On the relation between p-adic and ordinary strings

Debashis Ghoshal
Harish-Chandra Research Institute, Chhatnag Road, Allahabad 211019, India. ghoshal@mri.ernet.in

The amplitudes for the tree-level scattering of the open string tachyons, generalised to the field of p-adic numbers, define the p-adic string theory. There is empirical evidence of its relation to the ordinary string theory in the $p \to 1$ limit. We revisit this limit from a worldsheet perspective and argue that it is naturally thought of as a continuum limit in the sense of the renormalization group.

PACS numbers: 11.25.-w, 11.10.Hi

The p-adic string theory was proposed[1] with a mathematical motivation and with a hope that the amplitudes of these theories, considered for all primes, will relate to those of the ordinary strings through the adelic relation[2]. While this idea remains to be realized, the early papers[1]–[4] worked out the details of this theory. In particular, all the tree-level tachyon amplitudes were computed from which the spacetime effective theory of the tachyonic scalar was obatined and the solutions of its equation of motion were studied. Subsequently a 'worldsheet' understanding was developed[5]–[9] (see [10] for a review). More recently it has come into focus through the realization that the exact spacetime theory of its tachyon allows one to study the process of tachyon condensation. In Ref.[11], the solitons of the effective theory of the p-adic tachyon[3] were identified with the D-branes and shown that the tachyon behaves according to the conjectures by Sen[12]. (Henceforth we will refer to the p-adic string as p-string and its tachyon as p-tachyon for brevity.)

An unexpected relation emerges with the ordinary bosonic open string in the $p \to 1$ limit[13] (see also the prescient comments in Ref.[5]), when the effective action of the p-tachyon turns out to approximate that obtained from the boundary string field theory[14, 15] (BSFT) of ordinary strings. BSFT was useful in proving the Sen conjectures[13, 16, 17]. This correspondence remains even after a noncommutative deformation of the p-tachyon effective action. In fact, thanks to it one can find *exact* noncommutative solitons in BSFT (of the ordinary string theory) at *all* values of the deformation parameter[18].

However, this relation in the $p \to 1$ limit is empirical. Moreover, strictly p can only take discrete values. In this Letter, we consider the issue from a worldsheet point of view to advocate that the limit is to be understood in terms of a sequence of string theories based on (algebraic) extensions of increasing degree of the p-adic number field $\mathbf{Q}_p$. We argue that each of these provide a discretization of the ordinary worldsheet (the disk or UHP) and their effective actions relate to each other in terms of the renormalization group (RG). There is a natural continuum limit in which the RG transformed effective value of p tends to one. A preliminary version of these ideas was presented in the '12th Regional Conference on Mathematical Physics' held in Islamabad, Pakistan[19].

Recall that the tree-level scattering amplitude of N on-shell ordinary open-string tachyons of momenta k_i ($i = 1, \cdots, N$), $k_i^2 = 2, \sum k_i = 0$ is given by the Koba-Nielsen formula in which the integrals are over the real line $\mathbf{R}$ and the integrand only involves absolute values of real numbers:

$$\mathcal{A}_N = \int \prod_{i=4}^{N} d\xi_i |\xi_i|^{k_1\cdot k_i} |1-\xi_i|^{k_2\cdot k_i} \prod_{4\le i<j\le N} |\xi_i - \xi_j|^{k_i\cdot k_j} . \tag{1}$$

Except for $\mathcal{A}_4$, the rest cannot be computed analytically. Ref.[1] considered the above problem over the local field of p-adic numbers $\mathbf{Q}_p$, to which it admits a ready extension. In order to describe it, let us digress briefly.

On the field of rational numbers $\mathbf{Q}$, the familiar norm is the absolute value. The field $\mathbf{R}$ of real numbers arise as the completion of $\mathbf{Q}$ when we put in the limit points of all Cauchy sequences, in which convergence is decided by the absolute value norm. However, it is possible to define other norms on $\mathbf{Q}$ consistently. To this end, fix a *prime* number p and determine the highest powers n_1 and n_2 of p that divides respectively the numerator z_1 and denominator z_2 in a rational number z_1/z_2, (z_1, z_2 coprime). The *p-adic norm* of z_1/z_2, defined as: $|z_1/z_2|_p = p^{n_2-n_1}$, satisfies all the required properties, indeed even a stronger version of the triangle inequality. The field $\mathbf{Q}_p$ is obatined by completing $\mathbf{Q}$ using the p-adic norm. Any p-adic number $\xi \in \mathbf{Q}_p$ has a representation as a Laurent-like series in p:

$$\xi = p^N \left(\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots\right), \tag{2}$$

where, $N \in \mathbf{Z}$ is an integer, $\xi_n \in \{0, 1, \cdots, p-1\}$, $\xi_0 \neq 0$ and $|\xi|_p = p^{-N}$. Deatils of materia p-adica are available in *e.g.*, [20]–[22]; some essential aspects are reviewed in [10].

Coming back to the Koba-Nielsen amplitudes, Freund *et al* modified these by replacing the absolute values by p-adic norms and the real integrals by integrals over $\mathbf{Q}_p$. These are, by definition, the amplitudes for the scattering of N open p-string tachyons. The benefit is that all these integrals over $\mathbf{Q}_p$ can be evaluated analytically. Equivalently the tree level effective action of the open p-string tachyon T is known *exactly*[3, 4]: in terms of a rescaled and shifted field $\varphi = 1 + g_s T/p$:

$$\mathcal{L}_p \;=\; \frac{p^2}{g^2(p-1)}\left[-\frac{1}{2}\varphi\, p^{-\frac{1}{2}\Box}\varphi + \frac{1}{p+1}\,\varphi^{p+1}\right]. \qquad (3)$$

We emphasize that in the above, the boundary of the the open p-string worldsheet is valued in $\mathbf{Q}_p$, but the spacetime in which the p-string propagates is the usual one. Once one arrives at the spacetime action (3), however, it can be extrapolated to all integers. Incidentally, there is also another extrapolation, unrelated to this, in which the Veneziano amplitude (expressed in terms of the gamma function) is modified to be valued in $\mathbf{Q}_p$[23]–[25].

The equation of motion from (3) admits the constant solutions $\varphi = 1$ (unstable vacuum with the D-brane) and $\varphi = 0$. There is no perturbative open string excitation around the latter, and hence is to be identified as the (meta-)stable closed p-string vacuum. There are also soliton solutions. For any (spatial) direction, there is a localized gaussian lump[3]. When identified as the different D-m-branes, the descent relations between these confirm the Sen conjectures[11].

If one substitutes $p = 1 + \epsilon$ in (3) and takes the limit $\epsilon \to 0$, one obtains[13], after a field redefinition $\varphi = e^{-T/2}$, the effective action of the tachyon of the ordinary open string theory calculated from BSFT[14, 15]. After a noncommutative deformation of (3), the gaussian soliton of p-string theory generalizes to a one-parameter family of solitons, which are exact solutions to the equation of motion. In the limit $p \to 1$, one finds an exact solution to the ordinary string theory, where the noncommutativity comes from a constant B-field background[18]. (Refs.[26, 27] attempt to find the worldsheet origin of the noncommutativity in p-string theory.)

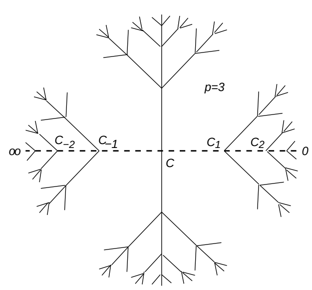

FIG. 1: The 'worldsheet' of the 3-adic string $\mathcal{B}_3$, $\partial\mathcal{B}_3 = \mathbf{Q}_3$. The dotted line is the path from the boundary points 0 to ∞.

At first sight the relation to the ordinary strings is all the more surprising and counter-intuitive from the point of view of the p-string 'worldsheet'. In fact, the 'worldsheet' itself, the boundary of which is $\mathbf{Q}_p$, is not in the least obvious[8, 9]. At tree level, the analog of the unit disk or the UHP of the usual theory, is an infinite lattice with no closed loops, *i.e.*, a uniform tree $\mathcal{B}_p$ in which $p+1$ edges meet at each vertex (see Fig. 1). This is the familiar Bethe lattice $\mathcal{B}_p$, known in the context of $\mathbf{Q}_p$ as the Bruhat-Tits tree. Its boundary, defined as the union of all infinitely remote vertices, can be identified with $\mathbf{Q}_p$. To see this, one may use *e.g.*, the representation (2) in which case, the integer N chooses a branch along the dotted path (in Fig. 1) and the infinite set of coefficients ξ_n determine the path to the boundary. On the other hand, the tree $\mathcal{B}_p$ is the (discrete) homogeneous space PGL(2,$\mathbf{Q}_p$)/PGL(2,$\mathbf{Z}_p$). This construction parallels the case of the ordinary string theory.

The Polyakov action on the 'worldsheet' $\mathcal{B}_p$ is the natural discrete lattice action for the free massless fields X^μ. The action of the laplacian at a site $z \in \mathcal{B}_p$ is $\nabla^2 X^\mu(z) = \sum_i X^\mu(z_i) - (p+1)X^\mu(z)$, where, z_i are the $p+1$ nearest neighbors of z. It was shown in [8] that starting with a finite Bethe lattice and inserting the tachyon vertex operators on the boundary, one recovers the prescription of [1, 3] in the thermodynamic limit.

Naively the lattice is one dimensional for $p = 1$. However, the relation to the ordinary string is through the limit $p \to 1$ and it is not apparent how to make sense of this for the discrete variable p. This is the problem we will address in the following. First, we claim that $\mathcal{B}_p$ gives a discretization of the disk/UHP. This does not seem possible because in $\mathcal{B}_p$, the number of sites upto some generation n from an origin C (say) grows exponentially for large n:

$$\mathcal{N}_n \sim \exp(n\,\ln p). \qquad (4)$$

Therefore, its formal dimension is infinite. Indeed, Bethe lattices are used in calculating the results in the upper critical dimension of model theories. For example, for a free scalar field theory with arbitrary interactions (from, say, vertex operators) the upper critical dimension is two. One would expect to get this from a Bethe lattice.

The tacit assumption above is that the embedding is in an *Euclidean* space. On the other hand, in a d-dimensional *hyperbolic* space with the metric $ds^2_H = dr^2 + R_0^2 \sinh^2\left(\frac{r}{R_0}\right) d\Omega^2_{d-1}$ the volume of a ball of radius R ($R >> R_0$, the radius of curvature) also grows exponentially for large R:

$$\mathrm{vol}_d(R) \sim \exp\left(\frac{d-1}{R_0}\,R\right). \qquad (5)$$

This suggests a natural embedding of $\mathcal{B}_p$ in hyperbolic spaces. Parametrizing

$$p = 1 + \frac{a}{R_0}(d-1), \qquad (6)$$

and considering the limit $a \to 0$ so that $p \to 1$, the formulas (4) and (5) agree for $\lim_{\substack{n\to\infty \\ a\to 0}} n\,a = R$, from which a is seen as the lattice spacing. Thus a uniform Bethe lattice $\mathcal{B}_p$ can be used to discretize a hyperbolic space of constant negative curvature. Moreover, $p \to 1$ provides a natural continuum limit. This is true, in particular, when the dimension $d = 2$, the case of our interest. In fact the embedding of $\mathcal{B}_p$ into the unit disk/UHP equipped with, say, the Poincaré metric), is *isometric*. It is related to the hyperbolic tessellation of the disk/UHP and often has interesting connection with the fundamental domains of the modular functions of SL(2,**C**) and its subgroups[28].

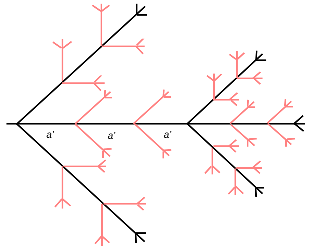

FIG. 2: The lattice with spacing a' leads to a coarse grained one with spacing $a = ma'$ ($m = 3$ here), when the 'grey' branches are integraded out.

The standard way to obtain a continuum limit from a lattice regularization is to go to lattices with smaller lattice spacings and eventually consider the limit in which this becomes vanishingly small. Suppose we start with the 'black' sublattice in Fig. 2, the boundary of which is $\mathbf{Q}_3$. In comparing this to the full lattice, we see that between two neighbouring 'black' nodes there are two 'grey' nodes, which in turn branch further so that the full lattice is similar to the 'black' one. What, if any, is the relation of the full lattice to $\mathbf{Q}_3$? To answer this question, we need to recall some facts about $\mathbf{Q}_p$.

The field $\mathbf{Q}_p$ (like $\mathbf{R}$) is not closed algebraically. That is, not all roots of polynomials with coefficients in the field belong to it. For $\mathbf{R}$, one can adjoin a root of $x^2+1 = 0$ and extend to the algebraically closed (and complete) field of complex numbers $\mathbf{C}$. It is said to be an *index* two extension, *i.e.*, $\mathbf{C}$ is a two dimensional vector space over $\mathbf{R}$. The story is more complex for $\mathbf{Q}_p$, for which there are infinitely many algebraic extensions none of which is closed. Now consider a finite extension $\overline{\mathbf{Q}}_p^{(n)}$ of index n. There are several such and an integer e, called the *ramification index* partially distinguishes between them. It turns out that e divides n, so that $f = n/e$ is again an integer[20]–[22]. First, we will consider a so called *totally ramified* extension for which $e = n$. The Bruhat-Tits tree for such extensions can be obtained from the original one of $\mathbf{Q}_p$ through the process described in the last paragraph. Namely, to get the tree for a totally ramified extension $\overline{\mathbf{Q}}_p^{(n)}$, start with the 'black' tree for $\mathbf{Q}_p$ and introduce $(e-1)$ new nodes between the exisiting ones. Connect (infinite) 'grey' branches to these so that the tree is uniform with coordination number p as before. In the other cases when $e < n$, one also needs to introduce an infinite number of new edges and nodes so that the resulting tree is uniform with coordination number p^f [8, 9].

In $\overline{\mathbf{Q}}_p^{(n)}$, there is a special element π, called the *uniformizer*, that plays the role of p for $\mathbf{Q}_p$. Specifically, any element of $\overline{\mathbf{Q}}_p^{(n)}$ can be expressed as a Laurent series in terms of π (just like (2)), and the norms of its elements are integer powers of π. In particular, for $p \in \overline{\mathbf{Q}}_p^{(n)}$:

$$p \simeq \pi^e, \qquad (7)$$

where the approximate equality indicates the leading term in the expansion. Parametrizing both p and π as in (6), a' of $\overline{\mathcal{B}}_p^{(n)}$ is related to a of $\mathcal{B}_p$ as $a \simeq na'$, as is apparent from the construction. Thus for larger and larger extensions $\pi \simeq p^{1/e}$ approaches the value 1 for any p. The corresponding lattices provide finer discretizations and a passage to the continuum limit.

The construction of the previous section suggests a way to understand the limit $p \to 1$ through a sequence of string theories based on the extensions of $\mathbf{Q}_p$. For simplicity let us consider a totally ramified extension. Apparently there is a puzzle. The tachyon amplitudes for the totally ramified extension $\overline{\mathbf{Q}}_p^{(e=n)}$, turn out to be exactly the same as those for $\mathbf{Q}_p$! This is because the *coefficients* in the Laurent expansions of both are from the same set; the trees are similar, therefore, the measures that affect the integrals work out to be identical[3]. Hence, the effective action of the tachyon of these two theories are identical. String theories based on extensions of $\mathbf{Q}_p$ were already considered in [1, 3], indeed the very first paper on p-adic string theory[1] dealt with the quadratic extensions of $\mathbf{Q}_p$. In analogy with ordinary strings, it was thought to be a theory of closed strings. The theories based on higher extensions were called *even more closed* strings! In hindsight, it is natural to think of all these as open strings.

Returning to the apparent paradox, the resolution comes from the following. In taking a continuum limit, one is not really interested in the results separately for the two theories, but rather in comparing the degrees of freedom of the coarse-grained lattice from the fine one from the perspective of a (real space) RG. In order to do this, only the degrees of freedom on the 'grey' nodes and branches (see Fig. 2) should be integrate out. This leaves one with the 'black' sublattice with some effective interaction between these residual degrees of freedom. A rescaling of the lattice so that the spacing $a \to ba = a'$ completes the RG transformation.

Let us see the effect of these on the Poisson kernel on the Bethe lattice. It is more transparent for the Dirichlet problem for which the Green's function is[8]

$$\mathcal{D}(z,w) = \frac{p}{p^2-1} p^{-d(z,w)}, \qquad (8)$$

where $d(z,w)$ is the number of steps in lattice units between the sites z, w. Since the spacing in $\mathcal{B}_p$ is $e = n$ times that in $\overline{\mathcal{B}}_p^{(n)}$, $d_{\mathcal{B}} = e d_{\overline{\mathcal{B}}^{(n)}} \equiv e\overline{d}$ and after integrating out the intermediate sites, $\mathcal{D}_{\text{eff}} = \frac{p}{p^2-1} p^{-e\overline{d}(z,w)}$. When the lattice is rescaled, the original form of the kernel is recovered with the substitution $p \to \pi = p^{1/e}$. The Green's function $\mathcal{N}(z,w)$ for the Neumann problem is roughly the logarithm of $\mathcal{D}(z,w)$[8], so the same argument holds there as well. Thus the effect of the RG transformation on the tachyon action (3) is to replace $p \to \pi = p^{1/e}$. The action for the ordinary bosonic string is obtained in the limit $e \to \infty$, which is a continuum limit in the sense of RG.

The above argument can be straightforwardly extended to any finite extension of $\mathbf{Q}_p$. Let us also note that only the unramified extension ($e = 1$) is unique; there are several partially and totally ramified extensions differing in the details of the structure of the field. However, the associated Bruhat-Tits trees, which are the objects of interest to us, are specified only by the values of e and f. It is not clear to us if the non-uniqueness has any role to play for the string theories based on these fields.

Further evidence comes from the problem of a random walk on a Bethe lattice, for which Ref.[29] found an exact solution. This goes over to the solution of the Brownian motion on a hyperbolic space of constant negative curvature in the (formal) limit $p \to 1$. Thus the Green's function for the diffusion equation on the hyperbolic disk/UHP can be obtained as a continuum limit from the Bethe lattice. The well known relation between the kernel of the diffusion equation and the Green's function of a free scalar field theory, can be to used obtain the latter. We are interested in a diffeomorphism and Weyl invariant free scalar field theory coupled to the metric on the disk/UHP. There are also marked points corresponding to asymptotic states given by vertex operators on its boundary. Only hyperbolic metrics can be consistently defined on such a surface. Further, with the freedom from diffeomorphism and Weyl invariance, the metric can be made one of constant negative curvature. In the worldsheet functional integral, therefore, the contribution is from such a surface. The continuum limit of a scalar field theory on a Bethe lattice would seem to give a good approximation.

In summary, we have argued that the observation that the effective field theory of the tachyon of the p-adic string approximates that of the ordinary string in the $p \to 1$ limit, can be understood in terms of RG flow on a sequence of open string theories based on (algebraic) extensions of increasing degree of the p-adic field. Each of these theories provides a discretization of the tree-level worldsheet of the ordinary string and the $p \to 1$ limit is a continuum limit in the sense of (real space) RG.

A few brief closing remarks. First, in the p-adic discretization, the 'worldsheet' is isometric to the disk/UHP with a metric of constant negative curvature. This is a solution to the equation of motion of Liouville field theory, and is interpreted as the D0-brane[31]. Secondly, there is a more standard discretization in terms of large N random matrices. The zeroes of the partition function of of the Ising and Potts models on random lattices from 1×1 matrices and on Bethe lattices are identical[30], suggesting some kind of complimentarity in the two discretizations. Finally, the $p \to 1$ limit in terms of a set of theories based on extensions of $\mathbf{Q}_p$ may be useful in finding the 'closed' strings of the p-adic theory.

It is a pleasure to thank Chandan Dalawat, Peter Freund and Stefan Theisen for useful discussions. Hospitality at the Albert Einstein Institute, Germany, where a part of the work was done, is acknowledged gratefully.

---

[1] P. Freund and M. Olson, Phys. Lett. B **199**, 186 (1987).
[2] P. Freund and E. Witten, Phys. Lett. B **199**, 191 (1987).
[3] L. Brekke, P. Freund, M. Olson and E. Witten, Nucl. Phys. B **302**, 365 (1988).
[4] P. Frampton and Y. Okada, Phys. Rev. D **37**, 3077 (1988); Phys. Rev. Lett. **60**, 484 (1988).
[5] B. Spokoiny, Phys. Lett. B **207**, 401 (1988).
[6] G. Parisi, Mod. Phys. Lett. **A3**, 639 (1988).
[7] R.B. Zhang, Phys. Lett. **B209**, 229 (1988).
[8] A. Zabrodin, Commun. Math. Phys. **123**, 463 (1989).
[9] L. Chekhov, A. Mironov and A. Zabrodin, Commun. Math. Phys. **125**, 675 (1989).
[10] L. Brekke and P. Freund, Phys. Rept. **233**, 1 (1993).
[11] D. Ghoshal and A. Sen, Nucl. Phys. B **584**, 300 (2000).
[12] A. Sen, hep-th/9904207.
[13] A. Gerasimov and S. Shatashvili, JHEP **0010**, 034 (2000).
[14] E. Witten, Phys. Rev. D **46**, 5467 (1992).
[15] S. Shatashvili, Phys. Lett. B **311**, 83 (1993); Alg. Anal. **6**, 215 (1994).
[16] D. Kutasov, M. Marino and G. Moore, JHEP **0010**, 045 (2000).
[17] D. Ghoshal and A. Sen, JHEP **0011**, 021 (2000). D. Ghoshal, in *Proceedings of Strings 2001*, A. Dabholkar *et al* (Eds.), pp 107 (AMS, Providence, 2002).
[18] D. Ghoshal, JHEP **0409**, 041 (2004).
[19] D. Ghoshal, Preprint MRI-P-0605001, AEI-2006-035.
[20] I.M. Gelfand, M.I. Graev and I.I. Pitaetskii-Shapiro, *Representation theory and automorphic functions* (Saunders, Philadelphia, 1969).
[21] N. Koblitz, *p-Adic numbers, p-adic analysis and zeta functions*, GTM 58 (Springer-Verlag, 1977).
[22] F. Gouvêa, *p-Adic numbers: an introduction* (Springer-Verlag, 1993).
[23] I.V. Volovich, Class. Quant. Grav. **4**, L83 (1987).
[24] B. Grossman, Phys. Lett. **B197**, 101 (1987).
[25] D. Ghoshal, Preprint AEI-2006-036, math-ph/0606003.
[26] D. Ghoshal and T. Kawano, Nucl. Phys. B **710**, 577 (2005).
[27] P. Grange, Phys. Lett. B **616**, 135 (2005).
[28] A. Comtet, S. Nechaev and R. Voituriez, J. Stat. Phys. bf 102, 203 (20001); S. Nechaev and O. Vasilyev, J. Phys. A **37**, 3783 (2004).
[29] C. Monthus and C. Texier, J. Phys. A **29**, 2399 (1996).
[30] B. Dolan, W. Janke, D. Johnston and M. Stathakopoulos, J. Phys. A34, 6211 (2001).
[31] A. Zamolodchikov and Al. Zamolodchikov, arXiv:hep-th/0101152.