M L 2000 *Phys. Rev.* D **62** 104013 (*Preprint* gr-qc/0004061)
Jing J L and Yan M L 2001 *Phys. Rev.* D **63** 024003 (*Preprint* gr-qc/0005105)

[6] Cvitan M, Pallua S and Prester P 2003 *Phys. Lett.* B **555** 248 (*Preprint* hep-th/0212029)
[7] Cvitan M, Pallua S and Prester P 2003 *Phys. Lett.* B **571** 217 (*Preprint* hep-th/0306021)
[8] Cvitan M, Pallua S and Prester P 2004 *Phys. Rev.* D **70** 084043 (*Preprint* hep-th/0406186)
[9] Cvitan M and Pallua S 2005 *Phys. Rev.* D **71** 104032 (*Preprint* hep-th/0412180)
[10] Medved A J M, Martin D and Visser M 2004 *Class. Quant. Grav.* **21** 3111 (*Preprint* gr-qc/0402069)
Medved A J M, Martin D and Visser M 2004 *Phys. Rev.* D **70** 024009 (*Preprint* gr-qc/0403026)