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HOW TO BYPASS BIRKHOFF THROUGH EXTRA DIMENSIONS: A SIMPLE FRAMEWORK FOR INVESTIGATING THE GRAVITATIONAL COLLAPSE IN VACUUM*

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It is fair to say that our current mathematical understanding of the dynamics of gravitational collapse to a black hole is limited to the spherically symmetric situation and, in fact, even in this case much remains to be learned. The reason is that Einstein's equations become tractable only if they are reduced to a $(1 + 1)$-dimensional system of partial differential equations. Owing to this technical obstacle, very little is known about the collapse of pure gravitational waves because by Birkhoff's theorem there is no spherical collapse in vacuum. In this essay, we describe a new cohomogeneity-two symmetry reduction of the vacuum Einstein equations in five and higher odd dimensions which evades Birkhoff's theorem and admits time-dependent asymptotically flat solutions. We argue that this model provides an attractive $(1+1)$-dimensional geometric setting for investigating the dynamics of gravitational collapse in vacuum.

Keywords: Gravitational collapse; Birkhoff's theorem.

The understanding of the nature of singularities that arise from gravitational collapse is one of the most challenging problems in classical general relativity. The key open issue is to resolve the so-called weak cosmic censorship hypothesis which says that in reasonable models of gravitational collapse, naked singularities do not generically develop from regular initial data.[1] If this hypothesis holds, the ignorance of

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laws of physics near the singularity does not prevent a distant observer from applying ordinary physics outside black holes.

An attempt to prove the cosmic censorship hypothesis in full generality is beyond the scope of existing mathematical techniques, so the attention of researchers has been focused on more tractable special cases, in particular spherically symmetric space–times. Although the assumption of spherical symmetry greatly simplifies the problem, it has a drawback of becoming trivial in vacuum since, by Birkhoff's theorem, the spherical gravitational field has no dynamical degrees of freedom. Thus, in order to generate dynamics, it is necessary to choose a matter model. A popular choice, which has led to important insights, is a real massless scalar field. For this matter model, Christodoulou showed that for small initial data the fields disperse to infinity,[2] while for large initial data black holes are formed.[3] The transition between these two scenarios was explored numerically by Choptuik[4] leading to the discovery of critical phenomena at the threshold of black hole formation. Later studies showed that the character of gravitational collapse and critical phenomena depends both qualitatively and quantitatively on a matter model[5] — for this reason it is difficult to determine which features of spherical collapse are model-dependent and which ones hold in general.

In this essay, we present a new framework which — at the expense of going to five (or higher odd) dimensions — evades Birkhoff's theorem and admits radially symmetric gravitational waves. For the sake of clarity, we shall illustrate the main idea in five dimensions and only at the end we shall briefly discuss higher dimensional generalizations.

We begin by recalling some elementary aspects of the geometry of spheres, in particular the three-sphere S^3. Usually, when one thinks of a sphere, one imagines a round sphere; however, there are other natural geometries with which a manifold S^n can be endowed. To see this, note that an n-sphere is a homogeneous manifold, that is a manifold which looks the same at every point. More formally, a manifold is said to be homogeneous if there is a Lie group which acts transitively on it. Any n-sphere is homogeneous because its group of rotations $SO(n+1)$ acts transitively on it. On S^2, there is no other transitive action than $SO(3)$; however, some higher dimensional spheres admit transitive actions of a proper subgroup of the rotation group. In particular, it is well known that the group $SU(2)$ acts transitively on S^3. This is most easily seen by considering a sequence of rotations acting on the standard three-sphere in four-dimensional Euclidean space. A simultaneous rotation in the xy and zw planes keeps no points on S^3 fixed. It is geometrically evident that this rotation, together with the simultaneous rotations in the xz, yw and xw, yz planes, forms a simply transitive group G acting on S^3. By computing the Lie algebra of G, it is easy to verify that G is isomorphic to $SU(2)$. Given a bare homogeneous manifold, it is natural to endow it with a metric which is invariant under the transitive symmetry group — such a metric is called homogeneous. It follows from the above that on S^3, there are two possible homogeneous metrics:

the standard maximally symmetric $SO(4)$-invariant metric and the less symmetric $SU(2)$-invariant metric. The latter one has the form

$$ds^2 = L_1^2\sigma_1^2 + L_2^2\sigma_2^2 + L_3^2\sigma_3^2, \tag{1}$$

where σ_k are left-invariant one-forms on $SU(2)$. In terms of Euler angles,

$$\begin{aligned} \sigma_1 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi, \\ \sigma_2 &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi, \\ \sigma_3 &= d\psi + \cos\theta\, d\phi. \end{aligned} \tag{2}$$

The constant coefficients L_k have the interpretation of the principal curvature radii of S^3. If all three L_k are equal, the metric (1) reduces to the standard round metric, otherwise the metric is anisotropic and the three-sphere is said to be "squashed." The metric (1) might be familiar from the mechanics of rigid bodies where it appears as the kinetic energy metric of an asymmetric top (with L_k^2 being the principal moments of inertia). In relativity, the metric (1) is well known from the studies of the Bianchi IX homogeneous cosmological models but, strangely enough, a possibility of using it in the context of gravitational collapse seems to have been overlooked.

Now, the key idea is to use (1) as the angular part of a cohomogeneity-two metric in five space–time dimensions. More precisely, we consider the five-dimensional space–time which is foliated by the squashed three-spheres endowed with the $SU(2)$-invariant homogeneous metrics proportional to (1). Using the coordinate freedom in the two-space orthogonal to the S^3 group orbit of $SU(2)$, we choose the volume radial coordinate $r = (\mathrm{vol}(S^3)/2\pi^2)^{1/3}$, and write the space–time metric in the following form:

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{1}{4}r^2\left[e^{2B}\sigma_1^2 + e^{2C}\sigma_2^2 + e^{-2(B+C)}\sigma_3^2\right], \tag{3}$$

where A, δ, B, and C are functions of t and r. Substituting the ansatz (3) into the vacuum Einstein equations in five dimensions, we get equations of motion for the functions A, δ, B, and C. These equations are rather complicated, so here we make an additional simplifying assumption that $B = C$ which means that the ansatz (3) has an extra $U(1)$ symmetry.

Using the mass function $m(t, r)$, defined by $A = 1 - m/r^2$, we obtain the following equations

$$m' = 2r^3(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2) + \frac{2}{3}r(3 + e^{-8B} - 4e^{-2B}), \tag{4a}$$

$$\delta' = -2r(e^{2\delta}A^{-2}\dot{B}^2 + B'^2), \tag{4b}$$

$$(e^{\delta}A^{-1}r^3\dot{B})^{\cdot} - (e^{-\delta}Ar^3B')' + \frac{4}{3}e^{-\delta}r(e^{-2B} - e^{-8B}) = 0, \tag{4c}$$

where primes and dots denote derivatives with respect to r and t, respectively. It is evident from these equations that the only dynamical degree of freedom is the field B which plays a role similar to a matter field in the spherically symmetric Einstein-matter systems in four dimensions. If $B = 0$, we have spherical symmetry and, in agreement with Birkhoff, the only solutions are Minkowski ($\delta = 0$, $m = 0$) and Schwarzschild ($\delta = 0$, $m = \text{const} > 0$).

The system (4) provides the simplest possible setting for investigating the dynamics of gravitational collapse in vacuum. Numerical simulations indicate that the spherically symmetric solutions, Minkowski and Schwarzschild, play the role of attractors in the evolution of generic regular initial data (small and large ones, respectively) and the transition between these two outcomes of evolution exhibits the type II discretely self-similar critical behavior.[6] These results strongly suggest that the above-mentioned Christodoulou's proof of the weak cosmic censorship hypothesis for the self-gravitating scalar field can be repeated for our system. Actually, the first step in this direction has already been done by Dafermos and Holzegel[7] who proved nonlinear stability of the Schwarzschild solution for perturbations respecting the ansatz (3).

There are many other natural problems that can be addressed within the proposed framework; due to space limitations we mention only some of them:

- *Asymptotic stability of the Schwarzschild black hole*: It is well known from heuristic and numerical studies that convergence to the Schwarzschild black hole proceeds first through exponentially damped oscillations (quasinormal modes) and then through a polynomial decay (tail); however, a mathematical description of this relaxation process remains elusive. Since dissipation by dispersion is much more effective in higher dimensions, the system (4) provides a promising setup for studying this problem.
- *Critical collapse*: The most mysterious aspect of Choptuik's discovery[4] is the discrete self-similarity of the critical solution. The fact that the critical solution in our model also has this strange symmetry lends support to the belief that it is an intrinsically gravitational phenomenon and the typical feature of critical collapse. We believe that our model might be helpful in unraveling a mechanism which is responsible for the discrete self-similarity of the critical solution. We remark in passing that the system (4) does not admit regular continuously self-similar solutions — this indicates that the naked singularities of this kind found by Christodoulou for the self-gravitating massless scalar field[8] are, in a sense, matter-generated and as such much less interesting.

  Obviously, the relevance of critical phenomena depends on how generic they are. Our numerical results[6] show that they are not restricted to spherically symmetric Einstein-matter systems but occur also in the collapse of pure gravitational waves.[a] It is worth to point out that an extra $U(1)$ symmetry that we have assumed to simplify the ansatz (3) does not affect (except for very special

[a]The evidence for critical behavior in the collapse of axisymmetric gravitational waves given by Abrahams and Evans in Refs. 13 and 14 is much less compelling because of numerical difficulties.

codimension-two solutions) the critical collapse: the evolution of critical initial data with two radiative degrees of freedom B and C tends to the same critical solution as in the simplified model; in other words the $U(1)$ symmetry is recovered dynamically.[9] The next important step in determining the genericity of the critical behavior would be to analyze the linear stability of the critical solution against general perturbations.

- *Non-asymptotically flat solutions*: The ansatz (3) is very rich and incorporates as special cases many non-asymptotically flat metrics.[b] In particular, some gravitational instantons, i.e. four-dimensional Riemannian Ricci-flat metrics (like for instance the Taub-NUT metric), can be viewed as static solutions in our model. In a sense, one can say that these solutions were waiting for the equations — having the system (4), it becomes possible to ask questions about the dynamical role of gravitational instantons and their stability properties.[10]
- *Higher-dimensional generalizations*: It should be clear from the derivation of the ansatz (3) that similar models can be formulated in higher $D = n+2$ dimensions as long as the corresponding n-sphere admits a non-round homogeneous metric, i.e. there exists a proper subgroup of the rotation group $SO(n+1)$ which acts transitively on S^n. According to the classification given by Besse,[11] such transitive actions exist on all odd-dimensional spheres. For example, the group $SU(n+1)$ acts transitively on S^{2n+1} and the group $Sp(n+1)$ acts transitively on S^{4n+3}. It is natural to ask whether the properties of gravitational collapse found in Ref. 6 are typical for this class of models or whether new phenomena appear in higher dimensions. Having this motivation in mind, together with our collaborators we have recently studied the nine-dimensional cohomogeneity-two model based on the squashed seven-sphere with $SO(8)$ isometry broken to $SO(5) \times SU(2)$.[12] We found that the overall picture of gravitational collapse and critical phenomena is qualitatively the same as in five dimensions, except for one surprising difference: while in five (and, of course, four) dimensions the mass function $m(t,r)$ is monotone increasing with r (as follows immediately from Eq. (4a)), in nine dimensions the mass function can decrease. This means that the energy density of the gravitational field may be locally negative and suggests a possibility of violating the weak cosmic censorship. This intriguing problem is being investigated. Preliminary studies show no evidence for naked singularities — the negative energy regions seem to be always eventually shielded by horizons. If this happens in general, it would indicate that the cosmic censor is more powerful than it has been thought.

To summarize, we have proposed a new theoretical framework for investigating gravitational collapse in a clean matter-free environment. We think that this framework is rich enough to capture, at least in part, the general features of vacuum Einstein's equations. We would like to remark that although our own motivation for going to higher dimensions was mathematical in nature, nowadays extra dimensions are

[b]We thank Gary Gibbons for pointing this out.

very common in physics, in particular the string theory predicts that our world has more than four dimensions. In order to understand the phenomenology of various string-related models (such as the braneworld scenario) and their experimental predictions, like production of mini black holes in the next generation of colliders, it is important to study solutions of Einstein's equations in higher dimensions. From this perspective, the ideas presented in this essay may turn out to be quite useful.

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