Rigid upper bounds for the angular momentum and centre of mass of non-singular asymptotically anti-de Sitter space-times

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Abstract

We prove upper bounds on angular momentum and centre of mass in terms of the Hamiltonian mass and cosmological constant for non-singular asymptotically anti-de Sitter initial data sets satisfying the dominant energy condition. We work in all space-dimensions larger than or equal to three, and allow a large class of asymptotic backgrounds, with spherical and non-spherical conformal infinities; in the latter case, a spin-structure compatibility condition is imposed. We give a large class of non-trivial examples saturating the inequality. We analyse exhaustively the borderline case in space-time dimension four: for spherical cross-sections of Scri, equality together with completeness occurs only in anti-de Sitter space-time. On the other hand, in the toroidal case, regular non-trivial initial data sets saturating the bound exist.

1 Introduction

In recent work [35] one of us (DM) has proved an inequality satisfied by the global charges for three-dimensional asymptotically anti-de Sitter initial data sets with spherical conformal infinity. In this paper we consider initial data sets

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$(\mathscr{S}, g, K)$ on a spin manifold $\mathscr{S}$ and we extend that work in several directions: First, in dimension $3+1$ we derive the corresponding inequality for a toroidal Scri under a condition of compatibility of spin structures, and we analyse the equality case. We prove that black hole solutions saturating the inequality do not exist. When the associated space-time has a complete $\mathscr{I}$ with spherical cross-sections, we prove that equality happens only in anti-de Sitter space-time. Next, we prove the corresponding inequalities in dimensions $n+1$, $n \geq 3$. For spherical Scris we obtain optimal inequalities for n equal to four and five, as well as some natural but non-optimal inequalities for all n. For toroidal Scris, again under a spin structure condition, we obtain optimal inequalities for all n, and we point out the existence of large families of non-singular (non-vacuum) initial data sets which saturate the inequality. Finally, we obtain an angular-momentum bound for general conformal boundaries at infinity with covariantly constant spinors.

2 Global charges and their positivity

In this work we consider n-dimensional general relativistic data sets $(\mathscr{S}, g, K)$, which are asymptotically anti-de Sitter (adS) in the following sense: there exists a Riemannian background metric b which, in the asymptotic region, is of the form

$$b = dr^2 + \mathring{f}(r)\check{h} \ ,$$

where $\check{h}$ is *either* a unit round metric on S^{n-1} (then with the cosmological constant Λ normalised to $-n$, and up to change of origin in r, $\mathring{f}(r) = \sinh^2 r$), *or* $\check{h}$ is a Ricci flat metric on a $(n-1)$-dimensional compact manifold ${}^{n-1}M$ (then, again up to these choices, $\mathring{f}(r) = e^{2r}$), where the space-dimension n is greater than or equal to 3. By [10], with those f, the initial data $(\mathscr{S}, b, 0)$ arise from static solutions of vacuum Einstein equations with a negative cosmological constant[1]. Note that in the spherical case, or if $({}^{n-1}M, \check{h})$ is a flat torus T^{n-1}, then $(\mathscr{S}, b, 0)$ are initial data for anti-de Sitter space-time, or a quotient thereof. In all cases $(\mathscr{S}, b, 0)$ provide initial data for a static Einstein metric.

We shall assume that there exist constants $k \geq 1$, $\alpha > n/2$ and $C > 0$ such that for large r we have[2]

$$|g-b|_b + |\mathring{D}g|_b + \cdots + |\underbrace{\mathring{D}\cdots\mathring{D}}_{k \text{ factors}} g|_b + |K|_b + \cdots + |\underbrace{\mathring{D}\cdots\mathring{D}}_{k-1 \text{ factors}} K|_b \leq Ce^{\alpha r} \ . \quad (2.1)$$

Here $|\cdot|_b$ denotes the norm of a tensor field with respect to the metric b, and $\mathring{D}$ is the covariant derivative of b. These decay conditions have been chosen because of simplicity of the analysis involved; it should be recognised that they are restrictive, and a completely satisfactory treatment should allow weaker

[1] It might seem natural also to allow $({}^{n-1}M, \check{h})$ to be a negatively curved Einstein manifold [10]. However, we shall see shortly that such solutions do not seem to fit into a Witten-type positivity argument, which is the main concern of this work.

[2] In *many* of our arguments it is sufficient to assume the weaker, integral-type, decay conditions of [15, Section 3], but we have not checked whether *all* the calculations go through under such conditions.

boundary conditions, compare [15] for a related analysis in the context of a vanishing cosmological constant $\Lambda = 0$.

Let X be a Killing vector in the asymptotic region of the associated background space-time. It is well known that each such X defines a Hamiltonian associated with the flow along X [13, 15, 16, 25], as follows: Let V be the normal component of X with respect to the space-time background metric, and let Y be the tangential component thereof; when defined along a spacelike hypersurface, such pairs (V, Y) are called KIDs (Killing Initial Data). Then the Hamiltonian $H(V, Y)$ corresponding to X (which we identify with the pair (V, Y)) takes the form:

$$H(V,Y) := \lim_{R\to\infty} \frac{1}{16\pi} \int_{r=R} \left(\mathbb{U}^i(V) + \mathbb{V}^i(Y)\right) dS_i \ , \tag{2.2}$$

where

$$\mathbb{U}^i(V) := 2\sqrt{\det g}\ \left(V g^{i[k} g^{j]l} \mathring{D}_j g_{kl} + D^{[i} V g^{j]k}(g_{jk} - b_{jk}\right) \ , \tag{2.3}$$

$$\mathbb{V}^i(Y) := 2\sqrt{\det g}\ \left(K^i{}_j - K^k{}_k \delta^i_j\right) Y^j \ . \tag{2.4}$$

Here all indices are space indices, running from 1 to n, and $\mathring{D}$ is the Levi-Civita derivative of the space background metric b.

The normalisation constant $1/16\pi$ in (2.2) is convenient in dimension $3+1$ when $\mathscr{I}$ has spherical cross-section, this choice being rather arbitrary in higher dimensions, or when non-spherical cross-sections are considered.

We wish to establish conditions under which a Witten-type proof of positivity of global charges [14, 15, 35, 40] applies. (Examples in which *it does not* are given in [18, 28].) For this one needs, first, a positivity hypothesis on the energy-momentum of the initial data set. Denoting the cosmological constant by Λ, we set

$$\rho := R - |K|^2 + |\mathrm{tr}_g K|^2 - 2\Lambda \ , \quad J^i = D_j K^i{}_j - D^i K^j{}_j \ ,$$

where R is the scalar curvature of g. The dominant energy condition reads then

$$\rho \geq |J|_g \ . \tag{2.5}$$

Next, we need an *imaginary Killing spinor for the background metric* b; by definition, this is a spinor field ψ in the asymptotic region $\mathscr{S}_{\mathrm{ext}} := [R_0, \infty) \times {}^{n-1}M$ solving the set of equations

$$\forall\ X \in T\mathscr{S} \quad \mathring{D}_X \psi = -i\sqrt{\frac{-\Lambda}{2n(n-1)}} X \cdot \psi \ , \tag{2.6}$$

where $X\cdot$ denotes the Clifford product of X, and $\mathring{D}$ is the usual Riemannian spinorial connection associated with the metric b. Such spinor fields are known to exist when ${}^{n-1}M$ is a sphere; we point out several (well known [8, 11, 23, 31, 34]) further examples, with alternative topologies at infinity, in Section 4 below. The field ψ is a section of a bundle of spinors[3] which we will denote by $\mathfrak{S}'$.

It is worthwile pointing out that *manifolds* $\mathscr{S}_{\rm ext}$*, with* $({}^{n-1}M,\check{h})$ *having negative Ricci curvature, do not admit imaginary Killing spinors*. This can be seen as follows: first, any imaginary Killing spinor leads to a Killing vector in $\mathscr{S}_{\rm ext}$. But it is known (see, e.g., the analysis in [2, Appendix A]) that there are no Killing vectors on $\mathscr{S}_{\rm ext}$ in this case. Thus, no lower bounds on the mass can be obtained by Witten-type techniques when, e.g., ${}^{n-1}M$ is a two-dimensional higher genus surface.

To continue, we need to assume that $\mathscr{S}$ admits a spin structure. Note that the spinor field ψ already singles out a spin structure on $\mathscr{S}_{\rm ext}$, which is necessarily compatible with the one of $\mathscr{S}$ when ${}^{n-1}M$ is simply connected. However, those spin structures might be incompatible when ${}^{n-1}M$ is *not* simply connected. A key, rather restrictive, hypothesis in our work is that

$$\textit{the bundle } \mathfrak{S}' \textit{ over } \mathscr{S}_{\rm ext} \textit{ extends to a bundle of spinors } \mathfrak{S} \textit{ over } \mathscr{S}\ . \qquad (2.7)$$

A short discussion of the hypothesis (2.7) is in order. First, (2.7) is satisfied by all product topologies $\mathscr{S}=\mathbb{R}\times{}^{n-1}M$, or $\mathscr{S}=[0,\infty)\times{}^{n-1}M$. Those examples include the hyperbolic-cusp solutions (4.1) below, or the Kottler black holes [32] with toroidal topology at infinity. On the other hand, (2.7) is *not* satisfied by the Horowitz-Myers solutions [28]. Now, in that last example $\mathscr{S}$ is the union of a compact set and of the asymptotic region $\mathscr{S}_{\rm ext}$, and in such a context we have the following[4]: If ${}^{n-1}M=\mathbb{T}^2$, the two-dimensional torus, and $\mathscr{S}$ has no boundary (other than the conformal boundary at infinity), then the trivial spin structure on $\mathbb{T}^2$, which does admit parallel spinors, *never* extends [33, p. 91] when compactness of the conformal completion of $\mathscr{S}$ is imposed. On the other hand, for all higher-dimensional toroidal boundaries at infinity ${}^{n-1}M=\mathbb{T}^{n-1}$, $n\geq 4$, compact boundaryless fillings for the trivial spin structure of $\mathbb{T}^{n-1}$ exist [33, p. 92]. All this leads to a large class of examples where (2.7) holds.

For the analytical arguments to go through, we need further to assume that $(\mathscr{S},g)$ is *complete*, either without boundary, or with a *compact* boundary satisfying the following: Let λ be the extrinsic curvature tensor of $\partial\mathscr{S}$ (considered as a submanifold of $\mathscr{S}$, recall that there is no space-time involved at this stage) with respect to an inward-pointing unit normal ν, let h be the metric induced on $\partial\mathscr{S}$ by g. The boundary contribution which arises in the Witten argument with a spinor field satisfying the boundary condition of [21] (compare [26]) will have the favorable sign provided that the boundary is either *weakly future trapped*,

$$\mathrm{tr}_h\lambda + h^{ab}K_{ab}\leq 0\ , \qquad (2.8)$$

or *weakly past trapped*, which corresponds to changing the sign in front of the K term in (2.8). An alternative condition which allows one to conclude is that

[3]By a "spinor field" we mean a section of a hermitian bundle associated to the Spin principal bundle over $\mathscr{S}$, equipped with an action of the Clifford algebra of $\mathscr{S}$ via anti-hermitian bundle-morphisms. In what follows we shall freely make use of "doubling constructions" such as the one in (2.11) below, and therefore we do *not* impose the often-implicitly-used condition that the representation of the Clifford algebra carried by the spinor bundle is irreducible.

[4]We are grateful to M. Stern for discussions and references concerning the compatibility of spin structures.

considered in [14, 35]. Setting $k(\nu) = K_{ia}\nu^i dx^a$, where the x^a's are coordinates on $\partial\mathscr{S}$, we then assume that

$$\mathrm{tr}_h\lambda + |k(\nu)|_h \leq \sqrt{\frac{-2(n-1)\Lambda}{n}} \tag{2.9}$$

(see [14, Remark 4.8] for a discussion of (2.9) when $k(\nu) = 0$).

Next, in the construction we will need a bundle isomorphism $\gamma^0 : \mathfrak{S} \to \mathfrak{S}$ with the following properties:

$$(\gamma^0)^2 = 1 \,, \tag{2.10a}$$

$$\forall X \in T\mathscr{S} \quad \gamma^0 X \cdot = -X \cdot \gamma^0 \,, \tag{2.10b}$$

$$(\gamma^0)^\dagger = \gamma^0 \,, \tag{2.10c}$$

$$D\gamma^0 = \gamma^0 D \,, \tag{2.10d}$$

where $(\gamma^0)^\dagger$ denotes the conjugate of γ^0 with respect to the hermitian product $\langle\cdot,\cdot\rangle$. Such a map always exists if $\mathfrak{S}$ is obtained by pulling-back to $\mathscr{S}$ a space-time spinor bundle, provided one has an externally oriented isometric embedding of $(\mathscr{S}, g)$ in a Lorentzian space-time at disposal. Then the Clifford product $N\,\cdot$, where N is the field of Lorentzian unit normals to the image of $\mathscr{S}$, has the required properties. Regardless of whether or not such a map exists, one can *always* replace $\mathfrak{S}$ by a direct sum of two copies of $\mathfrak{S}$; then, for $X \in T\mathscr{S}$, we let $X\cdot$ denote the Clifford action of X and we set

$$\gamma^0(\psi_1,\psi_2) := (\psi_2,\psi_1) \,, \tag{2.11a}$$

$$X\cdot(\psi_1,\psi_2) := (X\cdot\psi_1, -X\cdot\psi_2) \,, \tag{2.11b}$$

$$D_X(\psi_1,\psi_2) := (D_X\psi_1, D_X\psi_2) \,. \tag{2.11c}$$

One checks that (2.11b) defines a representation of the Clifford algebra of $(\mathscr{S}, b)$ on $\mathfrak{S}\oplus\mathfrak{S}$, and that (4.6) holds.

One use of γ^0 is to construct Killing vectors for the metric b out of imaginary Killing spinors. Indeed, if ψ is such a spinor, and e_i is a (locally defined) ON basis of $T\mathscr{S}$, then the vector field

$$Y = \langle\psi, \gamma^0 e^i\cdot\psi\,\rangle e_i$$

is a Killing vector of the metric b. Furthermore, the pair (V, Y), where $V = \langle\psi,\psi\rangle$, defines a KID of $(\mathscr{S}, b, 0)$.

We now have:

Theorem 2.1 (Positive charges theorem) *Consider an initial data set $(\mathscr{S}, g, K)$ satisfying the positivity and fall-off conditions (2.5) and (2.1), with $(\mathscr{S}, g)$ complete, and with finite total matter energy: $\rho \in L^1(\mathscr{S})$. We assume that either $\mathscr{S}$ has no boundary, or $\partial\mathscr{S}$ is compact and then either (2.8) (changing K to $-K$ if necessary) or (2.9) holds. Suppose that the Riemannian background metric b admits imaginary Killing spinors in the asymptotic region, with respect to a spin structure which extends to the interior of $\mathscr{S}$. Let $\mathscr{K}_o$ be the subset of the* *set of b-KIDs which are of the form* $(\langle\psi,\psi\rangle, \langle\psi,\gamma^0 e^i\cdot\psi\rangle e_i)$ *for some b-imaginary Killing spinor* ψ*. Then for all* $X=(V,Y)\in\mathscr{K}_0$ *we have*

$$H(V,Y)\geq 0\ ,$$

with equality if and only if ψ *asymptotes to a imaginary Killing spinor of* $(\mathscr{S},g,K)$ *associated with* ∇*.*

Remark 2.2 It should be emphasised that the imaginary Killing spinors provided by Theorem 2.1 are only defined along $\mathscr{S}$, and not in an associated space-time if there is one.

Remark 2.3 The bundle of spinors which is used in the proof is arbitrary. We will freely make use of this fact in our analysis in subsequent sections.

Proof: We use a Witten-type argument, as follows. Let $(\mathfrak{S},\langle\cdot,\cdot\rangle)$ be any Riemannian bundle of spinors over (M,g) with hermitian product $\langle\cdot,\cdot\rangle$, such that Clifford multiplication (which we denote by "$\cdot$") is anti-hermitian, and with a map γ^0 satisfying (2.10).

Given an initial data set $(\mathscr{S},g,K)$, a vector field X, and a spinor field ξ we set

$$\begin{aligned} K(X) &:= K_i{}^j X^i e_j\cdot\ , && (2.12)\\ \nabla_X\xi &:= D_X\xi+\frac{1}{2}K(X)\gamma^0\xi\ . && (2.13)\end{aligned}$$

Here e_i is a local orthonormal basis of TM; it is straightforward to check that (2.12) does not depend upon the choice of this basis.

The argument now has two main steps. First, one shows existence of a spinor χ satisfying a modified Dirac equation,

$$e^j\cdot\Big(\nabla_j+i\sqrt{\frac{-\Lambda}{2n(n-1)}}e_j\cdot\Big)\chi=0\ , \tag{2.14}$$

and which asymptotes to ψ, where ψ is an imaginary Killing spinor of the background metric. This can be done by rather obvious modifications of the arguments in [14], compare [35], see also [6, 26] for the treatment of the boundary terms arising from a non-empty $\partial\mathscr{S}$. Let us simply point out that one of the ingredients of the proof is a weighted Poincaré inequality, established e.g. in [6] for the metrics of interest. This proves positivity of the boundary integral in the Witten identity. The next step is to prove that this boundary integral coincides with the Hamiltonian $H(V,Y)$. This is done by following the calculations in [3] and [14]. We note that the relevant part of those calculations does not use the explicit form of the imaginary Killing spinors, but only the equation satisfied by those. □

3 Spherical conformal infinity

A preferred set of background Killing vector fields is provided by those which are b-normal to the initial data surface. The resulting Hamiltonians are usually interpreted as energies. In contradistinction with the asymptotically flat case, where only one normal background Killing vector field exists, if one assumes that conformal infinity has spherical space-like sections, then there are several normal background Killing vector fields. This implies that there is not a *single* energy, but rather an *energy functional* M. This functional M is uniquely characterised by $n+1$ numbers $m_{(\mu)}$, $\mu = 0, 1, \ldots, n$, which transform as a Lorentz covector under asymptotic isometries[5] of g, see [16, 40]. (The component $m_{(0)}$ coincides with the Abbott-Deser mass under appropriate restrictions [16].) It follows that the Lorentzian length of $m_{(\mu)}$ is a geometric invariant of $(\mathscr{S}, g)$.

We start by reviewing the known $3+1$ results. The asymptotically-adS-positive-energy theorem implies that $m_{(\mu)}$ is causal, future pointing [21, 22, 35] (compare [14, 40, 41]). If it vanishes, then $(\mathscr{S}, g, K)$ are initial data for anti-de Sitter space-time.[6]

Quite generally, one can view the hyperbolic space as a unit spacelike hyperboloid in $\mathbb{R}^{n+1}$, the latter equipped with the Minkowski metric. If one assumes that $m_{(\mu)}$ is timelike, after applying an asymptotic isometry to obtain $m_{(\mu)} = (m, 0, \cdots, 0)$, the background Killing vector fields tangent to $\mathscr{S}$ can now be split into rotations and "boosts". In space-time dimension four it is customary to define the rest-frame angular momentum as

$$j_{(i)} := H(0, \beta_{(i)}) ,$$

where the $\beta_{(i)}$'s are the generators of rotations of S^2, when embedded in $\mathbb{R}^3$:

$$\beta_{(i)} = \epsilon_{ijk} x^j \partial_k .$$

The numerical values of the remaining three Hamiltonians generating boost transformations (see (3.10) below) will be denoted by $c_{(i)}$. In the asymptotically flat case the $c_{(i)}$'s have the interpretation of the centre of mass of the system, and can always be set to zero by a translation of the coordinates. This freedom does not exist in the asymptotically adS situation. We will retain the name *centre of mass* for the vector $\vec{c} = (c_{(i)})$.

It does not appear to be widely known that the positive energy theorem for asymptotically adS initial data implies an upper bound on the center of mass and the angular momentum in terms of m. This should be contrasted with the asymptotically Minkowskian positive energy theorem, which bounds the space-momentum in terms of the energy, but does not impose constraints either on angular momentum or on centre of mass.[7] Recall that with our choices

[5] Those isometries are, essentially, characterised by conformal isometries of the conformal boundary at infinity (in the current case the sphere).

[6] In fact, the proof of this in [35] contains a gap which we fill, see the proof of Theorem 3.8, end of Section 3.1 below.

[7] Schoen (seminar at the ESI, summer 2003) has shown that there is no bound on the ratio $|\vec{j}|/m$ for vacuum initial data sets with $\Lambda = 0$.

so far the energy-momentum vector $m_{(\mu)}$ lies along the time axis. A rotation of the coordinate system aligns the angular momentum vector $\vec{j}$ along the first coordinate axis. One can then rotate $\vec{c} = (c_{(i)})$ to lie in the x–y plane. It is shown in [35] that the positivity theorem 2.1 implies the following inequality

$$m \geq \sqrt{-\Lambda/3}\sqrt{(|j_{(1)}| + |c_{(2)}|)^2 + c_{(1)}^2}\ , \tag{3.1}$$

with vanishing m if and only if the initial data set arises from anti-de Sitter space-time.[8]

The inequality (3.1) can be rewritten in the manifestly rotation-invariant form

$$m \geq \sqrt{-\Lambda/3}\sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|}\ , \tag{3.2}$$

where $\vec{c} \times \vec{j}$ is the vector product, while $|\vec{j}| = \sqrt{j_{(1)}^2 + j_{(2)}^2 + j_{(3)}^2}$, etc. In particular we have the striking upper bounds

$$m \geq \sqrt{-\Lambda/3}|\vec{j}|\ , \quad m \geq \sqrt{-\Lambda/3}|\vec{c}|\ . \tag{3.3}$$

Thus, both the length of the angular momentum vector and that of the centre of mass vector are bounded by (a multiple of) the invariant norm of the mass functional M.

The first inequality in (3.3) is a familiar condition in the explicit family of Kerr-adS metrics (see, e.g., [25]). Thus, the restriction on the range of parameters stemming from the Kerr-adS family is not a result of our incomplete knowledge of the set of all solutions, but a necessary property of non-singular asymptotically adS space-times satisfying the dominant energy condition.

The above leaves several questions unanswered: is there an equivalent of (3.2) when $m_{(\mu)}$ is null? what happens if the inequalities are equalities? what if ${}^{n-1}M$ is a two-dimensional torus? What happens in higher dimensions? In this work we give partial or complete answers to those questions. We start with the following:

THEOREM 3.1 *Let $n \geq 3$ and ${}^{n-1}M = S^{n-1}$ (then the spin structures on $\mathscr{S}$ and $\mathscr{S}_{\rm ext}$ are necessarily compatible). Under the remaining hypotheses of Theorem 2.1, $m_{(\mu)}$ is causal future[9] directed, or vanishes. Furthermore,*

1. *In every conformal frame[10] it holds that*

$$\ell m_{(0)} \geq |\omega_{(1)}| + |\omega_{(2)}| + \ldots + |\omega_{(l)}| \tag{3.4}$$

[8] The normalisations of the Hamiltonians are a matter of conventions, ours are as follows: the mass $m_{(0)}$ is the numerical value of the Hamiltonian associated with the background Killing vector ∂_t when the background adS metric is written in the form $-(-\Lambda r^2/n(n-1)+1)dt^2 + (-\Lambda r^2/n(n-1)+1)^{-1}dr^2 + r^2 d\Omega^2$, where $d\Omega^2$ is the unit round metric on the $(n-1)$-dimensional sphere. This normalisation is convenient for comparison with the $\Lambda = 0$ limit. Next, the angular momentum is the numerical value of the Hamiltonian associated with the rotations of S^{n-1} normalised so that a rotation by 2π is the identity. Finally, the center of mass is normalised to make the right-hand-side of our inequalities look simple.

[9] The notion of causality of $m_{(\mu)}$ is determined by a Lorentzian metric with signature $(1, n)$ defined by the group of isometries of hyperbolic space [16], with "future" defined as $m_{(0)} > 0$.

[10] Recall that the decomposition of g as a background plus a correction term involves a choice, and that two such choices can be related to each other by a conformal transformation of the conformal boundary at infinity, plus higher order corrections [16]. We use the term "conformal frame" to emphasise the fact that such a choice has been made.

(see (3.13) below for the definition of the $\omega_{(i)}$'s), where

$$\ell := \sqrt{-\frac{n(n-1)}{2\Lambda}} \,. \tag{3.5}$$

2. *If $m_{(\mu)}$ is null, then the space of ∇–imaginary Killing sections of $\mathfrak{S} \oplus \mathfrak{S}$ over $\mathscr{S}$ (as defined in (2.11)) is at least $\dim \mathfrak{S}$–dimensional.*

3. *In dimension $3+1$ the energy-momentum vector $m_{(\mu)}$ cannot be null.*

4. *When $m_{(\mu)}$ is timelike we also have, in a frame where $m_{(i)} = 0$,*

$$\ell m_{(0)} \geq \sqrt{c_{(1)}^2 + \cdots + c_{(n)}^2} \,, \tag{3.6}$$

   (see below for the definition of $c_{(i)}$'s.)

5. *If $m_{(0)}$ vanishes in some conformal frame, then all global charges vanish. If one moreover assumes that $(\mathscr{S}, g)$ is conformally compactifiable, then $(\mathscr{S}, g, K)$ arises from a hypersurface in anti-de Sitter space-time.*

6. *In dimension $5+1$, in a frame as defined in the proof below, we have the stronger, optimal inequality*

$$\ell m \geq \sqrt{c_{(1)}^2 + c_{(3)}^2 + c_{(5)}^2 + \omega_{(1)}^2 + \omega_{(2)}^2 + 2\sqrt{(\omega_{(1)} c_{(1)})^2 + (\omega_{(2)} c_{(3)})^2 + (\omega_{(1)}\omega_{(2)})^2}} \,. \tag{3.7}$$

7. *Inequality (3.7) remains valid and optimal in dimension $4+1$ after setting $c_{(5)} = 0$.*

8. *Similarly (3.7) remains valid and optimal in dimension $3+1$ after setting $c_{(5)} = \omega_{(2)} = 0$, and is then identical to (3.2). Furthermore, equality in (3.2) together with $\vec{j} \times \vec{c} = 0$ (equivalently, $\omega_{(1)} c_{(1)} = 0$) occurs if and only if $(\mathscr{S}, g, K)$ can be obtained from a hypersurface in anti-de Sitter space-time.*

REMARK 3.2 Equation (3.7) suggests that in all dimensions one should have the (non-optimal) inequality

$$\ell m_{(0)} \geq \sqrt{\sum_i c_{(i)}^2 + \Big(\sum_{i<j} |J_{(i)(j)}|\Big)^2} \,.$$

REMARK 3.3 A class of $4+1$ dimensional examples saturating the bound (3.4) is given by the metrics in [24] with $F^I_{\mu\nu} = 0$, or the metrics in [19].

PROOF: To avoid annoying multiplicative factors involving the dimension and the cosmological constant, all calculations that follow are done assuming $\Lambda = -n(n-1)/2$, so that the background hyperbolic metric has all sectional curvatures equal to one. This can be achieved by a scaling of the metric; the general result is then obtained by rescaling back.

We view the hyperbolic space as the open unit ball $B^n(1) \subset \mathbb{R}^n$ equipped with the metric $b = {}^nb = \omega^{-2}\delta$, where δ is the standard flat metric on $\mathbb{R}^n$, and

$$\omega = \frac{1-|x|^2}{2} \ .$$

In the obvious spin frame associated with this conformal representation[11], the imaginary Killing spinors of ${}^nb$ take the form

$$\psi_u = \omega^{-1/2}(1 - ix^k\gamma^k)u \tag{3.8}$$

(summation over k), where u is a spinor with constant entries, while the anti-hermitian matrices γ^k with constant entries satisfy the flat space Clifford relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij} \ .$$

(The fact that the ψ_u's exhaust the space of imaginary Killing spinors follows e.g. from the fact that the map which assigns u to $\psi_u(0)$ is a bijection). As already mentioned, we will also need a hermitian matrix γ^0, with constant entries, satisfying

$$(\gamma^0)^2 = 1 \ , \qquad \gamma^0\gamma^j + \gamma^j\gamma^0 = 0 \ .$$

(If such a matrix does not exist we first make a doubling construction on the u's as in (2.11).) The KID (N_u, Y^i_u) associated to ψ_u takes the form

$$N_u \ := \ \langle\psi_u,\psi_u\rangle = 2\Big(|u|^2 \underbrace{\frac{1+|x|^2}{1-|x|^2}}_{=:V_{(0)}} + \langle u, i\gamma^k u\rangle \underbrace{\frac{(-2)x^k}{1-|x|^2}}_{=:V_{(k)}}\Big) \ , \tag{3.9}$$

$$\begin{aligned} Y^i_u\partial_i \ &:= \ \langle\psi_u, \gamma^0\gamma^i\psi_u\rangle e_i \\ &= \ 2\langle u, \gamma^0\gamma^k u\rangle \underbrace{\Big(\frac{1+|x|^2}{2}\delta^i_k - x^ix^k\Big)\partial_i}_{:=C_{(k)}} + \frac{1}{2}\langle u, i\gamma^0(\gamma^k\gamma^i - \gamma^i\gamma^k)u\rangle \underbrace{(x_k\partial_i - x_i\partial_k)}_{:=\Omega_{(k)(i)}} \ . \end{aligned} \tag{3.10}$$

The KIDs $(V_{(\mu)},0)$, $\mu = 0,\dots,n$, together with $(0,C_{(k)})$, $k = 1,\dots n$, and $(0,\Omega_{(i)(j)})$, $1 \le i < j \le n$, span the space of KIDs of $(B(1),b,0)$. The $\Omega_{(i)(j)}$'s obviously generate rotations, and therefore it is natural to use the name *angular momenta* for the corresponding global charges; those will be denoted by $J_{(i)(j)}$. As shown in [16, 40], the collection of functions $(V_{(0)}, V_{(1)}, \dots, V_{(n)})$, transforms as a Lorentz covector under conformal isometries of the boundary at infinity. This is at the origin of the name *energy-momentum vector*, denoted by $m_{(\mu)}$, for the associated charges. As already mentioned at the beginning of this section, the $C_{(k)}$'s generate Lorentz boosts, when the hyperbolic space is embedded as a hyperboloid into $(n+1)$–dimensional Minkowski space; the associated charges will be denoted by $c_{(k)}$, and called *center of mass*.

[11]More precisely, we take a spin frame which projects to the frame $\theta^i = \omega^{-1}dx^i$, and a local basis of the spinor bundle in which the γ^μ's are constant matrices.

We have

$$
\begin{aligned}
H(N_u, Y_u^i) &= 2H\Big(|u|^2 (V_{(0)}, 0) + \langle u, i\gamma^k u\rangle (V_{(k)}, 0) \\
&\qquad + \langle u, \gamma^0\gamma^k u\rangle (0, C_{(k)}) + \frac{1}{4}\langle u, i\gamma^0 (\underbrace{\gamma^k\gamma^j - \gamma^j\gamma^k}_{=:2\gamma^{kj}}) u\rangle (0, \Omega_{(k)(j)})\Big) \\
&= 2\Big(|u|^2 \underbrace{H(V_{(0)}, 0)}_{m_{(0)}} + \langle u, i\gamma^k u\rangle \underbrace{H(V_{(k)}, 0)}_{m_{(k)}} \\
&\qquad + \langle u, \gamma^0\gamma^k u\rangle \underbrace{H(0, C_{(k)})}_{c_{(k)}} + \frac{1}{2}\langle u, i\gamma^0\gamma^{kj} u\rangle \underbrace{H(0, \Omega_{(k)(j)})}_{J_{(k)(j)}}\Big) \\
&= 2\langle u, \underbrace{\Big(m_{(0)} + i\gamma^k m_{(k)} + \gamma^0\gamma^k c_{(k)} + \frac{1}{2} i\gamma^0\gamma^{kj} J_{(k)(j)}\Big)}_{=:Q} u\rangle\ .
\end{aligned}
$$

By the positivity theorem 2.1 the matrix Q must be positive semi-definite. Let us explore the consequences thereof.

We start by restricting our considerations to spinors u satisfying

$$\gamma^0 u = \pm u \tag{3.11}$$

and $|u|^2 = 1$ (recall that γ^0 is hermitian, and its eigenvalues are plus or minus one since its square is one). As γ^i anti-commutes with γ^0, it maps (± 1)–eigenspinors of γ^0 to (∓ 1)–eigenspinors; thus $\gamma^i u$ and $\gamma^0\gamma^i u$ are each orthogonal to u. It follows that on the eigenspaces of γ^0 we have

$$\langle u, Qu\rangle = \langle u, \Big(m_{(0)} + \frac{i}{2}\gamma^0\gamma^{kj} J_{(k)(j)}\Big) u\rangle\ .$$

As the matrix $J_{(k)(j)}$ is anti-symmetric, there exists a ON-frame in which $J_{(k)(j)}$ is block-diagonal, built out of two-by-two blocks of the form

$$\begin{bmatrix} 0 & \omega_{(i)} \\ -\omega_{(i)} & 0 \end{bmatrix}\ ,$$

with furthermore a last column of zeros in odd space-dimension. Thus for $n = 3$ we have

$$\frac{1}{2} i\gamma^0\gamma^{kj} J_{(k)(j)} = \omega_{(1)} i\gamma^0\gamma^1\gamma^2\ , \tag{3.12}$$

while in higher dimensions $2l \le n \le 2l + 1$ we can write

$$\frac{1}{2} i\gamma^0\gamma^{kj} J_{(k)(j)} = \omega_{(1)} i\gamma^0\gamma^1\gamma^2 + \omega_{(2)} i\gamma^0\gamma^3\gamma^4 + \ldots + \omega_{(l)} i\gamma^0\gamma^{2l-1}\gamma^{2l}\ . \tag{3.13}$$

The matrices $i\gamma^0\gamma^{2k-1}\gamma^{2k}$ are hermitian, with square one, therefore their eigenvalues are plus or minus one. We will need the following:

Lemma 3.4 *For every collection $\{\epsilon_a\}_{a=0,\ldots,l}$, with $\epsilon_a^2 = 1$, after performing a doubling of $\mathfrak{S}$ if necessary as in (2.11), there exists u satisfying $\gamma^0 u = \epsilon_0 u$ and*

$$\forall a \qquad i\gamma^0\gamma^{2a-1}\gamma^{2a} u = \epsilon_a u\ .$$

REMARK 3.5 The result is wrong without the doubling in general, which can be seen by taking $n = 2$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^2$, and $\gamma^0 = \sigma^3$, where the σ^i's are the usual two-by-two Pauli matrices.

PROOF: The matrix $i\gamma^{2l-1}\gamma^{2l}$ is hermitian, with square one, therefore its eigenvalues are plus or minus one. The matrix γ^{2l} defines a bijection between the $(+1)$–eigenspace and the (-1)–eigenspace, so that each of those spaces is non-empty. Let X_l denote the $\epsilon_0\epsilon_l$–eigenspace of $i\gamma^{2l-1}\gamma^{2l}$. For $0 \leq \mu \leq 2l-2$ the matrices γ^μ commute with $i\gamma^{2l-1}\gamma^{2l}$, which implies that X_l is invariant under their action. For $l \geq 3$ we repeat this construction to obtain a subspace $X_{l-1} \subset X_l$ on which $i\gamma^{2l-3}\gamma^{2l-2} = \epsilon_0\epsilon_{l-1}$. After l steps we obtain a space $X_0 \subset X_1 \subset \ldots \subset X_l$ which is invariant under γ^0. If there exists a spinor u in X_0 such that $\gamma^0 u = \epsilon_0 u$, the result immediately follows. Otherwise we double $\mathfrak{S}$ as in (2.11), we take $\hat{u}$ to be any non-zero element of X_0, and we set $u = (\hat{u}, \epsilon_0\hat{u})$. □

Let u be given by Lemma 3.4 with $\epsilon_a = -\text{sgn}\omega_{(a)}$. We obtain

$$0 \leq \langle u, Qu\rangle = \Big(m_{(0)} - |\omega_{(1)}| - \ldots - |\omega_{(l)}|\Big)|u|^2 ,$$

proving that

$$m_{(0)} \geq |\omega_{(1)}| + \ldots + |\omega_{(l)}| ,$$

In particular $m_{(0)}$ is non-negative. Since conformal transformations of the sphere at infinity induce Lorentz transformations of $m_{(\mu)}$ we obtain that $m_{(\mu)}$ is causal future directed, or vanishes. Equality implies that the boundary integral in the Witten identity vanishes, and the volume integral shows that u is an imaginary Killing spinor (on $\mathscr{S}$) for the modified connection (2.13).

If $m_{(\mu)}$ is timelike we clearly also have

$$m \geq |\omega_{(1)}| + \ldots + |\omega_{(l)}| , \tag{3.14}$$

where

$$m := \sqrt{|\eta^{(\mu)(\nu)}m_{(\mu)}m_{(\nu)}|} ,$$

and the $\omega_{(i)}$'s in (3.14) are the angular momenta in a Lorentz frame in which $m_{(\mu)}$ is aligned along the time axis.

Still assuming timelikeness of $M := (m_{(\mu)})$, and choosing an ON frame in which M is aligned along $e_{(0)}$, we now drop the condition (3.11) and assume that $n = 3$. We retain (3.12), and make a rotation in the $\{e_1, e_2\}$ plane so that $c_{(2)} = 0$. Since the hermitian matrices $\gamma^0\gamma^1$ and $i\gamma^0\gamma^1\gamma^2$ commute, and square to one, we can choose u_1 such that $|u_1|^2 = 1$ and, replacing γ^1 by $-\gamma^1$ and γ^2 by $-\gamma^2$ if necessary,

$$i\gamma^0\gamma^1\gamma^2 u_1 = u_1 , \quad \gamma^0\gamma^1 u_1 = u_1 .$$

Set

$$u_2 := \gamma^0\gamma^3 u_1 , \quad u_3 := \gamma^0\gamma^2 u_1 , \quad u_4 := \gamma^0\gamma^3 u_3 = -\gamma^3\gamma^2 u_1 . \tag{3.15}$$

From the Clifford relations one easily finds that

$$\begin{pmatrix} Qu_1 \\ Qu_2 \\ Qu_3 \\ Qu_4 \end{pmatrix} = \begin{pmatrix} m + (c_{(1)} + \omega_{(1)}) & c_{(3)} & 0 & 0 \\ c_{(3)} & m - (c_{(1)} + \omega_{(1)}) & 0 & 0 \\ 0 & 0 & m + (-c_{(1)} + \omega_{(1)}) & c_{(3)} \\ 0 & 0 & c_{(3)} & m - (-c_{(1)} + \omega_{(1)}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

One can further check that the u_i's form an ON basis as follows: u_1 is orthogonal to u_2 because both are eigenvectors of the hermitian matrix $i\gamma^0\gamma^1\gamma^2$ with different eigenvalues. (This can be verified by inspecting the sign in front of $\omega_{(1)}$ in the matrix above.) For the same reason u_1 is orthogonal to u_4, and u_2 is orthogonal to u_3. It remains to justify orthogonality of the elements of the pair (u_1, u_3), similarly for (u_2, u_4). Those follow from the fact that the first spinor in each of those pairs is an eigenvector of $\gamma^0\gamma^1$ with an eigenvalue different from the second one in the pair.[12]

Thus, the u_i's form an ON basis of Vect$\{u_i\}$, so that the positivity properties of Q, when restricted to this subspace, can be read off by calculating the eigenvalues of the matrix above. Those are easily found to be

$$m \pm \sqrt{(c_{(1)} \pm \omega_{(1)})^2 + c_{(3)}^2}\ .$$

In particular we have rederived the property that Q is positive if and only if Maerten's inequality (3.2) holds. Furthermore, there will be at least two linearly independent imaginary Killing spinors if and only if the kernel of Q is at least two-dimensional. Under the current hypotheses, and assuming an irreducible representation of the Clifford algebra, this will happen if and only if

$$c_{(1)}\omega_{(1)} = 0 \quad \Longleftrightarrow \quad J_{(i)(j)}c^j = 0 \quad \Longleftrightarrow \quad \vec{j} \times \vec{c} = 0\ . \tag{3.16}$$

Theorem 3.8 below shows that, under suitable regularity conditions, this is only possible for initial data sets which arise from hypersurfaces in anti-de Sitter space-time.

We now return to general dimension, dropping the assumption that $m_{(\mu)}$ is timelike. We use spinors obtained by the "doubling" technique as in (2.11); it then follows that the matrix Q has the following block structure:

$$Q = \begin{pmatrix} m_{(0)} + i\gamma^k m_{(k)} & \underbrace{-\gamma^k c_{(k)} + \frac{i}{2} J_{(k)(l)}\gamma^k\gamma^l}_{=:B} \\ \gamma^k c_{(k)} + \frac{i}{2} J_{(k)(l)}\gamma^k\gamma^l & m_{(0)} - i\gamma^k m_{(k)} \end{pmatrix} \tag{3.17}$$

(Positivity of Q when restricted to spinors of the form $(u, 0)$ gives immediately that $m_{(\mu)}$ is causal future pointing, which we already know.)

Suppose that $m_{(\mu)}$ is null, then there exists a $(\frac{1}{2}\dim \mathfrak{S})$–dimensional space of $u \in \mathfrak{S}$ such that $(m_{(0)} + i\gamma^k m_{(k)})u = 0$. Likewise there exists a $(\frac{1}{2}\dim \mathfrak{S})$–dimensional space of $v \in \mathfrak{S}$ such that $(m_{(0)} - i\gamma^k m_{(k)})v = 0$. Applying Q to a

[12] If one uses a space of spinors which carries an irreducible representation of the Clifford algebra, than the above matrix describes Q completely. Otherwise one can, using descending induction, find an ON basis in which Q is block-diagonal, with blocks as above.

pair $(u, \lambda v)$, where $\lambda \in \mathbb{C}$, with such an u and v, we obtain

$$0 \le \langle (u,\lambda v), Q(u,\lambda v)\rangle = 2\Re\Big(\langle u, \lambda B v\rangle\Big) .$$

Since λ is arbitrary we conclude that $\langle (u,v), Q(u,v)\rangle = 0$. Thus, the space of pairs (u,v) which lead to a zero Hamiltonian charge H equals at least $\frac{1}{2}\dim \mathfrak{S} + \frac{1}{2}\dim\mathfrak{S} = \dim\mathfrak{S}$. Witten's identity shows that each such u leads to an imaginary ∇-Killing spinor of $(\mathscr{S}, g, K)$, section of $\mathfrak{S}\oplus\mathfrak{S}$.

In space-dimension three this implies that there exist at least two linearly-independent imaginary Killing spinors, and the property that the initial data set arises from one in anti-de Sitter space-time follows from Theorem 3.8.

Suppose, next, that $m_{(\mu)}$ is timelike, and let us use a conformal frame in which $m_{(k)} = 0$. Using a spinor of the form (u, iu) one obtains instead

$$0 \le \langle (u,iu), Q(u,iu)\rangle = 2\Re\Big(\langle u, (m_{(0)} + iB)u\rangle\Big) = 2\langle u, (m_{(0)} - i\gamma^k c_{(k)})u\rangle$$

for all u, proving (3.6).

We describe now an attempt to obtain a simple form of Q in higher dimensions. While part of the calculation that follows can be done in any dimension, we have only been able to carry it out completely in dimensions $4+1$ and $5+1$. We assume that $m_{(k)}$ is timelike, and we use an ON frame adapted to $m_{(k)}$ in which (3.13) holds. In each plane $\mathrm{Vect}\{e_{2j-1}, e_{2j}\}$ we further make a rotation so that $c_{(2j)} = 0$. Let l be such that $2l \le n \le 2l+1$, for $1 \le j \le l$ set

$$B_j := i\gamma^0\gamma^{2j-1}\gamma^{2j} , \quad A_j = \gamma^0\gamma^{2j-1} ,$$

then the A_i's and B_i's are hermitian, with square one, and satisfy the commutation relations

$$B_iB_j = B_jB_i , \quad B_iA_j = \begin{cases} -A_jB_i, & i \ne j; \\ A_jB_i, & i = j, \end{cases} , \quad A_iA_j = \begin{cases} -A_jA_i, & i \ne j; \\ A_jA_i, & i = j. \end{cases} \tag{3.18}$$

Changing some of the γ^k's to $-\gamma^k$'s if necessary, we can find a spinor u such that

$$\forall i \quad B_i u = u .$$

Setting

$$u_i := A_i u ,$$

one easily obtains the $B_j u_i$'s using (3.18):

$$B_j u_i = \begin{cases} -u_i, & i \ne j; \\ u_i, & i = j. \end{cases}$$

For $n = 6$ we can enlarge $\mathrm{Vect}\{u_0 := u, u_1, u_2, u_3\}$ to a space which is invariant under the action of the A_i's by adding, to the generating family $\{u_\mu\}$, the spinors $u_4 := A_1A_2u$, $u_5 := A_1A_3u$, $u_6 := A_2A_3u$, and $u_7 := A_1A_2A_3u$. It is then easy to work out the matrix of Q in that basis (by considerations similar to the ones after (3.15) one checks that the u_μ's form an orthonormal family);

we only report the result for $\omega_{(3)} = 0$; for typesetting reasons we write b_i for $\omega_{(i)}$ and a_i for $c_{(2i-1)}$:

$$\begin{bmatrix} m+b_1+b_2 & a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ a_1 & m+b_1-b_2 & 0 & 0 & -a_2 & -a_3 & 0 & 0 \\ a_2 & 0 & m-b_1+b_2 & 0 & a_1 & 0 & -a_3 & 0 \\ a_3 & 0 & 0 & m-b_1-b_2 & 0 & a_1 & a_2 & 0 \\ 0 & -a_2 & a_1 & 0 & m-b_1-b_2 & 0 & 0 & a_3 \\ 0 & -a_3 & 0 & a_1 & 0 & m-b_1+b_2 & 0 & -a_2 \\ 0 & 0 & -a_3 & a_2 & 0 & 0 & m+b_1-b_2 & a_1 \\ 0 & 0 & 0 & 0 & a_3 & -a_2 & a_1 & m+b_1+b_2 \end{bmatrix}$$

One can use Maple or Mathematica to compute the eigenvalues of Q without assuming $\omega_{(3)} = 0$, but this does not lead to useful expressions. However, suppose that $n = 5$; after embedding the five-dimensional Clifford algebra into a six dimensional one, this form of Q still holds in the basis above. A Maple calculation shows then that the eigenvalues of Q on this subspace all have multiplicity two, and are equal to

$$m\pm\sqrt{c_{(1)}^2 + c_{(3)}^2 + c_{(5)}^2 + \omega_{(1)}^2 + \omega_{(2)}^2 \pm 2\sqrt{(\omega_{(1)}c_{(1)})^2 + (\omega_{(2)}c_{(3)})^2 + (\omega_{(1)}\omega_{(2)})^2}}\ .$$

This gives (3.7).

Specialising further to $c_{(5)} = 0$, a similar argument gives the inequality for $n = 4$.

It remains to prove point 5. Suppose that $m_{(0)}$ vanishes, then $m_{(k)} = 0$ by causality of $m_{(\mu)}$, further $J_{(k)(\ell)}$ vanishes by (3.4), Applying (3.17) to spinors of the form $(u, \pm v)$, positivity of Q implies $c_{(k)} = 0$. Thus Q vanishes, and one could quote now [35, Theorem 1.4], except for the following: the proof there requires geodesic completeness of the Killing development of $(\mathscr{S}, g, K)$, which is justified in [35] by invoking [1, Lemma 1.1]. However, that last lemma is incorrect,[13] so we give a proof of geodesic completeness under the current hypotheses here.

Now, the arguments of the proof of Theorem 3.6 show that there exists a Killing development $(\mathscr{M} \approx \mathbb{R} \times \mathscr{S}, {}^{n+1}g)$ of the initial data with a globally timelike Killing vector field X. Set

$$e^{\mu} := {}^{n+1}g(X, X)\ ;$$

by the asymptotic conditions μ tends to infinity as one recedes to infinity on $\mathscr{S}_{\text{ext}}$. By hypothesis $\mathscr{S}$ is conformally compactifiable, so that μ is bounded from below.

[13] A counter-example is given by the domain of outer communications of an extreme Reissner-Nordström black hole.

Let us write

$$
{}^{n+1}g = -e^{\mu}(dt + \underbrace{\theta_i dx^i}_{=:\theta})^2 + h \;,
$$

where h is a Riemannian metric on $\mathscr{S}$. By the asymptotic conditions (2.1) $e^{\mu/2}|\theta|_g$ approaches zero in (each of) the asymptotic regions, hence h asymptotes to b, and since $\mathscr{S}$ is a union of a compact set and a finite number of compactifiable ends we conclude that $(\mathscr{S}, h)$ is a complete manifold.

Let $\Gamma(s) = (t(s), \lambda(s))$ be an affinely parameterised maximally extended geodesic in $(\mathscr{M}, {}^{n+1}g)$, set

$$
\epsilon := {}^{n+1}g(\dot\Gamma, \dot\Gamma) \;, \quad p := {}^{n+1}g(\dot\Gamma, X) = -e^{\mu}(\dot t + \theta(\dot\lambda)) \;,
$$

thus ϵ and p are constant along Γ. Hence

$$
h(\dot\lambda, \dot\lambda) = \epsilon + e^{-\mu}p^2 \le C
$$

for some constant C, and then

$$
|\dot t| = |e^{-\mu}p - \theta(\dot\lambda)| \le C' \;,
$$

for some other constant C'. This implies that for any bounded interval $I \subset \mathbb{R}$ the closure $\overline{\Gamma(I)}$ of the image $\Gamma(I) \subset \mathscr{M}$ is compact, and completeness of $(\mathscr{M}, {}^{n+1}g)$ readily follows. □

3.1 Impossibility of null energy-momentum when $n = 3$

Under the hypotheses of Theorem 2.1, equality in (3.2) leads to the existence of imaginary ∇-Killing spinors on $\mathscr{S}$. We have the following:

THEOREM 3.6 *Let* $\dim \mathscr{S} = 3$, *and suppose that* $(\mathscr{S}, g, K)$ *admits a non-trivial imaginary Killing spinor for the connection (2.13). Then:*

1. *The Killing development of* $(\mathscr{S}, g, K)$ *admits an imaginary Killing spinor.*
2. *If there are two linearly independent such spinors on* $\mathscr{S}$, *then the Killing development of* $(\mathscr{S}, g, K)$ *is vacuum and has vanishing Weyl tensor.*

REMARK 3.7 In higher dimensions, the minimal number of Killing spinors which enforces the vanishing of the Weyl tensor does not appear to be known. For example, consider a five-dimensional Lorentzian Einstein-Sasaki manifold (all regular types can be constructed as S^1-bundles over Kähler-Einstein manifolds with negative scalar curvature, see [8, 11, 31]). A Lorentzian Einstein-Sasaki space is not conformally flat and has (if it is simply connected) at least two linearly independent imaginary Killing spinors.[14] In those examples we can choose $\dim \mathfrak{S} = 4$, leading to dimension four of the space of doubled imaginary Killing spinors in point (2) of Theorem 2.1. Restricting to an irreducible subrepresentation of the Clifford algebra will presumably lead to a two-dimensional

[14]We are grateful to Helga Baum for those remarks.

space, so that our constraints on a null $m_{(\mu)}$ do not exclude such non-trivial geometries. In fact, five-dimensional examples with a two-dimensional space of imaginary Killing spinors can be found within the family described in Section 4.4, with a toroidal Scri; but note that these do not have a null $m_{(\mu)}$.

Before proving Theorem 3.6 we note the following corollary thereof:

THEOREM 3.8 *Under the hypotheses of Theorem 2.1 let* $\dim \mathscr{S} = 3$, *assume that* $\mathscr{S}$ *is conformally compactifiable and suppose that* $(\mathscr{S}, g, K)$ *admits two linearly independent imaginary Killing spinors for the connection (2.13). If the conformal boundary at infinity has spherical topology, then the initial data set can be isometrically embedded into anti-de Sitter space-time. More generally, the conclusion remains valid for the universal cover of the initial data set.*

REMARK 3.9 In view of Section 3.3 below, Theorem 3.8 is of interest only for solutions without "black hole boundaries" (as defined by the hypotheses of Theorem 2.1).

PROOF OF THEOREM 3.8: Theorem 3.6 shows that the Weyl tensor vanishes, so that [27] the Witten boundary term is identically zero. Hence the matrix Q vanishes, and the result follows from point 5 of Theorem 2.1. □

PROOF OF THEOREM 3.6: The method will be to show that existence of *space* imaginary Killing spinors for (2.13),

$$\widehat{\nabla}_X \psi \equiv D_X \psi + \frac{1}{2} K(X) \cdot \gamma^0 \psi + i \sqrt{\frac{-\Lambda}{2n(n-1)}} X \cdot \psi = 0 \, , \quad X \in T\mathscr{S} \, , \quad (3.19)$$

implies that of *space-time* imaginary Killing spinors in the Killing development of $(\mathscr{S}, g, K)$. We begin with some generalities. Our convention will be to use Greek indices for space-time, preserving Latin indices for some of the lower dimensional situations which follow; two-component spinor indices will be capital Latin indices as usual.

We will prove the result using Dirac spinors on $\mathscr{S}$, and especially their decomposition into two component spinors. A space-time imaginary Killing spinor ψ can then be represented by a pair of spinor fields $(\alpha_A, \beta_{A'})$ satisfying the following coupled system of equations (compare [36, Section 2]):

$$\begin{aligned} \nabla_{AA'} \alpha_B &= b \epsilon_{AB} \beta_{A'} \, , \\ \nabla_{AA'} \beta_{B'} &= b \epsilon_{A'B'} \alpha_A \, , \end{aligned} \quad (3.20)$$

where b is a constant (not to be confused with the background metric of Section 2), which without loss of generality may be assumed real, and is then related to the cosmological constant by $\Lambda = -6b^2$.

Saturation of (3.2) implies that the data $(\mathscr{S}, g, K)$ admits a spinor field ψ satisfying the projection into $\mathscr{S}$ of (3.20), say

$$\Pi^\alpha_\gamma S_\alpha = 0, \quad (3.21)$$

where Π^α_γ is the tensor projecting tangentially to $\mathscr{S}$ and S_α stands for:

$$S_\alpha := \begin{pmatrix} \nabla_{AA'}\alpha_B - b\epsilon_{AB}\beta_{A'} \\ \nabla_{AA'}\beta_{B'} - b\epsilon_{A'B'}\alpha_A \end{pmatrix} . \tag{3.22}$$

Given a solution $(\alpha_A, \beta_{A'})$ of (3.20), another solution is provided by $(\overline{\beta}_A, \overline{\alpha}_{A'})$. The two solutions are linearly independent unless α_A and $\overline{\beta}_A$ are proportional, say $\alpha_A = f\overline{\beta}_A$ for some function f. In this case, it follows from (3.20) that f is a complex constant, of modulus one, and it can then be absorbed into a redefinition of $\beta_{A'}$. Thus, given a solution of (3.20), we necessarily have at least a two-dimensional space of solutions unless we have a solution of

$$\nabla_{AA'}o_B = b\epsilon_{AB}\overline{o}_{A'}. \tag{3.23}$$

For reasons which will appear, we shall call this the null case, while a solution of (3.20) not of this form we call the non-null case.

Assuming that the full (as opposed to (3.21)) system (3.20) holds, by commuting derivatives one finds

$$\psi_{ABCD}\alpha^D = 0 = \overline{\psi}_{A'B'C'D'}\beta^{D'}$$

$$\phi_{ABA'B'}\alpha^B = 0 = \phi_{ABA'B'}\beta^{B'}$$

where ψ_{ABCD} is the Weyl spinor, the spinor representing the Weyl tensor, and $\phi_{ABA'B'}$ is the Ricci spinor, representing the trace-free part of the Ricci tensor. In the non-null case, when α_A and $\overline{\beta}_A$ are linearly independent, it follows from this that the Weyl and trace-free Ricci tensors both vanish and the space-time is locally anti-de Sitter.

For non-trivial examples, therefore, we need to be in the null case. From (3.23) by differentiating again we obtain

$$\psi_{ABCD}o^D = 0 = \phi_{ABA'B'}o^B,$$

so that

$$\psi_{ABCD} = \Psi o_A o_B o_C o_D \quad , \phi_{ABA'B'} = \Phi o_A o_B \overline{o}_{A'} \overline{o}_{B'},$$

for complex functions Ψ and Φ.

Even in the null case, if there are two linearly-independent such solutions, we shall again have just anti-de Sitter space (since ψ_{ABCD} and $\phi_{ABA'B'}$ cannot take this form for two independent spinors).

We return now to (3.21). Suppose first that we are in the non-null case. The vector K^α constructed according to

$$K^\alpha = \alpha^A\overline{\alpha}^{A'} + \overline{\beta}^A\beta^{A'} , \tag{3.24}$$

will give [35] 'Killing Initial Data' at $\mathscr{S}$. In the Killing development of $(\mathscr{S}, g, K)$, K^α will be a future-pointing, time-like Killing vector. From (3.24) we see, at $\mathscr{S}$,

$$K^\alpha K_\alpha = 2V\overline{V}, \tag{3.25}$$

where

$$V = \alpha_A \overline{\beta}^A .$$

By (3.21) we have

$$\Pi^\alpha_\mu \nabla_\alpha V = \Pi^\alpha_\mu b(\alpha_A \overline{\alpha}_{A'} - \overline{\beta}_A \beta_{A'}), \tag{3.26}$$

which is real so that the imaginary part of V, say I, is necessarily a constant along $\mathscr{S}$.

Recall that the Lie-derivative of a spinor field α_A along a Killing vector X^a is defined as

$$\mathscr{L}_X \alpha_A := X^\mu \nabla_\mu \alpha_A + \Phi_A{}^M \alpha_M \tag{3.27}$$

where the symmetric spinor Φ_{MN} is defined by

$$\nabla_\mu X_\nu = \Phi_{MN}\epsilon_{M'N'} + \overline{\Phi}_{M'N'}\epsilon_{MN} \ , \tag{3.28}$$

see e.g. [29, p. 40]. For K^α, from (3.24) and (3.21) we find, at $\mathscr{S}$,

$$\Pi^\mu_\alpha \nabla_\mu K_\beta = 2b\Pi^\mu_\alpha(\alpha_{(M}\overline{\beta}_{B)}\epsilon_{M'B'} + \overline{\alpha}_{(M'}\beta_{B')}\epsilon_{MB}),$$

so that, at $\mathscr{S}$ in the Killing development,

$$\nabla_\alpha K_\beta = 2b(\alpha_{(A}\overline{\beta}_{B)}\epsilon_{A'B'} + \overline{\alpha}_{(A'}\beta_{B')}\epsilon_{AB}) + n_\alpha v_\beta,$$

for some vector field v_β where n_α is the (unit, time-like) normal to $\mathscr{S}$. Symmetrising over the indices α and β the left-hand-side vanishes, thus so does the right-hand-side, which implies $v_\alpha = 0$. Thus the derivative at $\mathscr{S}$ of K is

$$\nabla_\alpha K_\beta = 2b(\alpha_{(A}\overline{\beta}_{B)}\epsilon_{A'B'} + \overline{\alpha}_{(A'}\beta_{B')}\epsilon_{AB}) \ , \tag{3.29}$$

so that

$$\Phi_{AB} = 2b\alpha_{(A}\overline{\beta}_{B)} \ .$$

We impose

$$\mathscr{L}_K \alpha_A - 2ibI\alpha_A = 0 = \mathscr{L}_K \beta_{A'} - 2ibI\beta_{A'} \tag{3.30}$$

in the Killing development, with α_A and $\beta_{A'}$ known on $\mathscr{S}$ and I the value of the (constant) imaginary part of V at $\mathscr{S}$. This determines the spinors throughout the Killing development. Note also that now $\mathscr{L}_K V = 0$ in the Killing development, so that I equals $\Im V$ throughout. Furthermore, it follows that

$$\mathscr{L}_K(\alpha_{(A}\overline{\beta}_{B)}) = 0,$$

so that the Lie derivative along K of both sides of (3.29) vanishes, and therefore this equation holds throughout the Killing development.

From (3.30), (3.29) and (3.27), we now have

$$K^\alpha S_\alpha = 0$$

with S_α as in (3.22), and from (3.30)

$$\mathscr{L}_K S_\alpha = 2ibIS_\alpha.$$

Since K^α is transversal to $\mathscr{S}$, this with (3.21) gives $S_\alpha = 0$ at $\mathscr{S}$, and therefore throughout the Killing development. Now we have a solution of (3.20) in the Killing development, which is therefore locally anti-de Sitter.

The null case is very similar: now we have a solution of

$$\Pi^\alpha_\gamma S_\alpha = 0, \tag{3.31}$$

where this time S_α stands for

$$S_\alpha := \nabla_{AA'} o_B - b\epsilon_{AB}\overline{o}_{B'}. \tag{3.32}$$

We define the Killing vector by

$$K^\alpha = o^A \overline{o}^{A'}. \tag{3.33}$$

This is a future-pointing null vector (which is why we called this the null case). Since V is now zero, (3.30) becomes

$$\mathscr{L}_K o_A = 0$$

and we proceed as before.

For the derivative of K we find in this case

$$\nabla_\alpha K_\beta = b o_A o_B \epsilon_{A'B'} + b\overline{o}_{A'}\overline{o}_{B'}\epsilon_{AB}. \tag{3.34}$$

It follows that

$$K^\alpha S_\alpha = 0 \tag{3.35}$$

at $\mathscr{S}$ but now $\mathscr{L}_K S_\alpha = 0$. We conclude as required that S_α vanishes in the Killing development.

If we have a second independent solution of (3.31) at $\mathscr{S}$, say ι_A, then we set

$$\alpha_A = o_A + i\iota_A\ , \quad \beta_{A'} = \overline{o}_{A'} + i\overline{\iota}_{A'}\ ,$$

and we are back in the non-null case, so that the Killing development is again anti-de Sitter. □

For nontrivial examples therefore we need just a one-dimensional family of solutions of (3.23). Metrics with this property will be briefly described in Section 3.2. We will see in Section 3.3 that, subject to some natural restrictions, such examples do not include black hole solutions. In fact, we shall see in Section 3.4 that such examples are not possible at all in three space dimensions if we further assume that $\mathscr{I}$ has spherical cross-sections and is "large enough".

3.2 Siklos waves

The "Lobatchevski plane waves" of Siklos [37], which we propose to call Siklos waves, are precisely characterised by the existence of a spinor satisfying (3.23). Siklos shows that it is possible to introduce coordinates so that the metric may be written as

$$g = \frac{1}{2b^2x^2}(dx^2 + dy^2 - 2dudv - H(u,x,y)du^2)\ . \tag{3.36}$$

Here $K = \partial/\partial v$. (The signature of (3.36) is reversed as compared to [37].) The Weyl and Ricci spinors are

$$\phi_{ABA'B'} = \Phi o_A o_B \overline{o}_{A'} \overline{o}_{B'} , \tag{3.37}$$
$$\psi_{ABCD} = \Psi o_A o_B o_C o_D , \tag{3.38}$$

where Φ and Ψ are given in terms of H by[15]

$$\Phi = -b^4 x^4 (H_{xx} + H_{yy} - 2H_x/x) . \tag{3.39}$$
$$\Psi = -b^4 x^4 (H_{xx} - H_{yy} - 2iH_{xy}) , \tag{3.40}$$

The cosmological constant is $\Lambda = -6b^2$ (this is not the Λ of the Newman-Penrose formalism which would be $24b^2$).

The Killing vector K Lie drags the Weyl spinor (since it is a symmetry) and the Lie derivative defined by (3.27) commutes with contractions and tensor products, so that from (3.38)

$$K^\alpha \partial_\alpha \Psi = 0. \tag{3.41}$$

If H is zero, then (3.36) is the metric of anti-de Sitter space with $\mathscr{I}$ at $x = 0$. A variety of other choices for the function H also lead to anti-de Sitter space, in particular a constant, say $H = H_0$, as the coordinate transformation

$$dv \to dV = dv + \frac{1}{2} H_0 du$$

demonstrates.

3.3 Non-existence of black hole solutions saturating the equality, $n = 3$

Whatever the dimension $n \geq 3$, there exist higher-genus Kottler black hole space-times with zero Hamiltonian mass. One could naively think of those as saturating our positivity bounds. However, it should be borne in mind that, for reasons already explained, those solutions (as well as any solutions with the same asymptotic behavior) do not possess imaginary Killing spinors, so our inequality does not apply.

We wish to show, under a natural supplementary assumption, non-existence of $(3 + 1)$-dimensional *black hole* solutions, with toroidal conformal infinity, *saturating* the angular momentum bound. To be precise, in addition to the hypotheses of the positivity theorem 2.1, we will assume that $\mathscr{S}$ is the union of an asymptotically hyperbolic region $\mathscr{S}_{\rm ext}$ and of a compact set, with non-empty smooth boundary. Moreover, we suppose that the space-time $(\mathscr{M}, {}^4g)$ is *not* conformally flat. The hypothesis that the bound is saturated implies existence of a Killing spinor, and thus also of the associated Killing vector which we call K, which must be null by the analysis of Section 3.1. The hypothesis of existence of a black hole will be encoded in the assumption that the boundary of $\mathscr{S}$, when

[15]The multiplicative factor 1/16 in the equation for Φ_{22} in [37, p. 254] should be 1/4.

moved by the flow of the Killing vector field K, forms[16] a *null hypersurface* $\mathscr{H}$. Then K is necessarily tangent to the generators of $\mathscr{H}$, with zero surface gravity since K is null everywhere. It follows from (3.34) that the solution is static in the sense that $K^\flat \wedge dK^\flat = 0$, where $K^\flat = {}^4g(K,\cdot)$. In vacuum this implies [17] that the horizon has higher genus topology, contradicting [20, Theorem 4.1], and thus proving our claim. However, the reader will easily check that the hypothesis that the space-time is vacuum plays no role in this argument, because the energy-momentum tensor of the space-time metric is proportional to $K\otimes K$, and such a tensor does not affect those equations in [17] which are relevant to the problem at hand.

3.4 Rigidity in the $n = 3$ spherical case

In Section 3.1 we have shown that a null $m_{(\mu)}$ cannot occur. In this section we wish to show that the remaining possibilities for equality in (3.2) only occur in anti-de Sitter space-time, under the supplementary condition[17] that the initial data set arises from a space-time with a conformal completion at infinity which is "sufficiently large in time". By this we mean that the interval of the t–coordinate below has length at least π.

A (four-dimensional) space-time $(\mathscr{M},\ {}^4g)$ is said to be asymptotically-anti-de Sitter if it is smoothly conformal to a manifold $\widetilde{\mathscr{M}}$ with boundary $\partial\widetilde{\mathscr{M}} \equiv \mathscr{I} \approx \mathbb{R}\times S^2$, with the usual condition that the conformal factor Ω, relating the metrics as ${}^4g = \Omega^{-2}\,{}^4\tilde g$, vanishes on $\mathscr{I}$ precisely at order one. It is further assumed that the restriction of ${}^4\tilde g$ to the conformal boundary at infinity equals

$$\mathring{h}_{ij}dx^idx^j = d\theta^2 + \sin^2\theta d\phi^2 - dt^2. \tag{3.42}$$

It is then possible to introduce[18] coordinates (R, x^i) for $i = 1, 2, 3$ so that the space-time metric ${}^4g = g$ can be written in the form

$$g = \frac{1}{R^2}(dR^2 + h_{ij}(R, x^k)dx^idx^j) \tag{3.43}$$

with

$$h_{ij} = \mathring{h}_{ij}(x^k) + O(R^2) \tag{3.44}$$

The metric (3.36) with $2b^2 = 1$ and $H = 0$ takes this form, though the metric of anti-de Sitter space is more commonly written as

$$g = d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) - \cosh^2\psi dt^2, \tag{3.45}$$

when the substitution $R = \mathrm{e}^{-\psi}$ will cast it in the form of (3.43).

Our aim now is to show that the metric of the Siklos wave (3.36) cannot be written in the asymptotically-anti-de Sitter form (3.43) unless it is exactly

[16] Note that the level sets of u *are* null hypersurfaces generated by K, but with non-compact intersection with $\mathscr{S}$.

[17] This hypothesis can often be removed by using the Killing development. This is, unfortunately, not the case for the problem at hand because of the zeros of the Killing vector at $\mathscr{I}$.

[18] In this section, and only in this section, we use the convention that $x^0 = R$; the reader should not confuse this with a time-coordinate.

anti-de Sitter. Our technique will be, first to obtain an asymptotic form of the Killing vector K and then to show that the equation (3.41) is incompatible with (3.43) unless $\Psi = 0$.

We suppose then that X is a Killing vector for the metric (3.43) and write it in the form

$$X = A\frac{\partial}{\partial R} + B^i\frac{\partial}{\partial x^i}. \tag{3.46}$$

The Killing equation may be written as

$$X^\gamma \partial_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\partial_\beta X^\gamma + g_{\gamma\beta}\partial_\alpha X^\gamma = 0.$$

Substituting from (3.43) we obtain for the (00) component of this

$$R\frac{\partial A}{\partial R} - A = 0$$

so that

$$A = RV(x^i) \tag{3.47}$$

for some $V(x^i)$, to be found. For the $(0i)$ components we find

$$\frac{\partial A}{\partial x^i} + h_{ij}\frac{\partial B^j}{\partial R} = 0,$$

so that

$$B^j(R, x^k) = B_0^j(x^k) + O(R^2). \tag{3.48}$$

Finally, for the (ij) components we find

$$\mathscr{L}_B h_{ij} = 2Vh_{ij} - RV\frac{\partial}{\partial R}h_{ij}.$$

The leading term in this equation, with what we have already, requires

$$\mathscr{L}_{B_0}\mathring{h}_{ij} = 2V\mathring{h}_{ij}\ . \tag{3.49}$$

We are interested in null Killing vectors, so that by (3.46)

$$g(K, K) := \frac{1}{R^2}(A^2 + h_{ij}B^iB^j) = 0,$$

which implies in particular that

$$\mathring{h}_{ij}B_0^iB_0^j = 0. \tag{3.50}$$

Thus B_0 is a null conformal Killing vector on $\mathscr{I}$, and our next task is to find these. We proceed as before, by setting

$$B_0^i\frac{\partial}{\partial x^i} = \beta\frac{\partial}{\partial t} + A^a\frac{\partial}{\partial y^a} \tag{3.51}$$

where $a = 2, 3$ and $(y^2, y^3) = (\theta, \phi)$. This is to be a conformal Killing vector of the metric (3.42) which we write as

$$\mathring{h}_{ij}dx^idx^j = \eta_{ab}dy^ady^b - dt^2.$$

The conformal Killing equation (3.49) may be written in the form

$$B_0^k \partial_k \mathring{h}_{ij} + \mathring{h}_{ik} \partial_j B_0^k + \mathring{h}_{kj} \partial_i B_0^k = 2V \mathring{h}_{ij}$$

from which, as before, we obtain the system of equations

$$\frac{\partial \beta}{\partial t} = V \tag{3.52}$$

$$\frac{\partial \beta}{\partial y^a} = \eta_{ab} \frac{\partial A^b}{\partial t} \tag{3.53}$$

$$\mathscr{L}_A \eta_{ab} = 2V \eta_{ab}. \tag{3.54}$$

To solve these, we need to know some facts about conformal Killing vectors on S^2 (which, by (3.54), $A = A^a \partial_a$ is). The general solution of (3.54) is of the form:

$$A^a = Z^a - \eta^{ab} \frac{\partial \alpha}{\partial y^b} \tag{3.55}$$

where Z^a is a Killing vector for η and α is a conformal scalar, by which we mean a solution of the equation:

$$\mathscr{D}_a \mathscr{D}_b \alpha = -\alpha \eta_{ab} \tag{3.56}$$

where $\mathscr{D}_a$ is the Levi-Civita covariant derivative for η. Thus, from (3.54), $V = \alpha$. Next, it now follows from (3.53) that

$$\frac{\partial}{\partial y^a}(\beta + \frac{\partial \alpha}{\partial t}) = \eta_{ab} \frac{\partial Z^b}{\partial t}.$$

Taking the divergence of this we find that

$$\frac{\partial Z^a}{\partial t} = 0; \quad \beta = \beta_0 - \frac{\partial \alpha}{\partial t} \tag{3.57}$$

for some β_0 independent of y^a. Finally, integrating (3.52) over $\mathbf{S}^2$, and noting that α integrates to zero because of the equation $\Delta \alpha = -2\alpha$, shows that β_0 is actually constant and α satisfies

$$\frac{\partial^2 \alpha}{\partial t^2} = -\alpha \tag{3.58}$$

which is readily solved.

We may write out solutions explicitly by regarding the $\mathbf{S}^2$ as the unit sphere in $\mathbf{R^3}$ with Cartesian coordinates $\mathbf{X} = (X^\mathbf{i})$, $\mathbf{i} = 1, 2, 3$. Then α is linear in $X^\mathbf{i}$ and, taking account of (3.58), may be written in the form

$$\alpha = -(\mathbf{a} \cdot \mathbf{X}) \cos t - (\mathbf{b} \cdot \mathbf{X}) \sin t \tag{3.59}$$

in terms of a pair of constant vectors $\mathbf{a}$ and $\mathbf{b}$. By (3.57) we obtain

$$\beta = \beta_0 - (\mathbf{a} \cdot \mathbf{X}) \sin t + (\mathbf{b} \cdot \mathbf{X}) \cos t, \tag{3.60}$$

while Z is a Killing vector, so that

$$Z^a \frac{\partial}{\partial y^a} = M_{\mathbf{ij}} X^{\mathbf{i}} \frac{\partial}{\partial X^{\mathbf{j}}} \tag{3.61}$$

for a constant, antisymmetric matrix $M_{\mathbf{ij}}$ (where necessary, indices $\mathbf{i}, \mathbf{j}$ can be raised or lowered with $\delta_{\mathbf{ij}}$).

Now we must impose the condition (3.50), that B_0 is null. From (3.51) and (3.55) this is the condition

$$-\beta^2 + \eta^{ab}(Z_a - \partial_a \alpha)(Z_b - \partial_b \alpha) = 0.$$

Substituting into this from (3.59), (3.60) and (3.61), we obtain a series of algebraic equations by equating to zero coefficients of 1, $\sin t$, $\cos t$ and $\cos 2t$. These are

$$|\mathbf{a}|^2 = |\mathbf{b}|^2; \quad \mathbf{a} \cdot \mathbf{b} = 0 \tag{3.62}$$

then

$$\begin{aligned} \beta_0 a_{\mathbf{i}} + M_{\mathbf{ij}} b_{\mathbf{j}} &= 0 \\ -\beta_0 b_{\mathbf{i}} + M_{\mathbf{ij}} a_{\mathbf{j}} &= 0 \end{aligned}$$

so that

$$M_{\mathbf{ij}} = \epsilon (a_{\mathbf{i}} b_{\mathbf{j}} - a_{\mathbf{j}} b_{\mathbf{i}}) \tag{3.63}$$

for constant ϵ, and finally

$$-\beta_0^2 + M_{\mathbf{ik}} M_{\mathbf{j}k} X^{\mathbf{i}} X^{\mathbf{j}} + |\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{X})^2 - (\mathbf{b} \cdot \mathbf{X})^2 = 0\ ,$$

which implies just

$$\beta_0 = -\epsilon |\mathbf{a}|^2 \tag{3.64}$$

with $\epsilon^2 |\mathbf{a}|^2 = 1$.

We have found the general form of any null Killing vector in any asymptotically adS space-time, so that K of Section 3.2 must have this form, in any Siklos wave which is asymptotically adS. There are two families depending on the sign of ϵ and the six real parameters $(\mathbf{a}, \mathbf{b})$ subject to (3.62). Replacing the Killing spinor by a multiple thereof if necessary, we can without loss of generality assume $|\mathbf{a}| = 1$. All choices are equivalent up to rotation and the discrete symmetry $t \to t + \pi/2$. We make the choices

$$a_1 = b_2 = \epsilon = 1$$

with other terms zero, then with $\mathbf{X} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ we obtain

$$\alpha = -\sin\theta \cos(\phi - t) \tag{3.65}$$

and

$$\begin{aligned} B_0 &= (-1 + \sin\theta \sin(\phi - t)) \frac{\partial}{\partial t} + \cos\theta \cos(\phi - t) \frac{\partial}{\partial \theta} \\ &\quad + (1 - \sin(\phi - t) \csc\theta) \frac{\partial}{\partial \phi}. \end{aligned} \tag{3.66}$$

Now we have the Killing vector K, at least asymptotically, we need to solve (3.41). For this we need the asymptotic form of Ψ compatible with (3.43). We recall some of the conventions associated with conformal rescaling in asymptotically adS space-times. The unphysical metric is

$$\tilde{g}_{\alpha\beta} = R^2 g_{\alpha\beta}$$

with R and $g_{\alpha\beta}$ as in (3.43). From (3.46) and (3.47), we see that the Killing vector K extends to a smooth vector field on $\mathscr{I}$, and we have

$$\nabla_\mu K_\nu = \frac{2}{R^3}\tilde{K}_{[\mu}\partial_{\nu]}R + O(R^{-2}) , \tag{3.67}$$

where $\tilde{K}_\mu := \tilde{g}_{\mu\nu}K^\nu$; here and below $O(R^k)$ refers to components in the coordinate system (R, x^i). From (3.29), where now $b = 1/\sqrt{2}$, and (3.38), the Weyl tensor W equals

$$W = A(dK \otimes dK - (*dK) \otimes (*dK)) + B(dK \otimes (*dK) + (*dK) \otimes dK) , \tag{3.68}$$

where $*$ is the space-time Hodge-dual, while

$$\Psi = A - iB$$

and Ψ is as in (3.38) and (3.40). Let $N = R\partial_R$ be a unit normal to the level sets of R. The rescaled electric Weyl tensor, finite on $\mathscr{I}$ (see [27, Lemma 3.1]) is, by equation (2.14) of [4] and by (3.67)-(3.68)

$$\begin{aligned} \tilde{E}_{ij} &= R^{-1}W_{i\gamma j\delta}N^\gamma N^\delta \\ &= R^{-5}\left(A\tilde{K}_i\tilde{K}_j + O(\Psi R)\right) . \end{aligned}$$

Similarly the magnetic part of the Weyl tensor is

$$\begin{aligned} \tilde{B}_{ij} &= R^{-1} *W_{i\gamma j\delta}N^\gamma N^\delta \\ &= R^{-5}\left(B\tilde{K}_i\tilde{K}_j + O(\Psi R)\right) . \end{aligned}$$

It follows that $\lim_{R\to 0} R^{-5}\Psi$ exists, and

$$\lim_{R\to 0}\tilde{E}_{ij} = M\tilde{K}_i\tilde{K}_j , \tag{3.69}$$

where $M = \lim_{R\to 0} R^{-5}\Re\Psi$. Up to a multiplicative factor, M is the integrand for the asymptotically defined, Ashtekar-Magnon global charges [4, 5]. If M is zero on $\mathscr{I}$ then all global charges are zero. M is bounded on $\mathscr{I}$ away from the zeroes of K, but could be singular where K is zero.

Now from (3.41)

$$K^\alpha\partial_\alpha(MR^5) = 0 ,$$

so with K given by (3.46), (3.65) and (3.66), we need on $\mathscr{I}$ that

$$B_0^i\frac{\partial M}{\partial x^i} - 5\sin\theta\cos(\phi - t)M = 0. \tag{3.70}$$

On the equator (3.70) can be integrated to give

$$M(t,\theta=\pi/2,\phi)=(1-\sin(\phi-t))^{-5/2}f(\phi+t)\ , \tag{3.71}$$

for some function f of $\phi+t$. From (3.69) one then has

$$\tilde{E}_{tt}=(1-\sin(\phi-t))^{-1/2}f(\phi+t)\ ,$$

so f vanishes if a smooth global $\mathscr{I}$ exists.

For $\cos\theta\neq 0$ introduce F by

$$M=(\cos\theta)^{-5}F(t,\theta,\phi) \tag{3.72}$$

then (3.70) becomes

$$(-1+\sin\theta\sin(\phi-t))\frac{\partial F}{\partial t}+\cos\theta\cos(\phi-t)\frac{\partial F}{\partial\theta}+(1-\sin(\phi-t)\csc\theta)\frac{\partial F}{\partial\phi}=0.$$

i.e. F is constant on the integral curves of the vector field B_0, which we need to consider. We shall find that $\cos\theta$ is either zero or asymptotic to zero along every integral curve, and that the components of K are asymptotic to zero along every integral curve. From the rate at which these quantities vanish, it will follow from (3.69) and (3.72) that, on the curves with $\cos\theta\neq 0$, F vanishes on $\mathscr{I}$, and therefore so does M, while it will follow from (3.71) that M is zero on the curves with $\cos\theta=0$. Note that the Ashtekar-Magnon mass equals the boundary term which arises in Witten's positive energy argument by [27]. Hence, under the hypotheses of the positive energy theorem of [35], we can then conclude that the initial data set can be embedded into anti-de Sitter space-time.

The integral curves of B_0 satisfy the system of equations

$$\frac{dt}{d\lambda} = (-1+\sin\theta\sin(\phi-t)) \tag{3.73}$$

$$\frac{d\theta}{d\lambda} = \cos\theta\cos(\phi-t) \tag{3.74}$$

$$\frac{d\phi}{d\lambda} = (1-\sin(\phi-t)\csc\theta) \tag{3.75}$$

in terms of a real parameter λ along the curves.

Equating the right-hand-sides to zero, we see that the fixed points of B_0 lie on the curve Γ defined by $\theta=\pi/2$, $\phi-t=2k\pi+\pi/2$, for integer k, which is a helix on $\mathscr{I}$.

We first investigate integral curves with constant θ. Any curve on which θ is constant must, by (3.74), have $\cos\theta$ or $\cos(\phi-t)$ vanishing, but in the second case, by (3.73) and (3.75), we arrive again at $\cos\theta=0$. Thus the only integral curves with θ constant have $\theta=\pi/2$. Going further with these, we find that $\phi+t$ must be constant on them. Introduce γ by $\phi-t=2\gamma+\pi/2$ then we find an equation for γ which integrates to give

$$\cot\gamma=2(\lambda_0-\lambda).$$

for some constant λ_0, and so

$$1 - \sin(\phi - t) = 2\sin^2\gamma = 2(1 + 4(\lambda_0 - \lambda)^2)^{-1}.$$

Now suppose we have an integral curve with a point where $\cos\theta \neq 0$. It is straightforward to check that the following are constants along the integral curve:

$$a := \frac{\sin\theta\sin\phi - \cos t}{\cos\theta} \tag{3.76}$$

$$b := \frac{\sin\theta\cos\phi + \sin t}{\cos\theta} \tag{3.77}$$

and then that

$$\frac{d}{d\lambda}\left(\frac{\sin t}{\cos\theta}\right) = a$$

$$\frac{d}{d\lambda}\left(\frac{\cos t}{\cos\theta}\right) = b,$$

so that

$$\frac{\sin t}{\cos\theta} = a\lambda + c \tag{3.78}$$

$$\frac{\cos t}{\cos\theta} = b\lambda + d, \tag{3.79}$$

for constants c and d. Squaring and adding these we find

$$\sec^2\theta = (a^2 + b^2)\lambda^2 + 2(ac + bd)\lambda + (c^2 + d^2). \tag{3.80}$$

If $a^2 + b^2 = 0$ then θ would be constant, but we have just seen that the only integral curves with constant θ have $\cos\theta = 0$ at all points. Thus $a^2 + b^2 \neq 0$, but now as λ goes to plus or minus infinity, $\sec\theta$ is unbounded, so that θ must tend to $\pi/2$ on each integral curve on which $\cos\theta$ is not always zero. More precisely, (3.80) shows that

$$\theta - \pi/2 \sim \lambda^{-1} \quad \Longleftrightarrow \quad \cos\theta \sim \lambda^{-1}$$

at infinity, in the sense that there exists a constant C such that $C^{-1}\lambda^{-1} \leq \cos\theta \leq C\lambda^{-1}$.

We now look at the components of B_0 from (3.66) along the integral curve. From (3.76), (3.77), (3.78) and (3.79) we have

$$\tan\theta\cos\phi = -a\lambda + (b - c); \quad \tan\theta\sin\phi = b\lambda + (a + d)$$

so that

$$\sin\theta\sin(\phi - t) - 1 = (ad - bc)\cos^2\theta \sim \lambda^{-2}\ . \tag{3.81}$$

To justify "$\sim$", we note that the cofficient of λ^{-2} cannot be zero as

$$ad - bc = \frac{a\cos t - b\sin t}{\cos\theta}$$

and if this were zero then t would be constant along the integral curve, which is inconsistent with (3.73). In the same manner we find that

$$\begin{aligned}\cos\theta\cos(\phi-t) &\sim \lambda^{-2}\,,\\ \frac{\sin(\phi-t)}{\sin\theta}-1 &\sim \lambda^{-2}\,,\end{aligned}$$

so that, by (3.66), all components of B_0 vanish at the rate $O(\lambda^{-2})$ *and no faster* along the integral curve. Putting this with (3.80) we find that $(\cos\theta)^{-5}K\otimes K$ is $O(\lambda)$ along the integral curve and therefore F, which is constant along the integral curve, must vanish. Thus F vanishes on $\mathscr{I}$, therefore so does M and all the asymptotically-defined momenta.

4 Ricci-flat conformal infinity

4.1 Toroidal infinity

We suppose that conformal infinity has toroidal topology

$$\mathbb{T}^{n-1}:=S^1\times\ldots\times S^1$$

with a flat metric $\check h$. The space-time metric

$$^{n+1}b=-\frac{r^2}{\ell^2}dt^2+\frac{\ell^2}{r^2}dr^2+r^2\check h\,, \tag{4.1}$$

where ℓ is related to the cosmological constant Λ by the formula $2\Lambda\ell^2=-n(n-1)$, provides a static vacuum example satisfying all the conditions of the positivity theorem. The slices $t=$ const. have complete induced metric, with one conformally compactifiable end where $r\to\infty$, as well as a "cuspidal end" where $r\to 0$. The toroidal Kottler black holes [32] also belong to this class. Note that the coordinate r in (4.1) can be rescaled by a constant factor, a subsequent redefinition of $\check h$ and of t preserves then the general form of the metric. A natural way of getting rid of this freedom is to assume that the volume of $(\mathbb{T}^{n-1},\check h)$ equals 16π. Alternatively, one can assume that this volume equals one, and remove the normalisation constant $1/16\pi$ in front of (2.2).

We consider the following, trivial spin structure over $\mathbb{T}^{n-1}$: Let $\mathfrak{S}''$ be a product Hermitian bundle of spinors over $\mathbb{T}^{n-1}$, with a representation of the Clifford algebra of $(\mathbb{T}^{n-1},\check h)$ via anti-hermitian matrices. On $\mathbb{T}^{n-1}$ we use manifestly flat local coordinates x^a, $a=1,\ldots,n-1$, ranging from 0 to 2π, and we choose a spin frame so that all connection coefficients vanish, with the Clifford action of parallel vectors represented by constant matrices.

The Witten-type proof of the positive energy theorem requires imaginary Killing spinors in the asymptotic region $\mathscr{S}_{\rm ext}$ "near conformal infinity"; in the current case this is the region $r\ge r_0$ for some large r_0, with the initial data metric g approaching the space-part of (4.1) as in (2.1), and with K_{ij} approaching $\mathring{K}_{ij}=0$ as required there. To construct those spinors we first consider $\mathfrak{S}'=\mathfrak{S}''\oplus\mathfrak{S}''$, the direct sum of two copies of $\mathfrak{S}''$, equipped with the direct-sum sesquilinear product $\langle\cdot,\cdot\rangle_\oplus$:

$$\langle(\psi_1,\psi_2),(\varphi_1,\varphi_2)\rangle_\oplus:=\langle\psi_1,\varphi_1\rangle+\langle\psi_2,\varphi_2\rangle\,. \tag{4.2}$$

For $X \in T\mathbb{T}^{n-1}$ we let $X\cdot$ denote the Clifford action of X and, similarly to (2.11), for $\psi_1, \psi_2 \in \mathfrak{S}''$ we set

$$\gamma^0(\psi_1,\psi_2) := (\psi_2,\psi_1) \ , \tag{4.3a}$$
$$X\cdot(\psi_1,\psi_2) := (X\cdot\psi_1, -X\cdot\psi_2) \ , \tag{4.3b}$$
$$D_X(\psi_1,\psi_2) := (D_X\psi_1, D_X\psi_2) \ . \tag{4.3c}$$

One checks that (4.3b) defines a representation of the Clifford algebra of $(\mathbb{T}^{n-1},\check{h})$ on $\mathfrak{S}'$. Further

$$(\gamma^0)^2 = 1 \ , \tag{4.4a}$$
$$\forall X \in T\mathbb{T}^{n-1} \quad \gamma^0 X\cdot = -X\cdot\gamma^0 \ , \tag{4.4b}$$
$$(\gamma^0)^\dagger = \gamma^0 \ , \tag{4.4c}$$
$$D\gamma^0 = \gamma^0 D \ , \tag{4.4d}$$

Next, it is convenient to pass to yet another direct sum bundle $\mathfrak{S} = \mathfrak{S}' \oplus \mathfrak{S}'$, equipped with the direct-sum Hermitian product which will be denoted by $\langle\cdot,\cdot\rangle_{\oplus\oplus}$. We define, for $\psi_1,\psi_2 \in \mathfrak{S}'$, $X \in T^{n-1}M$ and $a \in \mathbb{C}$,

$$\gamma^n(\psi_1,\psi_2) := (-\psi_2,\psi_1) \ , \tag{4.5a}$$
$$(X\cdot + a\gamma^0)(\psi_1,\psi_2) := ((X\cdot + a\gamma^0)\psi_1, -(X\cdot + a\gamma^0)\psi_2) \ , \tag{4.5b}$$
$$D_X(\psi_1,\psi_2) := (D_X\psi_1, D_X\psi_2) \ . \tag{4.5c}$$

This provides one more representation[19] of the Clifford algebra of $(\mathbb{T}^{n-1},\check{h})$, on $\mathfrak{S}$, with moreover

$$(\gamma^n)^2 = -1 \ , \tag{4.6a}$$
$$\forall X \in T\mathbb{T}^{n-1} \ , \ a \in \mathbb{C} \quad \gamma^n(X\cdot + a\gamma^0) = -(X\cdot + a\gamma^0)\gamma^n \ , \tag{4.6b}$$
$$(\gamma^n)^\dagger = -\gamma^n \ , \tag{4.6c}$$
$$D\gamma^n = \gamma^n D \ . \tag{4.6d}$$

We assume that on $\mathscr{S}_{\rm ext}$ the background metric b takes the form

$$b = (dx^n)^2 + e^{4\mu x^n}\check{h} \ ; \tag{4.7}$$

this corresponds to the space-part of the metric (4.1) when $\mu = 1/(2\ell)$. The conformal boundary at infinity is constructed by multiplying by $e^{-4\mu x^n}$, and replacing x^n by $y = e^{-2\mu x^n}$; the boundary is then the set $\{y = 0\}$. We note that (4.7) is a complete space-form metric.

Any vector $Y \in T\mathscr{S}_{\rm ext}$ can be written in form $Y = Y^n\partial_n + X^a e_a$, where $e_a = e^{-2\mu x^n} f_a$, and where the f_a's form a $\check{h}$–ON basis. Note that $\{\partial_n, e_a\}$ form a b–ON basis. We define the b–Clifford action of Y on $\mathfrak{S}$ as

$$Y\cdot = Y^n\gamma^n + X^a f_a\cdot \ .$$

[19] This representation will not be irreducible, but this is irrelevant for the positivity argument. In fact, already the doubling (4.3) will lead to a reducible representation of the $(\mathbb{T}^{n-1},\check{h})$–Clifford algebra extended by adding γ^0 when n is odd.

Let the co-frame $\theta^i = (dx^n, \theta^a)$ be dual to (∂_n, e_a), then the only non-vanishing connection coefficients are $-\omega_{anb} = \omega_{nab} = -2\mu\check{h}_{ab}$. One then has

$$\mathring{D}_k = \partial_k - \frac{1}{4}\omega_{ijk} e^i \cdot e^j \cdot = \begin{cases} \partial_n, & k = n; \\ \partial_b + \mu\gamma^n e_b \cdot, & k = b. \end{cases}$$

It follows (compare [7]) that for any $\chi \in \mathfrak{S}'$, with constant entries, the spinor field

$$\psi := \frac{e^{\mu x^n}}{\sqrt{2}}(i\chi, \chi) , \tag{4.8}$$

defined over $\mathscr{S}_{\rm ext}$, is an imaginary Killing spinor for b; by definition,

$$\mathring{D}_Y \psi = -\mu i Y \cdot \psi , \tag{4.9}$$

where $\mathring{D}$ denotes the covariant derivative operator of b. One also has

$$\forall\ Z \in T\mathscr{S}_{\rm ext} \qquad \mathring{\nabla}_Z \hat{\psi} := \Big(\mathring{D}_Z - \frac{1}{2}\mathring{K}_i{}^j Z^i e_j \cdot \gamma^0\Big)\hat{\psi} = -\mu i Z \cdot \hat{\psi} , \tag{4.10}$$

because the background extrinsic curvature $\mathring{K}_{ij}$ of the slices $t = 0$ for the associated space-time background metric ${}^{n+1}b$ vanishes.

Let $\mathscr{K}$ denote the space of imaginary Killing spinors $\psi \in \Gamma\mathfrak{S}$ constructed so far. As already mentioned in Section 2, to any element of $\mathscr{K}$ one can associate a KID of the background initial data $(b, 0)$ as follows

$$\mathscr{K} \ni \psi \to \Big(V = \langle \psi, \psi \rangle_{\oplus\oplus}\ , Y = \langle \psi, \gamma^n \cdot \gamma^0 \psi \rangle_{\oplus\oplus} \partial_n + \sum_a \langle \psi, f_a \cdot \gamma^0 \psi \rangle_{\oplus\oplus}\ e_a\Big) . \tag{4.11}$$

Chasing through the definitions we find

$$\begin{aligned} V &= \langle \psi, \psi \rangle_{\oplus\oplus} = e^{2\mu x^n} \langle \chi, \underbrace{\chi}_{=(\chi_1,\chi_2)} \rangle_{\oplus} \\ &= e^{2\mu x^n}\Big(\langle \chi_1, \chi_1 \rangle + \langle \chi_2, \chi_2 \rangle\Big) , \end{aligned} \tag{4.12}$$

$$\begin{aligned} Y &= \underbrace{\langle \psi, \gamma^n \cdot \gamma^0 \psi \rangle_{\oplus\oplus}}_{=0} \partial_n + \sum_a \langle \psi, f_a \cdot \gamma^0 \psi \rangle_{\oplus\oplus}\ e_a \\ &= e^{2\mu x^n} \sum_a \langle \chi, f_a \cdot \gamma^0 \chi \rangle_{\oplus}\ e_a = \sum_a (\langle \chi_1, f_a \cdot \chi_2 \rangle - \langle \chi_2, f_a \cdot \chi_1 \rangle) f_a \\ &= 2\sum_a \Re\Big(\langle \chi_1, f_a \cdot \chi_2 \rangle\Big) f_a . \end{aligned} \tag{4.13}$$

Let m denote the value of H corresponding to the background-KID $V = e^{2\mu x^n}/\ell$, $Y = 0$. This last KID corresponds to the Killing vector ∂_t of the metric (4.1), so that m has the interpretation as energy. Similarly let $j_{(b)}$ be the value of H corresponding to f_b; thus $V = 0$ and $Y^a\partial_a = f_b$. Clearly each $j_{(b)}$ has a natural interpretation of angular momentum.

Under the hypotheses of Theorem 2.1, one concludes that the composition of (4.11) with the Hamiltonian map (2.2) defines a *positive* Hermitian form on

$\mathscr{K}$. We have

$$\begin{aligned} H(V,Y) &= H\Big((\langle\chi_1,\chi_1\rangle + \langle\chi_2,\chi_2\rangle)(e^{2\mu x^n},0) + 2\sum_a \Re(\langle\chi_1, f_a\cdot\chi_2\rangle)(0,f_a)\Big) \\ &= (\langle\chi_1,\chi_1\rangle + \langle\chi_2,\chi_2\rangle)\ell m + 2\sum_a \Re(\langle\chi_1, f_a\cdot\chi_2\rangle) j_{(a)} \geq 0 \,, \end{aligned}$$

for all constant spinors (χ_1,χ_2). This is possible if and only if[20]

$$m \geq \sqrt{-\frac{2\Lambda}{n(n-1)}}|\vec{j}| \,, \qquad |\vec{j}| := \sqrt{j_{(1)}^2 + \ldots + j_{(n)}^2} \,. \tag{4.14}$$

We have thus derived the toroidal equivalent of Maerten's inequality (3.2); we emphasise the spin-structure compatibility condition (2.7).

In space-dimension three (4.14) can be viewed as the special case $\vec{c}=0$ of (3.2), but the justification of this appears to require the work above.

Let j_a be the angular momentum associated with the Killing vector ∂_a. It should be clear that with this definition the inequality in (4.14) remains valid if $|\vec{j}|$ is taken to be $\sqrt{\check{h}^{ab} j_a j_b}$, where $\check{h}^{ab}$ is the inverse matrix to $\check{h}_{ab} := \check{h}(\partial_a,\partial_b)$.

4.2 General conformal infinities with parallel spinors

We now consider a metric (4.1), *without* assuming that $\check{h}$ is flat: instead we assume that the manifold $({}^{n-1}M,\check{h})$ carries a non-trivial covariantly constant spinor χ, section of a spinor bundle $\mathfrak{S}''$. (Such manifolds are necessarily Ricci flat, compare [9, 12, 30, 38, 39].) The construction of the background imaginary Killing spinors of the previous section carries over with only trivial modifications to such a setting. Under the hypotheses of Theorem 2.1 we then obtain a positive definite quadratic functional on the space of covariantly constant spinors of $({}^{n-1}M,\check{h})$. It appears that an optimal form of the resulting constraints has to be analysed case-by-case. Here we only note the following: For every $\check{h}$–parallel χ the norm squared $\langle\chi,\chi\rangle$ is constant over ${}^{n-1}M$. It follows that we can normalise χ to obtain two KIDs as in (4.11) with $\chi_2 = \pm\chi_1 = \chi$ in (4.12)-(4.13), and with time component of the associated KIDs equal to one. The positivity of H for both the plus and minus signs then gives

$$\ell m \geq |j| \,,$$

where j is the angular momentum associated with the b-Killing vector Y corresponding to χ, and ℓ has been defined in (3.5). We thus obtain positivity of m, together with an upper bound on $|j|$ in terms of m. The result is optimal if the space of covariantly constant spinors of $({}^{n-1}M,\check{h})$ is one-dimensional. Otherwise we clearly also have the non-optimal inequality

$$\ell m \geq \sup_\psi |j(X_\psi)| \,,$$

where the supremum is taken over the covariantly Killing spinors ψ normalised as described above, and $j(X_\psi)$ denotes the angular momentum along the Killing vector X_ψ associated to ψ.

[20] Indeed, if $|\vec{j}|=0$ the inequality (4.14) is clear. Otherwise choose $\chi_2 = \sum_a j_{(a)} f_a\cdot\chi_1/|\vec{J}|$ to conclude that (4.14) is necessary. The proof of sufficiency is left to the reader.

4.3 Nonrigidity in the toroidal case for $n = 3$

By Section 3.1 equality in (4.14) leads, locally, to space-forms or to Siklos waves. In order to see that those are compatible with the toroidal topology at infinity note, first, that the metric (3.36) with $2b^2 = 1$ and $H = 0$ gives anti-de Sitter, by introducing $t = (u+v)/\sqrt{2}$ and $z = (v-u)/\sqrt{2}$:

$$g = \frac{1}{x^2}(dx^2 + dy^2 + dz^2 - dt^2) .$$

This metric covers part of anti-de Sitter space-time. However, we now impose a periodic identification in y and in z. Then this is a metric with a $\mathscr{I}$ which is topologically $\mathbb{T}^2 \times \mathbb{R}$ at $x = 0$ and a 'hyperbolic cusp' as $x \to \infty$, as in (4.1). We can retain these asymptotics with a nonzero $H(u, x, y)$ which is suitably periodic in u and y and decays appropriately in x as x goes to zero. A simple class of examples may be generated as follows: take

$$H = xf'(x) - f(x),$$

then from (3.39) we find

$$\Phi = -\frac{x^6}{4}\left(\frac{f''}{x}\right)' . \tag{4.15}$$

while from (3.40)

$$\Psi = -\frac{x^4}{4}(xf'')' \tag{4.16}$$

For the dominant energy condition ($T_{ab}v^a v^b \geq 0$ for time-like v^a) to hold we need Φ to be non-negative so set $\Phi = \frac{x^6}{4}\rho(x)$ for a non-negative function ρ. For simplicity, we suppose that ρ has compact support, and then we solve for f'' as

$$f''(x) = x\int_x^\infty \rho(y)dy. \tag{4.17}$$

Suppose ρ is supported in $0 < a < x < b < \infty$, then so is Φ (and so also is the energy-momentum tensor). For $x < a$, we find $f'' = mx$ with $m = \int_a^b \rho dx$ and then $\Psi = -mx^5/2$, which is the rate of decay we found we required in (3.4). Thus $\mathscr{I}$ exists at $x = 0$ with this H, as with $H = 0$. Letting t and z be as at the beginning of this section, we require the level sets of t to be spacelike. This is equivalent to

$$H < 2 , \tag{4.18}$$

which can be arranged by the choice of $f(0)$ for any ρ as above. We note the following formulae for the metric and second fundamental form of the hypersurface $\mathscr{S} := \{t = 0\}$

$$g_{ij}dx^i dx^j = x^{-2}\Big\{dx^2 + dy^2 + \Big(1 - \frac{H}{2}\Big)dz^2\Big\} , \quad \sqrt{\det g_{ij}} = x^{-3}\sqrt{1 - \frac{H}{2}} , \tag{4.19}$$

$$g_{tt} = -x^{-2}\Big(1 + \frac{H}{2}\Big) , \quad g_{zt} = x^{-2}\frac{H}{2} , \tag{4.20}$$

$$K_{ij}dx^i dx^j = -\frac{H'}{x\sqrt{4 - 2H}}dx\, dz , \tag{4.21}$$

$$|K|_g^2 = \frac{(xH')^2}{2(2 - H)^2} , \tag{4.22}$$

which shows that K satisfies the decay conditions needed for a well-defined mass (recall that ρ vanishes near $x = 0$). One can check that

$$H(x) = H(0) + \int_0^x \frac{y^3}{3}\rho(y)dy + \frac{x^3}{3}\int_x^\infty \rho(y)dy \ ,$$

so that H is a non-decreasing function of x for non-negative ρ and, subsequently, that (4.18) will hold if and only if

$$3H(0) + \int_0^\infty x^3\rho(x)dx < 6 \ . \tag{4.23}$$

Assuming this condition, and a compact support of ρ in $(0,\infty)$, the hypersurface $\mathscr{S} = \{t = 0\}$ with the induced fields provides an example of non-trivial initial data set which saturates the inequality (4.14), and satisfies all the hypotheses of the positive energy theorem listed in the previous section.

For $x > b$, $\Psi = 0$ and the space-time is locally anti-de Sitter. For example, if we choose $H(0) = 0 = f(0)$ (note that the choice of $f'(0)$ is irrelevant as it does not change H) then for $x > b$, H is constant and equal to $H_\infty = (\int_a^b x^3\rho dx)/3$. The metric is

$$g = \frac{1}{x^2}(dx^2 + dy^2 + dz^2 - dt^2 - \frac{H_\infty}{2}(dt - dz)^2)$$

but the coordinate transformation

$$\begin{aligned} d\tilde{z} &= dz(1 - \frac{H_\infty}{4}) + \frac{H_\infty}{4}dt \\ d\tilde{t} &= -\frac{H_\infty}{4}dz + (1 + \frac{H_\infty}{4})dt \end{aligned}$$

has the effect of setting H_∞ to zero.

Another interesting example is $f = C \sinh x \sin y \sin u$, where C is a constant. Now $H = xf_x - f$ gives vacuum. It satisfies the asymptotic conditions, with $\Psi = O(x^5)$ for small x, but because the solution is exponentially large for large x the existence of globally regular spacelike surfaces is not clear. This leads naturally to the question of existence of non-trivial *vacuum* initial data sets saturating the angular momentum inequality (4.14). Recall that no such black hole solutions exist by the results in Section 3.3, but the general result is not known.

4.4 Higher dimensional examples

Gibbons and Ruback [23] have presented some metrics which are generalisations of the Siklos metrics to higher dimensions (compare [8, p. 14]). In space dimension n (so space-time dimension $(n + 1)$), the metrics can be written in the form

$$g^{GR} = \frac{1}{2b^2x^2}(dx^2 + h_{ab}dy^a dy^b - 2dudv - H(u, x, y^a)du^2) \ . \tag{4.24}$$

where $h = h_{ab}dy^a dy^b$ is a Ricci-flat, Riemannian metric on an $(n-2)$-dimensional manifold ${}^{n-2}M$ (compare (3.36)). To analyse the imaginary Killing spinor equation we use the frame

$$\theta^0 = \frac{du}{x}\ , \quad \theta^1 = \frac{1}{x}(dv + \frac{H}{2}du)\ , \quad \theta^2 = \frac{dx}{x}\ , \quad \theta^a = \frac{1}{x}\check{\theta}^a\ ,$$

where $\check{\theta}^a$ is an ON-frame for $({}^{n-2}M, h)$. A somewhat lengthy calculation shows that if ψ_h is a covariantly constant spinor for h then, in a basis of the spinor bundle where the γ–matrices are independent of x, u and v, the spinor field $\psi = x^{-1/2}\psi_h$ is an imaginary Killing spinor for (4.24) and, in fact:

PROPOSITION 4.1 *The metrics (4.24) admit non-trivial imaginary Killing spinors if and only if* $({}^{n-2}M, h)$ *admits non-zero covariantly constant spinors.*

So, when such spinors exist, the volume integral in the Witten identity vanishes, therefore so does the boundary integral. Assuming the asymptotic conditions permit the existence of $\mathscr{I}$, the metrics (4.24) will therefore saturate the n-dimensional version of our bounds.

The Ricci tensor Ric^{GR} for g^{GR} may be written

$$\mathrm{Ric}^{GR} = -2b^2 n g^{GR} + 2\Phi K \otimes K$$

where $K = \partial/\partial v$ and Φ is the function

$$\Phi = -(bx)^4 \left(\frac{\partial^2 H}{\partial x^2} - \frac{(n-1)}{x}\frac{\partial H}{\partial x} + \Delta_h H \right)\ ,$$

where Δ_h is the Laplacian for h. The dominant energy condition again requires Φ to be positive and as in the three dimensional case we can readily find solutions with H independent of u and y^a: set $\Phi = \frac{1}{2}b^4 x^{n+3}\rho(x)$ then solve

$$(x^{1-n}H'(x))' = -\rho,$$

where prime denotes d/dx, to find

$$H'(x) = x^{n-1}\int_x^\infty \rho(s)ds, \tag{4.25}$$

(compare (4.17)). Now if we assume that ${}^{n-2}M$ is compact and that u is periodic, we obtain a solution with a $\mathscr{I}$ located at $x = 0$, whose cross-sections are ${}^{n-2}M \times S^1$. The discussion around (4.18) goes through as before: set $2b^2 = 1$ then $K_{ij}dx^i dx^j$ is as in (4.21), where now $x^i = (x, y^a, z)$, and (4.19) is replaced by

$$g_{ij}dx^i dx^j = x^{-2}\Big\{dx^2 + h_{ab}dy^a dy^b + \Big(1 - \frac{H}{2}\Big)dz^2\Big\}.$$

Therefore, under (4.18), and assuming that ρ is non-negative and compactly supported, these solutions will satisfy the global and asymptotic conditions of the positivity theorem.

For the counterpart of (3.69) we obtain, with conventions as above and in [4] and with $R = bx$,

$$\begin{aligned}\tilde{E}_{ij} &= \frac{1}{(n-2)}(bx)^{2-n} W_{i\alpha j\beta} N^\alpha N^\beta \\ &= \frac{-2}{(n-1)}(bx)^{2-n} H'' \tilde{K}_i \tilde{K}_j\end{aligned}$$

which, by (4.25), has a finite limit on $\mathscr{I}$ where $x = 0$.

The imaginary Killing spinors described immediately before the statement of Proposition 4.1 have the property that

$$K \cdot \psi = 0 \; , \tag{4.26}$$

where, as before, $K = \langle \psi, \gamma^0 \gamma^\mu \psi \rangle \partial_\mu$, and $\cdot$ denotes the space-time Clifford multiplication. An analysis similar to that in Section 3.3 applies, whatever $n \geq 3$, as follows: Differentiating (4.26) one finds, for all K,

$$(\nabla_X K) \cdot \psi \sim \psi \; .$$

By e.g. [34, Lemma 2.1, point 2] we then have

$$\nabla_X K \sim K \; ,$$

which immediately implies staticity:

$$K_{[\mu} \nabla_\nu K_{\rho]} = 0 \; .$$

The remaining arguments of Section 3.3 show that the space-metric on the event horizon has a Ricci tensor proportional to the metric, with negative proportionality constant. The consequences of this are not clear, as the constraints imposed by topological censorship [20] are less stringent in higher dimensions.

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