Ann. Phys. (Leipzig) **15**, No. 1 – 2, 84 – 108 (2006) / **DOI** 10.1002/andp.200510173

Is general relativity 'essentially understood' ? *

Helmut Friedrich**

Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Golm, Germany

Received 3 August 2005, revised 3 September 2005, accepted 12 September 2005
Published online 23 December 2005

Key words Classical general relativity, singularities and cosmic censorship, initial value problem for Einstein's field equations, asymptotic structure, numerical relativity, classical black holes.

PACS 04.20.-9, 04.20 Dw, 04.20 Ex, 04.20 Ha, 04.25 Dm, 04.70 Bw

The content of Einstein's theory of gravitation is encoded in the properties of the solutions to his field equations. There has been obtained a wealth of information about these solutions in the ninety years the theory has been around. It led to the prediction and the observation of physical phenomena which confirm the important role of general relativity in physics. The understanding of the domain of highly dynamical, strong field configurations is, however, still quite limited. The gravitational wave experiments are likely to provide soon observational data on phenomena which are not accessible by other means. Further theoretical progress will require, however, new methods for the analysis and the numerical calculation of the solutions to Einstein's field equations on large scales and under general assumptions. We discuss some of the problems involved, describe the status of the field and recent results, and point out some open problems.

1 Introduction

The fascinating reports on the exciting new theories, which propose to unite our present and all future ideas about space-time, gravitation, and quantum physics into one coherent scheme from which general relativity will be derived in the end as a particular limit, may suggest that Einstein's classical general relativity is essentially understood. This impression is easily corroborated by the amazing successes of general relativity. Einstein concludes his synopsis from 1916 of the theory of general relativity [58] with three predictions: the red shift, the bending of light rays, and the precession of the perihelion in planetary motion. In view of the present observational facts the revolutionary character these implications may have had at the time falls into oblivion. The red shift can be measured today in terrestrial experiments [116], the bending of light rays is used under the heading 'gravitational lensing' as an effective astrophysical tool [27], and various double star systems provide laboratories for relativistic gravity which let the minute advance of the periastron of Mercury (42.98 arcsec/century) fade into insignificance: in the case of the binary pulsar PSR 1913 + 16 the observed advance of periastron is 4.226595(5) deg/yr [141] and for the recently discovered double pulsar J0737-3039 it even amounts to 16.9 deg/yr [97].

In the articles [59, 60] Einstein discusses 'gravitational radiation' and derives the famous quadrupole formula. Doubts have been raised subsequently whether the notion of gravitational radiation referred to a real physical phenomenon (cf. [109]), but again the prediction has been confirmed convincingly. Using Einstein's quadrupole formula to calculate the rate of period decrease of the system PSR 1913 + 16 due to its emission of gravitational radiation, one obtains, after taking into account certain small corrections, a curve which shows an uncanny agreement (to within about 0.2 percent) with the data gathered over the last 30 years [141].

* Extended version of a talk which was delivered at the DPG Frühjahrstagung in Berlin, 5 March 2005.
** E-mail: hef@aei.mpg.de

The global studies of general relativity starting in the second half of the last century, led, together with unexpected observations, to concepts which went far beyond what had been envisioned by Einstein. Among those the notion of a black hole, a pure space-time structure which has no place in special relativity, is certainly the most remarkable one. Though the derivation of detailed observational information still poses difficulties, the present situation suggests that black holes have to be accepted as part of our reality [39,119]. More could be said in support of general relativity but I shall leave it at that.

The overwhelming success of general relativity alluded to above may suggest a clear and simple answer to the question posed in the title of the present article. Also, we understand, of course, what it means that the geometry of the world is modelled by a Lorentz metric $g_{\mu\nu}$ on a 4-dimensional manifold M and that this structure is governed by Einstein's field equations

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2}\, T\, g_{\mu\nu} \right) + \lambda\, g_{\mu\nu}, \tag{1}$$

with cosmological constant λ, together with equations for matter fields which define the energy-momentum tensor $T_{\mu\nu}$ and its trace T. In fact, the predictions referred to above have been derived from these requirements. There is, however, still a large and potentially most important part of the theory we do not have access to, neither mathematically, nor theoretically, nor observationally.

On the observational side this situation may change soon. With the gravitational wave experiments presently becoming operational, we may well enter one of the most important eras of experimental relativity. The mathematical or theoretical side seems to be lacking behind, however. In spite of a few general and a rich collection of specific results our qualitative and quantitative understanding of *highly dynamical processes, strong field situations, and Einstein evolution over long time scales under general assumptions* is still quite limited.

This may sound more pessimistic than it should. The field has seen a substantial progress in recent years and for young researchers there is a unique opportunity to contribute to the theoretical investigation of an unexplored domain of fundamental physics. Confronted with the gravitational wave measurements these studies may even result in the discovery of new phenomena which might give missing clues in which directions to search in the development of 'new theories'.

Because the content of the theory is essentially defined by Einstein's equations it is not surprising that the main problem in the field is the analysis of these equations and their solutions. The following is meant to give an outline of the present situation and some insight into the questions to be dealt with. It is definitely not to be considered a survey of a field which has been growing since 1915. When I try to explain the situation I shall avoid technicalities to the extent to which this is possible without getting too vague in a field which abounds with technical questions by its very nature. References will be given mainly to illustrate a point or to direct the reader to more precise statements about concepts and results. The given references to specialized survey articles should allow the reader to get a more complete picture.

2 Gravitational radiation, singularities, and black holes

Einstein's analysis of gravitational radiation was based on the linearized field equations and the quadrupole formula. As indicated above, this gives essentially correct results in the 'weak' field of the binary pulsar and it is in fact still the basis for many calculations of gravitational wave emission in gravitational collapse scenarios. It can, however, hardly be expected to provide reliable answers in situations involving strong and highly dynamical fields, it is useless for calculating wave signals resulting from the coalescence of black holes.

The reconsideration of the idea of gravitational radiation in the non-linear theory led in the 1960's to a concept of gravitational radiation, which does not require mathematical approximations but relies on the idealization of *asymptotic flatness* in null directions [26]. The latter is based on a picture of the overall behaviour of the gravitational field of an isolated self-gravitating system which assumes the field to become weaker and weaker in a characteristic way along null geodesics running out to null infinity. One might expect that 'close to null infinity linearized gravity takes over', but the situation turned out to be more subtle than that. Penrose has given a geometric and particularly elegant characterization of asymptotic flatness in terms of the extendibility of the conformal structure through null infinity with a certain degree of smoothness [110]. This suggestion associates with the far field of a selfgravitating isolated system a natural concept of *radiation field* but it poses at the same time difficult questions about the long time behaviour of gravitational fields and the nature of the field equations. It will be referred to in the following as *Penrose proposal* (cf. [63, 71] for further discussion and references).

As it turned out, this was only a first realization of the importance of global considerations in the analysis of the non-linear field equations and the introduction of various general relativistic concepts. The dominating topics of the following years were concerned with physical situations such as *gravitational collapse* and *cosmological singularities*, where fields could be expected to show even unlimited growth over finite times. Some of the main results of that period of research were (i) the singularity theorems of Penrose [111] and Hawking and Penrose [85], which showed that the occurrence of space-time singularities is a stable feature of gravitational fields, (ii) the analysis by Belinskii, Khalatnikov, and Lifshitz [19,20,100], which led to the *BKL conjecture* about the general (oscillatory) behaviour of the field near cosmological singularities, and (iii) the development of the theory of black holes [79, 84].

It may be noted that 'singularity' is defined in the singularity theorems as the existence of a causal geodesic which is non-extendible and non-complete. It is left open, for which reason the curve should be non-extendible. In the case of specific explicit solutions more could be said concerning the nature of their singularities (cf. [84] for the discussion of examples) but in general the methods given at the time were not sufficient to supply more information.

Among the questions raised by gravitational collapse theory the most important and still unsolved one is whether the evolution of gravitational fields admits a *cosmic censorship principle* [112] which excludes in *generic* circumstances the existence of *naked singularities*, i.e. singularities which could be seen by (possibly distant) observers. If such singularities existed stably under small perturbations this would reduce the predictive power of the theory. Giving a precise meaning to this principle is part of the problem. There exist different formulations in the literature such as the *weak cosmic censorship principle*, which asserts that singularities are hidden within black holes [140], and the *strong cosmic censorship principle*, which asserts that maximally extended space-times arising from *generic* non-singular initial data are globally hyperbolic (a notion explained below) [107, 114].

There have been given fascinating accounts of these exciting developments by some of those involved (cf. [31, 46, 92, 136]) and I shall not try to repeat any of it here. Naturally, any research into a theory as complicated as general relativity starts from what could be called its 'fringes', defined by situations close to Newtonian ones, by static or stationary model situations, by configurations with other symmetry or other simplifying features, etc. While such studies, combined with perturbative calculations, have led to impressive results and far reaching extrapolations, the end of the story certainly still needs be told. Moreover, problems which may be considered by the early pioneers as having been settled a long time ago, may require new considerations in the darkness of new questions.

The problem of cosmic censorship, questions about the strength of the singularity, and the more specific but not unrelated questions raised by the BKL conjecture and the Penrose proposal represented the *guiding problems* for much of the subsequent research on the global evolution problem.

We may well only have scratched the surface of the domain of highly dynamical and strong field configurations and it is far from clear that we have exhausted the content of the theory. The general relativistic phenomena mentioned above had been predicted as theoretical consequences of the theory before they have been confirmed by carefully directed observations. Why should we have reached the end of it? With our restricted theoretical and observational access to the domains alluded to above we may be missing out fundamental facts and unexpected phenomena. The results by Choptuik [36] on the phenomena occurring at the threshold of black hole formation may just give us a glimpse at things to come.

The coalescence of black holes is expected to represent one of the strongest sources of gravitational radiation. Even in the clean cut pure vacuum situation still little can be said about this process under general assumptions, though reliable quantitative results are needed for analysing the data of the gravitational wave experiments.

In recent years huge strides have been taken towards the goal of controlling the solutions to Einstein's field equations on large scale space-time domains. The results available now go, in any sense of the word, far beyond what was known around 1980. Nevertheless, *getting qualitative and quantitative (resp. analytical, theoretical, and numerical) control on the long time evolution of gravitational fields under general assumptions is still the most important open problem in classical general relativity.*

3 The exploration of the solution manifold

One of the main problems in controlling the behaviour of solutions to Einstein's field equations is posed by the Ricci-operator $R_{\nu\rho}[g]$ on the left hand side of (1). In general coordinates $(x^\lambda)_{\lambda=0,\dots,3}$ the unknown metric g is given by a symmetric 4×4 matrix $(g_{\mu\nu})_{\mu,\nu=0,\dots,3}$ whose entries are functions of the coordinates. In terms of these unknowns the Ricci-operator reads

$$R_{\nu\rho}[g] = \frac{1}{2} \sum_{\delta,\eta=0}^{3} \left\{ \frac{\partial}{\partial x^\delta} \left(g^{\delta\eta} \left[-\frac{\partial}{\partial x^\eta} g_{\rho\nu} + \frac{\partial}{\partial x^\rho} g_{\nu\eta} + \frac{\partial}{\partial x^\nu} g_{\rho\eta} \right] \right) - \frac{\partial}{\partial x^\rho} \left(g^{\delta\eta} \frac{\partial}{\partial x^\nu} g_{\delta\eta} \right) \right\}$$
$$+\frac{1}{4} \sum_{\lambda,\delta,\eta,\pi=0}^{3} \left\{ g^{\lambda\pi} \left(\frac{\partial}{\partial x^\lambda} g_{\delta\pi} + \frac{\partial}{\partial x^\delta} g_{\lambda\pi} - \frac{\partial}{\partial x^\pi} g_{\lambda\delta} \right) g^{\delta\eta} \left(\frac{\partial}{\partial x^\rho} g_{\nu\eta} + \frac{\partial}{\partial x^\nu} g_{\rho\eta} - \frac{\partial}{\partial x^\eta} g_{\rho\nu} \right) \right.$$
$$\left. -g^{\lambda\eta} \left(\frac{\partial}{\partial x^\rho} g_{\delta\eta} + \frac{\partial}{\partial x^\delta} g_{\rho\eta} - \frac{\partial}{\partial x^\eta} g_{\rho\delta} \right) g^{\delta\pi} \left(\frac{\partial}{\partial x^\lambda} g_{\nu\pi} + \frac{\partial}{\partial x^\nu} g_{\lambda\pi} - \frac{\partial}{\partial x^\pi} g_{\lambda\nu} \right) \right\}, \tag{2}$$

where we use the coefficients $g^{\delta\eta}$ of the matrix $(g^{\delta\eta})$ inverse to $(g_{\mu\nu})$, which are of the form $g^{\delta\eta} = \left(\det(g_{\mu\nu})\right)^{-1} p^{\delta\eta}$ with polynomials $p^{\delta\eta}$ of degree 3 in the $g_{\mu\nu}$.

In the case of a vacuum field with vanishing cosmological constant the matter fields and the energy-momentum tensor vanish and the various difficulties arising from matter models and matter equation are not present. Eqs. (1) then simplifies to the *vacuum field equations*

$$R_{\nu\rho}[g] = 0, \tag{3}$$

and the operator (2) is all one needs to consider. But even somebody working in PDE theory for many years may just notice that (3) is an equation of second order which is linear in the second derivatives of $g_{\mu\nu}$ (i.e. quasi-linear), quadratic in the derivatives of $g_{\mu\nu}$ and rational in the unknown $g_{\mu\nu}$ but then may feel lost. Working out the content of the field equations without taking into account their geometric background is a hopeless task. Much of the richness of the system and important paths towards analysing the structure of its solutions will remain hidden.

In general one has to deal with various matter models and thus needs to consider in addition to Eqs. (1) the resultant *matter equations*. For physical reasons it is clearly most important to understand the behaviour of the resulting coupled systems. Matter equations import, however, their own specific difficulties (cf. [121]). It is only because we do not want to be distracted by these individual properties that almost nothing will be said about results on solutions with matter fields.

For several reasons the cosmological constant attracted increasing interest in recent years and there are available quite a few results on the corresponding solutions. Nevertheless, only the case $\lambda = 0$ will be considered in the following.

Under *general assumptions* (no symmetries, algebraically non-restricted curvature tensors, no approximation requirements like low speeds etc.) qualitative results on the solutions to (3), in particular on large space-time domains, can only be obtained by applying geometric and abstract PDE methods to the analysis of the field equations, subject to various suitably chosen boundary conditions. Before formulating boundary value problems and proving anything, one needs ideas about fruitful problems and a hunch of what might be provable. Physical questions usually lead to deep mathematical problems whose analysis requires a considerable amount of ingenuity and the invention of new methods. But in the end they often allow for natural answers which illustrate the remarkable coherence of the theory.

In the last 90 years there has been developed a host of methods to deduce information and heuristic results from the equations. The study of certain explicit solutions exhibiting unexpected features, formal expansion type analyses of the solutions based on various representations of the field equations, studies of perturbations away from well-understood situations, and the use of topological, differential geometric, and PDE methods gave rise to far reaching generalizations.

In recent years numerical calculations have proven a powerful extension of the arsenal. A remarkable example is provided by the countable family of smooth, static, spherically symmetric solutions to the Einstein-Yang-Mills equations discovered by numerical methods by Bartnik and McKinnon [14]. These solutions certainly would not have been discovered by purely analytical methods up to this day (cf. [132] for the complications of the analytic proof).

A closer interaction between the analytical and numerical relativists appears to have a huge potential for further progress in the domain of relativity we are discussing here. In the following I shall try to point out some domains where such collaborations have been successful and also questions where a collaboration would be desirable and fruitful or even necessary. For a discussion of questions more specific to numerical calculations I refer to the article [99] and the literature given there.

The abstract analysis of Einstein's field equations developed slowly. That these equations themselves satisfy the requirement of local causality was shown by Stellmacher [133] only in 1938. The first general local existence proof for solutions to Einstein's equations was given in 1952 by Choquet-Bruhat [62]. Until 1980 only existence locally in time was considered (cf. [35, 61] for surveys). Choquet-Bruhat and Geroch obtained, however, an important uniqueness result, which also sheds light on the dangers of using the word 'global' [34]. It states that an initial data set for Einstein's vacuum field equations determines a *maximal*, globally hyperbolic time evolution *uniquely up to isometries*.

Here a space-time (not necessarily solving any equations) (M, g) is called *globally hyperbolic* if it contains a space-like hypersurface S such that each causal curve which cannot be extended as such meets S in precisely one point. In that case S is called a *Cauchy hypersurface*. Let (M, g) be a globally hyperbolic space-time and S a Cauchy hypersurface in M. Cauchy data for the linear wave equation $\Box_g \phi = 0$ on S then determine a solution which exists everywhere on M and which is unique. It will be shown below that the vacuum field equations can be understood as a system of (non-linear) wave equations. This makes the global uniqueness result plausible, but there are subtleties.

There are known maximal globally hyperbolic solutions to Einstein's equations which can locally be extended (as solutions) but the extension will not be unique. The hypersurface across which the extension takes place is called a *Cauchy horizon*. The presence of a Cauchy horizon signals a *loss of uniqueness*, which is certainly something to worry about if the field equations are to predict the future. There arise delicate points here. So far we assumed everything to be smooth. Should we take the extension seriously if it can only be performed with low smoothness? Which smoothness requirements will still make sense from the PDE point of view? When do extensions still admit reasonable physical interpretations? Where is the dividing line between 'extension of low smoothness' and 'weak singularity'?

One reason for the slow start of the general, abstract analysis of Einstein's equations is certainly the fact that already in the vacuum case the investigation of boundary problems for Einstein's field equations requires the study of four different differential systems:

– the *system of constraints*,

– the *gauge system*,

– the *main evolution system* or *reduced system*,

– the *subsidiary system*.

The need for this is due to the diffeomorphism invariance of Einstein's field equations. In an initial value problem Eq. (3) cannot have a unique solution in the sense of PDE theory. There exists a large set of diffeomorphisms which leave the metric invariant up to second order on the given initial time-slice. Therefore one has to take special measures to reduce a problem for Einstein's equations to a standard PDE problem. Moreover, without further assumptions the operator (2) is not of the hyperbolic type which one might expect for an equation required to respect local causality.

The systems above are not independent of each other. While some are fixed by the given boundary problem, there is a large freedom in the choices of others and changes of one of them usually entails considerable changes in the others. A deeper understanding of their meaning, freedom, and interaction is required each time new problems are analysed. This will be illustrated below in the case of the subsidiary system. While it is of marginal interest in the analytical arguments, it seems to acquire an important role in the numerical construction of space-times.

In the following the Cauchy problem will be considered most often. Of the many boundary problems studied for Einstein's equations, this is the most important but not the only relevant one. Under specific circumstances or as auxiliary problems other boundary problems may be equally important. To illustrate some of the questions which need to be considered in the context of existence theorems or in the numerical calculation of space-times, I begin with some remarks on the systems mentioned above and point out some old and new results.

3.1 The constraint equations

In the Cauchy problem for Einstein's field equations a prospective solution (M, g) of Eq. (3) is characterized in terms of data prescribed on a 3-manifold S which is envisioned as being embedded as a space-like hypersurface into M. Since the initial data determine the solution near the initial hypersurface uniquely, they contain the basic information on the solution. The construction, detailed understanding and interpretation of initial data is thus an important part of the Cauchy problem. Of the many results available now only a few can be discussed here and we refer to the recent survey [15] for more results and details of the methods only indicated in the following.

It turns out that the data which need to be prescribed in a Cauchy problem for the vacuum field equations (3) are given by symmetric tensor fields h_{ab} and χ_{ab} which by the embedding of S into M will acquire the meaning of the Riemannian metric and the second fundamental form induced on S by the prospective solution g. As a consequence of the covariance of the field equations the data need to satisfy the *vacuum constraint equations*

$$R[h] - \chi_{ab}\,\chi^{ab} + (\chi_a{}^a)^2 = 0, \qquad D_c\,\chi_a{}^c - D_a\,\chi_c{}^c = 0, \tag{4}$$

where D and $R[h]$ denote the covariant derivative and the Ricci-scalar of the metric h_{ab} and the latter is used to move indices. These four quasi-linear equations form an underdetermined elliptic system for the twelve components of h_{ab} and χ_{ab}. If one takes into account the freedom to perform transformations of the three coordinates and considers S as being determined in M essentially by the *mean extrinsic curvature* $\psi = \frac{1}{3}\,\chi_a{}^a$, the rough function counting gives two degrees of freedom for the gravitational field.

The analysis of these equations depends on conditions which are not or only partially controlled by Einstein's equations such as the topology of S or the fall-off behaviour at infinities. Of particular importance are the cases where S is *compact* with arbitrary topology, where (S, h_{ab}) is *asymptotically flat* or *asymptotically euclidean* (corresponding to a hypersurface extending to space-like infinity where the metric h_{ab} approaches an euclidean metric), or where (S, h_{ab}) is *hyperboloidal* (corresponding to a hypersurface extending to null infinity, like a space-like unit hyperbola on Minkowski space). In each of the non-compact cases one may consider $k \geq 1$ *asymptotic ends*, so that for some compact subset K of S the manifold $S \setminus K$ has k components each of which is diffeomorphic to $\mathbb{R}^3 \setminus B$ where B is a closed ball in $\mathbb{R}^3$.

In these cases the construction of solutions to the constraints is well understood if the mean extrinsic curvature ψ is assumed to be constant ($\psi = 0$ in the asymptotically flat, $\psi = const. \neq 0$ in the hyperboloidal case). Following a suggestion by Lichnerowicz and using the behaviour of the equations under conformal rescalings of the metric, one finds that the metric can be chosen in the form $h_{ab} = \phi^4 \, \bar{h}_{ab}$ with some positive scalar function ϕ which is to be determined by solving some equation and a metric $\bar{h}_{ab}$ which is to be prescribed on S. If the symmetric trace-free tensor $\bar{\chi}_{ab}$ then satisfies with respect to the metric $\bar{h}_{ab}$ the equation $\bar{D}_c \, \bar{\chi}_a{}^c = 0$ the fields h_{ab} and $\chi_{ab} = \phi^{-2} \bar{\chi}_{ab} + \psi \, h_{ab}$ satisfy the vacuum constraints. This method, referred to as the *conformal method*, reduces the problem of solving the constraints to a linear elliptic system to obtain $\bar{\chi}_{ab}$ and a decoupled semi-linear elliptic scalar equation for the conformal factor ϕ, called the *Lichnerowicz equation*.

The solvability conditions for the latter in the compact and to some extent in the asymptotically flat case came along with the complete clarification of the Yamabe problem (a long standing mathematical problem, the final step of which was taken by Schoen who, remarkably, used ideas introduced by general relativity ([130], cf. also [98])). The criterion is given in terms of the sign of the *Yamabe number*, an invariant of the conformal structure defined by $(S, \bar{h}_{ab})$. The general solvability condition in the asymptotically flat case has been given only recently by Maxwell [103]. It requires the (suitably generalized) Yamabe number to be positive. In the hyperboloidal case solvability conditions do not arise.

The conformal method provides large classes of solutions to the constraint equations. In the case of non-constant mean extrinsic curvature ψ the conformal method does not lead, however, to a decoupling of the equations and the resulting simplifications. The existence of solutions to the constraints can still only be shown under severe conditions on ψ. The resulting restriction on the class of space-times which can be constructed from such data may be quite serious. Isenberg, Mazzeo, and Pollack have shown the existence of asymptotically flat vacuum space-times which do not admit maximal (case $\psi = 0$) time slices [91] while Chruściel, Isenberg, and Pollack obtained a similar result for vacuum solutions with compact time slices [43]. Moreover, time slices with $\psi \neq const.$ are encountered quite often in discussions of the evolution problem and the conformal method can not be applied to discuss the data induced on such slices. If such slicings are used in numerical studies it appears difficult to replace such data by improved data which are close to the given ones and satisfy the constraints with higher accuracy.

We have restricted our discussion to the vacuum case for convenience only, there do exist methods to provide data for Einstein's equations coupled to various matter fields. In those cases the data often have a direct *physical interpretation*. If the data comprise, for instance, a ball of perfect fluid with a vacuum exterior (cf. [57] for a detailed discussion of this situation and its subtleties) we have a fairly clear idea about their meaning, though the large freedom to dispose of the exterior vacuum field still raises questions (cf. Sect. 3.5.2).

In the pure vacuum case the physical meaning of the data is not so obvious and can in general hardly be assessed without analysing their evolution in time (cf. Sect. 3.5.2). Some interpretation is obtained for vacuum data with special properties. The singularity theorems and the cosmic censorship principle suggest that asymptotically flat data containing a trapped surface, an embedded surface Σ which is characterized by certain convergence properties of the out- and ingoing family of light rays orthogonal to Σ, develop into space-times containing event horizons and black holes. Evolving such data, which arise, for instance, if several asymptotic ends are present, is thus the usual method to model black holes and their coalescence.

In numerical calculations, it may be advantageous to start from asymptotically flat initial data on a manifold S with an inner boundary $\Sigma = \partial S$ which represents a trapped surface. Depending on the precise conditions to be achieved on Σ, the conformal method leads in this case to various overdetermined elliptic boundary value problems. Recently the nature of such problems has been analysed and existence theorems have been proven by Dain [53], Dain et al. [56], and Maxwell [103]. The initial data sets so obtained comprise exterior data which arise from non-trivial topologies as considered above as well as exterior data extending to data on $S = \mathbb{R}^3$ as described in Sect. 3.5.2.

If asymptotically flat or hyperboloidal data are to be calculated numerically without imposing cut-offs at artificial finite boundaries, there arises the problem that the data can develop logarithmic singularities at the asymptotic ends. In the asymptotically flat case conditions on the 'free data' under which logarithmic singularities do or do not occur at space-like infinity have been discussed by Dain and Friedrich [55]. The analogous question for hyperboloidal data has been discussed by Andersson et al. [8] and in great generality by Andersson and Chruściel [6,7].

Data with quite unexpected properties have been obtained recently by *gluing techniques*. A particularly remarkable idea has been introduced by Corvino in [47]. The underlying method to exploit the under-determinedness of the constraint equations to obtain *smooth, localized deformations of solutions to the constraints* has been extended by Chruściel and Delay [41] and Corvino and Schoen [48]. It provides a deeper understanding of the *constraint map* Φ, which maps the fields h_{ab}, χ_{ab} onto the expressions on the left hand sides of Eqs. (4) and it allows one to construct solutions to the constraints which are not accessible by the conformal method. In particular, it enables one to deform given asymptotically flat solutions to the vacuum constraints outside a given compact set to solutions which are *exactly* static or stationary *in a neighbourhood of space-like infinity* or to solutions which are *asymptotically* static or stationary *at space-like infinity up to a given order or at all orders*.

As discussed below, the surprising freedom in modifying data in their asymptotic domain sheds a new light on the Penrose proposal and it raises subtle conceptional questions about the calculation of wave forms characterizing isolated self-gravitating systems.

Quite different aspects of initial data sets are addressed in the investigations of *Penrose inequalities*. With any asymptotically flat initial data set can be associated a certain invariant called the *total mass* or ADM-mass [13]. It is obtained by performing a certain integral over a large sphere and taking its limit when this sphere is pushed to infinity so as to encompass the whole manifold S. In view of the weak conditions imposed on the initial data by the constraints it is quite remarkable that this mass could be shown to be non-negative and to be zero only for flat data. The Penrose inequalities may be considered as extensions of this positive mass theorem by Schoen and Yau [131] and Witten [142]. In the case of a space-like hypersurface S embedded in a space-time with event horizons so that the induced initial data set is asymptotically flat and S intersects the event horizon in a 2-surface Σ, these inequalities are expected to give a lower bound for the total mass of the initial data set in terms of the square root of the area of Σ. Remarkably, under the condition $R[h] \geq 0$ on the Ricci scalar and certain assumptions which simplify the identification of the surface Σ, Penrose inequalities have been proven by Huisken and Ilmanen [89] and by Bray [28] (cf. [29] for a survey).

A related class of problems is that of associating with an extended but finite space-time domain a notion of energy or energy-momentum. While some of the suggestions considered here played an important role in the above proofs of the Penrose inequalities, there does not exist a general agreement on the ultimate notions of quasi-local energy-momentum and other quasi-local quantities (cf. [134] for a detailed survey).

It is interesting that Penrose arrives at the type of inequality named after him by invoking the 4-dimensional space-time picture and using a chain of arguments each of which raises questions itself [113]. He assumes in particular a version of weak cosmic censorship and makes use of the idea that after developing (something which is to become) an event horizon the space-time will settle down 'in some appropriate but as yet ill-defined sense' to become a Kerr black hole. This idea is supported by the results on the black hole equilibrium problem (cf. [31]) but a proof would require control on the long time evolution and estimates which describe in detail how the corresponding member of the Kerr family will be approached.

Once the Penrose inequality can be derived by relying only on properties of initial data sets, one may ask whether the argument could be turned around and the inequalities or the techniques underlying their proofs could be used to obtain estimates to control the evolution of black holes. This will, however, almost certainly require a proof of the appropriate Penrose inequalities under sufficiently general assumptions (including perhaps hyperboloidal data). As discussed above, general mean extrinsic curvatures create difficulties in solving the constraints. They also create difficulties in the present context. The first of Eqs. (4) shows that the condition $R[h] \geq 0$ may be violated in the case of an unrestricted mean extrinsic curvature. New methods may be needed to obtain the inequality in such cases.

3.2 The gauge system

To apply PDE techniques to the local evolution problem, one has to impose *gauge conditions*, restrictions on the freedom to perform diffeomorphisms or coordinate transformations. In the numerical calculation of space-times a number of unsolved questions are related to the gauge problem. To illustrate the general argument and related problems without being too vague, I indicate one specific *reduction procedure* by which the initial value problem for Einstein's field equations is cast into a Cauchy problem for a hyperbolic system. The one chosen here yields the most concise expressions.

In the given local coordinates x^μ on M the expression (2) can be rewritten in the form

$$R_{\mu\nu} = -\frac{1}{2}\, g^{\lambda\rho}\, g_{\mu\nu,\lambda\rho} + \nabla_{(\mu}\Gamma_{\nu)} + \Gamma_\lambda{}^\eta{}_\mu\, g_{\eta\delta}\, g^{\lambda\rho}\, \Gamma_\rho{}^\delta{}_\nu + 2\,\Gamma_\delta{}^\lambda{}_\eta\, g^{\delta\rho}\, g_{\lambda(\mu}\,\Gamma_{\nu)}{}^\eta{}_\rho. \tag{5}$$

Here the comma indicates partial derivatives, the $\Gamma_\nu{}^\mu{}_\eta = 1/2\, g^{\mu\lambda}\,(g_{\lambda\eta,\nu} + g_{\nu\lambda,\eta} - g_{\nu\eta,\lambda})$ are the Christoffel symbols, ∇ is the covariant derivative operator of $g_{\mu\nu}$, and the summation rule applies. The Γ^μ denote the contracted Christoffel symbols $\Gamma^\mu = g^{\nu\eta}\,\Gamma_\nu{}^\mu{}_\eta$ which (together with the functions $F_\nu = g_{\nu\mu}\,F^\mu$ considered in the following) are being formally treated as if they defined a vector field (which, of course, they do not). Thus $\Gamma_\nu = g_{\nu\mu}\,\Gamma^\mu$ and $\nabla_\mu\Gamma_\nu = \partial_\mu\Gamma_\nu - \Gamma_\mu{}^\lambda{}_\nu\,\Gamma_\lambda$.

The form (5) emphasizes the first term on the right hand side of (2) which is obtained by applying to the unknown $g_{\mu\nu}$ a wave operator, a type of differential operator for which a good theory is available. The following three terms of second order in (2) prevent the direct application of PDE results. They are hidden in the representation (5) in the second term on the right hand side. It turns out that the apparent difficulties dissolve once the role of the contracted Christoffel symbols Γ^ν is recognized [65].

Let (M, g) denote some Lorentz manifold and let $S = \{t = 0\}$, with some coordinate function t, be some space-like hypersurface of it. Consider a map $\mathbb{R}^4 \ni x^\lambda \to F^\mu(x^\lambda) \in \mathbb{R}^4$. Ignoring subtleties arising from differentiability questions, everything is assumed to be smooth. Local Cauchy data x^λ, $\partial_t x^\lambda$ on S determine a unique local solution to the Cauchy problem for the semi-linear system of wave equations

$$\Box_g\, x^\mu = -F^\mu(x^\lambda), \tag{6}$$

where $\Box_g = \nabla_{\nu'}\,\nabla^{\nu'}$ denotes the scalar wave operator defined by g. If the dx^μ are chosen initially to be pointwise linearly independent, the solution provides a local coordinate system x^μ. In terms of these coordinates the system above simply takes the form

$$\Gamma^\mu(x^\lambda) = F^\mu(x^\lambda). \tag{7}$$

As a consequence

- by a suitable choice of coordinates the contracted Christoffel symbols can locally be made to agree with any prescribed set of functions F^μ,
- in turn, these *gauge source functions* and the initial data determine the coordinates uniquely,
- for a given metric g any coordinate system is characterized by suitable gauge source functions and initial data.

This suggests replacing the functions Γ^ν in (5) by freely chosen gauge source functions F^ν. The vacuum field equations then take the form

$$0 = R^F_{\mu\nu} \equiv -\frac{1}{2}\, g^{\lambda\rho}\, g_{\mu\nu,\lambda\rho} + \nabla_{(\mu}\, F_{\nu)} + \Gamma_\lambda{}^\eta{}_\mu\, g_{\eta\delta}\, g^{\lambda\rho}\, \Gamma_\rho{}^\delta{}_\nu + 2\,\Gamma_\delta{}^\lambda{}_\eta\, g^{\delta\rho}\, g_{\lambda(\mu}\,\Gamma_{\nu)}{}^\eta{}_\rho, \tag{8}$$

of a system of quasi-linear wave equations for the $g_{\mu\nu}$, which represents the *main evolution or reduced system* of our procedure. For this system the local Cauchy problem with appropriate data on a space-like hypersurface S is well posed, which means that there can be shown the existence and the uniqueness of solutions and their stable dependence on the initial data (cf. [77] for a detailed discussion). A few interesting observations are to be made about this procedure.

The metric coefficients $g_{\mu\nu}$ obtained as solution to (8) are given in terms of coordinates x^{μ} which are determined implicitly by the gauge source functions, the initial data, and Eq. (8). Do these coordinates really satisfy the *gauge system* (6) resp. its implicit coordinate expression (7)? For the moment we assume this to be the case and consider this question again in Sect. (3.4).

The domain on which the x^{μ} form a good coordinate system depends on the initial data, the gauge source functions, and on the solution g itself. Since information on g is only acquired by solving (8), little can be said a priori on the domain of existence of the coordinates. In this respect there is in general no difference between *harmonic coordinates* (more appropriately called now *wave coordinates*), characterized by $F^{\mu} = 0$, and coordinates defined by other gauge source functions. Without special precautions nothing prevents the dx^{μ} to become linearly dependent, the slices $x^0 = t = const.$ with $\{t = 0\} = S$ to turn time-like, the coordinates to develop an undesirable asymptotic behaviour, etc. If a coordinate system turns bad, one will have to construct further coordinates and may end up with a collection of overlapping coordinate patches which serve to define the manifold M underlying the solution space-time.

In practice, in particular in numerical calculations of space-times, one would like to avoid such situations and try to find coordinates which cover the entire solution. An interesting way of using gauge source functions to control the evolution of the slicing in a numerical code has recently been put forward by Pretorius in his work[5] following up [117]. It remains to be seen whether the method will allow him to steer the slicing unscathed through the dangers of sufficiently general long time evolutions.

Lindblad and Rodnianski discuss the small data, global existence for the Einstein vacuum equations in wave coordinates and it turns out that the coordinates can be arranged to cover the complete solution space-time [101]. These solutions have an asymptotic structure qualitatively similar to that of Minkowski space. If the data were slightly changed one would expect the coordinates to be still well behaved by stability considerations. Such considerations become quite delicate in global problems, however, and unexpected things may happen when the data are increased to admit the development of singularities and black holes. In any case the behaviour of the coordinates has to be controlled in the context of the evolution problem, in which little is known about the metric a priori.

In [73] have been discussed coordinates which cover the entire Schwarzschild-Kruskal space-time up to the singularity and even extend smoothly through the null infinities. Again, it is unclear how they will behave if the underlying solution space-time is perturbed. This general class of coordinates is based on certain geometrically distinguished curves called *conformal geodesics* and the defining equations of these curves contain elements which might allow one to control the behaviour of the coordinates over long time intervals. While these coordinates can be characterized *in principle* in terms of the gauge source functions considered above, there is no way of identifying these functions without knowing the solution g. Therefore one needs a different reduction procedure to incorporate these coordinates into hyperbolic evolution equations [69]. In this reduction the coordinates are *characterized by explicit conditions on the unknowns* and one might hope that the question which we left open above does not arise. It comes back as the question of certain constraints being satisfied during the evolution.

The last example shows that it is not easy to identify good gauge source functions. It shows also that there may be good reasons for analysing reductions different from the one indicated above. The role of the gauge source functions may be assumed then by quite different quantities. Motivated by problems in numerical relativity there has been considered a large variety of different reductions in recent years, based on different representations of the field equations, different unknowns, and different types of gauge conditions. Many of these reductions lead to hyperbolic main evolution systems.

[5] In preparation

3.3 The main evolution system

The main evolution system is clearly most important for working out local existence, uniqueness, smoothness, and more specific properties of solutions and for calculating solutions numerically. To avoid entering technicalities I shall only make a few general remarks about it.

In analytical work the main goal in choosing this system is to be able to exploit the intrinsic hyperbolicity of the equations. This does not mean that the system needs to be hyperbolic. Some useful gauge conditions are elliptic in nature and there have been studied for instance hyperbolic-elliptic reduced systems by Andersson and Moncrief [11].

In recent years there has been a tendency in the general investigation of non-linear evolution equations to study solutions of low smoothness [135]. Klainerman and Rodnianski [96] study Einstein evolution under smoothness assumptions which are weaker than those considered up to a few years ago (cf. [88]). Maxwell [104] discusses the existence of rough solutions to the constraints equations.

If one is mainly interested in physical phenomena, these activities may appear quite esoteric and it will have to be seen to which extent solutions so obtained admit a physical interpretation. As indicated above, questions about precise and low smoothness requirements can become, however, unavoidable in the discussion of singularities and Cauchy horizons. Moreover, trying to push to their lower limits the smoothness requirements under which the Einstein equations still make sense forces one to explore the specific structure of the equations much more carefully and it is bound to lead to more precise information on the evolution.

Once local existence has been treated, there may be used other methods to get control on the long time evolution. While some hyperbolic main evolution systems imply energy estimates involving the Bel-Robinson tensor, this tensor may be used independent of any reduction to derive estimates on the solution.

In any case there needs to be erected some kind of rigid space-time structure, a foliation by space-like or null slices or a fixed coordinate system, relative to which estimates of the metric field are expressed and the evolution of this structure itself can be controlled. In the case of the global or semi-global non-linear stability results mentioned below one can lean back on information supplied by an explicit reference solution, choose on it a foliation determined by some suitable (evolution) law, and construct a foliation on the perturbed solution governed by the appropriately perturbed law. If the coalescence of two black holes is to be modelled under general assumptions, however, there are no reference solutions available and one has to develop an intuition for foliations with long life times and good evolution properties in the context of an existence proof or in the course of a numerical calculation.

These remarks also show that the needs of analytical and numerical work are different. In the latter one has to rely on an explicit main system for all times of the evolution (or at least change the system only a finite number of times). For the choice of this system the main requirement is the stability of the numerical evolution. So far this has essentially been a matter of trial and error and there appears to be no way to translate this requirement into a precise criterion in terms of algebraic or other properties of the system. Manifest hyperbolicity is reasonable but apparently not sufficient for that purpose. There is in use a class of systems, the so-called BSSN systems ([16], cf. also [1, 77, 78] for hyperbolic versions) some of which are not manifestly hyperbolic but nevertheless seem to lead to stable numerical evolutions. Understanding why this should be so or for which class of problems stability fails for these systems is a theoretical challenge.

3.4 The subsidiary system

In our discussion of the gauge system we left open the question whether the implicit gauge condition (7) is preserved under the evolution defined by (8). Here is the analytical argument. Equation (8) is of the form

$$R_{\mu\nu} = \nabla_{(\mu} Q_{\nu)}, \tag{9}$$

where $Q_\mu = \Gamma_\mu - F_\mu$ with the Γ_μ calculated from the solution $g_{\mu\nu}$. The requirement that $Q_\mu = 0$ may be considered as a kind of constraint for (9) [it is in fact related to the constraints considered in Sect. (3.1)]. The twice contracted Bianchi identity, which holds for any metric, reads $\nabla^\mu (R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu}) = 0$. Applying this to the equation above gives

$$\nabla_\mu \nabla^\mu Q_\nu + R^\mu{}_\nu Q_\mu = 0, \tag{10}$$

and thus a *subsidiary system* of wave equations for the quantities Q_ν.

It turns out that for data satisfying the constraints and the gauge condition $Q_\mu = 0$ on S the solution to (8) satisfies $\partial_t Q_\mu = 0$ on S. Since we have thus vanishing Cauchy data for the hyperbolic system (10), it follows that Q_μ vanishes and $g_{\mu\nu}$ does solve (3). This closes the argument in the continuum model. Note that only the homogeneity and the uniqueness property of (10) are being used here. There is no further role for (10) in analytical studies.

The situation is quite different in the discrete model. Most numerical calculations of solutions to Einstein's equations are being plagued by an undesirably fast growth of constraint violations. In fact, many workers in the field report on seemingly unmotivated catastrophic blow-ups of numerical calculations at a stage of the numerical time evolution where coordinate or curvature singularities were not to be expected. Understanding this situation requires and deserves a major effort. In the following I shall not discuss any of the remedies which have been suggested (cf. [30,82]), or the stability analyses of the subsidiary system, considered as a linear system on a given background (cf. [64,143]). I would like to identify instead possible sources of the problem in the analytical structure of the equations.

Analytically, (10) is simply a differential identity implied by (9). It is hard to see, however, how to devise a numerical scheme for the second order wave equations (9) for which the subsidiary system (10), which is of third order in the metric, could be identified as an identity. The relations between the two systems will therefore become obscured, and the development of the constraint violation is not easy to analyse.

If we observe (9) in (10), the latter takes the form

$$\nabla_\mu \nabla^\mu Q_\nu + Q^\mu \nabla_{(\mu} Q_{\nu)} = 0, \tag{11}$$

of a manifestly non-linear wave equation. In the continuum model this equation would still imply $Q_\nu = 0$. In the discrete model the quantity Q_ν comes, however, with an error initially and develops further errors during the evolution. The detailed propagation properties of (11) therefore become important. If (10) were a linear system on a given background, standard energy estimates would admit and cannot exclude an exponential growth of the unknown Q_μ but would admit nothing faster than that. The non-linearity of (11) might induce, however, a much faster growth of the constraint violation.

Analysing this situation is not easy. While (9) can be studied independently, Eq. (11) does not decouple from (9). The latter supplies the metric defining the background for (11) and the growth of Q_μ has an effect on the evolution of the metric by (9). The coupling of (11) to (9) defines a second, though less direct, non-linearity in the evolution of Q_μ. Nevertheless, considering the background metric in (11) as given, should provide some understanding of the propagation properties of that equation.

If one considers (11) as an equation on Minkowski space, it turns out that it is easy to find Cauchy data Q_μ and $\partial_t Q_\mu$ on $\{t = 0\}$ for which the solutions develop poles after finite coordinate times $t_* > 0$ (cf. [75] for details). These data can be chosen pointwise as small as one likes, though $t_* \to \infty$ if the data approach zero. This suggests that one may have a relatively stable numerical evolution for a while but at a certain stage effects due to the non-linearity in (11) take over and induce a catastrophic collapse of the calculation after some finite time.

There are quite a few interesting questions to be answered and the situation certainly deserves closer analytical and numerical study. We have discussed here the non-linearity of the subsidiary system in a specific example. Most likely, non-linearities are found in all subsidiary systems. To what extent their effects are different and whether there exist preferred cases remains to be seen.

3.5 Global studies

In the following we want to present some global or semi-global results on the existence of solutions to Einstein’s field equations. The field has seen a rapid development in recent years. It will not be possible here to give due reference to all the important contributions and we will have to make a somewhat arbitrary choice. In the following it will be convenient to make a distinction between vacuum solutions which are *asymptotically flat* and solutions with *compact time sections*. In spite of the fact that there are important cosmological models with non-compact time slices, by a *cosmological space-time* will be meant in the following always a vacuum solution with compact time slices.

3.5.1 Cosmological space-times

In the context of cosmological solutions there exists a large variety of possible assumptions on the (local) symmetries and the topologies of space-sections. The tendency of much of the recent work has been to analyse the global behaviour of the solutions under strongly simplifying assumptions and then, relaxing them step by step, to work ones way up to develop in the process the insight and the technical means to analyse in the end also quite general classes of solutions. Of the many results obtained in this program only a few can be considered in the following.

What are the questions to be asked? If one tries to work out a ‘global’ existence result by analysing a Cauchy problem for a certain class of solutions and it can be shown that the maximal globally hyperbolic solutions determined by the data are all geodesically complete in the past and in the future one may still be interested in the precise asymptotic behaviour near past and future time-like infinity but the essential goal has been reached.

In general the situation will be more complicated. It may happen instead that the solution will be geodesically complete in one time direction (or not at all) but in the other direction there are obstructions to completeness. Extreme (and, in a sense, the clearest) possibilities which can occur are that the solution approaches a smooth Cauchy horizon or that it develops of a curvature singularity at which an invariant built from the curvature tensor such as the Kretschmann scalar $R_{\mu\nu\lambda\rho}\, R^{\mu\nu\lambda\rho}$ becomes unbounded. This may happen globally when a family of space-like hypersurfaces approaches the corresponding end of the evolution or it may happen only locally. There are possible all kinds of situations in between and combinations thereof. One would like to have a detailed description of the respective behaviour and relate specific types of behaviour to the ‘size’ of the corresponding subset of the space of initial data.

Since any statement about cosmic censorship must refer to *generic* classes of initial data, the analysis of solutions with symmetries cannot give a final answer to the question of cosmic censorship. It can, however, provide insight into the nature of the problem to analyse the question in the class of space-times with a given symmetry. Moreover, if the subset of solutions which do not admit Cauchy horizons contains a dense open subset of the data set one would be prepared to consider this as evidence for cosmic censorship (*restricted cosmic censorship*).

The ‘simplest’ and still quite interesting case to consider is given by space-times which are *spatially homogeneous* in the sense that they admit a 3-dimensional space of Killing fields which at each point generate the tangent space of a foliation by (compact) space-like hypersurfaces. In the universal covering space the (not necessarily compact) leaves of the foliation can then be considered as orbits of a 3-dimensional Lie group G of isometries. These *Bianchi space-times* have been classified in terms of the Lie algebras of their isometry groups.

For spatially homogeneous space-times the Einstein vacuum equations essentially reduce to systems of ODE’s. Nevertheless, their analysis turns out quite difficult, not because of the various cases in the Bianchi classification but because of the various phenomena which can occur. The Bianchi classification consists of two classes. We shall consider only one of them, in which the structure constants of the Lie algebras can be specified in terms of a symmetric matrix which can be assumed to be diagonalized (case A). In that case the Einstein vacuum equations have been written by Hsu and Wainwright [139] as an ODE and a constraint for an unknown $u = (N_1, N_2, N_3, \Sigma_+, \Sigma_-)$ of the form

$$\frac{d}{dt} u = f(u), \quad q(u) = 1,$$

with q a quadratic polynomial in the components of u. The first three components of u contain essentially the information on the Lie algebra and the other two information on the second fundamental form on the leaves of the foliation.

When the Lie algebra is commutative (Bianchi I) only Σ_+, Σ_- do not vanish and the constraint reduces to the equations $\Sigma_+^2 + \Sigma_-^2 = 1$, which defines the *Kasner circle* parameterizing these solutions. The fields Σ_+, Σ_- are in fact constant, i.e. the solutions are fixed points of the dynamical system above. The solutions are given explicitly by the *Kasner metrics*

$$ds^2 = -dt^2 + \sum_{a=1}^{3} t^{2p_a} (dx^a)^2,$$

on $]0, \infty[\times\mathbb{R}^3$ or, after a periodic identification, on $]0, \infty[\times T^3$. The real numbers p_a satisfy the Kasner relations $\sum_{i=1}^{3} p_a = 1$ and $\sum_{i=1}^{3} p_a^2 = 1$ and are related to the two unknowns above by $\Sigma_+ = \frac{3}{2}(p_2+p_3)-1$ and $\Sigma_- = \frac{\sqrt{3}}{2}(p_2 - p_3)$.

The three points $(-1, 0)$, $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ on the Kasner circle correspond to flat solutions, which in the case without identifications are isometric to open subsets of Minkowski space given by the future of the intersection of two null hyperplanes. In the non-flat case the causal geodesics are incomplete in the past (where $t \downarrow 0$) and approach a curvature singularity there. Note that the volume of the time slices grows monotonically with t but the growth happens locally in an anisotropic way because the p_a can not all have the same sign.

Of particular interest is the case where the group G is $SU(2)$ (Bianchi IX). It contains a subclass, the Taub type IX solutions, characterized by linear conditions on the components of u. These solutions can also be given explicitly and it turns out that they admit in the past (compact) Cauchy horizons and smooth extensions (there do exist in fact non-isometric extensions [42]) across it, resulting in the Taub-NUT space-times.

The remaining 'generic Bianchi IX solutions' or *proper Mixmaster solutions* [105] show an interesting behaviour. They have a curvature singularity which is approached by the solutions with an oscillatory behaviour. The projections of their trajectories into the (Σ_+, Σ_-) - plane moves into the Kasner circle and stays there, approaching subsequently different points on the circle (cf. [22] and [120, 127] for numerical and analytical studies). The detailed studies of the Bianchi models show that curvature blow-up and the non-existence of Cauchy horizons in the contracting direction is a feature of generic Bianchi space-times, which shows that restricted strong cosmic censorship holds in this class of models [127], cf. also [45].

While being interesting in themselves, the results on the Mixmaster solutions are expected to be of a much wider significance. The BKL conjecture mentioned in section 2 suggests that the behaviour of these solutions near the singularity provides a model for the local behaviour of general cosmological solutions in the neighbourhood of singularities.

Moncrief initiated with the article [106] the global study of a class of vacuum solutions with two commuting space-like Killing fields and compact space sections called *Gowdy space-times*. With the spatial topology being that of the 3-torus T^3 the metric can be written in the form

$$g = \sqrt{\tau\, e^{-\lambda}}\, (-dt^2 + dx^2) + t\, (e^P (dy + Q\, dz)^2 + e^{-P}\, dz^2),$$

with x, y, z each being a coordinate on the circle S^1, the coordinate t taking values in $\mathbb{R}_+$, and the coefficients λ, P, Q depending only on t and x.

For these metrics the Einstein vacuum equations read

$$P_{,tt} - P_{,xx} = -\frac{1}{t}\, P_{,t} + e^P (Q_{,t}^2 - Q_{,x}^2),$$

$$Q_{,tt} - Q_{,xx} = -\frac{1}{t}\,Q_{,t} - 2\,(P_{,t}\,Q_{,t} - P_{,x}\,Q_{,x}),$$

and

$$\lambda_{,x} = -2\,t\,(P_{,t}\,P_{,x} + e^{2P}Q_{,t}\,Q_{,x}),$$

$$\lambda_{,t} = -t\,(P_{,t}^2 + P_{,x}^2 + e^{2P}(Q_{,t}^2 + Q_{,x}^2)).$$

If Cauchy data for P and Q are given for which the integral over the circle of the expression on the right hand side of the third equation vanishes, the discussion reduces to the analysis of the *semi-linear* system of wave equations for the functions P and Q, which essentially represent the two polarizations states of the gravitational field.

After Moncrief had given the first global existence proof for the solutions, these solutions and the nature of their singularities were studied by many authors. The analysis was largely assisted by the numerical work initiated by Berger and Moncrief [23]. It showed that the solutions tend to develop strong gradients (*spikes*) in the approach towards the singularity, which brought an important aspect into the analytical discussion. Kichenassamy and Rendall constructed families of solutions with singularities by Fuchsian methods (cf. the survey [124] and also the work by Chae and Chruściel [32]). Rendall and Weaver constructed families of Gowdy solutions with spikes from solutions without spikes, and distinguished 'true' spikes, which have a geometric meaning, from 'false' spikes [125]. Building on this and earlier work, Ringström recently showed that for a 'generic' set of initial data the corresponding solutions exhibit a curvature blow up on dense open subsets of the singularity [128]. Combining this result with the work by Chruściel and Lake on the occurrence of Cauchy horizons in Gowdy space-times [44], he concludes that this class of space-times satisfies a restricted strong cosmic censorship principle.

This short discussion hardly gives credit to the many important contributions which led to the 'final' answers. A more complete picture of these interesting developments can be obtained from the articles [5,9,21]. The latter is also particularly interesting because it highlights the remarkable, successful, and still ongoing interplay between numerical and analytical studies in this field.

The results mentioned above do not finish the analysis of solutions with two Killing fields. In fact, the Gowdy metrics considered here only define a 'negligible' subset of the set of all solutions with two Killing fields.

As a further step in the program solutions with only one Killing field should be analysed. A semi-global, non-linear stability result in this directions is obtained by Choquet-Bruhat, who generalizes previous work with Moncrief to show future completeness for a class of $U(1)$-symmetric vacuum solutions on manifolds of the form $M = \mathbb{R} \times \Sigma \times S^1$ with Cauchy hypersurface diffeomorphic to $\Sigma \times S^1$, where Σ is an orientable, compact surface of genus greater than 1 and the space-like Killing fields are assumed to be tangent to the fibres of the fibration defined by the projection $M \to \mathbb{R} \times \Sigma$ [33].

Generalizing even further, Andersson and Moncrief obtain a semi-global, non-linear stability result for vacuum solutions without imposing any symmetry conditions [12] (the solutions of [33] are not included). Denote by V the interior of the future light cone at the origin in Minkowski space and by τ the Minkowskian distance from the origin. Identification of points of V by the action of a suitable discrete subgroup of the Lorentz group yields a *reference space-time* of the form

$$M = \mathbb{R} \times S, \quad g = -d\tau^2 + \tau^2\,h,$$

where (S, h) denotes a 3-dimensional, compact hyperbolic space of sectional curvature -1. We assume ∂_τ to be future directed. The authors consider cases where (S, h) satisfies a certain *rigidity* condition. Identifying S with the set $\{\tau = 1\}$, the metric above induces on S the *reference data* $(h_{ab}, \chi_{ab} = h_{ab})$. It is shown in [12] that vacuum data on S sufficiently close to rigid reference data develop into solutions of the vacuum field equations for which the causal geodesics are future complete. The authors also describe the asymptotic decay of their solutions towards the reference solution.

For recent attempts to control the behaviour near the singularity under 'general' assumptions, to give precise meaning to the BKL conjecture, and to develop tools which would allow one to decide on its validity, we refer to the work by Andersson et al. [10] and Garfinkle [80]. There appears to be a general expectation that the BKL conjecture will turn out to be basically correct.

3.5.2 Asymptotically flat space-times

Asymptotically flat space-times provide the basic model of isolated self-gravitating systems such as stars, star systems, black holes, etc. and as such they are important for discussing many observable general relativistic phenomena, the analysis of radiative phenomena being at present the most important and urgent one. The modelling of stars is clearly an important task. Nevertheless we shall concentrate again on the vacuum case or situations with field theoretical matter models. On the one hand, progress in these cases will provide important insights into the behaviour of dynamical black holes, the coalescence of black holes, the resulting radiation fields etc., and possibly quantitative results about the latter, while on the other hand even these configurations still present major challenges.

Asymptotically flat space-times pose problems which do not occur in the cosmological context. The space-like slices are of infinite extent and on an asymptotically flat slice the fall-off behaviour near space-like infinity can, in terms of coordinates x^μ which realize the asymptotic flatness conditions, not be faster then

$$g_{\mu\nu} - \delta_{\mu\nu} = \frac{m}{2\,|x|}\,\delta_{\mu\nu} + o\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty,$$

unless the total mass m vanishes, in which case the space-time is flat by the positive mass theorem. This poses major problems for global existence proofs and, in particular, for the verification of the Penrose proposal under general assumptions.

The infinite extent of space-like slices also requires additional considerations in the numerical study of space-times, because numerical calculations need to be performed on finite computational grids. Quite a few technical considerations and also conceptional questions, some of which will be indicated below, are related to that fact.

Since gravitational radiation can either escape to null infinity or fall through an event horizon into a black hole, which is possibly (cf. the discussion below) generated by the radiation itself, the completeness properties of the space-times in the future and the structure of their time-like infinities can vary considerably. Already in stationary examples such as the Schwarzschild-Kruskal, the Reissner-Nordström, and the Kerr solution there arise extreme and quite distinct situations near time-like infinity, where singularities, event horizons, null infinities, and Cauchy horizons seem to meet in the standard causal pictures [84]. Hardly anything is known about the possibilities under general assumptions.

A central question in the field is again whether cosmic censorship is a valid principle. Since any precise statement about that principle will use the word generic, the general answer can only be obtained by analytical methods. Numerical methods are likely to play, however, a major role in the investigation of the possibilities.

At present the main question related to physical observations is concerned with concepts of radiation, the precise asymptotic behaviour at null infinity, and methods to derive quantitative results about radiative properties. Here numerical methods are bound to play a dominant role in the end, but before that quite a few analytical questions will need to be answered.

The two questions above are not independent of each other. Even if one only seeks to calculate the radiation field, if one were not interested in the interior of black holes and the structure of singularities, and if one knew somehow that singularities were always hidden behind event horizons, one could hardly ignore in the analysis of the long time evolution of gravitational fields the tendency of solutions to develop singularities. Furthermore, since the location of an event horizon is not known at a finite stage of the evolution, the domain of outer communication, comprising the far fields, can in the usual approach to initial value problems not be cleanly separated from the interior of a black hole. Present attempts in numerical calculations to cut out the singularity from the computational domain may become a delicate matter in long time calculations.

In the case of asymptotically flat solutions the number of interesting simplifying assumptions is much smaller than in the cosmological context. If stationary solutions are excluded and one insists on complete and regular far fields, only spherical or axial symmetry represent admissible symmetries. Additional fields need to be considered to analyse any dynamical behaviour in a spherically symmetric setting, because spherically symmetric vacuum solutions are, by the Birkhoff theorem, locally isometric to patches of the Schwarzschild-Kruskal solution. Of the large body of work dealing with spherically symmetric situations the following three contributions are particularly important.

In a remarkable series of articles Christodoulou analyses the formation of black holes and singularities for the spherically symmetric Einstein-scalar field equations. He gives conditions on the data for the avoidance and for the development of singularities, respectively. He further gives conditions under which the singularity will be hidden by an event horizon but he also finds solutions whose singularities can be seen by distant observers, thus showing the occurrence of naked singularities. Finally, he shows that the existence of naked singularities is in fact an unstable property for the spherically symmetric Einstein-scalar field equations, which supports the cosmic censorship hypothesis (cf. [37] and the references given there).

In a seminal article [36] and subsequent work Choptuik studies numerically one-parameter families of spherically symmetric Einstein-scalar fields for which the solutions disperse for values of the parameter below but form black holes for values of the parameter beyond a certain threshold value. This allows him to discover phenomena which would have been difficult to find by purely analytic methods. In particular, he finds self-similar critical solutions, finite or arbitrarily small black hole masses as the parameter approaches its critical value, and a certain scaling of the black hole masses with universal critical exponents. This work initiated quite a number of further investigations involving various different matter models (cf. [81] for a survey). It also led to systematic studies of the blow-up behaviour of solutions to other non-linear equations of mathematical physics ([24, 25]) which caused analysts to revise some of their previous conjectures concerning the global existence of solutions.

Dafermos and Rodnianski study the spherically symmetric Einstein-Maxwell scalar field equations, assuming that a regular event horizon has formed [51]. They give rigorous proof to Price’s result [118] that perturbations of gravitational fields show in terms of a suitably chosen advanced time coordinate a polynomial decay on the event horizon near time-like infinity. Moreover, they confirm results by Israel and Poisson [115] concerning the occurrence of weakly singular Cauchy horizons (cf. also [93, 108] for the general background and [21] for numerical studies of the interior of black holes). The significance of this result concerning the question of strong cosmic censorship remains an open question as long as only spherically symmetric situations are considered (fortunately some kind of censorship is enforced already by the presence of the event horizon).

Compared with the richness of these detailed results and observed phenomena, the study of the large scale structure of asymptotically flat vacuum solutions *without symmetries* is still in its infancy. Compared, however, with the technical difficulties to be overcome, quite some progress has been achieved in the last twenty years. All the global or semi-global results concerning asymptotically flat solutions without symmetries which have been obtained so far are stability results which show the existence of geodesically complete or future complete solutions for *small data*, i.e. for data which are in suitable norms close to the data of a well understood reference solution, provided by Minkowski space or parts of it. Analogous results for comparison solutions with black holes are not available yet.

An early semi-global existence result was obtained in the article [67], where it was shown that smooth hyperboloidal data sufficiently close to Minkowskian hyperboloidal data develop into solutions of the vacuum field equations which have a smooth future complete structure at null infinity for which time-like infinity is represented in suitable conformal extensions by a regular point. The analysis uses the conformal behaviour of the Einstein equations in an explicit way and it is carried out in terms of a conformally rescaled metric with respect to which null infinity and time-like infinity are at a finite location. The solutions show in particular the *peeling behaviour*, so that along outgoing null geodesics with affine parameter r the components of the conformal Weyl tensor satisfy in a suitable orthonormal frame a fall-off behaviour near null infinity which can be expressed in terms of certain entire powers of r^{-1}.

The particular initial value problem was considered to avoid in a first step the complications at space-like infinity. While the setting was thus intended as a preparation for a global study, it acquired in the meantime some interest for the numerical calculation of radiation fields based on the underlying conformal field equations [63, 87, 90].

Soon afterwards attempts were made to use this result to construct complete solutions. The idea was to avoid the complications at space-like infinity by constructing Cauchy data for which the evolution near space-like infinity could be controlled explicitly and the existence of smooth hyperboloidal slices close to Minkowskian ones could be shown. Cutler and Wald managed to construct non-trivial data for the Einstein-Maxwell equations which are spherically symmetric outside a compact set [49]. Thus they were able to establish the existence of geodesically complete solutions to the Einstein-Maxwell equations with smooth and complete structures and non-trivial radiation fields at future and past null infinity (cf. [68]). More recently, using the result of Corvino mentioned above, Chruściel and Delay showed the existence of a large class of vacuum solutions with these properties [40]. To some extent this justifies the Penrose proposal but the fact that these solution are exactly Schwarzschild near space-like infinity clearly leaves space for generalizations.

The first global existence result for a general class of asymptotically flat data was obtained by Christodoulou and Klainerman [38]. They exploit the conformal behaviour of the solutions only indirectly. With a considerable effort they skillfully manage to control the behaviour of the fields near space-like infinity. They show that vacuum data satisfying certain regularity conditions near space-like infinity (which include the vanishing of the linear momentum) and suitable smallness conditions on the initial slice develop into a unique, globally hyperbolic solution of Einstein's vacuum field equations which is geodesically complete and asymptotically flat in the sense that the Riemann curvature approaches zero along any geodesic if the affine parameter tends to infinity. The global structure of their solutions is qualitatively similar to that of the space-times considered above. Without further conditions on the data, however, the peeling behaviour in its usual form is not satisfied and the smoothness of the conformal structure at null infinity is thus weaker than that required by the Penrose proposal.

Using instead of a maximal foliation near space-like infinity a double null foliation, Klainerman and Nicolò prove a similar though technically simplified result [94]. Revisiting their proof in the light of the results discussed above they give asymptotic conditions on the data near space-like infinity which allow them to verify the peeling behaviour [95].

More recently Lindblad and Rodnianski obtained a technically much more simplified global existence proof by using the Einstein vacuum equations in wave coordinates [101]. In this article they avoid the difficulties near space-like infinity by starting from Cauchy data which are exactly Schwarzschild near space-like infinity. In a subsequent article global stability of Minkowski space is obtained under quite general assumptions on the asymptotic behaviour of the initial data [102]. It is not clear yet to what extent their method will allow them to gain precise control on the smoothness resp. peeling behaviour of the fields near null infinity.

Nevertheless, any analysis of the evolution of the fields near space-like infinity will need to impose some restrictions on the initial data to obtain control on the evolution and on the behaviour of the fields near null infinity. In the articles [69] and [70] has been developed a setting which allows one to analyse under suitable regularity conditions on the data at space-like infinity the evolution of the fields near *the critical sets where space-like infinity touches the null infinities* (a notion made precise in [70]) at all orders. As a result it is shown that even for (conformal) data of maximal smoothness the solution can develop at all orders logarithmic singularities at the critical set and consequently on null infinity (cf. [72]). Furthermore, there is obtained for the first time a series of analytical expressions which relates a certain class of obstructions to the smoothness at null infinity to certain specific fall-off properties of the initial data. While the analysis is carried out in an algorithmic way, the complexity of certain expressions grows so quickly with increasing order that the possible existence of a further class of obstructions was left open.

Recently the remaining case was studied by Valiente Kroon by using an algebraic computer program [137]. He did find further obstructions and remarkably, up to the order to which the calculation could be performed, there is now evidence that in the case of time reflection symmetric data smoothness (resp. C^k) at null infinity requires the data to be *asymptotically static* (resp. asymptotically static up to an order p for a certain integer $p = p(k)$ which still needs to be determined). Note that this is much weaker than 'static in a neighbourhood of space-like infinity', in which case smoothness at null infinity is easily shown. For more general data [138] the situation is not so clear yet but *asymptotic stationarity* may play an important role. A more detailed discussion of the situation is given in [74].

Since the work referred to above relies on a conformal representation of the Einstein equations which only seems to be useful in 4 dimensions (cf. [71]), it may be worth mentioning here that other conformal field equations have been suggested recently by Anderson which work in all *even* space-time dimensions [3]. It can be expected that many of the results obtained in 4 dimensions can be generalized to all even dimensions [4]. Recent results by Hollands and Wald [86] and Rendall [122] suggest, however, that conformal equations with similar properties do not exist in odd space-time dimensions.

Besides clarifying the asymptotic behaviour of gravitational fields one of the main motivations for the work in [70] was to provide a setting which would allow one to *calculate numerically entire space-times on finite grids, including their asymptotic structure and radiation fields*. Once the numerical evolution can be pushed past the critical set the solution will contain hyperboloidal slices and earlier analytical and numerical results can be applied. The numerical implementation for this program has not been given yet. As shown by the following example, there are interesting question which would be difficult to study by other numerical approaches.

Beig and O'Murchadha describe in [17] an interesting construction of asymptotically flat initial data on $S = \mathbb{R}^3$ with trapped surfaces and suggest that these surfaces are due to concentration of gravitational radiation (these data are not obtained by using initial hypersurfaces with non-trivial topology but by analysing sequences h_n of conformal metrics with positive Yamabe number on S^3 for which the Yamabe number approaches zero as $n \to \infty$). Since the data are time reflection symmetric, the singularity to be expected in the future must be considered as a reflection of the singularity in the past and cannot be interpreted as being due to radiation. In [18] similar data are constructed without the time reflection symmetry, but this by itself does not preclude the existence of a singularity in the past.

Dafermos has recently shown the existence of maximal developments arising from asymptotically flat Cauchy data for the spherically symmetric Einstein-scalar field equations, which contain an event horizon in the future but for which all causal geodesics are complete in the past [50]. One would consider a solution to the Einstein *vacuum* field equations with these global properties as presenting a black hole due to a collapse of gravitational radiation. Clearly, it is an interesting question whether such solutions do exist or whether they are excluded by the field equations.

This question cannot be answered by analysing spherically symmetric situations. Since the techniques used in [50] only apply to wave equations in two space-time dimensions, the answer is not known. If one could perform numerical calculations which cover the maximal globally hyperbolic solution space-time one might be able to show the existence of solutions as indicated above. A natural problem to consider here is the characteristic initial value problem for the conformal vacuum field equations where data are prescribed on a cone representing past null infinity (cf. [66]). In that case one would have perfect control on the past but there arise other difficulties (non-smoothness of the initial hypersurface, how to prescribe the data on the cone to obtain an asymptotically flat solution, development of caustics, the transition of the numerical evolution process through space-like infinity, etc.) which let the setting indicated above look more attractive.

We end this article by pointing out a problem which did not receive much attention yet but which will become important as soon as certain technical questions, which are still under investigation, will be understood. Depending on the way it is approached, it poses itself differently, but it is most severe if radiation fields are to be calculated numerically. There are essentially three different approaches to the numerical calculation of radiation fields: (i) the standard approach based on Cauchy data and the introduction of an artificial time-like boundary to make the computational grid finite, (ii) semi-global approaches based either on characteristic foliations and characteristic initial hypersurfaces extending to null infinity (possibly combined in a 'Cauchy-characteristic matching' with the standard approach) or based on the conformal field equations and hyperboloidal hypersurfaces, (iii) global approaches, like the one indicated above, which aspire to calculate entire space-times (possibly including their asymptotics). In all three cases inner boundaries may be considered to avoid the approach to singularities, but such boundaries will not be considered here.

In the standard approach particular technical problems are introduced by the presence of the time-like boundary, which, being in general not distinguished geometrically, is somewhat unnatural. Nevertheless, it has been shown by Nagy and Friedrich that the vacuum field equations admit *well posed initial-boundary value problems* (which includes, of course, that all constraints be satisfied). In the article [76] has also been discussed the principal freedom to prescribe boundary conditions and data and the main difficulties and specific features, such as a certain non-covariance of the problem under general assumptions on the boundary, have been pointed out.

It turns out that three real functions can be prescribed on a time-like boundary. In the setting considered in [76] these are the mean extrinsic curvature, which can be understood as controlling the evolution of the boundary in the solution space-time (it does not suffice to say 'the boundary is the hypersurface $\{x = 0\}$' for some coordinate x), and the other two are components of the conformal Weyl tensor which may be interpreted as controlling the two radiative degrees of freedom, though in general a fully satisfactory physical interpretation of these data does not exist.

As pointed out in Sect. 3.1, a non-constant mean extrinsic curvature creates problems in the analysis of the constraints on space-like hypersurfaces. Therefore is is worth mentioning that the choice of a non-constant mean extrinsic curvature on the time-like boundary also creates certain difficulties (cf. [76]). Though these are quite different from the ones encountered on space-like hypersurfaces, it seems to indicate that something important about the mean extrinsic curvature is not understood yet. We shall not be concerned here with this question, however, because it does not prevent us from solving the initial-boundary value problem in all generality.

While the representation of the field equations considered in [76] has been used in numerical calculations before (cf. [63]), it is not the one used in the majority of general relativistic numerical codes and there is now a considerable amount of work being done to derive similar results based on other main evolution equations (cf. [126,129] and the references given there). It can therefore be expected that the basic numerical problems arising from the initial-boundary value problem will soon be overcome. Moreover, numerical experiments will also show how the boundary data must be prescribed to ensure a regular long time evolution of the boundary.

One will then have to provide a meaningful concept of *outgoing radiation* in an initial-boundary value problem. A well-defined rigorous definition is not in sight since in general there does not exist a distinguished outgoing null vector field transverse to the boundary (the spherically symmetric case is trivial and in no way representative). We shall not be concerned with this question here, pretending that some answer can be given. Then there will still remain the question: *How should one dispose of the freedom to prescribe boundary data?* This problem has hardly been considered so far and physical intuition is not likely to give an answer.

That I am not raising here a purely academic question is illustrated by some recent calculations. Allen et al. intend to study in the article [2] radiation tails for black hole evolutions by solving numerically initial boundary value problems for wave equations on a Schwarzschild background. The problem is readily reduced to a problem for a wave equation of the type $\partial_u \, \partial_v \, \psi = U_l(r) \, \psi$, where u and v denote the standard retarded and advanced Schwarzschild time coordinates. The authors prescribe certain initial data and impose the boundary condition

$$\partial_v \, \psi = 0 \quad \text{on} \quad T = \{r = r_0\}, \tag{12}$$

for some suitable value $r_0 > 2\,m$ of the standard Schwarzschild coordinate r. It turns out that radiation tails as predicted by Price's law [118] cannot be identified in the subsequent calculations. The authors conclude in their summary: *We have shown that finite-radius boundary conditions prevent the formation of power-law tails*.

The results of these numerical calculations have been confirmed analytically. It has been shown by Dafermos and Rodnianski [52] that in the setting of [2] any tails vanish on the event horizon faster than $p(v)^{-1}$ as $v \to \infty$, where p is any polynomial of the advanced time coordinate v.

One should not think, however, that these rigorous results confirm the conclusion above. Gundlach et al. successfully verify Price's law numerically by solving characteristic initial value problems on the same background [83]. This is in accordance with the analytical results of [51], where the characteristic initial value problem is analysed with data prescribed on a pair of intersecting null hypersurfaces N_{out}, N_{in} which extend to future null infinity and across the horizon respectively.

It follows now from the general theory of initial-boundary value problems that the data given on N_{in} and the data induced on a time-like hypersurface T which extends from $N_{out} \cap N_{in}$ to future time-like infinity determine the solutions considered in [83] and [51] uniquely in the future domain of dependence D^+ of $T \cup N_{in}$ (the set of points p for which any inextensible past directed causal curve through p meets $T \cup N_{in}$). There is no reason why the solution induced on D^+ by the numerical results of [83] should not be reproducible by a numerical calculation based on the data on T and N_{in}. For this purpose one would have to require the boundary condition

$$\partial_v \, \psi = d \quad \text{on} \quad T, \tag{13}$$

where the function d on T would have to be read off from the solution given by [83].

The calculations in [2] thus do not indicate that power-law tails cannot be calculated by solving initial-boundary value problems, they just confirm that there is a serious problem: *In general, one does not know the 'correct' boundary data on the right hand side of* (13).

It may be said that the radiation signals one is really interested in will not be as delicate as the radiation tails. But this does not tell us how strongly changes in the boundary data will affect wave forms and, in particular, what will happen in a long time calculation. By a wrong choice of boundary data, the system which was to be modelled may be affected so drastically that the approximate radiation field calculated in the end has little do with the system envisaged originally. To what extent this can be the case can probably only be explored by numerical experiments.

The discussion above seems to indicate that the calculation of wave signals based on characteristic, hyperboloidal, or standard Cauchy problems will more robust than those based on initial boundary value problems. But whatever one does, there will always be an arbitrariness in the choice of data. The work initiated by Corvino, discussed in Sect. 3.1, clearly illustrates the large freedom to deform the data outside compact sets, or, in other words, how to choose the extension near space-like infinity of the data which characterize 'our system' on a compact set.

This leaves one with the task of minimizing the import of accidental information. The notion of *spurious radiation* may have an intuitive meaning but it cannot be well defined. There are two analytical suggestions, however, which may supply useful criteria. Dain was able to associate with a given Cauchy data set a certain number, related to the *global structure* in a similar sense as the total mass but defined in quite a different way, which vanishes if and only if the data are stationary [54]. This number may thus be considered as a measure for the *radiation content* of the data. It remains to be seen, how changes in the data are reflected in changes of this number and whether minimizing this quantity in suitable classes of data leads to applicable results.

The second suggestion follows from the analysis of asymptotic smoothness properties. As discussed above, smoothness requirements at null infinity imply *asymptotic conditions* on the Cauchy data near space-like infinity like asymptotic staticity or asymptotic stationarity. These may be interpreted as suppressing 'spurious radiation' near space-like infinity.

In any case there will remain some freedom and one needs to assess the effect of changes of the data near space-like infinity on the wave forms. In the end one may have to look for *features of radiation signals which are stable under the remaining admissible changes of the data*. Identifying such features and understanding to what extent they characterize the type of the source is certainly an important task. It will require numerical as well as analytical input.

References

[1] M. Alcubierre, B. Brügmann, M. Miller, and W.-M. Suen, Phys. Rev. D **60**, 064017 (1999).

[2] E.W. Allen, E. Buckmiller, L.M. Burko, and R.H. Price, Phys. Rev. D **70**, 044038 (2004).

[3] M.T. Anderson, Existence and stability of even dimensional asymptotically de Sitter spaces. http://xxx.lanl.gov/abs/gr-qc/0408072.

[4] M.T. Anderson and P.T. Chruściel, Asymptotically simple solutions of the vacuum Einstein equations in even dimensions. http://xxx.lanl.gov/abs/gr-qc/0412020.

[5] L. Andersson, The global existence problem in general relativity, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).

[6] L. Andersson and P.T. Chruściel, Commun. Math. Phys. **161**, 533–568 (1994).

[7] L. Andersson, P.T. Chruściel, Solutions of the constraint equations in general relativity satisfying hyperboloidal boundary conditions. Dissertationes Mathematicae Polska Akademia Nauk, Inst. Matem., Warszawa, 1996.

[8] L. Andersson, P.T. Chruściel, and H. Friedrich, Commun. Math. Phys. **149**, 587–612 (1992).

[9] L. Andersson, H. van Elst, and C. Uggla, Class. Quantum. Gravity **21**, S29–S57 (2004).

[10] L. Andersson, H. van Elst, W. Lim, and C. Uggla, Phys. Rev. Lett. **94**, 051101 (2005).

[11] L. Andersson and V. Moncrief, Ann. Henri Poincaré **4**, 1–34 (2003).

[12] L. Andersson, V. Moncrief, Future complete vacuum space-times, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).

[13] R. Bartnik, Commun. Pure Appl. Math. **39**, 661–693 (1986).

[14] R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141–144 (1988).

[15] R. Bartnik and J. Isenberg, The constraint equations, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).

[16] T. Baumgarte and S. Shapiro, Phys. Rev. D **59**, 024007 (1999).

[17] R. Beig and N. O'Murchadha, Phys. Rev. Lett. **66**, 2421–2424 (1991).

[18] R. Beig and N. O'Murchadha, Class. Quantum Gravity **11**, 419–430 (1994).

[19] V. Belinskii, I. Khalatnikov, and E. Lifshitz, Adv. in Phys. **19**, 525–573 (1970).

[20] V. Belinskii, I. Khalatnikov, and E. Lifshitz, Adv. in Phys. **31**, 639–667 (1982).

[21] B. Berger, Numerical Approaches to Space-Time Singularities. Living Reviews in Relativity 2002.

[22] B. Berger, D. Garfinkle, and E. Strasser, Class. Quantum Gravity **14** L29–L36 (1997).

[23] B. Berger and V. Moncrief, Phys. Rev. D **48**, 4676–4687 (1993).

[24] P. Bizoń Acta Phys. Polonica B **33**, 1893–1922 (2002).

[25] P. Bizoń and Z. Tabor, Phys. Rev. D **64**, 121701-1–121701-4 (2001).

[26] H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner, Proc. Roy. Soc. A **269**, 21–52 (1962).

[27] T. Brainerd and C. Kochanek (eds.), Gravitational Lensing: Recent Progress and Future Go. ASP Conference Series, Vol. CS-237, 2001.

[28] H.L. Bray, J. Diff. Geom. **59**, 177–267 (2001).

[29] H. Bray and P. Chruściel, The Penrose inequality, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).

[30] O. Brodbeck, S. Fritelli, P. Hübner, and O. Reula, J. Math. Phys. **40**, 909–923 (1999).

[31] B. Carter, Has the black hole equilibrium problem been solved?, in: The Eighth Marcel Grossmann Meeting, edited by T. Piran and R. Ruffini (World Scientific, Singapore, 1999).

[32] M. Chae and P. Chruściel, Commun. Pure Appl. Math. **57**, 1015–1074 (2004).

[33] Y. Choquet-Bruhat, Future complete $U(1)$ symmetric Einsteinian space-times, the unpolarized case, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).

[34] Y. Choquet-Bruhat and R. Geroch, Commun. Math. Phys. **14**, 329–335 (1969).
[35] Y. Choquet-Bruhat and J.W. York, The Cauchy problem, in: General Relativity and Gravitation, edited by A. Held, Vol. 1 (New York, Plenum, 1980).
[36] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[37] D. Christodoulou, Ann. Math. **149**, 183–217 (1999).
[38] D. Christodoulou and S. Klainerman, The Global Nonlinear Stability of the Minkowski Space (Princeton University Press, Princeton, 1993).
[39] P.T. Chruściel, Black holes, in: The Conformal Structure of Space-Time, edited by J. Frauendiener and H. Friedrich (Springer, Berlin, 2002).
[40] P.T. Chruściel and E. Delay, Class. Quantum Gravity **19**, L71–L79 (2002); Erratum, Class. Quantum Gravity **19**, 3389 (2002).
[41] P.T. Chruściel and E. Delay, On mapping properties of the general relativistic constraints operator in weighted function spaces, with application. Mém. Soc. Math. France, submitted. http://xxx.lanl.gov/abs/gr-qc/0301073.
[42] P.T. Chruściel and J. Isenberg, Phys. Rev. D **48**, 1616–1628 (1993).
[43] P.T. Chruściel, J. Isenberg, and D. Pollack, Phys. Rev. Lett. **93**, 081101 (2004).
[44] P.T. Chruściel and K. Lake, Class. Quantum Gravity **21**, S153–S170 (2004).
[45] P.T. Chruściel and A. Rendall, Ann. Phys. **242**, 349–385 (1995).
[46] C. Clarke, G. Ellis, and F. Tipler, Singularities and Horizons – A Review Article, in: General Relativity and Gravitation, Vol. 2, edited by A. Held (Plenum, New York, 1980).
[47] J. Corvino, Commun. Math. Phys. **214**, 137–189 (2000).
[48] J. Corvino and R. Schoen, On the Asymptotics for the Vacuum Einstein Constraint Equations. http://xxx.lanl.gov/abs/gr-qc/0301071.
[49] C. Cutler and R.M. Wald, Class. Quantum Gravity **6**, 453–466 (1989).
[50] M. Dafermos, Black hole formation from a complete past. arXiv:gr-qc/0310040.
[51] M. Dafermos and I. Rodnianski, A Proof of Price's law for the collapse of a self-gravitating scalar field. arXiv:gr-qc/0309115.
[52] M. Dafermos and I. Rodnianski, A note on boundary value problems for black hole evolutions. http://xxx.lanl.gov/abs/gr-qc/0403034.
[53] S. Dain, Class. Quantum Gravity **21**, 555–573 (2004).
[54] S. Dain, Phys. Rev. Lett. **93**, 231101 (2004).
[55] S. Dain and H. Friedrich, Commun. Math. Phys. **222**, 569–609 (2001).
[56] S. Dain, J. Jaramillo, and B. Krishnan, On the existence of initial data containing isolated black holes, gr-qc/0412061 (2004).
[57] S. Dain and G. Nagy, Phys. Rev. D **65**, 084020-1 (2002).
[58] A. Einstein Ann. Phys. **49**, 769–822 (1916).
[59] A. Einstein, Näherungsweise Integration der Feldgleichungen der Gravitation. Sitzungsberichte Königl. Preuss. Akademie der Wiss. (1916), pp. 688–696.
[60] A. Einstein, Über Gravitationswellen. Sitzungsberichte Königl. Preuss. Akademie der Wiss. (1918), pp. 154–167.
[61] A. Fischer and J. Marsden, The initial value problem and the dynamical formulation of general relativity, in: General relativity. An Einstein centenary survey, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
[62] Y. Fourès-Bruhat, Acta Mathematica **88**, 141–225 (1952).
[63] J. Frauendiener, Conformal infinity. Living reviews, 2002. http://www.livingreviews.org/lrr-2004-1.
[64] J. Frauendiener and T. Vogel, Class. Quantum Gravity **22**, 1769–1793 (2005).
[65] H. Friedrich, Commun. Math. Phys. **100**, 525–543 (1985).
[66] H. Friedrich, Commun. Math. Phys. **103**, 35–65 (1986).
[67] H. Friedrich, Commun. Math. Phys. **107**, 587–609 (1986).
[68] H. Friedrich, J. Differ. Geom. **34**, 275–345 (1991).
[69] H. Friedrich, J. Geom. Phys. **17**, 125–184 (1995).
[70] H. Friedrich, J. Geom. Phys. **24**, 83–163 (1998).
[71] H. Friedrich, Conformal Einstein evolution, in: The Conformal Structure of Spacetime: Geometry, Analysis, Numerics, edited by J. Frauendiener and H. Friedrich (Springer, Berlin, 2002).
[72] H. Friedrich, Class. Quantum. Gravity **20**, 101–117 (2003).

[73] H. Friedrich, Commun. Math. Phys. **235**, 513–543 (2003).
[74] H. Friedrich, Smoothness at null infinity and the structure of initial data, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).
[75] H. Friedrich, Class. Quantum Gravity **22**, L77–L82 (2005).
[76] H. Friedrich and G. Nagy, Commun. Math. Phys. **201**, 619–655 (1999).
[77] H. Friedrich and A. Rendall, The Cauchy Problem for the Einstein Equations, in: Einstein's Field Equations and Their Physical Implications, edited by B. Schmidt (Lecture Notes in Physics, vol. 540, Springer, Berlin 2000).
[78] S. Frittelli and O. Reula, J. Math. Phys. **40**, 5143–5156 (1999).
[79] V. Frolov and I. Novikov, Black Hole Physics (Kluwer, Dordrecht, 1998).
[80] D. Garfinkle, Numerical simulations of generic singularities. arXiv: gr-qc/0312117 v2.
[81] C. Gundlach, Phys. Rep. **367**, 339–405 (2003).
[82] C. Gundlach, G. Calabrese, and I. Hinder, Constraint damping in the Z4 formulation and harmonic gauge. http://xxx.lanl.gov/abs/gr-qc/0504114.
[83] C. Gundlach, R. Price, and J. Pullin, Phys. Rev. D **49**, 883 (1994); Phys. Rev. D **49**, 890 (1994).
[84] S. Hawking and G. Ellis, The large scale structure of space-time (Cambridge University Press, Cambridge, 1973).
[85] S. Hawking and R. Penrose, Proc. Roy. Soc. A **314**, 529–548 (1970).
[86] S. Hollands and R. Wald, Conformal infinity does not exist for radiating solutions in odd space-time dimensions. gr-qc/0407014.
[87] P. Hübner, Class. Quantum Gravity **18**, 1871–1884 (2001).
[88] T. Hughes, T. Kato, and J. Marsden, Arch. Ration. Mech. Anal. **63**, 273–294 (1977).
[89] G. Huisken and T. Ilmanen, J. Diff. Geom. **59**, 353–437 (2001).
[90] S. Husa, Problems and successes in the numerical approach to the conformal field equations, in: The Conformal Structure of Spacetime: Geometry, Analysis, Numerics, edited by J. Frauendiener and H. Friedrich (Springer, Berlin, 2002).
[91] J. Isenberg, R. Mazzeo, and D. Pollack, Ann. Henri Pioncaré **4**, 369–383 (2003).
[92] W. Israel, Dark stars: the evolution of an idea, in: Three hundred years of gravitation, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, 1989).
[93] W. Israel, The internal structure of black holes, in: Black Holes and Relativistic Stars, edited by R.M. Wald (University of Chicago Press, Chicago, 1998).
[94] S. Klainerman and F. Nicolò, The Evolution Problem in General Relativity. (Birkhäuser, Basel, 2003).
[95] S. Klainerman and F. Nicolò, Class. Quantum Gravity **20**, 3215–3257 (2003).
[96] S. Klainerman and I. Rodnianski Causal geometry of Einstein vacuum space-times with finite curvature flux. math.ap/0308123 (2003).
[97] M. Kramer et al., The double pulsar – A new testbed for relativistic gravity. Binary Radio Pulsars, ASP Conference Series, Vol. TBD, 2004, edited by F.A. Rasio and I. H. Stairs, arXiv: astro-ph/0405179.
[98] J. Lee and T. Parker, Bull. Am. Math. Soc. **17**, 37–91 (1987).
[99] L. Lehner and O. Reula, Status quo and open problems in the numerical construction of space-times, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P. T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).
[100] E. Lifshitz and I. Khalatnikov, Adv. in Phys. **12**, 525–573 (1963).
[101] H. Lindblad and I. Rodnianski, Global existence for the Einstein vacuum equations in wave coordinates. arXiv:math.AP/0312479.
[102] H. Lindblad and I. Rodnianski, The global stability of Minkowski space-time in harmonic gauge. arXiv:math.AP/0411109.
[103] D. Maxwell, Commun. Math. Phys. (2004).
[104] D. Maxwell, J. Hyp. Diff. Equ. **2**, 521–546 (2005).
[105] C. Misner, Phys. Rev. Lett. **29**, 1071–1074 (1969).
[106] V. Moncrief, Ann. Phys. **132**, 87–107 (1981).
[107] V. Moncrief and D. Eardley, Gen. Rel. Grav. **13**, 887–892 (1981).
[108] A. Ori, Phys. Rev. D **61**, 024001 (1999).
[109] A. Pais, Subtle is the Lord (Oxford University Press, 1982).
[110] R. Penrose, Phys. Rev. Lett. **10**, 66–68 (1963).

[111] R. Penrose, Phys. Rev. Lett. **14**, 57–59 (1965).
[112] R. Penrose, Rev. Nuovo Cimento **1**, 252–276 (1969).
[113] R. Penrose, Ann. N.Y. Acad. Sci. **224**, 125–134 (1973).
[114] R. Penrose, Singularities and time-asymmetry, in: General relativity. An Einstein centenary survey, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
[115] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796–1809 (1990).
[116] R. Pound and G. Rebka, Phys. Rev. Lett. **4**, 337–341 (1960).
[117] F. Pretorius, Class. Quantum Gravity **22**, 425–451 (2005).
[118] R. Price, Phys. Rev. D **5**, 2419–2438 (1972).
[119] M. Rees, Black holes in the real universe and their prospects as probes of relativistic gravity, in: The Future of Theoretical Physics and Cosmology, edited by G. Gibbons, E. Shellard, and S. Rankin (Cambridge University Press, Cambridge, 2003).
[120] A. Rendall, Class. Quantum Gravity **14**, 2341–2356 (1997).
[121] A. Rendall, Theorems on Existence and Global Dynamics for the Einstein Equations. Living Reviews in Relativity (2002). http://www.livingreviews.org/lrr-2002-6.
[122] A. Rendall, Asymptotics of solutions of the Einstein equations with positive cosmological constant. gr-qc/0312020.
[123] A. Rendall, The Einstein-Vlasov system, in: The Einstein equations and the large scale behaviour of gravitational fields, edited by P.T. Chruściel and H. Friedrich (Birkhäuser, Basel, 2004).
[124] A. Rendall, Class. Quantum Gravity **21**, S295–S304 (2004).
[125] A. Rendall and M. Weaver, Class. Quantum Gravity **18**, 2959–2975 (2001).
[126] O. Reula and O. Sarbach, J. Hyp. Diff. Equ. **2**, 397–435 (2005).
[127] H. Ringström, Class. Quantum Gravity **17**, 713–731 (2000); Ann. Inst. Henri Poincare **2**, 405–500 (2000); Class. Quantum Gravity **20**, 1943–1989 (2003).
[128] H. Ringström, J. Hyp. Diff. Equ. **2**, 547–564 (2005).
[129] O. Sarbach and M. Tiglio, J. Hyp. Diff. Equ. (to appear).
[130] R. Schoen, J. Diff. Geom. **20**, 479–495 (1984).
[131] R. Schoen and S.-T. Yau, Commun. Math. Phys. **79**, 231–260 (1981).
[132] J. Smoller, A. Wasserman, S.-T. Yau, and J. McLeod, Commun. Math. Phys. **143**, 115–147 (1991).
[133] K. Stellmacher, Math. Ann. **115**, 136–152 (1938).
[134] L. Szabados, Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article. Living Reviews in Relativity (2004). http://www.livingreviews.org/lrr-2004-4.
[135] D. Tataru, Nonlinear wave equations, in: Proceedings of the International Congress of Mathematicians (Beijing, 2002), Vol. III, pp. 209–220. (Higher Ed. Press, Beijing, 2002).
[136] K. Thorne, Black Holes and Time Warps: Einstein's Outrageous Legacy (Norton, New York, 1994).
[137] J. Valiente Kroon, Commun. Math. Phys. **251**, 211–234 (2004).
[138] J. Valiente Kroon, Class. Quantum Gravity **22**, 1683–1707 (2005).
[139] J. Wainwright and L. Hsu, Class. Quantum Gravity **6**, 1409–1431 (1989).
[140] R.M. Wald, Gravitational collapse and cosmic censorship, in: Black Holes, Gravitational Radiation and the Universe, edited by B.R. Iyer and B. Bhawal (Dordrecht, Kluwer Academic Publishers, 1999).
[141] J. Weisberg and J. Taylor, Relativistic Binary Pulsar B1913+16: Thirty Years of Observation and Analysis. Binary Radio Pulsars. ASP Conference Series, Vol. TBD, 2004, edited by F.A. Rasio and I.H. Stairs (arXiv: astro-ph/0407149).
[142] E. Witten, Commun. Math. Phys. **80**, 381–402 (1981).
[143] G. Yoneda and H. Shinkai, Class. Quantum Gravity **20**, L31–L36 (2003).

