

GEOMETRIC ANALYSIS AND GENERAL RELATIVITY

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Geometric analysis can be said to originate in the 19'th century work of Weierstrass, Riemann, Schwarz and others on minimal surfaces, a problem whose history can be traced at least as far back as the work of Meusnier and Lagrange in the 18'th century. The experiments performed by Plateau in the mid-19'th century, on soap films spanning wire contours, served as an important inspiration for this work, and led to the formulation of the Plateau problem, which concerns the existence and regularity of area minimizing surfaces in \mathbb{R}^3 spanning a given boundary contour. The Plateau problem for area minimizing disks spanning a curve in \mathbb{R}^3 was solved by Jesse Douglas (who shared the first Fields medal with Lars V. Ahlfors) and Tibor Rado in the 1930's. Generalizations of Plateau's problem have been an important driving force behind the development of modern geometric analysis. Geometric analysis can be viewed broadly as the study of partial differential equations arising in geometry, and includes many areas of the calculus of variations, as well as the theory of geometric evolution equations. The Einstein equation, which is the central object of general relativity, is one of the most widely studied geometric partial differential equations, and plays an important role in its Riemannian as well as in its Lorentzian form, the Lorentzian being most relevant for general relativity.

The Einstein equation is the Euler-Lagrange equation of a Lagrangian with gauge symmetry and thus in the Lorentzian case it, like the Yang-Mills equation, can be viewed as a system of evolution equations with constraints. After imposing suitable gauge conditions, the Einstein equation becomes a hyperbolic system, in particular using spacetime harmonic coordinates (also known as wave coordinates), the Einstein equation becomes a quasilinear system of wave equations. The constraint equations implied by the Einstein equations can be viewed as a system of elliptic equations in terms of suitably chosen variables. Thus the Einstein equation leads to both elliptic and hyperbolic problems, arising from the constraint equations and the Cauchy problem, respectively. The groundwork for the mathematical study of the Einstein equation and the global nature of spacetimes was laid by, among others, Choquet-Bruhat who proved local well-posedness for the Cauchy problem, Lichnerowicz, and later York who provided the basic ideas for the analysis of the constraint equations, and Leray who formalized the notion of

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global hyperbolicity, which is essential for the global study of spacetimes. An important framework for the mathematical study of the Einstein equations has been provided by the singularity theorems of Penrose and Hawking, as well as the cosmic censorship conjectures of Penrose.

Techniques and ideas from geometric analysis have played, and continue to play, a central role in recent mathematical progress on the problems posed by general relativity. Among the main results are the proof of the positive mass theorem using the minimal surface technique of Schoen and Yau, and the spinor based approach of Witten, as well as the proofs of the (Riemannian) Penrose inequality by Huisken and Illmanen, and Bray. The proof of the Yamabe theorem by Schoen has played an important role as a basis for constructing Cauchy data using the conformal method.

The results just mentioned are all essentially Riemannian in nature, and do not involve study of the Cauchy problem for the Einstein equations. There has been great progress recently concerning global results on the Cauchy problem for the Einstein equations, and the cosmic censorship conjectures of Penrose. The results available so far are either small data results, among these the nonlinear stability of Minkowski space proved by Christodoulou and Klainerman, or assume additional symmetries, such as the recent proof by Ringström of strong cosmic censorship for the class of Gowdy spacetimes. However, recent progress concerning quasilinear wave equations and the geometry of spacetimes with low regularity due to, among others, Klainerman and Rodnianski, and Tataru and Smith, appears to show the way towards an improved understanding of the Cauchy problem for the Einstein equations.

Since the constraint equations, the Penrose inequality and the Cauchy problem are discussed in separate articles, the focus of this article will be on the role in general relativity of “critical” and other geometrically defined submanifolds and foliations, such as minimal surfaces, marginally trapped surfaces, constant mean curvature hypersurfaces and null hypersurfaces. In this context it would be natural also to discuss geometrically defined flows such as mean curvature flows, inverse mean curvature flow, and Ricci flow. However, this article restricts the discussion to mean curvature flows, since the inverse mean curvature flow appears naturally in the context of the Penrose inequality and the Ricci flow has so far mainly served as a source of inspiration for research on the Einstein equations rather than an important tool. Other topics which would fit well under the heading “General Relativity and Geometric Analysis” are spin geometry (the Witten proof of the Positive mass theorem), the Yamabe theorem and related results concerning the Einstein constraint equations, gluing and other techniques of “spacetime engineering”. These are all discussed in other articles. Some techniques which have only recently come into use and for which applications in general relativity have not been much explored, such as Cheeger-Gromov compactness, are not discussed.

1. MINIMAL AND RELATED SURFACES

Consider a hypersurface N in Euclidean space \mathbb{R}^n which is a graph $x_n = u(x_1, \dots, x_{n-1})$ with respect to the function u . The area of N is given by $\mathcal{A}(N) = \int \sqrt{1 + |Du|^2} dx^1 \dots dx^{n-1}$. N is stationary with respect to \mathcal{A} if u satisfies the equation

$$\sum_i D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0 \tag{1}$$

A hypersurface N defined as a graph of u solving (1) minimizes area with respect to compactly supported deformations, and hence is called a *minimal surface*. For $n \leq 7$, a solution to equation (1) defined on all of \mathbb{R}^{n-1} must be an affine function. This fact is known as a Bernstein principle. Equation (1), and more generally, the prescribed mean curvature equation which will be discussed below, is a quasilinear, uniformly elliptic second order equation. The book [14] is an excellent general reference for such equations.

The theory of rectifiable currents, developed by Federer and Fleming, is a basic tool in the modern approach to the Plateau problem and related variational problems. A rectifiable current is a countable union of Lipschitz submanifolds, counted with integer multiplicity, and satisfying certain regularity conditions. Hausdorff measure gives a notion of area for these objects. One may therefore approach the study of minimal surfaces via rectifiable currents which are stationary with respect to variations of area. Suitable generalizations of familiar notions from smooth differential geometry such as tangent plane, normal vector, extrinsic curvature can be introduced. The book by Federer [11] is a classic treatise on the subject. Further information concerning minimal surfaces and related variational problems can be found in [17, 20]. Note, however, that unless otherwise stated, all fields and manifolds considered in this article are assumed to be smooth. For the Plateau problem in a Riemannian ambient space, we have the following existence and regularity result.

Theorem 1 (Existence of embedded solutions for Plateau problem). *Let M be a complete Riemannian manifold of dimension $n \leq 7$ and let Γ be a compact $n-2$ dimensional submanifold in M which bounds. Then there is an $n-1$ dimensional area minimizing hypersurface N with Γ as its boundary. N is a smooth, embedded manifold in its interior.*

If the dimension of the ambient space is greater than 7, solutions to the Plateau problem will in general have a singular set of dimension $n-8$. Let N be an oriented hypersurface of a Riemannian manifold M with covariant derivative D . Let η be the unit normal of N and define the second fundamental form and mean curvature of N by $A_{ij} = \langle D_{e_i} \eta, e_j \rangle$ and $H = \text{tr} A$. Define the action functional $\mathcal{E}(N) = \mathcal{A}(N) - \int_{M;N} H_0$, where H_0 is a function defined on M , and $\int_{M;N}$ denotes the integral over the volume bounded

by N in M . The problem of minimizing \mathcal{E} is a useful generalization of the minimization problem for \mathcal{A} .

Theorem 2 (Existence of minimizers in homology). *Let M be a compact Riemannian manifold of dimension ≤ 7 , and let α be an integral homology class on M of codimension one. Then there is a smooth minimizer for \mathcal{E} representing $[\alpha]$.*

Again, in higher dimensions, the minimizers will in general have singularities. The general form of this result deals with elliptic functionals. For surfaces in 3-manifolds, the problem of minimizing area within homotopy classes has been studied. Results in this direction played a central role in the approach of Schoen and Yau to manifolds with non-negative scalar curvature.

If M is not compact, it is in general necessary to use barriers to control the minimizers, or consider some version of the Plateau problem. Barriers can be used due to the strong maximum principle, which holds for the mean curvature operator since it is quasilinear elliptic. Consider two hypersurfaces N_1, N_2 which intersect in a point p and assume that N_1 lies on one side of N_2 with the normal pointing towards N_1 . If the mean curvatures H_1, H_2 of the hypersurfaces, defined with respect to consistently oriented normals, satisfy $H_1 \leq \lambda \leq H_2$ for some constant λ , then N_1 and N_2 coincide near p and have mean curvatures equal to λ . This result requires only mild regularity conditions on the hypersurfaces. Generalizations hold also for the case of spacelike or null hypersurfaces in a Lorentzian ambient space, see [2, 12].

Let ϕ be a smooth compactly supported function on N . The variation $\mathcal{E}' = \delta_{\phi\eta}\mathcal{E}$ of \mathcal{E} under a deformation $\phi\eta$ is

$$\mathcal{E}' = \int_N \phi(H - H_0)$$

Thus N is stationary with respect to \mathcal{E} if and only if N solves the prescribed mean curvature equation $H(x) = H_0(x)$ for $x \in N$. Supposing that N is stationary and H_0 is constant, the second variation $\mathcal{E}'' = \delta_{\phi\eta}\mathcal{E}'$ of \mathcal{E} is of the form

$$\mathcal{E}'' = \int_N \phi(J\phi)$$

where J is the second variation operator, a second order elliptic operator. A calculation, using the Gauss equation and the second variation equation shows

$$J\phi = -\Delta_N\phi - \frac{1}{2}[(\text{Scal}_M - \text{Scal}_N) + H^2 + |A|^2]\phi, \quad (2)$$

where $\Delta_N, \text{Scal}_M, \text{Scal}_N$ denote the Laplace-Beltrami operator of N , and the scalar curvatures of M and N , respectively. If J is positive semidefinite, N is called stable.

To set the context where we will apply the above, let (M, g_{ij}) be a connected, asymptotically Euclidean three-dimensional Riemannian manifold

with covariant derivative, and let K_{ij} be a symmetric tensor on M . Suppose (M, g_{ij}, K_{ij}) is imbedded isometrically as a spacelike hypersurface in a spacetime $(V, \gamma_{\alpha\beta})$ with g_{ij}, K_{ij} the first and second fundamental forms induced on M from V , in particular $K_{ij} = \langle D_{e_i} T, e_j \rangle$ where T is the timelike normal of M in the ambient spacetime V , and D is the ambient covariant derivative. We will refer to (M, g_{ij}, K_{ij}) as a Cauchy data set for the Einstein equations. Although many of the results which will be discussed below generalize to the case of a nonzero cosmological constant Λ , we will discuss only the case $\Lambda = 0$ in this article. $G_{\alpha\beta} = \text{Ric}_{V\alpha\beta} - \frac{1}{2}\text{Scal}_V\gamma_{\alpha\beta}$ be the Einstein tensor of V , and let $\rho = G_{\alpha\beta}T^\alpha T^\beta$, $\mu_j = G_{j\alpha}T^\alpha$. Then the fields (g_{ij}, K_{ij}) satisfy the Einstein constraint equations

$$R + \text{tr}K^2 - |K|^2 = 2\rho \quad (3)$$

$$\nabla_j \text{tr}K - \nabla^i K_{ij} = \mu_j \quad (4)$$

We assume that the dominant energy condition (DEC)

$$\rho \geq \left(\sum_i \mu_i \mu^i \right)^{1/2} \quad (5)$$

holds. We will sometimes make use of the null energy condition (NEC), $G_{\alpha\beta}L^\alpha L^\beta \geq 0$ for null vectors L , and the strong energy condition (SEC), $\text{Ric}_{V\alpha\beta}v^\alpha v^\beta \geq 0$ for causal vectors v . M will be assumed to satisfy the fall-off conditions

$$g_{ij} = \left(1 + \frac{2m}{r}\right)\delta_{ij} + O(1/r^2) \quad (6a)$$

$$K_{ij} = O(1/r^2) \quad (6b)$$

as well as suitable conditions for the fall-off of derivatives of g_{ij}, K_{ij} . Here m is the ADM (Arnowitt, Deser, Misner) mass of (M, g_{ij}, K_{ij}) .

1.1. Minimal surfaces and positive mass. Perhaps the most important application of the theory of minimal surfaces in general relativity is in the Schoen-Yau proof of the positive mass theorem, which states that $m \geq 0$, and $m = 0$ only if (M, g, K) can be embedded as a hypersurface in Minkowski space. Consider an asymptotically Euclidean manifold (M, g) with g satisfying (6a) and with non-negative scalar curvature. By using Jang's equation, see below, the general situation is reduced to the case of a time symmetric data set, with $K = 0$. In this case the DEC implies that (M, g) has nonnegative scalar curvature.

Assuming $m < 0$ one may, after applying a conformal deformation, assume that $\text{Scal}_M > 0$ in the complement of a compact set. Due to the asymptotic conditions, level sets for sufficiently large values of one of the coordinate functions, say x^3 , can be used as barriers for minimal surfaces in M . By solving a sequence of Plateau problems with boundaries tending to infinity, a stable entire minimal surface N homeomorphic to the plane is

constructed. Stability implies using (2),

$$\int_N \left(\frac{1}{2} \text{Scal}_M - \kappa + \frac{1}{2} |A|^2 \right) \leq 0,$$

where $\kappa = \frac{1}{2} \text{Scal}_N$ is the Gauss curvature of N . Since by construction $\text{Scal}_M \geq 0$, $\text{Scal}_M > 0$ outside a compact set, this gives $\int_N \kappa > 0$. Next, one uses the identity, related to the Cohn-Vossen inequality

$$\int_N \kappa = 2\pi - \lim_i \frac{L_i^2}{2A_i}$$

where A_i, L_i are the area and circumference of a sequence of large discs. Estimates using the fact that M is asymptotically Euclidean show that $\lim_i \frac{L_i^2}{2A_i} \geq 2\pi$ which gives a contradiction and shows that the minimal surface constructed cannot exist. It follows that $m \geq 0$. It remains to show that the case $m = 0$ is rigid. To do this proves that for an asymptotically Euclidean metric with non-negative scalar curvature, which is positive near infinity, there is a conformally related metric with vanishing scalar curvature and strictly smaller mass. Applying this argument in case $m = 0$ gives a contradiction to the fact that $m \geq 0$. Therefore $m = 0$ only if the scalar curvature vanishes identically. Suppose now that (M, g) has vanishing scalar curvature but non-vanishing Ricci curvature Ric_M . Then using a deformation of g in the direction of Ric_M , one constructs a metric close to g with negative mass, which leads to a contradiction.

This technique generalizes to Cauchy surfaces of dimension $n \leq 7$. The proof involves induction on dimension. For $n > 7$ minimal hypersurfaces are singular in general and this approach runs into problems. The Witten proof using spinor techniques does not suffer from this limitation but instead requires that M be spin.

1.2. Marginally trapped surfaces. Consider a Cauchy data set (M, g_{ij}, K_{ij}) as above and let N be a compact surface in M with normal η , second fundamental form A and mean curvature H . Then considering N as a surface in an ambient Lorentzian space V containing M , N has two null normal fields which after a rescaling can be taken to be $L_{\pm} = T \pm \eta$. Here T is the future directed timelike unit normal of M in V . The null mean curvatures (or null expansions) corresponding to L_{\pm} can be defined in terms of the variation of the area element μ_N of N as $\delta_{L_{\pm}} \mu_N = \theta_{\pm} \mu_N$ or

$$\theta_{\pm} = \text{tr}_N K \pm H,$$

where $\text{tr}_N K$ denotes the trace of the projection of K_{ij} to N . Suppose L_+ is the outgoing null normal. N is called outer trapped (marginally trapped, untrapped) if $\theta_+ < 0$ ($\theta_+ = 0$, $\theta_+ > 0$). An asymptotically flat spacetime which contains a trapped surface with $\theta_- < 0$, $\theta_+ < 0$ is causally incomplete. In the following we will for simplicity drop the word outer from our terminology.

Consider a Cauchy surface M . The boundary of the region in M containing trapped surfaces is, if it is sufficiently smooth, a marginally trapped surface. The equation $\theta_+ = 0$ is an equation analogous to the prescribed curvature equation, in particular it is a quasilinear elliptic equation of second order. Marginally trapped surfaces are not variational in the same sense as minimal surfaces. Nevertheless, they are stationary with respect to variations of area within the outgoing light cone. The second variation of area along the outgoing null cone is given, in view of the Raychaudhuri equation, by

$$\delta_{\phi L_+} \theta_+ = -(G_{++} + |\sigma_+|^2)\phi, \quad (7)$$

for a function ϕ on N . Here $G_{++} = G_{\alpha\beta} L_+^\alpha L_+^\beta$, and σ_+ denotes the shear of N with respect to L_+ , i.e. the trace-free part of the null second fundamental form with respect to L_+ . Equation (7) shows that the stability operator in the direction L_+ is not elliptic.

In case of time-symmetric data, $K_{ij} = 0$, the dominant energy condition implies $\text{Scal}_M \geq 0$ and marginally trapped surfaces are simply minimal surfaces. A stable compact minimal 2-surface N in a 3-manifold M with nonnegative scalar curvature must satisfy

$$2\pi\chi(N) = \int \kappa \geq \frac{1}{2} \int_N \text{Scal}_M + |A|^2 \geq 0$$

and hence by the Gauss-Bonnet theorem, N is diffeomorphic to a sphere or a torus. In case N is a stable minimal torus, the induced geometry is flat and the ambient curvature vanishes at N . If, in addition, N minimizes, then M is flat [7].

For a compact marginally trapped surface N in M , analogous results can be proved by studying the stability operator defined with respect to the direction η . Let J be the operator defined in terms of a variation of θ_+ by $J\phi = \delta_{\phi\eta}\theta_+$. Then

$$J\phi = -\Delta_N \phi + 2s^A D_A \phi + \left(\frac{1}{2} \text{Scal}_N - s_A s^A + D_A s^A - \frac{1}{2} |\sigma_+|^2 - G_{+-} \right) \phi.$$

Here $s_A = -\frac{1}{2} \langle L_-, D_A L_+ \rangle$ and G_{+-} is the Einstein tensor evaluated on L_+, L_- . We may call N stable if the real part of the spectrum of J is nonnegative. A sufficient condition for N to be stable is that N is locally outermost. This can be formulated for example by requiring that a neighborhood of N in M contains no trapped surfaces exterior to N . In this case, assuming that the dominant energy condition holds, N is a sphere or a torus, and if the real part of the spectrum of J is positive then N is a sphere. If N is a torus, then the ambient curvature and shear vanishes at N , s_A is a gradient, and N is flat. One expects that in addition, global rigidity should hold, in analogy with the minimal surface case. This is an open problem. If N satisfies the stronger condition of strict stability, which corresponds to the spectrum of J having positive real part, then N is in the interior of a hypersurface H of the ambient spacetime, with the property

that it is foliated by marginally trapped surfaces [3]. If the NEC holds and N has nonvanishing shear, then H is spacelike at N . A hypersurface H with these properties is known as a dynamical horizon.

1.3. Jang's equation. Consider a Cauchy data set (M, g_{ij}, K_{ij}) . Extend K_{ij} to a tensor field on $M \times \mathbb{R}$, constant in the vertical direction. Then the equation for a graph

$$N = \{(x, t) \in M \times \mathbb{R}, \quad t = f(x)\}$$

such that N has mean curvature equal to the trace of the projection of K_{ij} to N with respect to the induced metric on N , is given by

$$\sum_{i,j} \left(K^{ij} - \frac{\nabla^i \nabla^j f}{(1 + |\nabla f|^2)^{1/2}} \right) \left(g_{ij} - \frac{\nabla_i f \nabla_j f}{1 + |\nabla f|^2} \right) = 0, \quad (8)$$

an equation closely related to the equation $\theta_+ = 0$. Equation (8) was introduced by P. S. Jang [16] as part of an attempt to generalize the inverse mean curvature flow method of Geroch from time-symmetric to general Cauchy data.

Existence and regularity for Jang's equation were proved by Schoen and Yau [19] and used to generalize their proof of the positive mass theorem from the case of maximal slices to the general case. The solution to Jang's equation is constructed as the limit of the solution to a sequence of regularized problems. The limit consists of a collection N of submanifolds of $M \times \mathbb{R}$. In particular, component near infinity is a graph and has the same mass as M . N may contain vertical components which project onto marginally trapped surfaces in M , and in fact these constitute the only possibilities for blow-up of the sequence of graphs used to construct N . If the DEC is valid, the metric on N has non-negative scalar curvature in the weak sense that

$$\int_N \text{Scal}_N \phi^2 + 2|\nabla \phi|^2 \geq 0$$

for smooth compactly supported functions ϕ . If the DEC holds strictly, the strict inequality holds and in this case the metric on N is conformal to a metric with vanishing scalar curvature.

Jang's equation can be applied to prove existence of marginally trapped surfaces, given barriers. Let (M, g_{ij}, K_{ij}) be a Cauchy data set containing two compact surfaces N_1, N_2 which together bound a compact region M' in M . Suppose the surfaces N_1 and N_2 have $\theta_+ < 0$ on N_1 and $\theta_+ > 0$ on N_2 . Schoen recently proved the following result.

Theorem 3 (Existence of marginally trapped surfaces). *Let M', N_1, N_2 be as above. Then there is a finite collection of compact, marginally trapped surfaces $\{\Sigma_a\}$ contained in the interior of M' , such that $\cup \Sigma_a$ is homologous to N_1 . If the DEC holds, then $\{\Sigma_a\}$ is a collection of spheres and tori.*

The proof proceeds by solving a sequence of Dirichlet boundary value problems for Jang's equation with boundary value on N_1, N_2 tending to $-\infty$

and ∞ respectively. The assumption on θ_+ is used to show the existence of barriers for Jang's equation. Let f_k be the sequence of solutions to the Dirichlet problems. Jang's equation is invariant under renormalization $f_k \rightarrow f_k + c_k$ for some sequence c_k of real numbers. A Harnack inequality for the gradient of the solutions to Jang's equation is used to show that the sequence of solutions f_k , possibly after a renormalization has a subsequence converging to a vertical submanifold of $M' \times \mathbb{R}$, which projects to a collection Σ_a of marginally trapped surfaces. By construction, the zero sets of the f_k are homologous to N_1 and N_2 . The estimates on the sequence $\{f_k\}$ show that this holds also in the limit $k \rightarrow \infty$. The statement about the topology of the Σ_a follows by showing, using the above mentioned inequality for Scal_N , that if DEC holds, the total Gauss curvature of each surface Σ_a is non-negative.

1.4. Center of mass. Since by the positive mass theorem $m > 0$ unless the ambient spacetime is flat, it makes sense to consider the problem of finding an appropriate notion of center of mass. This problem was solved by Huisken and Yau who showed that under the asymptotic conditions (6) the isoperimetric problem has a unique solution if one considers sufficiently large spheres.

Theorem 4 (Huisken and Yau [15]). *There is a $H_0 > 0$ and a compact region B_{H_0} such that for each $H \in (0, H_0)$ there is a unique constant mean curvature sphere S_H with mean curvature H contained in $M \setminus B_{H_0}$. The spheres form a foliation.*

The proof involves a study of the evolution equation

$$\frac{dx}{ds} = (H - \bar{H})\eta \tag{9}$$

where \bar{H} is the average mean curvature. This is the gradient flow for the isoperimetric problem of minimizing area keeping the enclosed volume constant. The solutions in Euclidean space are standard spheres. Equation (9) defines a parabolic system, in particular we have

$$\frac{d}{ds}H = \Delta H + (\text{Ric}(\eta, \eta) + |A|^2)(H - \bar{H}).$$

It follows from the fall-off conditions (6) that the foliation of spheres constructed in Theorem 4 are untrapped surfaces. They can therefore be used as outer barriers in the existence result for marginally trapped surfaces, Theorem 3.

The mean curvature flow for a spatial hypersurface in a Lorentz manifold is also parabolic. This flow has been applied to construct constant mean curvature Cauchy hypersurfaces in spacetimes.

2. MAXIMAL AND RELATED SURFACES

Let N be the hypersurface $x_0 = u(x_1, \dots, x_n)$ in Minkowski space \mathbb{R}^{1+n} with line element $-dx_0^2 + dx_1^2 + \dots + dx_n^2$. Assume $|\nabla u| < 1$ so that N is

spacelike. Then N is stationary with respect to variations of area if u solves the equation

$$\sum_i \nabla_i \left(\frac{\nabla_i u}{\sqrt{1 - |\nabla u|^2}} \right) = 0 \quad (10)$$

N maximizes area with respect to compactly supported variations, and hence is called a *maximal surface*. As in the case of the minimal surface equation, equation (10) and more generally the Lorentzian prescribed mean curvature equation, is quasilinear elliptic, but it is not uniformly elliptic, which makes the regularity theory more subtle.

A Bernstein principle analogous to the one for the minimal surface equation holds for the maximal surface equation (10). Suppose that u is a solution to (10) which is defined on all of \mathbb{R}^n . Then u is an affine function [8]. An important tool used in the proof is a Bochner type identity, originally due to Calabi, for the norm of the second fundamental form. For a hypersurface in a flat ambient space, the Codazzi equation states $\nabla_i A_{jk} - \nabla_j A_{ik} = 0$. This gives the identity

$$\Delta A_{ij} = \nabla_i \nabla_j H + A_{km} R_{i \ j}^m \ ^k + A_{mi} \text{Ric}^m_j \quad (11)$$

The curvature terms can be rewritten in terms of A_{ij} if the ambient space is flat. Using (11) to compute $\Delta |A|^2$ gives an expression which is quadratic in ∇A , and fourth order in $|A|$, and which allows one to perform maximum principle estimates on $|A|$. Generalizations of this technique for hypersurfaces in general ambient spaces play an important role in the proof of regularity of minimal surfaces, and in the proof of existence for Jang's equation, see section 1.3, as well as in the analysis of the mean curvature flow used to prove existence of round spheres, see section 1.4. The generalization of equation (11) is known as a Simons identity.

For the case of maximal hypersurfaces of Minkowski space, it follows from further maximum principle estimates, that a maximal hypersurface of Minkowski space is convex, in particular it has non-positive Ricci curvature. Generalizations of this technique allows one to analyze entire constant mean curvature hypersurfaces of Minkowski space.

Consider a globally hyperbolic Lorentzian manifold (V, γ) . A C^0 hypersurface is said to be weakly spacelike if timelike curves intersect it in at most one point. Call a codimension two submanifold $\Gamma \subset V$ a weakly spacelike boundary if it bounds a weakly spacelike hypersurface N_0 .

Theorem 5 (Existence for Plateau problem for maximal surfaces [5]). *Let V be a globally hyperbolic spacetime and assume the causal structure of V is such that the domain of dependence of any compact domain in V is compact. Given a weakly spacelike boundary Γ in V , there is a weakly spacelike maximal hypersurface N with Γ as its boundary. N is smooth except possibly on null geodesics connecting points of Γ .*

Here maximal hypersurface is understood in a weak sense, referring to stationarity with respect to variations. Due to the non-uniform ellipticity

for the maximal surface equation, the interior regularity which holds for minimal surfaces fails to hold in general for the maximal surface equation.

A time oriented spacetime is said to have a crushing singularity to the past (future) if there is a sequence Σ_n of Cauchy surfaces so that the mean curvature function H_n of Σ_n , diverges uniformly to $-\infty$ (∞).

Theorem 6 (Gerhardt [13]). *Suppose that (V, γ) is globally hyperbolic with compact Cauchy surfaces and satisfies the SEC. Then if (V, γ) has crushing singularities to the past and future it is globally foliated by constant mean curvature hypersurfaces. The mean curvature τ of these Cauchy surfaces is a global time function.*

The proof involves an application of results from geometric measure theory to an action \mathcal{E} of the form discussed in section 1. A barrier argument is used to control the maximizers. Bartnik [4, Theorem 4.1] gave a direct proof of existence of a constant mean curvature (CMC) hypersurface, given barriers. If the spacetime (V, γ) is symmetric, so that a compact Lie group acts on V by isometries, then CMC hypersurfaces in V inherit the symmetry. Theorem 6 gives a condition under which a spacetime is globally foliated by CMC hypersurfaces. In general, if the SEC holds in a spatially compact spacetime, then for each $\tau \neq 0$, there is at most one constant mean curvature Cauchy surface with mean curvature τ . In case V is vacuum, $\text{Ric}_V = 0$, and 3+1 dimensional, then each point $x \in V$ is on at most one hypersurface of constant mean curvature unless V is flat and splits as a metric product.

There are vacuum spacetimes with compact Cauchy surface which contain no CMC hypersurface [10]. The proof is carried out by constructing Cauchy data, using a gluing argument, on the connected sum of two tori, such that the resulting Cauchy data set (M, g_{ij}, K_{ij}) has an involution which reverses the sign of K_{ij} . The involution extends to the maximal vacuum development V of the Cauchy data set. Existence of a CMC surface in V gives, in view of the involution, barriers which allow one to construct a maximal Cauchy surface homeomorphic to M . This leads to a contradiction, since the connected sum of two tori does not carry a metric of positive scalar curvature, and therefore, in view of the constraint equations, cannot be imbedded as a maximal Cauchy surface in a vacuum spacetime. The maximal vacuum development V is causally geodesically incomplete. However, in view of the existence proof for CMC Cauchy surfaces, cf. Theorem 6, these spacetimes cannot have a crushing singularity. It would be interesting to settle the open question whether there are stable examples of this type.

In the case of a spacetime V which has an expanding end, one does not expect in general that the spacetime is globally foliated by CMC hypersurfaces even if V is vacuum and contains a CMC Cauchy surface. This expectation is based on the phenomenon known as the collapse of the lapse; for example the Schwarzschild spacetime does not contain a global foliation by maximal Cauchy surfaces [6]. However, no counterexample is known in the spatially compact case. In spite of these caveats, many examples of spacetimes with

global CMC foliations are known and the CMC condition, or more generally prescribed mean curvature is an important gauge condition for general relativity.

Some examples of situations where global constant or prescribed mean curvature foliations are known to exist in vacuum or with some types of matter are spatially homogenous spacetimes, and spacetimes with two commuting Killing fields. Small data global existence for the Einstein equations with CMC time gauge have been proved for spacetimes with one Killing field, with Cauchy surface a circle bundle over a surface of genus > 1 , by Choquet-Bruhat and Moncrief. Further, for 3+1 dimensional spacetimes with Cauchy surface admitting a hyperbolic metric, small data global existence in the expanding direction has been proved by Andersson and Moncrief. See [1] and [18] for surveys on the Cauchy problem in general relativity.

3. NULL HYPERSURFACES

Consider an asymptotically flat spacetime containing a black hole, i.e. a region B such that future causal curves starting in B cannot reach observers at infinity. The boundary of the trapped region is called the event horizon \mathcal{H} . This is a null hypersurface, which under reasonable conditions on causality has null generators which are complete to the future. Due to the completeness, assuming that \mathcal{H} is smooth, one can use the Raychaudhuri equation (7) to show that the null expansion θ_+ of a spatial cross section of \mathcal{H} must satisfy $\theta_+ \geq 0$, and hence that the area of cross sections of \mathcal{H} grows monotonously to the future. A related statement is that null generators can enter \mathcal{H} but may not leave it. This was first proved by Hawking for the case of smooth horizons, using essentially the Raychaudhuri equation. In general \mathcal{H} can fail to be smooth. However, from the definition of \mathcal{H} as the boundary of the trapped region follows that it has support hypersurfaces, which are past lightcones. This property allows one to prove that \mathcal{H} is Lipschitz and hence smooth almost everywhere. At smooth points of \mathcal{H} , the calculations in the proof of Hawking applies, and the monotonicity of the area of cross sections follows.

Theorem 7 (Area theorem [9]). *Let \mathcal{H} be a black hole event horizon in a smooth spacetime (M, g) . Suppose that the generators are future complete and the N.E.C. holds on \mathcal{H} . Let S_a , $a = 1, 2$ be two spacelike cross sections of \mathcal{H} and suppose that S_2 is to the future of S_1 . Then $\mathcal{A}(S_2) \geq \mathcal{A}(S_1)$.*

The eikonal equation $\nabla^\alpha u \nabla_\alpha u = 0$ plays a central role in geometric optics. Level sets of a solution u are null hypersurfaces which correspond to wave fronts. Much of the recent progress on rough solutions to the Cauchy problem for quasilinear wave equations is based on understanding the influence of the geometry of these wave fronts on the evolution of high-frequency modes in the background spacetime. In this analysis many objects familiar

from general relativity, such as the structure equations for null hypersurfaces, the Raychaudhuri equation, and the Bianchi identities play an important role, together with novel techniques of geometric analysis used to control the geometry of cross sections of the wave fronts and to estimate the connection coefficients in a rough spacetime geometry. These techniques show great promise and can be expected to have a significant impact on our understanding of the Einstein equations and general relativity.

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