

# Motion of Isolated bodies

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## Abstract

It is shown that sufficiently smooth initial data for the Einstein-dust or the Einstein-Maxwell-dust equations with non-negative density of compact support develop into solutions representing isolated bodies in the sense that the matter field has spatially compact support and is embedded in an exterior vacuum solution.

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## 1 Introduction

While a detailed understanding of the motion of massive isolated bodies in general relativity is, for obvious physical reasons, of great interest, only a few special situations have been successfully analysed so far.

The initial value problem for the Einstein equations with suitable matter fields, such as a perfect fluid model with an appropriate equation of state, is well understood in domains where the total energy density is positive.

Domains with vacuum-matter interfaces, which define the boundaries of the bodies, still pose technical problems. While quite general initial data for the Einstein-Euler system have been constructed by Dain and Nagy ([2]), which describe compact fluid balls embedded in an exterior vacuum field, the evolution problem for these data has not been solved yet in the same generality.

If the density has a positive one-sided limit at the matter boundaries, where the pressure is required to vanish, the right hand side of Einstein's field equations will have discontinuities along the boundaries. Under the assumption of spherical symmetry, which is very special from the PDE point of view, the evolution problem has successfully been analysed in the presence of discontinuities by Ehlers and Kind ([5]). Nothing is known for situations with lower degrees of symmetry.

Situations where the density goes to zero at the boundaries with a certain smoothness have been discussed by Rendall ([7]). While no symmetries are assumed, his method requires the use of particular equations of state. These allow him to construct smooth solutions to the Einstein-Euler system with fluid bodies of spatially compact support, but, as pointed out by the author himself, the method he uses has undesirable properties such as a lack of uniqueness and the impossibility to discuss perturbations of static spherically symmetric solutions to the Einstein-Euler system.

The present article deals with a further case in which the density goes to zero at the boundaries of the bodies and where no space-time symmetries are required. It assumes pressure free matter (also referred to as 'dust') or charged pressure free matter as a matter model.

Two methods are known to write the Einstein-dust equations in hyperbolic form. In [4] have been employed the frame formalism and the Bianchi equations for the conformal Weyl tensor to combine the hyperbolicity of the equations with a Lagrangian representation of the fluid flow. This is clearly of interest if one wants to control the location of the boundaries under general assumptions on the fluid model. These equations appear to be useful, however, only in four space-time dimensions.

The representation of [1] starts from the Einstein equations in wave ('harmonic' in the older terminology) coordinates. To render the complete system hyperbolic, a further derivative is applied to the equations for the metric coefficients so that one ends up again with a system of third order in the metric. The equations so obtained are applicable in all space-time dimension. Since the location of the boundary is not an essential problem in the situation

considered here these equations will be used in the following.

In the existence proof we find it technically convenient to use a ‘non-physical’ extension of the matter flow vector field into the vacuum part of the solution. It is then shown that the solutions are in fact independent of these extensions and the latter can be ignored. Our main result concerning the motion of general relativistic bodies consisting of pressure free matter is then stated in the Theorem 2.3. The analogous result concerning charged pressure free matter is given in Theorem 3.3.

## 2 Solutions to the Einstein-dust equations

### 2.1 The equations

The stress energy tensor  $T$  of pressure-free matter is

$$T_{\alpha\beta} = ru_{\alpha}u_{\beta} \quad (2.1)$$

where  $r$  is the scalar matter density, and  $u$  is the (future directed) flow vector field which satisfies

$$u^{\alpha}u_{\alpha} = -1, \text{ hence } u^{\alpha}\nabla_{\beta}u_{\alpha} = 0, \quad (2.2)$$

with  $\nabla$  the covariant differential in the spacetime metric  $g$ .

The tensor  $T$  satisfies the conservation law

$$\nabla_{\alpha}T^{\alpha\beta} = 0. \quad (2.3)$$

Its projection orthogonal to  $u$ , which reads  $r u^{\alpha}\nabla_{\alpha}u_{\beta} = 0$ , implies near points where  $r \neq 0$  the geodesic equation

$$u^{\alpha}\nabla_{\alpha}u_{\beta} = 0. \quad (2.4)$$

Its projection into the direction of  $u$  gives the continuity equation

$$\nabla_{\alpha}(ru^{\alpha}) = u^{\alpha}\partial_{\alpha}r + r\nabla_{\alpha}u^{\alpha} = 0. \quad (2.5)$$

The Einstein equations with pressure-free matter source read in  $n + 1$  dimensional spacetime

$$R_{\alpha\beta} = r(u_{\alpha}u_{\beta} + \frac{1}{n-1}g_{\alpha\beta}). \quad (2.6)$$

We shall regard equations (2.4), (2.5), (2.6) as our basic system of differential equations.

## 2.2 An existence and uniqueness theorem.

The geometric initial data for the spacetime metric  $g$  on an initial manifold  $M$  are a Riemannian metric  $\bar{g}$  and a symmetric 2-tensor  $\bar{k}$ . The initial data for a dust source are a scalar function  $\bar{r}$  on  $M$  and a tangent vector field  $\bar{v}$  to  $M$ . A solution  $(V, g, r, u)$  of the coupled Einstein-dust equations is an Einsteinian development of the initial data set  $(M, \bar{g}, \bar{k}, \bar{r}, \bar{v})$  if  $M$  can be diffeomorphically identified with an embedded submanifold of  $(V, g)$  so that  $\bar{g}$  and  $\bar{k}$  are respectively the induced metric and second fundamental form on  $M$ , while  $\bar{r}$  is the function induced by  $r$  on  $M$  and  $\bar{v}$  is the value on  $M$  of the dust velocity with respect to the proper frame (and the proper time) of an observer with timelike vector orthogonal<sup>1</sup> to  $M$  in  $(V, g)$ . In local coordinates such that the values on  $M$  of the shift and the lapse of the development are respectively  $\bar{\beta} = 0$  and  $\bar{N} = 1$ , it holds that  $\bar{v}^i = (\bar{u}^0)^{-1} \bar{u}^i$ , where  $\bar{u}^\alpha$  are the components of  $u$  in the considered coordinate system at points of  $M$ .

If  $M$  is compact, we denote by  $H_s$  a usual Sobolev (Hilbert) space of functions or tensor fields on  $M$  which are square integrable together with their derivatives of order up to  $s$  in a given smooth Riemannian metric on the manifold  $M$ . The notation  $M_s$  stands for continuous and bounded Riemannian metrics with derivatives in  $H_{s-1}$ . On non compact manifolds suitable variants of these spaces can be used. Since the equations we consider are hyperbolic, the evolution problem can be localized. The discussion of the basic problem, namely the analysis of the evolution in domains containing boundaries of the bodies, will thus be essentially the same for compact or non-compact initial manifolds<sup>2</sup>.

A tensor field of degree  $p$  on a manifold  $M \times R$ , with  $R$  parametrized by  $t$ , can be decomposed, by projections on  $R$  or  $M$ , into a set of  $t$ -dependent tensor fields, of degree  $0, 1, \dots, p$  on  $M$ . For a  $t$ -dependent a tensor field on  $M$ ,  $t \in [0, T[$ , the space  $E_s(T)$  is defined as follows:

$$E_s(T) = C^0([0, T[, H_s) \cap C^1([0, T[, H_{s-1}) \cap C^2([0, T[, H_{s-2}).$$

We say that a tensor on  $M \times [0, T[$  is in  $E_s(T)$  if each of its projections is in  $E_s(T)$ . We denote by  $L_s(T)$ , respectively by  $U_s(T)$ , the space of Lorentzian

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<sup>1</sup>Two such timelike vectors corresponding to two developments isometric under a diffeomorphism  $f$  are mapped onto each other by  $f$ .

<sup>2</sup>One can replace, for instance, the spaces  $H_s$  by spaces  $H_s^{u.loc}$  of tensors which belong to  $H_s$  in the open sets of a locally finite covering of the manifold  $M$ , with uniformly bounded  $H_s$  norms.

metrics, respectively the space of unit vectors in the metric  $g$ , which belong to  $E_s(T)$ .

We will prove the following local existence and geometrical global uniqueness theorem.

**Theorem 2.1** *The Einstein equations in wave gauge with source a pressure free matter, form a hyperbolic Leray system for  $g$ ,  $u$  and  $r$ , which is causal as long as  $g$  is Lorentzian and  $u$  is timelike. There is an interval  $[0, T[ \subset \mathbb{R}$  such that the Cauchy problem for these equations with data on the manifold  $M$  with  $\bar{g} \in M_s$ ,  $\bar{k} \in H_{s-1}$ ,  $\bar{v} \in H_{s-1}$ ,  $\bar{r} \in H_{s-2}$ , where  $s > \frac{n}{2} + 2$ ,  $|\bar{v}|_{\bar{g}} < 1$ , has one and only one solution  $g \in L_s(T)$ ,  $u \in U_{s-1}(T)$ ,  $r \in \bar{E}_{s-2}(T)$ .*

**Corollary 2.2** *If the initial data satisfy the Einstein constraints, the solution obtained in wave gauge satisfy the original Einstein-dust system. The solution is globally hyperbolic. There is a unique solution, up to isometry, in the class of maximal globally hyperbolic spacetimes if  $s > \frac{n}{2} + 3$ .*

**PROOF.** The Einstein equations with dust source read in wave coordinates:

$$R_{\alpha\beta}^{(h)} \equiv -\frac{1}{2}g^{\lambda\mu}\partial_{\lambda\mu}^2 g_{\alpha\beta} + H_{\alpha\beta}(g, \partial g) = \rho_{\alpha\beta} \equiv r(u_\alpha u_\beta + \frac{1}{n-1}g_{\alpha\beta}). \quad (2.7)$$

They are to be coupled with equations (2.4), (2.5).

We think of  $M$  as being embedded into the solution manifold and the coordinate  $x^0$  to be chosen such that  $M = \{x^0 = 0\}$ . The solution manifold then takes close to  $M$  the form  $\mathbb{R}_0^+ \times M$ , with  $x^0$  inducing a coordinate on the first factor, and  $M \simeq \{0\} \times M$ . The initial values for the unknowns  $g_{\alpha\beta}$  and  $u^\alpha$  are deduced from the geometric initial data set by assuming the contracted Christoffel symbols  $\Gamma^\alpha = g^{\beta\gamma} \Gamma_\beta^\alpha{}_\gamma$  to vanish on  $M$  and the wave coordinates to be chosen such that the values on  $M$  of the lapse and the shift of the metric are respectively  $\bar{N} = 1$  and  $\bar{\beta} = 0$ .

We take a derivative of equations (2.7) in the direction of  $u$  and use equations (2.4), (2.5) to obtain equations which are of third order in  $g$  but still do not contain derivatives of  $r$ :

$$u^\gamma \nabla_\gamma R_{\alpha\beta}^{(h)} = -r(u_\alpha u_\beta + \frac{1}{n-1}g_{\alpha\beta}) \nabla_\gamma u^\gamma. \quad (2.8)$$

The initial values for the second derivatives of  $g$  are determined on  $M$  so that equations (2.7) will hold on  $M$ . This will imply that the derivatives of the

contracted Christoffel symbols  $\Gamma^\alpha$  calculated from the derivatives of  $g$  will also vanish on  $M$ .

To the equations (referred to below by their equation numbers) we assign the Leray-Volevic indices [6]

$$m(2.8) = 0, \quad m(2.4) = 1, \quad m(2.5) = 0,$$

and to the unknowns  $g$ ,  $u$ ,  $r$  the indices

$$\ell(g) = 3, \quad \ell(u) = 2, \quad \ell(r) = 1.$$

The matrix of principal parts of the various orders  $\ell - m$  is then diagonal and given by

$$\begin{pmatrix} -\frac{1}{2}g^{\lambda\mu}u^\gamma\partial_{\gamma\lambda\mu}^3g_{\alpha\beta} & 0 & 0 \\ 0 & u^\alpha\partial_\alpha u^\beta & 0 \\ 0 & 0 & u^\alpha\partial_\alpha r \end{pmatrix}.$$

The system is quasidiagonal with characteristic cotangent cone

$$g^{\lambda\mu}\xi_\lambda\xi_\mu u^\alpha\xi_\alpha = 0,$$

the union of the light cone and of a spacelike hyperplane exterior to it, if  $u$  is a timelike vector. The system is thus hyperbolic and causal. The coefficients of the principal terms are in  $C^1$  under the given hypothesis. The Leray-Dionne theory [3], [6] gives, after some work<sup>3</sup>, the existence of a number  $T > 0$  such that the considered Cauchy problem has a unique solution  $g \in L_s(T)$ ,  $u \in U_{s-1}(T)$ ,  $r \in E_{s-2}(T)$ .

Since equation (2.8) can then be rewritten as an ODE for  $R_{\alpha\beta}^{(h)} - \rho_{\alpha\beta}$  along the integral curves of  $u$  and we have (2.7) arranged to be satisfied on  $M$ , it follows that (2.7) is satisfied on the solution manifold.

The proof of the corollary follows the same lines as in the vacuum case.

■

### 2.3 The motion of isolated bodies.

We have just proven that the Cauchy problem with data  $\bar{g}, \bar{k}, \bar{v}, \bar{r}$  as in Theorem 2.1 is geometrically well posed and provides a unique solution  $g$ ,  $u$ ,  $r$  on  $V = [0, T[ \times M$ . There remains, however, an open problem.

<sup>3</sup>The lowering of the regularity of the data required by the general Leray theory of hyperbolic systems, which was worked out by Leray's student Dionne, was written up only in the case of one equation, though of arbitrary order.

Denote by  $\omega$  the maximal open subset of  $M$  (assumed to be non-empty) on which  $\bar{r} > 0$ . Suppose that its closure satisfies  $\bar{\omega} = \text{supp}(\bar{r})$  and, that  $\bar{r}$  vanishes on an open set so that  $M \setminus \bar{\omega} \neq \emptyset$ . We can think then of  $\bar{\omega}$  as a union of disjoint compact subsets of  $M$  which represent the space occupied by material bodies whose density tends continuously to zero at the boundary, since  $\bar{r} \in H_2$ . While  $\bar{v}$  represents the physically well defined ‘matter flow velocity’ at points of  $\omega$ , it has no physical meaning at points where  $\bar{r}$  vanishes (except, perhaps, on the boundary of  $\omega$  as limit of physical velocities). In fact, the  $H_{s-1}$  extension of  $\bar{v}$  from  $\omega$  to all of  $M$  has only be introduced as a convenient device to get the existence result.

Equation (2.5) implies that  $r = 0$  along geodesics which start with tangent vectors  $\bar{u}(p)$  with  $p \in M \setminus \omega$ . Along the geodesics with tangent vectors  $\bar{u}(p)$  with  $p \in \omega$  the function  $r$  will be positive and bounded as long as the convergence  $-\nabla_\alpha u^\alpha$  remains bounded. If  $-\nabla_\alpha u^\alpha \rightarrow \infty$ , which indicates the development of a caustic of the geodesic vector field, we can expect  $r \rightarrow \infty$  and the evolution comes to an end. For geometric reasons we cannot have  $\nabla_\alpha u^\alpha \rightarrow \infty$ , whence  $r \rightarrow 0$ , in the future development.

The closure of the ‘geodesic tube’  $\Omega$  of points in  $V$  swept out by the geodesics with origin in  $\omega$  coincides with the support of  $r$  in  $V$ . It represents the history of the material bodies. In  $V \setminus \Omega$  the metric  $g$  satisfies Einstein’s vacuum field equations.

**Theorem 2.3** *Suppose  $\bar{r} \geq 0$ . If  $s > \frac{n}{2} + 2$  the physical solution in wave gauge obtained in Theorem 2.1, which is given by the fields  $g$  and  $r$  on  $V$  and the field  $u$  on  $\Omega$ , is uniquely determined in a neighbourhood of  $M$  by the data  $\bar{g}$ ,  $\bar{k}$ ,  $\bar{r}$  on  $M$  and the datum  $\bar{v}$  on  $\omega$  with  $\bar{\omega} = \text{supp}(\bar{r})$ . It does not depend on the extension of  $\bar{v}$  to  $M$ .*

**Remark 2.4** *The following discussion will also show that the life-time of the solution considered in Theorem 2.1 might be increased by suitable redefinitions of the field  $u$  outside  $\Omega$ .*

**PROOF.** We note that the smoothness result of the theorem (ensured by choosing  $T$  small enough) excludes that any two of the geodesics corresponding to  $u$  cross each other on  $V$  (otherwise  $u$  would not even correspond to a well defined vector field). It follows in particular that none of the geodesics which start at points of  $M \setminus \omega$  enters the set  $\Omega$ . Because  $\Omega$  is generated by time-like geodesics starting at points of  $\omega$ , it has empty intersection with the

future  $g$ -domain of dependence  $D^+(M \setminus \bar{\omega})$  in  $V$  of the set  $M \setminus \bar{\omega}$ . It follows from general results about the Einstein equations that  $g$  is determined on  $D^+(M \setminus \bar{\omega})$  uniquely by the restriction of the data  $\bar{g}$  and  $\bar{k}$  to  $M \setminus \bar{\omega}$ .

Suppose,  $\bar{v}_*$  is a datum on  $M$  analogous to  $\bar{v}$  so that  $\bar{v}_* = \bar{v}$  (whence  $\bar{u}_* = \bar{u}$ ) on  $\omega$  and  $\bar{g}, \bar{k}, \bar{v}_*, \bar{r}$  satisfy the requirements of Theorem 2.1 with  $s > \frac{n}{2} + 2$ . Then there exists a number  $T_*$ ,  $0 < T_* \leq T$ , so that the tangent vectors of the  $g$ -geodesics in  $V$  with tangent vector  $\bar{u}_*$  on  $M$  define a non-vanishing vector field  $u_*$  on  $V_* = [0, T_*[ \times M \subset V$ .

It follows that for points  $p \in \omega$  the restrictions to  $V_*$  of the  $g$ -geodesics with initial vectors  $\bar{u}_*(p)$  resp.  $\bar{u}(p)$  coincide and are contained in  $V_* \cap \Omega$ . Equation (2.5) with  $u$  replaced by  $u_*$  then determines a unique solution  $r_*$  on  $V_*$  which agrees with  $\bar{r}$  on  $M$ . The functions  $r$  and  $r_*$  have their support in the closure of  $V_* \cap \Omega$ , and we have in fact  $r_* = r$  everywhere on  $V_*$ .

The expression on the right hand side of (2.8) remains unchanged if we replace  $u$  and  $r$  by  $u_*$  and  $r_*$ . The fields  $g, u_*, r$  thus define on  $V_*$  a solution to the Einstein-dust equations. By the uniqueness statement of Theorem 2.1 it must coincide with the solution  $g_*, u_*, r_*$  determined by the data  $\bar{g}, \bar{k}, \bar{v}_*, \bar{r}$  on  $M$ . Therefore  $g = g_*$  and  $r_* = r$  on  $V_*$ , and, as seen previously  $u = u_*$  on  $V_* \cap \Omega$ . The data  $\bar{g}, \bar{k}, \bar{v}, \bar{r}$  and  $\bar{g}, \bar{k}, \bar{v}_*, \bar{r}$  determine the same physical solution on  $V_*$ .

The argument would be complete if we had  $T_* = T$  for all possible extensions of  $v|_\omega$  to  $M$ . This can, however, not be expected. Since geodesic flows are defined by equations of second order, they tend to develop caustics. If this happens with the geodesics generating  $\Omega$  this might lead, as remarked above, to a blow up of the density  $r$  and to physical phenomena like shell crossings which have been studied in the literature (cf. [8]).

Whether geodesics outside  $\Omega$  show this behaviour or whether geodesics starting on  $M \setminus \bar{\omega}$  tend to approach  $\Omega$  depends on the metric  $g$  as well as on the choice of extension of  $v|_\omega$  to  $M$ , and the latter will control to some extent the location of the set where this is going to happen. By a more judicious choice of the extension or by subsequent redefinitions of the extensions on suitable slices  $\{t = \text{const.}\}$  this phenomenon, which is of no physical or geometrical relevance but may restrict the domain of existence in a similar way as a bad choice of gauge, can be avoided. ■

The Einstein-dust constraints on  $M$  do not depend on the extension of  $\bar{v}$  at points where  $\bar{r}$  vanishes. They depend only of  $\bar{g}, \bar{k}, \bar{r}$  on  $M$  and  $\bar{v}$  on  $\bar{\omega}$  if  $\text{supp}(\bar{r}) = \bar{\omega}$ . We can therefore state the following corollary.

**Corollary 2.5** *If the initial data  $\bar{g}, \bar{k}, \bar{r}$  on  $M$  and  $\bar{v}$  on  $\bar{\omega} = \text{supp}(\bar{r})$  satisfy the Einstein - dust constraints, then the physically unique local solution obtained in wave gauge in theorem 2.1 satisfies the full Einstein equations. It is locally geometrically unique.*

PROOF. Since any extension to  $M$  of the given initial data satisfies the constraints, Corollary 2.2 shows that the corresponding solution on  $V$  obtained in wave gauge satisfies the full Einstein-dust equations. The same holds a fortiori for the restriction of this solution to the physically unique solution.

To prove local geometric uniqueness we consider a solution of the Einstein-dust system  $g, r, u$ , in arbitrary coordinates, with Cauchy data  $\bar{g}, \bar{k}, \bar{r}$  on  $M$ , and  $\bar{v}$  on  $\bar{\omega} = \text{supp}(\bar{r})$ . One can show that there exist in a neighbourhood of  $M$  a change to harmonic coordinates reducing to the identity on  $M$ , hence preserving  $\bar{r}$  and  $\bar{v}$ , and such that  $\bar{g}, \bar{k}$ , are preserved. ■

### 3 Charged dust.

#### 3.1 The equations.

The stress energy tensor of charged pure matter (dust) is the sum of the stress energy tensor of the matter and the Maxwell tensor of the electromagnetic field  $F$  :

$$T_{\alpha\beta} = r u_\alpha u_\beta + \tau_{\alpha\beta}, \quad (3.1)$$

with

$$\tau_{\alpha\beta} = F_\alpha{}^\lambda F_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu}. \quad (3.2)$$

The Einstein equations read

$$R_{\alpha\beta} = \Phi_{\alpha\beta} + r(u_\alpha u_\beta + \frac{1}{n-1} g_{\alpha\beta}) \quad (3.3)$$

where we have set

$$\Phi_{\alpha\beta} := F_\alpha{}^\lambda F_{\beta\lambda} + c_n g_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu}, \quad c_n := \left(\frac{n-3}{n-1} - \frac{1}{4}\right). \quad (3.4)$$

The Maxwell equations are, with  $J$  the convection electric current of the density of charge  $q$

$$dF = 0 \quad \text{and} \quad \nabla \cdot F = J, \quad \text{i.e.} \quad \nabla_\alpha F^{\alpha\beta} = J^\beta := qu^\beta; \quad (3.5)$$

they imply the conservation of charge equation

$$\nabla_\alpha(qu^\alpha) = 0. \quad (3.6)$$

For simplicity we suppose the space-time to be simply connected so that there exists a 1-form  $A$ , the electromagnetic potential, with

$$F = dA. \quad (3.7)$$

We take  $A$  in the Lorentz gauge, i.e. so that

$$\delta A = 0, \quad \text{i.e.} \quad \nabla_\alpha A^\alpha = 0. \quad (3.8)$$

The Maxwell equations read then as a wave equation for  $A$ , namely

$$\nabla_\alpha \nabla^\alpha A^\beta - R^\beta{}_\lambda A^\lambda = J^\beta = qu^\beta, \quad (3.9)$$

where we can replace the Ricci tensor  $R_{\alpha\beta}$  by the right hand side of (3.3).

Modulo the Maxwell equations (and  $u^\alpha u_\alpha = -1$ ), the stress energy conservation equations are equivalent to

$$\nabla_\alpha(ru^\alpha) = 0 \quad (3.10)$$

and

$$ru^\alpha \nabla_\alpha u^\beta + qu_\lambda F^{\beta\lambda} = 0. \quad (3.11)$$

Near points where  $r \neq 0$  equations (3.6) and (3.10) imply

$$u^\alpha \partial_\alpha \left( \frac{q}{r} \right) = 0,$$

so that  $q/r$  is constant along the flow lines. It will be constant on the space-time to be constructed if it is constant initially. We will make this simplifying (though not necessary) assumption, and set

$$q = cr, \quad (3.12)$$

with  $c$  some given number. Equation (3.11) then implies near points where  $r > 0$  holds

$$u^\alpha \nabla_\alpha u^\beta + cu_\lambda F^{\beta\lambda} = 0. \quad (3.13)$$

## 3.2 Existence and uniqueness theorem.

**Theorem 3.1** *The Einstein equations in wave gauge with sources an electromagnetic field of potential in Lorentz gauge, together with a charged pressure free matter, form a hyperbolic Leray system for  $g$ ,  $A$ ,  $q$ ,  $u$  and  $r$ , which is causal as long as  $g$  is Lorentzian and  $u$  is timelike. There is an interval  $[0, T[ \subset \mathbb{R}$  such that the Cauchy problem for these equations with data on the manifold  $M$  such that  $\bar{g} \in M_s$ ,  $\bar{A} \in H_s$ ,  $\bar{k} \in H_{s-1}$ ,  $\bar{v} \in H_{s-1}$ ,  $\bar{r} \in H_{s-2}$ , where  $s > \frac{n}{2} + 2$ ,  $|\bar{v}|_{\bar{g}} < 1$  and  $\bar{q} = c\bar{r}$  with  $c = \text{const.}$ , has one and only one solution  $g \in L_s(T)$ ,  $A \in H_s(T)$ ,  $u \in U_{s-1}(T)$ ,  $q = kr \in E_{s-2}(T)$ .*

**Corollary 3.2** *If the initial data satisfy the Einstein and Maxwell constraints, the solution in wave and Lorentz gauges satisfies the original Einstein-Maxwell-dust system. It is globally hyperbolic. There is a unique, up to isometries, maximal, globally hyperbolic solution if  $s > \frac{n}{2} + 3$ .*

PROOF. As in the previous section we differentiate the Einstein equations in wave gauge in the direction of  $u$ . We obtain, using the equation (3.10) (we could replace  $u^\gamma \nabla_\gamma u_\lambda$  by its valued taken from (3.13), but it is not necessary for our application)

$$u^\gamma \nabla_\gamma R_{\alpha\beta}^{(h)} = f_{\alpha\beta}, \quad (3.14)$$

with

$$f_{\alpha\beta} := u^\gamma \nabla_\gamma \Phi_{\alpha\beta} - r \left\{ (u_\alpha u_\beta + \frac{1}{n-1} g_{\alpha\beta}) \nabla_\gamma u^\gamma + u^\gamma \nabla_\gamma (u_\alpha u_\beta) \right\}. \quad (3.15)$$

We also differentiate equations (3.9) in the direction of  $u$  to obtain

$$u^\gamma \nabla_\gamma \nabla_\alpha \nabla^\alpha A^\beta - f^\beta{}_\lambda A^\lambda - R_\lambda^\beta u^\gamma \nabla_\gamma A_\lambda = cr \{ -u^\beta \nabla_\gamma u^\gamma + u^\gamma \nabla_\gamma u^\beta \}. \quad (3.16)$$

We shall consider now the system for  $g$ ,  $A$ ,  $u$ , and  $r$  consisting of equations (3.14), (3.16), (3.13), and (3.10). To the equations we assign the Leray-Volevic indices

$$m(3.14) = 0, \quad m(3.16) = 0, \quad m(3.13) = 1, \quad m(3.10) = 0, \quad (3.17)$$

and to the unknowns  $g$ ,  $A$ ,  $u$ ,  $r$  the indices

$$\ell(g) = 3, \quad \ell(A) = 3, \quad \ell(u) = 2, \quad \ell(r) = 1. \quad (3.18)$$

We see that the matrices of principal parts is again diagonal, with the same kind of terms in the diagonal as in the previous section. The existence and uniqueness theorem in wave and Lorentz gauge follows from the Leray-Dionne theory.

The proof of the existence of a solution of the full Einstein-Maxwell-dust system when the initial data satisfy the Einstein and Maxwell constraints follows the same lines as in the vacuum Einstein-Maxwell case. Geometric global uniqueness is also proved similarly. ■

### 3.3 The motion of isolated bodies.

Again we denote by  $\omega$  the maximal open subset of  $M$  on which  $\bar{r} > 0$  and assume that its closure satisfies  $\bar{\omega} = \text{supp}(\bar{r})$ . If  $\bar{r}$  vanishes on an open set so that  $M \setminus \bar{\omega} \neq \emptyset$ , we denote by  $\Omega$  the subset of  $V = [0, T[ \times M$  which is generated by the integral curves of  $u$  which start at points of  $\omega$ . The support of  $r$  and  $q$  is then given by the closure  $\bar{\Omega}$  of  $\Omega$  in  $V$ . It follows that  $g$  and  $A$  satisfy the matter free Einstein-Maxwell equations in  $V \setminus \bar{\Omega}$ .

**Theorem 3.3** *Suppose that  $\bar{r} \geq 0$ . The physical solution  $g, A, r, q$  on  $V$  and  $u$  on  $\Omega$  obtained in wave and Lorentz gauges in theorem 4.1 with initial data, on  $M$ ,  $\bar{g}, \bar{k}, \bar{A}, \bar{r}, \bar{q} = c\bar{r}$  and an extension to  $M$  of the datum  $\bar{v}$  on  $\omega$  with  $\bar{\omega} = \text{supp}(\bar{r})$  is independent, in a neighbourhood of  $M$ , of the extension of  $\bar{v}$  to  $M$ .*

PROOF. The argument is the same as the proof of Theorem 2.3 with the geodesics curves considered there replaced by the integral curves of the vector field  $u$  defined by equation (3.13). ■

**Corollary 3.4** *If the initial data  $\bar{g}, \bar{k}, \bar{A}, \bar{r}, \bar{q} = c\bar{r}$  on  $M$ ,  $\text{supp}\bar{r} \subset \omega$  and  $\bar{v}$  on  $\bar{\omega}$ , satisfy the Einstein - Maxwell - dust constraints, then the physically unique local solution obtained in wave and Lorentz gauge satisfies the full Einstein- Maxwell - dust equations. It is locally geometrically unique.*

PROOF. The arguments are analogous to the proof of the corollary 2.5. ■

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