

Maximal supergravities and the E_{10} model

Axel Kleinschmidt and Hermann Nicolai

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Mühlenberg 1, D-14476
Potsdam, Germany

E-mail: `axel.kleinschmidt,hermann.nicolai@aei.mpg.de`

Abstract. The maximal rank hyperbolic Kac-Moody algebra \mathfrak{e}_{10} has been conjectured to play a prominent role in the unification of duality symmetries in string and M theory. We review some recent developments supporting this conjecture.

1. Introduction

Hidden symmetries of exceptional type in the reduction of supergravity theories were first discovered in maximal $N = 8$ supergravity in $d = 4$ [1, 2]. The unexpected emergence of the coset $E_7/SU(8)$ describing the scalar sector of the theory was soon generalised to other dimensions and other theories [3]. The most prominent example remains the chain of hidden symmetries occurring in the dimensional reduction of $d = 11$ maximal supergravity on a torus T^n . For $1 \leq n \leq 8$ the scalars always appear in group cosets which have become known as $E_n/K(E_n)$ where $K(E_n)$ designates the maximal compact subgroup of E_n . For $n > 8$ it was soon conjectured that the resulting symmetry groups become infinite-dimensional [4] and formulations using the centrally extended loop group E_9 [5, 6] and partial results on E_{10} [7] have since been obtained.

In an initially unrelated development, the study of the asymptotic behaviour of $d = 11$ supergravity (and IIA and IIB supergravity) near a space-like singularity also revealed evidence for infinite dimensional symmetries, and the hyperbolic Kac-Moody algebra E_{10} , in particular. Namely, in this limit, the dynamics can be asymptotically described by a cosmological billiard taking place in the fundamental Weyl chamber of E_{10} [8, 9]. The dynamical variables in this case are the spatial scale factors. This led the authors of [10] to propose a one-dimensional non-linear σ -model based on a coset $E_{10}/K(E_{10})$ and [10, 11] uncovered a remarkable dynamical equivalence between a truncation of the bosonic $d = 11$ supergravity equations and a truncated version of this infinite-dimensional coset model. It is the aim of the present contribution to review this correspondence and similar correspondences to the $d = 10$ maximal supergravities which were derived in [12, 13]. We will not present the relation of higher derivative corrections to the E_{10} model which can be found in [14]. Related work in [15, 16] discusses the role of imaginary roots of E_{10} from a brane point of view and orbifolds.

We note that already in [17, 18] it was proposed that $d = 11$ supergravity is a non-linear realisation of the bigger group E_{11} (and the conformal group via a Borisov-Ogievetsky-type construction [19]). The non-linear E_{11} model is thus supposed to operate directly in eleven (or even more) dimensions, and is therefore very different from the one-dimensional E_{10} model which we will present below. In addition, as will be discussed in section 5.3, space-time is thought to

emerge in the present scheme from the E_{10} model itself, whereas for E_{11} space-time is realised through an additional E_{11} invariant structure [20]. These two space-time concepts, and a third one based on a one-dimensional E_{11} model proposed in [21], were discussed in [22]. The relation of that latter E_{11} model to the model considered here¹ was studied in [26]. It was also shown in [27, 28] that E_{11} can unify the symmetries of the bosonic sectors type IIB and massive type IIA supergravity by use of the same techniques as in [18].

A most remarkable feature of the results obtained so far is that E_{10} implies several results on the bosonic sectors of maximal supergravity theories which were heretofore thought to require (maximal) local supersymmetry, to wit:

- the correct bosonic multiplets of all maximal supergravities in eleven and all lower dimensions, in particular for both (massive) type IIA and type IIB supergravity [27, 28, 12, 13];
- the self-duality of the 5-form field strength in IIB supergravity [13];
- the correct bosonic self couplings for all these theories, in particular of the $D = 11$ Chern Simons term [10];
- the vanishing of the cosmological constant in $D = 11$ supergravity [11], originally shown in [29];
- (possibly) restrictions on the form of the higher order corrections in M theory [14].

This casts some doubt on widely held expectations concerning the role of (local) supersymmetry as a fundamental symmetry, and may indicate that the concept of supersymmetry may have to be replaced by yet another, and in some sense, even more ‘fundamental’ symmetry concept² (possibly also involving quantisation). Moreover, in a scheme where space(-time) is treated as an ‘emergent’ phenomenon, the distinction between bosons and fermions may well disappear, too, and only ‘emerge’ together with space(-time) itself.

The structure of this contribution is as follows. Section 2 reviews some basic facts about the hyperbolic Kac–Moody algebra \mathfrak{e}_{10} underlying E_{10} , leading in particular to a spectral analysis of the $E_{10}/K(E_{10})$ model. The model itself is defined in section 3. The correspondences to the various maximal supergravity theories in $d = 10$ and $d = 11$ are derived or reviewed in section 4. In section 5 we discuss some cosmological applications and extensions of the model presented here and end with a few open problems.

2. The \mathfrak{e}_{10} Kac–Moody algebra

In this section we recall the definition of the hyperbolic Kac–Moody Lie algebra $\mathfrak{e}_{10} = \text{Lie}(E_{10})$ and present its basic properties required for the coset model which will be defined in section 3.

2.1. Definition of \mathfrak{e}_{10}

We use the Chevalley–Serre presentation also employed in [30, 31]. This definition starts from the generalised Cartan matrix or, equivalently, the \mathfrak{e}_{10} Dynkin diagram given in fig. 1.

The *rank* of \mathfrak{e}_{10} is ten since there are ten nodes in fig. 1. The Cartan matrix $A = (A_{ij})$ encoded in the \mathfrak{e}_{10} Dynkin diagram is given by $(i, j = 1, \dots, 10)$

$$A_{ij} = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{if there is a link between nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

¹ There exists a related ‘brane version’ of the E_{10} model for which the denominator is non-compact and has non-unique space-time signatures [23, 24] which are identical to those of the exotic M-theories of [25].

² Of course, we will have to await what LHC has to say on this issue!

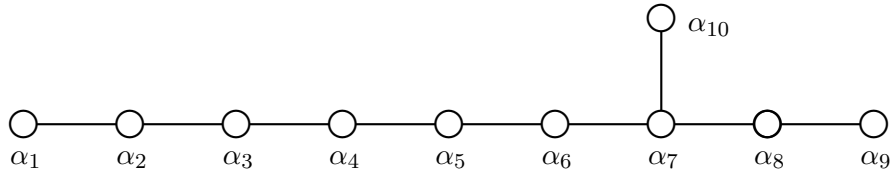


Figure 1. Dynkin diagram of \mathfrak{e}_{10} .

For the case of \mathfrak{e}_{10} the Cartan matrix is non-degenerate and indefinite (with nine positive eigenvalues and one negative eigenvalue). According to the general theory of Kac–Moody Lie algebras [30], the Chevalley–Serre presentation therefore starts from thirty *Chevalley generators*

$$e_i, f_i, h_i \quad (i = 1, \dots, 10) \quad (1)$$

subject to the relations

$$[h_i, e_j] = A_{ij}e_j, \quad [h_i, f_j] = -A_{ij}f_j, \quad [e_i, f_j] = \delta_{ij}h_i, \quad [h_i, h_j] = 0. \quad (2)$$

On top of the thirty generators of (1) one now considers multiple commutators of the *simple positive generators* e_i and of the *simple negative generators* f_i of the form

$$[e_{i_1}, [\dots[e_{i_{k-1}}, e_{i_k}]]] \quad \text{and} \quad [f_{i_1}, [\dots[f_{i_{k-1}}, f_{i_k}]]], \quad (3)$$

spanning free Lie algebras on $\{e_i\}$ and $\{f_i\}$ and then subjected to the *Serre relations* (for $i \neq j$)

$$(\text{ad } e_i)^{1-A_{ij}} e_j = 0 \quad \text{and} \quad (\text{ad } f_i)^{1-A_{ij}} f_j = 0, \quad (4)$$

and, of course, to the Jacobi and antisymmetry relations of the Lie bracket. The number of Chevalley generators in such a nested multiple commutator is called the *height* of the element.

The Lie algebra consisting of (1) and (Serre) non-trivial elements (3) is the infinite-dimensional Kac–Moody algebra \mathfrak{e}_{10} , which we will also sometimes denote by $\mathfrak{g} \equiv \mathfrak{e}_{10}$. We consider \mathfrak{e}_{10} in its *split real form* obtained by taking only real linear combinations of the basis elements. Like every Kac–Moody algebras, \mathfrak{e}_{10} possesses a triangular structure

$$\mathfrak{e}_{10} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad (5)$$

where \mathfrak{h} is the *Cartan subalgebra* (CSA) spanned by the elements h_i , and \mathfrak{n}_\pm have bases consisting of the positive simple generators e_i together with their multiple commutators and the negative simple generators f_i together with their multiple commutators, respectively. The positive half \mathfrak{n}_+ and the negative half \mathfrak{n}_- are exchanged by the *Chevalley involution* θ which acts on the Chevalley generators (1) by

$$\theta(e_i) = -f_i, \quad \theta(f_i) = -e_i, \quad \theta(h_i) = -h_i \quad (6)$$

and extends to all of \mathfrak{e}_{10} . The fixed point set

$$\mathfrak{k}_{10} := K(\mathfrak{e}_{10}) := \{x \in \mathfrak{e}_{10} : \theta(x) = x\} \quad (7)$$

will be called the *maximal compact subalgebra* in analogy with the finite-dimensional theory. The Lie algebra \mathfrak{k}_{10} is *not* a Kac–Moody algebra [32].

We will furthermore make use of the *root space decomposition* of $\mathfrak{g} \equiv \mathfrak{e}_{10}$ defined through

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad (8)$$

with

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}. \quad (9)$$

The non-trivial linear maps $\alpha : \mathfrak{h} \rightarrow \mathbb{R}$ are called *roots* and $\Delta = \Delta_+ \cup \Delta_-$ is the *space of all roots*. The *simple roots* are denoted by α_i and are defined by $\mathfrak{g}_{\alpha_i} = \langle e_i \rangle$. The *root lattice* Q is obtained as the integer lattice over the simple roots. All roots are linear combinations of simple roots with either all non-negative or all non-positive coefficients; the roots are then called ‘positive’ or ‘negative’, respectively. We thus write an arbitrary root $\alpha \in \Delta$ as

$$\alpha = \sum_{i=1}^{10} m_i \alpha_i \quad (10)$$

with either all $m_i \geq 0$ or all $m_i \leq 0$. The *height* of α is $\text{ht}(\alpha) = \sum_i m_i$.

For the hyperbolic³ Kac–Moody algebra \mathfrak{e}_{10} one can furthermore show [30] that the set of all roots $\Delta \subset Q$ is given by

$$\Delta = \{\alpha \in Q : \alpha^2 \leq 2\} \setminus \{0\}. \quad (11)$$

Here, the norm $\alpha^2 = \langle \alpha | \alpha \rangle$ is computed using the Cartan matrix as inner product (on both \mathfrak{h} and \mathfrak{h}^*), such that for all simple roots $\alpha_i^2 = \langle \alpha_i | \alpha_i \rangle = 2$ and for an arbitrary root (10) we have $\alpha^2 = \sum_{i,j} m_i A_{ij} m_j$. The symmetric form $\langle \cdot | \cdot \rangle$ on \mathfrak{h} can be extended to a symmetric invariant form on all of \mathfrak{e}_{10} by letting $\langle e_i | f_j \rangle = \delta_{ij}$ and then using the invariance to define $\langle \cdot | \cdot \rangle$ on multiple commutators. A final piece of terminology we require from the theory of Kac–Moody algebras is the notion of *real* and *imaginary* roots. A root α of \mathfrak{e}_{10} is said to be real if $\alpha^2 = 2$ and imaginary otherwise; in the latter case, one further distinguishes lightlike (null) and timelike roots, for which $\alpha^2 = 0$ and $\alpha^2 < 0$, respectively.

Further details can be found in [30].

2.2. Spectral analysis: Level decomposition

Among the basic quantities of interest of a Kac–Moody algebra are the *root multiplicities* $\text{mult}(\alpha) = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ for roots $\alpha \in \Delta$. In contradistinction to the finite-dimensional Lie algebras and affine Kac–Moody algebras there is no closed formula determining the root multiplicity of an arbitrary root α . Presently, only recursive techniques such as the Peterson formula can be used to calculate the multiplicities⁴ by working up in height, starting from the fact that for the simple roots one has $\text{mult}(\alpha_i) = 1$. By performing such calculations on a computer one can obtain a good picture of the root structure of a Kac–Moody algebra [33, 34]. Due to the Lorentzian structure of the Cartan matrix of \mathfrak{e}_{10} and the condition (11) we end up with the following picture: All lattice points α inside the solid hyperboloid $\{\alpha^2 \leq 2\} \subset Q$ in the Lorentzian space \mathfrak{h}^* are roots of \mathfrak{e}_{10} , and each such point represents the vector space \mathfrak{g}_α associated with the corresponding root α . We imagine the lattice points as being labelled in addition by the dimension (=multiplicity) of the root space.

³ The adjective ‘hyperbolic’ means that upon deletion of any single node from the Dynkin diagram, the remaining diagram consists only of diagrams of finite-dimensional or affine diagrams [30].

⁴ It is known that the root multiplicities grow exponentially in $-\alpha^2$.

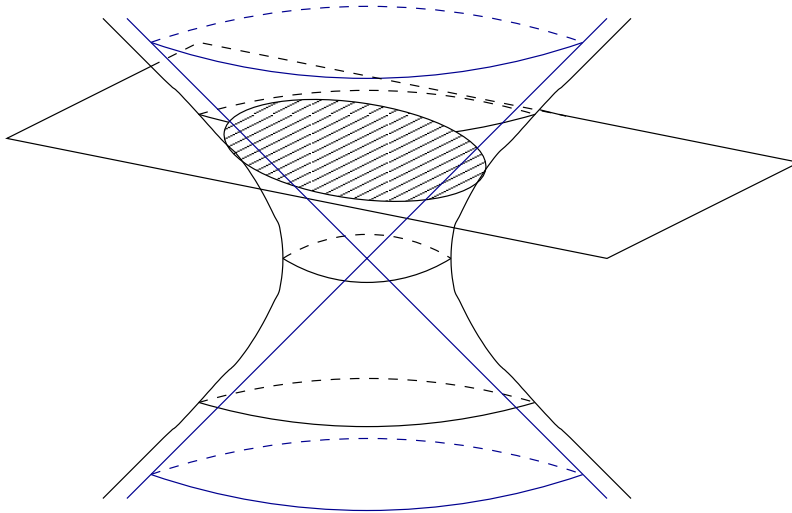


Figure 2. A sketch of the solid hyperboloid in root space with an elliptic slice corresponding to a finite-dimensional subalgebra of \mathfrak{e}_{10} . For the pictorial presentation the number of dimensions has been lowered.

An economical and physically motivated way to present the algebraic data is the *level decomposition* under a finite-dimensional regular subalgebra⁵ introduced in [10]. In the above picture this corresponds to an elliptic slicing of the solid hyperboloid. From this it is obvious that any given slice contains only a finite number of points representing a root space, see fig. 2. In this language, the algebra can be described by a stack of slices each of which contains a finite number of irreducible representations of the subalgebra (acting via the adjoint action). The *level* ℓ of such a slice is given as the vector of numbers of times the simple generators of the deleted nodes appears in the elements on the slice, and ℓ provides a grading of the algebra. Instead of giving the details of how this decomposition is done, we will give the relevant examples below and refer the reader to [10, 35, 36, 33, 37] for expositions of the general technique. We will only give levels $\ell \geq 0$ since the negative levels are contragradient to the positive levels due to the Chevalley involution θ .⁶

2.2.1. Decomposition under A_9 We first single out the ‘exceptional’ node 10 in diagram 1 and delete it together with its link to node 7. The remaining subalgebra is $A_9 \equiv \mathfrak{sl}(10)$. The level ℓ in this decomposition is the last entry m_{10} of a root $\alpha = \sum_{i=1}^{10} m_i \alpha_i$. The spectrum on levels $0 \leq \ell \leq 3$ in the A_9 decomposition was computed in [10] and is given in table 1.

The notation in table 1 is as follows. Irreducible $\mathfrak{sl}(10)$ representations are given in terms of their Dynkin labels, such that the entry in the first row of the table is the adjoint representation, whereas the second representation for $\ell = 0$ is the trivial one. They combine to give the generators K^a_b of $\mathfrak{gl}(10)$. Indices a, b etc. are $\mathfrak{sl}(10)$ vector indices and hence take values $1, \dots, 10$. Similarly, the representations on levels $\ell = 1$ and $\ell = 2$ are totally antisymmetric of rank three and six, respectively, and we have already introduced the symbols we will use below when referring to these representations. The $\ell = 3$ representation is of mixed symmetry type,

⁵ By this we mean a subalgebra obtained by deleting nodes from the Dynkin diagram. Examples will be given below.

⁶ Another reason for restricting the spectral analysis to $\ell \geq 0$ is that in section 3 we will use a triangular gauge which only employs $\ell \geq 0$ generators.

ℓ	A_9 representation	\mathfrak{e}_{10} root α	Generator
0	[100000001]	(1, 1, 1, 1, 1, 1, 1, 1, 0)	K^a_b (traceless)
	[000000000]	(0, 0, 0, 0, 0, 0, 0, 0, 0)	$K := \sum_{a=1}^{10} K^a_a$
1	[000000100]	(0, 0, 0, 0, 0, 0, 0, 0, 1)	$E^{a_1 a_2 a_3}$
2	[000100000]	(0, 0, 0, 0, 1, 2, 3, 2, 1, 2)	$E^{a_1 \dots a_6}$
3	[010000001]	(0, 0, 1, 2, 3, 4, 5, 3, 1, 3)	$E^{a_0 a_1 \dots a_8}$

Table 1. The first four levels in the A_9 decomposition of \mathfrak{e}_{10} .

the corresponding generator satisfies the irreducibility constraints

$$E^{a_0 | a_1 \dots a_8} = E^{a_0 | a_1 \dots a_8}, \quad E^{[a_0 | a_1 \dots a_8]} = 0. \quad (12)$$

These $0 \leq \ell \leq 3$ tensors will play a rôle below when we relate an E_{10} symmetric coset model to $d = 11$ supergravity in section 4.1. Let us also note that up to level $\ell \leq 3$, the decomposition is essentially the same for all E_n ; in particular, for E_{11} , the representations are analogous [18].

2.2.2. Decomposition under D_9 Another possible slicing of the hyperboloid is obtained by deleting the node 9 in diagram 1. The remaining subalgebra is $D_9 \equiv \mathfrak{so}(9, 9)$. The level decomposition was carried out in [12] with result for $\ell = 0, 1, 2$ reproduced in table 2. Here, the indices $I, J, K = 1, \dots, 18$ are vector indices of $\mathfrak{so}(9, 9)$, whereas $A = 1, \dots, 256$ is a spinor index. The representations on level $\ell = 0$ are the adjoint and the scalar representation, respectively. Level $\ell = 1$ contains the Dirac spinor, and $\ell = 2$ an anti-symmetric three form.⁷ In section 4.3 we will relate the corresponding tensors to quantities appearing in (massive) type IIA supergravity.

ℓ	D_9 representation	\mathfrak{e}_{10} root α	Generator
0	[010000000]	(1, 2, 2, 2, 2, 2, 2, 1, 0, 1)	M^{IJ}
	[000000000]	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	T
1	[000000010]	(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)	E_A
2	[001000000]	(0, 0, 0, 1, 2, 3, 4, 3, 2, 2)	E^{IJK}

Table 2. The first three levels in the D_9 decomposition of \mathfrak{e}_{10} .

2.2.3. Decomposition under $A_8 \oplus A_1$ The final choice of subalgebra we consider is obtained by deleting node 8 from fig. 1. The remaining subalgebra is now $A_8 \oplus A_1 \equiv \mathfrak{sl}(9) \oplus \mathfrak{sl}(2)$. The decomposition was carried out in [13] with the result for $\ell = 0, 1, 2, 3, 4$ reproduced in table 3. Here, the $\mathfrak{sl}(9)$ vector indices a, b , etc. range from 1 to 9, the indices $i = 1, 2, 3$ are $\mathfrak{so}(2, 1) \equiv \mathfrak{sl}(2)$ vector indices and $\alpha = 1, 2$ are $\mathfrak{so}(2, 1)$ spinor indices. These tensors will be related to IIB supergravity quantities in section 4.4.

2.3. Commutation relations

Once the representation content (spectrum) of the algebra has been determined to the required level one then needs to work out the commutation relations between the Lie algebra elements. Here the power of the level decomposition becomes evident since only a few structure constants

⁷ Generally, the representations occurring on even levels are tensor representations of $\mathfrak{so}(9, 9)$ and the odd level representations are spinor representations of $\mathfrak{so}(9, 9)$.

ℓ	$A_8 \oplus A_1$ representation	\mathfrak{e}_{10} root α	Generator
0	$([10000001], \mathbf{1})$	$(1, 1, 1, 1, 1, 1, 1, 0, 0, 1)$	K^a_b (traceless)
	$([00000000], \mathbf{1})$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	K
	$([00000000], \mathbf{3})$	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$	J_i
1	$([00000010], \mathbf{2})$	$(0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$	$E_\alpha^{a_1 a_2}$
2	$([00001000], \mathbf{1})$	$(0, 0, 0, 0, 0, 1, 2, 2, 1, 1)$	$E^{a_1 \dots a_4}$
3	$([00100000], \mathbf{2})$	$(0, 0, 0, 1, 2, 3, 4, 3, 1, 2)$	$E_\alpha^{a_1 \dots a_6}$
4	$([01000001], \mathbf{1})$	$(0, 0, 1, 2, 3, 4, 5, 4, 2, 2)$	$E^{a_0 a_1 \dots a_7}$
	$([10000000], \mathbf{3})$	$(0, 1, 2, 3, 4, 5, 6, 4, 1, 3)$	$E_i^{a_1 \dots a_8}$

Table 3. The first five levels in the $A_8 \oplus A_1$ decomposition of \mathfrak{e}_{10} .

appear between the representations. The number of \mathfrak{e}_{10} elements thus covered can still be quite large if the representations are large but covariance under the subalgebra action fixes all the structure constants within a representation. We again illustrate these facts in examples. A more general discussion on how to compute the commutation relations can be found in [38].

We consider the A_9 decomposition of \mathfrak{e}_{10} in some detail. As discussed in section 2.2.1, the $\ell = 0$ generators are K^a_b ($a, b = 1, \dots, 10$) with standard $\mathfrak{gl}(10)$ commutation relations

$$[K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b \quad (13)$$

and normalisation

$$\langle K^a_b | K^c_d \rangle = \delta_b^c \delta_d^a - \delta_b^a \delta_d^c. \quad (14)$$

Their identification with the Chevalley generators of \mathfrak{e}_{10} is

$$e_i = K^i_{i+1}, \quad f_i = K^{i+1}_i, \quad h_i = K^i_i - K^{i+1}_{i+1} \quad (i = 1, \dots, 9). \quad (15)$$

and

$$h_{10} = -\frac{1}{3}K + K^8_8 + K^9_9 + K^{10}_{10}. \quad (16)$$

This follows from demanding $[h_{10}, e_i] = A_{10i}e_i$. The ‘exceptional’ generators e_{10} and f_{10} belong to levels $\ell = 1$ and $\ell = -1$ respectively and will be identified below.

On $\ell = 1$, the generators K^a_b act by $\mathfrak{gl}(10)$ rotations⁸

$$[K^a_b, E^{c_1 c_2 c_3}] = \delta_b^{c_1} E^{a c_2 c_3} + \delta_b^{c_2} E^{c_1 a c_3} + \delta_b^{c_3} E^{c_1 c_2 a} \equiv 3\delta_b^{[c_1} E^{c_2 c_3]a}. \quad (17)$$

The next relation to be worked out involves the transposed $\ell = -1$ generator

$$F_{a_1 a_2 a_3} := (E^{a_1 a_2 a_3})^T := -\theta(E^{a_1 a_2 a_3}), \quad (18)$$

which transforms contragradiently under K^a_b

$$[K^a_b, F_{c_1 c_2 c_3}] = -3\delta_{[c_1}^a F_{c_2 c_3]b}. \quad (19)$$

Due to the grading property of the level the commutator $[(\ell = 1), (\ell = -1)]$ has to be contained in $(\ell = 0)$. The relations (17) and (19) are insensitive to the normalisations of $E^{a_1 a_2 a_3}$ (and

⁸ We use (anti-)symmetrizers of strength one.

hence $F_{a_1 a_2 a_3}$) but $[E^{a_1 a_2 a_3}, F_{b_1 b_2 b_3}]$ is not. We fix the norm in terms of the invariant bilinear form to be

$$\langle E^{a_1 a_2 a_3} | F_{b_1 b_2 b_3} \rangle = 3! \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3}. \quad (20)$$

The lowest element on $\ell = 1$ is E^{8910} and in view of $\langle e_{10} | f_{10} \rangle = 1$ and (20) we can identify

$$e_{10} = E^{8910}, \quad f_{10} = F_{8910}. \quad (21)$$

This also fixes the coefficients in the commutation relation

$$[E^{a_1 a_2 a_3}, F_{b_1 b_2 b_3}] = -2\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} K + 18\delta_{[b_1 b_2}^{[a_1 a_2} K_{b_3]}^{a_3]}, \quad (22)$$

which completes the full set of commutation relations on levels $\ell = -1, 0, 1$. Thus, in the above relation, $GL(10)$ covariance fixes all structure constants in terms of only *two* coefficients. More precisely, these two coefficients are fixed by invariance of the bilinear form and the normalisation (20). Similar remarks apply to higher level commutators.

Proceeding to $\ell = 2$, we again have to fix the normalisation of the relevant generator $E^{a_1 \dots a_6}$, which we do by *defining*

$$E^{a_1 \dots a_6} := [E^{a_1 a_2 a_3}, E^{a_4 a_5 a_6}], \quad (23)$$

which, using the invariance of the bilinear form, leads to

$$\langle E^{a_1 \dots a_6} | F_{b_1 \dots b_6} \rangle = 6! \delta_{b_1 \dots b_6}^{a_1 \dots a_6}. \quad (24)$$

Using Jacobi identities the remaining commutators for $|\ell| \leq 2$ can be worked out and are listed in [11].

For $\ell = 3$ we define

$$E^{[a_0 | a_1 a_2] a_3 \dots a_8} := [E^{a_0 a_1 a_2}, E^{a_3 \dots a_8}], \quad (25)$$

which is equivalent to

$$E^{a_0 | a_1 \dots a_8} = 4 [E^{a_0 [a_1 a_2]}, E^{a_3 \dots a_8}], \quad (26)$$

with normalisation

$$\langle E^{a_0 | a_1 \dots a_8} | F_{b_0 | b_1 \dots b_8} \rangle = 8 \cdot 8! \left(\delta_{b_0}^{a_0} \delta_{b_1 \dots b_8}^{a_1 \dots a_8} - \delta_{b_0}^{[a_1} \delta_{[b_1 \dots b_7}^{a_2 \dots a_8]} \delta_{b_8]}^{a_0} \right). \quad (27)$$

Again, Jacobi identities can be used to determine the remaining commutation relations. These were given in [11].

Similar considerations can be used for the D_9 and $A_8 \oplus A_1$ decompositions of sections 2.2.2 and 2.2.3. The results can be found in [12] and [13], respectively.

3. The E_{10} coset model

In this section, we present, following [10], a coset model with manifest E_{10} symmetry. This model is a null geodesic model on the coset space $E_{10}/K(E_{10})$, where E_{10} is the Kac–Moody group with Lie algebra \mathfrak{e}_{10} and $K(E_{10})$ its ‘maximal compact subgroup’ with Lie algebra $\mathfrak{k}_{10} \subset \mathfrak{e}_{10}$ fixed by

the Chevalley involution θ from (6). We take a time-dependent coset element $\mathcal{V}(t) \in E_{10}/K(E_{10})$ of the form

$$\mathcal{V}(t) = \exp\left(\sum_{i=1}^{10} \phi_i(t) h_i\right) \exp\left(\sum_{\alpha \in \Delta_+} \sum_{s=1}^{\text{mult}(\alpha)} A_\alpha^{(s)}(t) E_\alpha^{(s)}\right), \quad (28)$$

which is in the so-called Borel gauge, where only the CSA \mathfrak{h} and the upper triangular part \mathfrak{n}_+ are used. $E_\alpha^{(s)}$ label the independent generators in the root space of α . In the finite-dimensional situation the parametrisation (28) can be reached due to the Iwasawa decomposition. Here, we simply take (28) as definition for the coset $E_{10}/K(E_{10})$. We will sometimes also use a slight modification of this parametrisation where only levels $\ell \geq 0$ are used and the exponentials are separated differently, which is a non-linear change of coordinates on the coset.

The \mathfrak{e}_{10} -valued velocity (Cartan form) associated with the coset element (28) is

$$\partial_t \mathcal{V} \mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P}, \quad \mathcal{Q} \in \mathfrak{k}_{10}, \quad \mathcal{P} \in \mathfrak{e}_{10} \ominus \mathfrak{k}_{10}, \quad (29)$$

such that $\theta(\mathcal{Q}) = \mathcal{Q}$ and $\theta(\mathcal{P}) = -\mathcal{P}$. We define the generators along the coset and the subgroup by

$$S_\alpha = E_\alpha + F_\alpha, \quad J_\alpha = E_\alpha - F_\alpha, \quad (30)$$

where $F_\alpha := (E_\alpha)^T := -\theta(E_\alpha)$ denotes the transposed generator to E_α .⁹ Therefore, \mathfrak{k}_{10} consists of the anti-symmetric elements and the remaining symmetric elements belong to $\mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$.

The time reparametrisation invariant Lagrange function defining the dynamics of the geodesic model is given by

$$\mathcal{L} = \mathcal{L}(t) = \frac{1}{2n} \langle \mathcal{P} | \mathcal{P} \rangle, \quad (31)$$

where $n(t)$ is a Lagrange multiplier ('einbein') needed for reparametrisation invariance and $\langle \cdot | \cdot \rangle$ is the \mathfrak{e}_{10} invariant bilinear form discussed in section 2.1. The equations of motion following from this Lagrange function are

$$\partial_t (n^{-1} \mathcal{P}) = [\mathcal{Q}, n^{-1} \mathcal{P}], \quad \langle \mathcal{P} | \mathcal{P} \rangle = 0. \quad (32)$$

We will refer to the second equation (obtained by varying n) as the the Hamiltonian constraint. It expresses the light-like orientation of the geodesic.

The system is formally *integrable*, as is already evident from the Lax formulation of the geodesic equation in (32). It is easy to write down infinitely many conserved charges through the \mathfrak{e}_{10} -valued current

$$\mathcal{J} = n^{-1} \mathcal{V}^{-1} \mathcal{P} \mathcal{V}. \quad (33)$$

This is the Noether current associated with the global E_{10} invariance of (31). The transformation

$$\mathcal{V}(t) \rightarrow k(t) \mathcal{V}(t) g^{-1} \quad (34)$$

for constant $g \in E_{10}$ (and compensating $k(t) \in K(E_{10})$ to maintain the Borel triangular gauge) induces the transformations

$$\mathcal{P} \rightarrow k \mathcal{P} k^{-1}, \quad \mathcal{Q} \rightarrow k \mathcal{Q} k^{-1} + \partial_t k k^{-1}, \quad (35)$$

⁹ The multiplicity index will be suppressed for clarity of notation and is summed over implicitly.

so that \mathcal{P} transforms covariantly and \mathcal{Q} like a gauge connection. This is as expected since \mathcal{Q} is associated with the unbroken $K(E_{10})$ gauge invariance of the coset $E_{10}/K(E_{10})$. We define a $K(E_{10})$ covariant derivative by

$$\mathcal{D} := \partial_t - \mathcal{Q}. \quad (36)$$

The set of equations of motion (32) now can be expanded in the basis consisting of the various S_α . We will do this in a form adapted to a level decomposition, where we expand

$$\mathcal{P} = \sum_{\ell \geq 0} P^{(\ell)} * S^{(\ell)}, \quad \mathcal{Q} = \sum_{\ell \geq 0} Q^{(\ell)} * J^{(\ell)}, \quad (37)$$

where the asterisk indicates a contraction of the various irreducible generators (like $S^{(\ell)}$) occurring on a given level with coefficients (like $P^{(\ell)}$). It is an important result that we can consistently truncate the equations of motion by demanding

$$P^{(\ell)} = 0, \quad \text{for } \ell > \ell_0, \quad (38)$$

where ℓ_0 is some fixed but arbitrary level. The consistency of this truncation was proved in [11].

Although the equations (32) are written in first order form by use of \mathcal{Q} and \mathcal{P} , the equations of motion are, of course, second order because the components of \mathcal{P} must be ultimately expressed as first derivatives of coordinates (ϕ_i, A_α) on the coset space $E_{10}/K(E_{10})$ ¹⁰; in terms of such an explicit parametrisation, the equation of motion then indeed takes the form of the standard geodesic equation. Eqn. (38) then implies that the higher level coordinates are non-trivial but evolve in just the right manner prescribed by the first order equation (38). When such an explicit parametrisation is not required, it is convenient to work with the ‘tangent space’ objects \mathcal{P} rather than the coordinates because one always deals with Lie algebra valued quantities transforming as tangent vectors under $K(E_{10})$ (the use of tangent space quantities is furthermore indispensable once one introduces fermions). The choice of fields parametrising the triangular gauge corresponds to a choice of local coordinates on the coset manifold, and as such is subject to a huge variety of coordinate reparametrisations (field redefinitions).

4. Correspondence to maximal supergravities

We will now construct correspondences between the abstract E_{10} σ -model defined in the preceding section on the one hand, and the bosonic sectors of the maximal supergravity theories in ten and eleven space-time dimensions on the other hand. These correspondences consist of a *dictionary* between the coset fields and supergravity fields under which the null geodesic equation (32) become equivalent to the field equations of the bosonic fields of the supergravity theories in a truncation to first spatial gradients only.

It is important that in all cases we make use of the *same* E_{10} invariant model and the correspondences arise through reading that very same model in terms of different subalgebras used for writing the model by means of a level decomposition, see section 2.2. In other words, one and the same E_{10} model gives rise to the different (and suitably truncated) maximal supergravity equations of motion, depending on how one slices the lightcone in the space of E_{10} roots. Therefore the E_{10} σ -model realises one of the central *desiderata* of M-theory: to explain the known maximal theories (hence, all the maximal supergravity theories) in terms of a single ‘Ur’-theory [39, 40]. We refer to this aspect of the E_{10} model as *versatility*.

¹⁰ In triangular gauge there is no obstruction to working out \mathcal{P} to arbitrarily high orders by use of the Baker–Campbell–Hausdorff equalities.

In order to establish the correspondences it is necessary to make a few common gauge choices in the supergravity theories. These choices are¹¹

- A pseudo-Gaussian gauge for the vielbein where there are no mixed space-time components:

$$E_M^A = \left(\begin{array}{c|c} N & 0 \\ \hline 0 & e_m^a \end{array} \right).$$

Here, N is the (arbitrary) lapse function, and we denote $g = \det(e_m^a)$. In the cosmological applications of section 5.1 below we will choose the convenient gauge $N = \det(e_m^a)$.

- The spatial coefficients of anholonomy

$$\Omega_{abc} = -\Omega_{bac} = 2e_{[a}^m e_b]^n \partial_m e_{nc}$$

are fixed, by exploiting spatial diffeomorphism invariance, to have vanishing trace $\Omega_{abb} = 0$. Following [11, 13] we will denote the remaining traceless part as $\tilde{\Omega}_{abc}$. The spin connection ω_{abc} is defined in terms of Ω_{abc} by

$$\omega_{abc} = -\omega_{acb} = \frac{1}{2} (\Omega_{abc} + \Omega_{cab} - \Omega_{bca}),$$

with $\Omega_{abb} = 0$ implying $\omega_{bba} = 0$.

- Spatial frame derivatives of the lapse N , of ω and of the gauge invariant field strengths are neglected.
- The coset model time parameter t is chosen such that, near the cosmological singularity, we have $t \rightarrow \infty$ as $T \rightarrow 0$, where T is the proper time.
- All fermionic terms are set to zero.

Under these assumptions the spatial components of the Ricci tensor (in flat indices) splits as

$$R_{ab} = R_{ab}^{\text{temp}} + R_{ab}^{\text{spat}},$$

where we defined, following [11],

$$R_{ab}^{\text{temp}} := \partial_0 \omega_{ab0} + \omega_{c0} \omega_{ab0} - 2\omega_{0c(a} \omega_{b)c0}, \quad (39)$$

$$R_{ab}^{\text{spat}} := \frac{1}{4} \tilde{\Omega}_{cda} \tilde{\Omega}_{cdb} - \frac{1}{2} \tilde{\Omega}_{acd} \tilde{\Omega}_{bcd} - \frac{1}{2} \tilde{\Omega}_{acd} \tilde{\Omega}_{bdc} - \frac{1}{2} \partial_c \tilde{\Omega}_{c(ab)}. \quad (40)$$

In standard normalisation of the kinetic term for a p -form gauge field $A_{M_1 \dots M_p}$ with $(p+1)$ -form field strength $F_{M_1 \dots M_{p+1}} = (p+1) \partial_{[M_1} A_{M_2 \dots M_{p+1}]}$ the space-space components of the Einstein equation reads (again everything in flat indices and multiplied by N^2)¹²

$$N^2 R_{ab}^{\text{temp}} = -T_{ab}^{\text{el}} + N^2 T_{ab}^{\text{magn}} - N^2 R_{ab}^{\text{spat}}, \quad (41)$$

where the stress energy contribution of the p -form was split into ‘electric’ and ‘magnetic’ contributions according to

$$T_{ab}^{\text{el}} = \frac{p}{2p!} F_{tac_1 \dots c_{p-1}} F_{tb c_1 \dots c_{p-1}} - \frac{p}{2p!(D-1)} \delta_{ab} F_{tc_1 \dots c_p} F_{tc_1 \dots c_p}, \quad (42)$$

$$T_{ab}^{\text{magn}} = \frac{1}{2p!} F_{ac_1 \dots c_p} F_{bc_1 \dots c_p} - \frac{p}{2(p+1)!(D-1)} \delta_{ab} F_{c_1 \dots c_{p+1}} F_{c_1 \dots c_{p+1}}. \quad (43)$$

¹¹ Our index conventions for gravity in $d = (1+D)$ dimensions are such that capital latin indices run from 0 to D , with 0 corresponding to the time direction, whereas lower case latin indices run only over the spatial directions 1 to D . Letters from the beginning of the alphabet are flat (tangent space) indices and from the middle of the alphabet are curved (world) indices. We use the ‘mostly plus’ convention for the metric.

¹² The remaining components are the Hamiltonian and diffeomorphism constraints of the theory. Whereas the Hamiltonian constraint is linked to the constraint of the E_{10} σ -model, there is currently no understanding of the diffeomorphism constraint in general. However, in the truncations to be considered below, it is satisfied.

4.1. $d = 11$ supergravity

Using the results of sections 2.2.1 and 2.3, we first write the coset model equations (32) in A_9 decomposition up to $\ell = 3$. They are [10, 11]

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P_{ab}^{(0)}) &= -\frac{1}{4}(P_{acd}^{(1)}P_{bcd}^{(1)} - \frac{1}{9}\delta_{ab}P_{c_1\dots c_3}^{(1)}P_{c_1\dots c_3}^{(1)}) \\ &\quad -\frac{1}{2\cdot 5!}(P_{ac_1\dots c_5}^{(2)}P_{bc_1\dots c_5}^{(2)} - \frac{1}{9}\delta_{ab}P_{c_1\dots c_6}^{(2)}P_{c_1\dots c_6}^{(2)}) \\ &\quad +\frac{4}{9!}(P_{c_1|c_2\dots c_8a}^{(3)}P_{c_1|c_2\dots c_8b}^{(3)} + \frac{1}{8}P_{a|c_1\dots c_8}^{(3)}P_{b|c_1\dots c_8}^{(3)}) \\ &\quad -\frac{1}{8}\delta_{ab}P_{c_0|c_1\dots c_8}^{(3)}P_{c_0|c_1\dots c_8}^{(3)} \end{aligned} \quad (44)$$

$$n\mathcal{D}^{(0)}(n^{-1}P_{abc}^{(3)}) = -\frac{1}{6}P_{abcd_1\dots d_3}^{(2)}P_{d_1\dots d_3}^{(1)} + \frac{1}{3\cdot 5!}P_{d_1|d_2\dots d_6abc}^{(3)}P_{d_1\dots d_6}^{(2)} \quad (45)$$

$$n\mathcal{D}^{(0)}(n^{-1}P_{a_1\dots a_6}^{(2)}) = \frac{1}{6}P_{c_1|c_2c_3a_1\dots a_6}^{(3)}P_{c_1c_2c_3}^{(1)} \quad (46)$$

$$n\mathcal{D}^{(0)}(n^{-1}P_{a_0|a_1\dots a_8}^{(3)}) = 0. \quad (47)$$

These equations are $SO(10) = K(GL(10))$ covariant by construction, and $SO(10)$ is the spatial Lorentz group of an eleven-dimensional theory. We have put the A_9 level on the components $P^{(\ell)}$ to make the structure more transparent. The derivative operator $\mathcal{D}^{(0)}$ appearing here is only partly covariantised and defined by

$$\begin{aligned} \mathcal{D}^{(0)}(P^{(\ell)} * S^{(\ell)}) &= \partial_t(P^{(\ell)} * S^{(\ell)}) - [Q^{(0)} * J^{(0)}, P^{(\ell)} * S^{(\ell)}] \\ &\quad + [P^{(0)} * S^{(0)}, Q^{(\ell)} * J^{(\ell)}]. \end{aligned} \quad (48)$$

Maximal supergravity in $d = (1 + 10)$ dimensions [1] has as bosonic fields gravity and a three-form gauge potential $A_{M_1M_2M_3}$ with field strength $F_{M_1\dots M_4} = 4\partial_{[M_1}A_{M_2M_3M_4]}$. Since in this case $D - 1 = 10 - 1 = 9$ and $p = 3$ we see that the structure of the $\ell = 0$ equation (44) is very similar to the Einstein equation (41) given that we map

$$n\mathcal{D}^{(0)}(n^{-1}P_{ab}^{(0)})(t) \longleftrightarrow N^2R_{ab}^{\text{temp}}(t, \mathbf{x}_0), \quad (49)$$

where we chose a fixed but arbitrary spatial point \mathbf{x}_0 . The map can be rewritten in terms of maps for $P_{ab}^{(0)}$, $Q_{ab}^{(0)}$ and n which we present together with corresponding maps for the other fields¹³

$$\begin{aligned} n(t) &\longleftrightarrow Ng^{-\frac{1}{2}}(t, \mathbf{x}_0), \\ Q_{ab}^{(0)}(t) &\longleftrightarrow \omega_{tab}(t, \mathbf{x}_0), \\ P_{ab}^{(0)}(t) &\longleftrightarrow \omega_{abt}(t, \mathbf{x}_0), \\ P_{abc}^{(1)}(t) &\longleftrightarrow F_{tabc}(t, \mathbf{x}_0), \\ P_{a_1\dots a_6}^{(2)}(t) &\longleftrightarrow -\frac{N}{4!}\epsilon_{a_1\dots a_6b_1\dots b_4}F_{b_1\dots b_4}(t, \mathbf{x}_0), \\ P_{a_0|a_1\dots a_8}^{(3)}(t) &\longleftrightarrow \frac{3}{2}N\epsilon_{a_1\dots a_8b_1b_2}\tilde{\Omega}_{b_1b_2a_0}(t, \mathbf{x}_0). \end{aligned} \quad (50)$$

¹³ Note that the position of the flat indices does not matter any more since they are $SO(10) \subset GL(10)$ indices and we use the Euclidean flat δ_{ab} to raise and lower them. Similarly, the ten index ϵ -symbol appearing in equation (50) is the invariant tensor of $SO(10)$.

Using this correspondence the equation (44) is mapped to the relevant Einstein equation¹⁴, equation (45) is mapped to the suitably truncated equation of motion for the gauge field (which in flat indices reads $D_A F^{ABCD} = -\frac{1}{8 \cdot 144} \epsilon^{BCDE_1 \dots E_4 F_1 \dots F_4} F_{E_1 \dots E_4} F_{F_1 \dots F_4}$), equation (46) is the Bianchi identity for the gauge potential, and equation (47) expresses the existence of a factorisation of the space-time-dependent vielbein into a space-dependent times a time-dependent factor [11].

The correspondence (50) provides a map between the dynamics of two seemingly different systems: The E_{10} invariant geodesic model (31) (truncated beyond $\ell = 3$) and the bosonic sector of $d = 11$ supergravity with the gauge choices and truncations detailed above.

4.2. Degeneracies and root multiplicities

As discussed in section 2.2, a generic root α of \mathfrak{e}_{10} is degenerate with multiplicity $\text{mult}(\alpha) > 1$. The significance and proper physical interpretation of the imaginary roots and their multiplicities is far from understood. However, there is at least one non-trivial example where a multiplicity, which is larger than one, admits a proper physical interpretation. Namely, the roots associated with the various components of the spatial spin connection fall into two classes [9]. First of all, there are the gravitational roots (*alias* ‘gravitational walls’) associated with the components $\tilde{\Omega}_{abc}$ for which the indices a, b, c are all different: they correspond to level-3 roots α_{abc} with $\alpha_{abc}^2 = 2$, hence *real* roots, see [9] section 6.2 for explicit formulas. The corresponding components of the dual field $P_{a_0|a_1 \dots a_8}$ are the ones where a_0 is equal to one of the indices a_1, \dots, a_8 .

On the other hand, [9] identified ten subleading gravitational walls associated with ten null roots, designated as μ_a for $a = 1, \dots, 10$, cf. eqn. (6.16) there. These ten null roots can all be obtained by $\mathfrak{sl}(10)$ Weyl reflections (or, equivalently, by permuting the spatial coordinates) from the primitive (i.e. lowest height) null root

$$\delta = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 4\alpha_8 + 2\alpha_9 + 3\alpha_{10} \quad (51)$$

at height 30 which has $\delta^2 = 0$, $\text{mult}(\delta) = 8$ and is identical to the null root of the affine subalgebra $\mathfrak{e}_9 \subset \mathfrak{e}_{10}$ (in the notation of [9], we have $\delta = \mu_1$). This null root and its images under the $\mathfrak{sl}(10)$ Weyl group are the only imaginary roots appearing on levels $\ell \leq 3$ in the A_9 decomposition. The associated components of the dual field $P_{a_0|a_1 \dots a_8}$ belonging to these null roots are the ones for which the indices a_0, \dots, a_8 are all distinct. Using the correspondence (50) we can now give a physical interpretation to the multiplicity $\text{mult}(\delta)$. Since all indices on $P_{a_0|a_1 \dots a_8}$ are different, two indices on the dual coefficient of anholonomy $\tilde{\Omega}_{abc}$ must be equal, i.e. we must consider the components $\tilde{\Omega}_{abb}$ (no summation on b). As shown in [9], these components are then all associated with the null root μ_a , and it would thus appear that we have nine possible values for b . However, due to our gauge choice, there is one linear condition ($\sum_b \tilde{\Omega}_{abb} = 0$) on these nine elements, whence the number of independent field components associated to each null root μ_a is only eight — in agreement with the root multiplicity $\text{mult}(\delta) = 8$! This argument demonstrates again the key importance of the traceless gauge for Ω_{abc} . Let us also note that this multiplicity remains the same when δ is considered as a root of E_{11} , whereas a naive counting would suggest that there are *nine* independent components in that case.

4.3. $d = 10$ massive type IIA supergravity

The analysis for $d = 11$ can be repeated for massive type IIA supergravity in $d = 10$ [41]. This was carried out in [12] in a formalism which uses the D_9 subalgebra of \mathfrak{e}_{10} . The spatial Lorentz

¹⁴ One term in (40) does not fully match when plugging in the correspondence for the mixed symmetry generator: Although one recovers the first and second term in (40) and the last term vanishes in the truncation, the third term with ‘crossed index contraction’ is not reproduced. In the cosmological billiard picture [9], this term is subdominant.

group is now $SO(9)$ and is the diagonal of $SO(9) \times SO(9) = K(SO(9,9))$. In order to find the right representations under the Lorentz group one has to further decompose the $SO(9,9)$ representations of section 2.2.2.

The $(SO(9,9) \times GL(1))/(SO(9) \times SO(9))$ coset model which is the restriction of the $E_{10}/K(E_{10})$ model to $\ell = 0$ in the D_9 decomposition can be shown to be equivalent to the reduction of the bosonic sector of $d = 10$ type I supergravity by extending arguments of [42]. As supergravity the bosonic sector of type I is identical to the NSNS sector of type IIA supergravity.

Turning to the $\ell = 1$ contributions in the $E_{10}/K(E_{10})$ σ -model, we first decompose the **256**-component $SO(9,9)$ spinor under the diagonal $SO(9)$ Lorentz group with the result

$$\mathbf{256} \rightarrow \mathbf{16} \otimes \mathbf{16} = \mathbf{9} + \mathbf{84} + \mathbf{126} + \mathbf{36} + \mathbf{1} \quad (52)$$

These representation can be seen as anti-symmetric tensors of rank p for $p = 1, 3, 5, 7, 9$, which are exactly the RR potentials of massive type IIA supergravity [43].

The suitably truncated sector of massive type IIA supergravity was rewritten in terms of the underlying $SO(9,9)$ symmetry in [12] and shown to be equivalent to the truncated E_{10} model under the map

$$\begin{aligned} n(t) &\longleftrightarrow Ng^{-\frac{2}{3}}e^{-\frac{7}{3}\varphi}(t, \mathbf{x}_0), \\ \varphi^{(0)}(t) &\longleftrightarrow \varphi(t, \mathbf{x}_0), \\ Q_{ij}^{(0)}(t) &\longleftrightarrow \omega_{tij}(t, \mathbf{x}_0) + \frac{1}{2}H_{tij}(t, \mathbf{x}_0), \\ \bar{Q}_{\bar{i}\bar{j}}^{(0)}(t) &\longleftrightarrow \omega_{t\bar{i}\bar{j}}(t, \mathbf{x}_0) - \frac{1}{2}H_{t\bar{i}\bar{j}}(t, \mathbf{x}_0), \\ P_{ij}^{(0)}(t) &\longleftrightarrow \omega_{ijt}(t, \mathbf{x}_0) - \frac{1}{2}H_{tij}(t, \mathbf{x}_0), \\ P_{\alpha\bar{\alpha}}^{(1)}(t) &\longleftrightarrow g^{\frac{1}{4}} \left[ne^{-\varphi} M \delta_{\alpha\bar{\alpha}} + \frac{1}{4} \gamma_{\alpha\bar{\alpha}}^m F_{tm} + \frac{1}{8} ne^{-\varphi} \gamma_{\alpha\bar{\alpha}}^{mn} F_{mn} \right. \\ &\quad \left. + \frac{1}{12} \gamma_{\alpha\bar{\alpha}}^{mnp} G_{tmnp} - \frac{1}{48} ne^{-\varphi} \gamma_{\alpha\bar{\alpha}}^{mnpq} G_{mnpq} \right] (t, \mathbf{x}_0). \end{aligned} \quad (53)$$

Here, indices $i, j = 1, \dots, 9$ and barred indices refer to the second factor of the compact $SO(9) \times SO(9)$. The further supergravity quantities are: φ is the (redefined) dilaton, H_{tij} is the electric field strength of the NSNS two-form, M is the Romans mass parameter, F_{tm} and F_{mn} are the electric and magnetic field strengths of the RR vector potential, and G_{tmnp} and G_{mnpq} are electric and magnetic field strengths of the RR three-form potential. The fields have to be redefined from their standard form in order to make the $SO(9) \times SO(9)$ gauge symmetry manifest, the details can be found in [12].

As shown in [12], under the correspondence (53) there is again a dynamical agreement between the E_{10} coset model equations truncated beyond level $\ell = 1$ in the D_9 decomposition and the truncated massive IIA equations of motion. It was also partly shown that the $\ell = 2$ representation in \mathfrak{e}_{10} under D_9 contains the magnetic duals of the NSNS fields. [12] also contains a partial treatment of fermions and supersymmetry which were used to derive some of the redefinitions required to make $SO(9) \times SO(9)$ manifest. The relation between the bosonic fields of massive IIA supergravity transforming under $GL(10)$ and E_{11} was analysed earlier in [28].

4.4. $d = 10$ type IIB supergravity

Our final example we consider is chiral $d = 10$ IIB supergravity [44, 45]. The corresponding level decomposition of \mathfrak{e}_{10} is the $A_8 \oplus A_1$ decomposition discussed in section 2.2.3. The explicit factor of $A_1 \equiv \mathfrak{sl}(2)$ together with tentative spatial Lorentz group $\mathfrak{so}(9) = K(A_8) = K(\mathfrak{sl}(9))$ already

hints at a type IIB interpretation. That this is indeed true was shown in [13]. We briefly recall the correspondence and highlight the interesting new features.

The field content of table 3 is interpreted in terms of IIB quantities. On $\ell = 0$ we find the spatial vielbein coset $GL(9)/SO(9)$ and the axion-dilaton coset $SL(2)/SO(2)$. Level $\ell = 1$ contains the $SL(2)$ doublet of electric field strengths of the two-form potentials which correspond to the F- and the D-string [46]. Level $\ell = 2$ contains an $SL(2)$ singlet electric field strength of the four-form potential. Level $\ell = 3$ carries the $SL(2)$ doublet of magnetic field strengths of the two-form potentials. Finally, level $\ell = 4$ contains the dual graviton mixed symmetry tensor and the magnetic field strengths of the axion-dilaton pair, together with a gauge potential.

As shown in [13] there is again a dynamical match of the truncated IIB equations of motion and the truncated E_{10} σ -model under the following correspondence

$$\begin{aligned}
n(t) &\longleftrightarrow Ng^{-\frac{1}{2}}(t, \mathbf{x}_0), \\
P_{ab}^{(0)}(t) &\longleftrightarrow \omega_{abt}(t, \mathbf{x}_0), \\
Q_{ab}^{(0)}(t) &\longleftrightarrow \omega_{tab}(t, \mathbf{x}_0), \\
P^{(0)1}(t) &\longleftrightarrow e^\phi \partial_t \chi(t, \mathbf{x}_0), \\
P^{(0)3}(t) &\longleftrightarrow \partial_t \phi(t, \mathbf{x}_0), \\
Q^{(0)}(t) &\longleftrightarrow e^\phi \partial_t \chi(t, \mathbf{x}_0), \\
P_{a_1 a_2}^{(1)\alpha}(t) &\longleftrightarrow H_{ta_1 a_2, \alpha}(t, \mathbf{x}_0), \\
P_{a_1 \dots a_4}^{(2)}(t) &\longleftrightarrow H_{ta_1 \dots a_4}(t, \mathbf{x}_0), \\
P_{a_1 \dots a_6}^{(3)\alpha}(t) &\longleftrightarrow \frac{1}{3!} N \epsilon_{a_1 \dots a_6 c_1 \dots c_3} H_{a_1 \dots a_3, \alpha}(t, \mathbf{x}_0), \\
P_{a_1 \dots a_7 | a_8}^{(4)}(t) &\longleftrightarrow \frac{1}{2} N \epsilon_{a_1 \dots a_7 bc} \tilde{\Omega}_{bc | a_8}(t, \mathbf{x}_0), \\
P_{a_1 \dots a_8}^{(4)1}(t) &\longleftrightarrow N \epsilon_{a_1 \dots a_8 b} e^\phi \partial_b \chi(t, \mathbf{x}_0), \\
P_{a_1 \dots a_8}^{(4)2}(t) &\longleftrightarrow N \epsilon_{a_1 \dots a_8 b} e^\phi \partial_b \chi(t, \mathbf{x}_0), \\
P_{a_1 \dots a_8}^{(4)3}(t) &\longleftrightarrow N \epsilon_{a_1 \dots a_8 b} \partial_b \phi(t, \mathbf{x}_0).
\end{aligned} \tag{54}$$

Here, ϕ and χ are the dilaton and the axion, respectively. The α index is an $SO(2)$ vector index and components 1 and 3 of the $SL(2)$ triplet form a doublet under $SO(2)$, while the 2 component transforming as a singlet. In the correspondence, the mixed $SL(9)$ symmetry tensor on $\ell = 4$ has the same shortcomings as the one discussed for $d = 11$ supergravity. Furthermore, we see that not all components of the triplet of eight-forms on $\ell = 4$ are independent which is due to the gauge fixing for the axion-dilaton coset, see [13] for more details and precise conventions.

An important point here concerns the self-duality constraint of the five-form field strength in IIB supergravity. As explained in [13], the dynamical correspondence (54) is only valid if this self-duality is used on the supergravity side, eliminating all magnetic field strength component in favour of electric ones which appear in (54). The relation between IIB supergravity and E_{11} was analysed earlier in [27].

As also discussed in [13] there is a qualitative difference between E_{11} and E_{10} which manifests itself in the present analysis. The corresponding type IIB decomposition of E_{11} is under $A_9 = \mathfrak{sl}(9)$ which allows for anti-symmetric ten-form potentials, which in string theory are thought to be the source for nine-branes [47]. The analysis of [37, 13] revealed the existence of an $SL(2)$ *quadruplet* and a *doublet* in the E_{11} spectrum under A_9 . This somewhat unexpected $SL(2)$ structure was later confirmed to be compatible kinematically with an analysis of the IIB supersymmetry algebra [48].

4.5. Versatility summary

The results presented so far can be summarised as follows. Different level decompositions of E_{10} produce the correct spectra of the maximal supergravity theories, see section 2.2. These identifications provide a *dynamical* match between the geodesic $E_{10}/K(E_{10})$ model defined in section 3, and written according to these decompositions and truncated at prescribed levels, and the appropriately truncated bosonic equations of motion of all maximal supergravities. The relevant decompositions are summarised in fig. 3.

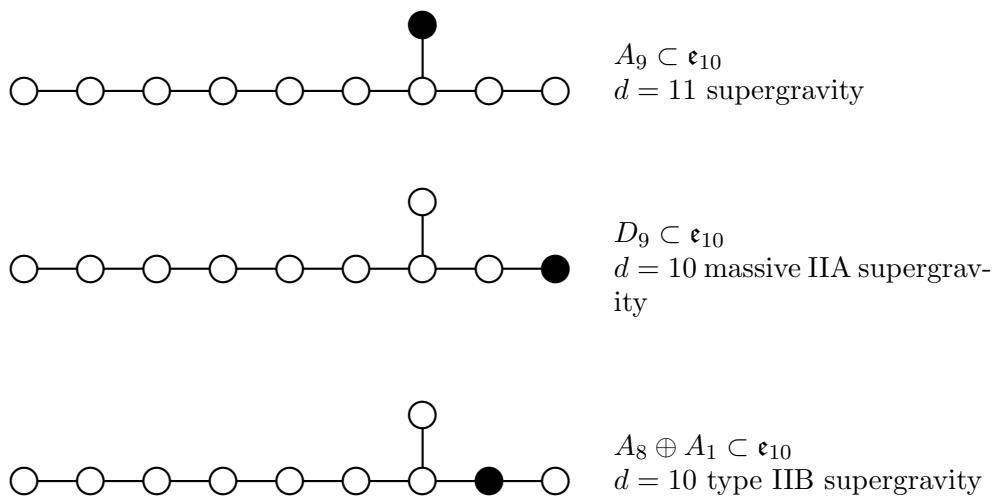


Figure 3. E_{10} versatility summarised. The filled nodes indicate the deleted nodes in the corresponding level decompositions.

5. Discussion and Outlook

Finally, we briefly discuss a few related topics and open problems in the E_{10} approach.

5.1. Cosmological solutions

Given that there is correspondence between two dynamical systems, as detailed in the maps (50), (53) and (54), one can map solutions of one system to solutions of the corresponding other system. Besides the fact that the E_{10} model might be simpler to solve in subsectors since it is integrable (as explained in section 3), constructing solutions of the E_{10} model has the additional advantage that one can map *one* solution of the E_{10} model to *three different* solutions of the maximal supergravity theories due to the versatility of the model.

The simplest way to obtain solutions to the E_{10} σ -model is by restricting to a submanifold which is a coset of a finite-dimensional subgroup of E_{10} . On such spaces the geodesic motion is known to be integrable of Toda type (see for example [49, 50] for overviews). The simplest example is $SL(2)/SO(2)$ where the problem reduces to solving the one-dimensional Liouville equation. In the E_{10} context this was studied in [51]. Using the $SL(2)$ generated by the \mathfrak{e}_{10} generators e_{10}, h_{10}, f_{10} , and one additional (orthogonal) CSA element to satisfy the Hamiltonian constraint of (32), the following solution to the bosonic sector of $d = 11$ supergravity was found in [51]

$$ds^2 = -e^{14\tilde{\phi}-2\phi/3} dt^2 + e^{4\phi/3} (dx^2 + dy^2 + dz^2) + e^{2\tilde{\phi}-2\phi/3} (dw_1^2 + \dots + dw_7^2),$$

$$F_{txyz} = \frac{E}{a \cosh^2(\sqrt{E}t)} = a e^{4\phi}, \quad (55)$$

with

$$\begin{aligned} \phi(t) &= -\frac{1}{2} \ln \left[\frac{a}{\sqrt{E}} \cosh(\sqrt{E}(t - t_0)) \right], \\ \tilde{\phi}(t) &= \frac{1}{42} \sqrt{\tilde{E}}(t - t_1). \end{aligned} \quad (56)$$

Here, x, y, z correspond to directions 8, 9, 10, and w_1, \dots, w_7 are seven transverse directions. The ‘energies’ E and \tilde{E} are related by $\tilde{E} = 21E$ as follows from the Hamiltonian constraint of the E_{10} model. Furthermore, the gauge $n = 1$ was chosen. This solution is known to have phases of accelerated and decelerated expansion [52, 53], and is identical in form to the SM2-brane solution, see [54] and references therein. Similar BPS solutions and their properties in the context of E_{11} were discussed earlier in [55, 56, 57, 58, 59, 60]. References to papers containing cosmological solutions to supergravity can be found in [51].

5.2. E_{10} , fermions and generalised holonomy

Recently progress was made on the important question of how to bring fermions into the picture [61, 62]. This amounts to adding spin to the massless particle moving on a geodesic trajectory in the $E_{10}/K(E_{10})$ coset space. In analogy with what happens for smaller hidden symmetries E_n ($n \leq 8$) one expects the fermions to transform under the denominator group $K(E_n)$. In the case of E_{10} this poses an algebraic problem since $K(\mathfrak{e}_{10})$ is *not* a Kac–Moody algebra as mentioned in section 2.1. In fact, hardly anything is known about the representation theory of this infinite-dimensional algebra.

The idea of [61] was to bring the fermionic equation of motion into the play *and* fix a supersymmetric gauge since for the bosonic correspondences one also needs to fix all gauges. The gauge chosen was $\psi_0 - \Gamma_0 \Gamma^a \psi_a = 0$ and to write the remaining **320** equations of motion for the $d = 11$ gravitino as¹⁵ $\mathcal{E}_a = 0 \iff \mathcal{D}\Psi = 0$, where Ψ is a $K(E_{10})$ spinor on which the $K(E_{10})$ -covariant derivative acts. The spinor Ψ is thought to be infinite-dimensional components and, if decomposed under the spatial Lorentz $SO(10)$, to contain the gravitino ψ_a . The analysis of [61] showed that by evaluating the equations of motion and using the dictionary (50) one can deduce the action of the $K(E_{10})$ generators up to ‘level’¹⁶ $\ell = 3$ and show consistency with the $K(E_{10})$ commutation relations. In fact, it can be proven that the **320** components of the supergravity fermion furnish a representation of $K(\mathfrak{e}_{10})$ by themselves. This representation is necessarily *unfaithful*.

Repeating the same analysis for Dirac fermions shows that the **32** representation of $SO(10)$ is also an unfaithful representation of $K(\mathfrak{e}_{10})$.¹⁷ The action on the Dirac spinor is in terms of anti-symmetric 32×32 matrices, hence fundamental $SO(32)$ -matrices. This is somewhat along the lines of proposals for a generalised holonomy of M-theory [64, 65, 66] but with a very important difference. The Dirac spinor representation of $SO(1, 10)$ is *not* turned into a representation of $SO(32)$ but into an unfaithful representation of $K(E_{10})$. This way one circumvents global problems with $SO(32)$ pointed out in [67].¹⁸ Similar remarks could very well apply to $SL(32)$ which was discovered in the M5-brane equations [68] and discussed in the context of E_{11} in [20].

¹⁵ The index a runs over the spatial directions and hence takes 10 values, the 32 component spinor index has been suppressed.

¹⁶ Level here is not meant as a grading of $K(\mathfrak{e}_{10})$ which does not exist.

¹⁷ See also [63] for low level results on Dirac fermions and E_{10} .

¹⁸ The gravitino representation could never have been formed an $SO(32)$ representation.

5.3. Open problems

Despite all the encouraging results presented in this contribution, there remain a number of very important open problems with the E_{10} model.

- The most pressing questions is probably the following: *What rôle do the higher levels play?* When establishing the correspondences to the supergravity theories in section 4, we truncated the σ -model equations of motion after a fixed level. However, there are infinitely many higher levels whose contributions to the dynamics can be determined in principle but whose physical interpretation is not clear. The original paper [10] made the conjecture that their effect could be to re-introduce the full space dependence from supergravity and thereby turn space-time into an emergent concept. This so-called *gradient conjecture* is based on the observation that in the infinite list of representations in the A_9 level decomposition there exist the following representations for $k \geq 0$

ℓ	A_9 Dynkin	Generator	Interpretation ?
$3k + 1$	$[k00000100]$	$E_{a_1 \dots a_k}{}^{b_1 b_2 b_3}$	$\partial_{a_1} \dots \partial_{a_k} F_{b_1 b_2 b_3}$
$3k + 2$	$[k00100000]$	$E_{a_1 \dots a_k}{}^{b_1 \dots b_6}$	$\partial_{a_1} \dots \partial_{a_k} \epsilon_{b_1 \dots b_6 c_1 \dots c_4} F_{c_1 \dots c_4}$
$3k + 3$	$[k10000001]$	$E_{a_1 \dots a_k}{}^{b_0 b_1 \dots b_8}$	$\partial_{a_1} \dots \partial_{a_k} \epsilon_{b_1 \dots b_8 c_1 c_2} \Omega_{c_1 c_2 b_0}$

The first Dynkin label entry k translates into k symmetric sets of 9 anti-symmetric indices which each have been lowered using the invariant $\mathfrak{sl}(10)$ ϵ -symbol¹⁹. The generators are therefore symmetric in the lower indices, which makes the suggested interpretation possible at least in principle.²⁰ Of course, the term ‘Taylor expansion’ here must be interpreted *cum grano salis*, since it is very well known from previous results on the cascades of dual potentials appearing for the affine Geroch group (see e.g. [69]) that these higher order spatial gradients involve *non-local* relations between the fields in the coset model, see also [32]. There is an infinity of additional representations besides the gradient representations and these have been conjectured to be associated partly with new M-theoretic degrees of freedom.

- There are very few results concerning involutory subgroups of infinite dimensional Kac–Moody groups, an $K(E_{10})$ in particular. For instance, is it possible to construct *faithful* spinor representations of $K(E_{10})$ which could also accommodate the spatial dependence of the fermionic fields? This hinges largely on a better understanding of the relevant representation theory.
- Can one construct similar correspondences for other Kac–Moody cosets and other theories? This seems likely and has already been done for the hyperbolic extension of G_2 [70].

Acknowledgements

Many of the results reported here have been obtained in collaboration with Thibault Damour and Marc Henneaux. We are grateful to them, as well as to Christian Hillmann and Peter West, for discussions and correspondence.

References

- [1] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. B **76** (1978) 409–412
- [2] E. Cremmer and B. Julia, *The $SO(8)$ Supergravity*, Nucl. Phys. B **159** (1979) 141
- [3] B. Julia, *Application of supergravity to gravitation theories*, in: Unified Field Theories of more than 4 Dimensions, eds. V. De Sabbata and E. Schmutzer, World Scientific (Singapore, 1983)
- [4] B. Julia, in: Lectures in Applied Mathematics, Vol. 21 (1985), AMS-SIAM, p. 335; preprint LPTENS 80/16

¹⁹ As $\mathfrak{gl}(10)$ representations this requires a compensating term in the transformation rules [32].

²⁰ There is an additional tracelessness constraint on the generators following from the Young symmetries of the $\mathfrak{sl}(10)$ representations.

- [5] H. Nicolai, *D = 11 Supergravity With Local SO(16) Invariance*, Phys. Lett. B **187** (1987) 316
- [6] H. Nicolai, *The Integrability Of N = 16 Supergravity*, Phys. Lett. B **194** (1987) 402
- [7] S. Mizoguchi, *E₁₀ symmetry in one-dimensional supergravity*, Nucl. Phys. B **528** (1998) 238–264, [hep-th/9703160](#)
- [8] T. Damour and M. Henneaux, *E₁₀, BE₁₀ and arithmetical chaos in superstring cosmology*, Phys. Rev. Lett. **86** (2001) 4749, [hep-th/0012172](#)
- [9] T. Damour, M. Henneaux and H. Nicolai, *Cosmological Billiards*, Class. Quant. Grav. **20** (2003) R145–R200, [hep-th/0212256](#)
- [10] T. Damour, M. Henneaux and H. Nicolai, *E₁₀ and a "small tension expansion" of M-theory*, Phys. Rev. Lett. **89** (2002) 221601, [hep-th/0207267](#)
- [11] T. Damour and H. Nicolai, *Eleven dimensional supergravity and the E₁₀/K(E₁₀) σ -model at low A₉ levels*, in: Group Theoretical Methods in Physics, Institute of Physics Conference Series No. 185, IoP Publishing, 2005, [hep-th/0410245](#)
- [12] A. Kleinschmidt and H. Nicolai, *E₁₀ and SO(9,9) invariant supergravity*, JHEP **0407** (2004) 041, [hep-th/0407101](#)
- [13] A. Kleinschmidt and H. Nicolai, *IIB supergravity and E₁₀*, Phys. Lett. B **606** (2005) 391, [hep-th/0411225](#)
- [14] T. Damour and H. Nicolai, *Higher order M theory corrections and the Kac-Moody algebra E₁₀*, Class. Quant. Grav. **22** (2005) 2849–2879, [hep-th/0504153](#)
- [15] J. Brown, O. J. Ganor and C. Helfgott, *M Theory and E₁₀: Billiards, Branes and Imaginary Roots*, [hep-th/0401053](#)
- [16] J. Brown, S. Ganguli, O. J. Ganor and C. Helfgott, *E₁₀ orbifolds*, JHEP **0506** (2005) 057, [hep-th/0409037](#)
- [17] P. C. West, *Hidden superconformal symmetry in M theory*, JHEP **0008** (2000) 007, [hep-th/0005270](#)
- [18] P. C. West, *E₁₁ and M theory*, Class. Quant. Grav. **18** (2001) 4443–4460, [hep-th/0104081](#)
- [19] A. B. Borisov and V. I. Ogievetsky, *Theory of dynamical affine and conformal symmetries as gravity theory*, Theor. Math. Phys. **23** (1976) 1179–1188
- [20] P. West, *E₁₁, SL(32) and central charges*, Phys. Lett. B **575** (2003) 333–342, [hep-th/0307098](#)
- [21] F. Englert and L. Houart, *G⁺⁺⁺ invariant formulation of gravity and M-theories: exact BPS solutions*, JHEP **0401** (2004) 002, [hep-th/0311255](#)
- [22] A. Kleinschmidt and P. West, *Representations of G⁺⁺⁺ and the role of space-time*, JHEP **0402** (2004) 033, [hep-th/0312247](#)
- [23] A. Keurentjes, *E₁₁: Sign of the times*, [hep-th/0402090](#)
- [24] S. de Buyl, L. Houart and N. Tabti, *Dualities and signatures of G⁺⁺ invariant theories*, JHEP **0506** (2005) 084, [hep-th/0505199](#)
- [25] C. M. Hull, *Duality and the signature of space-time*, JHEP **9811**, 017 (1998), [hep-th/9807127](#)
- [26] F. Englert, M. Henneaux and L. Houart, *From very-extended to overextended gravity and M-theories*, JHEP **0502** (2005) 070, [hep-th/0412184](#)
- [27] I. Schnakenburg and P. C. West, *Kac-Moody symmetries of 2B supergravity*, Phys. Lett. B **517** (2001) 421–428, [hep-th/0107081](#)
- [28] I. Schnakenburg and P. C. West, *Massive IIA supergravity as a nonlinear realization*, Phys. Lett. B **540** (2002) 137–145, [hep-th/0204207](#)
- [29] K. Bautier, S. Deser, M. Henneaux and D. Seminara, Phys. Lett. B **B406** (1997) 49–53, [hep-th/9704131](#)
- [30] V. G. Kac, *Infinite dimensional Lie algebras*, 3rd edition, Cambridge University Press (Cambridge, 1990)
- [31] R. V. Moody and A. Pianzola, *Lie algebras with triangular decomposition*, John Wiley&Sons (New York, 1995)
- [32] A. Kleinschmidt and H. Nicolai, *Gradient representations and affine structures in AE_n*, Class. Quant. Grav. **22** (2005) 4457, [hep-th/0506238](#)
- [33] H. Nicolai and T. Fischbacher, *Low level representations of E₁₀ and E₁₁*, Contribution to the Proceedings of the Ramanujan International Symposium on Kac-Moody Algebras and Applications, ISKMAA-2002, Chennai, India, 28–31 January, [hep-th/0301017](#)
- [34] A. Kleinschmidt, *E₁₁ as E₁₀ representation at low levels*, Nucl. Phys. B **677** (2004) 553–586, [hep-th/0304246](#)
- [35] M. R. Gaberdiel, D. I. Olive and P. West, *A class of Lorentzian Kac-Moody algebras*, Nucl. Phys. B **645** (2002) 403–437, [hep-th/0205068](#)
- [36] P. West, *Very extended E₈ and A₈ at low levels, gravity and supergravity*, Class. Quant. Grav. **20** (2003) 2393–2406, [hep-th/0212291](#)
- [37] A. Kleinschmidt, I. Schnakenburg and P. West, *Very-extended Kac-Moody algebras and their interpretation at low levels*, Class. Quant. Grav. **21** (2004) 2493 [[arXiv:hep-th/0309198](#)].
- [38] T. Fischbacher, *The structure of E₁₀ at higher A₉ levels: A first algorithmic approach*, JHEP **0508** (2005) 012, [hep-th/0504230](#)
- [39] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B **438** (1995) 109–137,

- hep-th/9410167
- [40] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. B **443** (1995) 85–126, hep-th/9503124
 - [41] L. J. Romans, *Massive $N = 2a$ supergravity in ten dimensions*, Phys. Lett. B **169** (1986) 374–380
 - [42] J. Maharana and J. H. Schwarz, *Noncompact Symmetries in String Theory*, Nucl. Phys. B **390** (1993) 3–32, hep-th/9207016
 - [43] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos and P. K. Townsend, *Duality of Type II 7-branes and 8-branes*, Nucl. Phys. B **470**, 113 (1996), hep-th/9601150
 - [44] J. H. Schwarz and P. C. West, *Symmetries And Transformations Of Chiral $N=2$ $D = 10$ Supergravity*, Phys. Lett. B **126** (1983) 301.
 - [45] J. H. Schwarz, *Covariant Field Equations Of Chiral $N=2$ $D = 10$ Supergravity*, Nucl. Phys. B **226** (1983) 269.
 - [46] J. H. Schwarz, *An $SL(2, \mathbb{Z})$ multiplet of type IIB superstrings*, Phys. Lett. B **360**, 13 (1995) [Erratum-ibid. B **364**, 252 (1995)] [arXiv:hep-th/9508143].
 - [47] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724, hep-th/9510017
 - [48] E. A. Bergshoeff, M. de Roo, S. F. Kerstan and F. Riccioni, *IIB supergravity revisited*, JHEP **0508**, 098 (2005), hep-th/0506013
 - [49] A. V. Razumov and M. V. Saveliev, *Lie Algebras, Geometry and Toda-type Systems*, Cambridge Lecture Notes in Physics, Cambridge University Press (Cambridge, 1997)
 - [50] O. Babelon, D. Bernard and M. Talon, *Introduction to classical integrable systems*, Cambridge University Press (Cambridge, 2003)
 - [51] A. Kleinschmidt and H. Nicolai, *E_{10} cosmology*, hep-th/0511290
 - [52] N. Ohta, *Accelerating cosmologies from S-branes*, Phys. Rev. Lett. **91** (2003) 061303, hep-th/0303238
 - [53] M. N. R. Wohlfarth, *Accelerating cosmologies and a phase transition in M-theory*, Phys. Lett. B **563** (2003) 1, hep-th/0304089
 - [54] N. Ohta, *Accelerating cosmologies and inflation from M/superstring theories*, Int. J. Mod. Phys. A **20** (2005) 1, hep-th/0411230
 - [55] F. Englert, L. Houart, A. Taormina and P.C. West, *The symmetry of M theories*, JHEP **0309** (2003) 020, hep-th/0304206
 - [56] F. Englert, L. Houart and P.C. West, *Intersection rules, dynamics and symmetries*, JHEP **0308** (2003) 025, hep-th/0307024
 - [57] P.C. West, *The IIA, IIB and eleven-dimensional theory and their common E_{11} origin*, hep-th/0402140
 - [58] F. Englert and L. Houart, *\mathcal{G}^{+++} invariant formulation of gravity and M-theories: exact intersecting brane solutions*, JHEP **0405** (2004) 059, hep-th/0405082
 - [59] P. P. Cook and P. C. West, *\mathcal{G}^{+++} and Brane Solutions*, hep-th/0405149
 - [60] L. Houart, *Kac-Moody algebras in gravity and M-theories*, hep-th/0511009
 - [61] T. Damour, A. Kleinschmidt and H. Nicolai, *Hidden symmetries and the fermionic sector of eleven-dimensional supergravity*, hep-th/0512163
 - [62] S. de Buyl, M. Henneaux and L. Paulot, *Extended E_8 Invariance of 11-Dimensional Supergravity*, hep-th/0512292
 - [63] S. de Buyl, M. Henneaux and L. Paulot, *Hidden symmetries and Dirac fermions*, Class. Quant. Grav. **22** (2005) 3595, hep-th/0506009
 - [64] M. J. Duff and K. S. Stelle, *Multimembrane solutions of $D = 11$ supergravity*, Phys. Lett. B **253** (1991) 113–118
 - [65] M.J. Duff and J. T. Liu, *Hidden space-time symmetries and generalized holonomy in M theory*, Nucl. Phys. B **674** (2003) 217–230, hep-th/0303140
 - [66] C. Hull, *Holonomy and symmetry in M theory*, hep-th/0305039
 - [67] A. Keurentjes, *The topology of U Duality (sub)groups*, Class. Quantum. Grav. **21** (2004) 1695-1708, hep-th/0309106
 - [68] O. Bärwald and P. C. West, *Brane rotating symmetries and the five-brane equations of motion*, Phys. Lett. B **476** (2000) 157–164, hep-th/9912226
 - [69] P. Breitenlohner and D. Maison, *On the Geroch group*, Ann. Poincaré Phys. Theor. **46** (1987) 215–246
 - [70] S. Mizoguchi, K. Mohri and Y. Yamada, *Five-dimensional supergravity and hyperbolic Kac-Moody algebra G_2^H* , hep-th/0512092