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Hot Bang States of Massless Fermions

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Abstract. According to the characterization of local thermal equilibrium states in Local Quantum Physics proposed by Buchholz et al. microscopic and corresponding macroscopic observables are computed for the model of massless, free fermions on Minkowski space. An example for a local equilibrium state describing a hot bang is given, the main step being the proof of its positivity.

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1. Introduction

In the framework of Local Quantum Physics, states that describe global thermodynamic equilibrium are characterized by the KMS condition [7]. For many models, the KMS states are relatively simple to compute, but it is a nontrivial problem to obtain states that describe systems which are only locally in thermodynamic equilibrium, such as hydrodynamic flows. This is due to the fact that such systems behave significantly different at small and large scales.

In [5] Buchholz et al. proposed a method to characterize such states. This characterization uses the KMS states as a tool to compare any given state with the global equilibrium states. If the given state coincides at some spacetime point with a thermal equilibrium state, one can assign to it thermodynamic properties. As this comparison is point-dependent, it delivers a way to describe notions of temperature or entropy density that may vary from point to point.

This method has been applied to the model of massless, free bosons on $\mathbb{R}^4$ in [4, 5] and has led to interesting results. Firstly, the microscopic dynamics induces macroscopic transport equations e.g. for the phase-space density akin to the collisionless Boltzmann equation. Secondly, one finds that states that have a nontrivial thermodynamic interpretation in a sufficiently large region break time-reversal symmetry, thus implementing a thermodynamic arrow of time. Thirdly, an example for a local equilibrium state was given describing a hot bang, i.e. the evolution of a temperature singularity at some spacetime point.

Most of these results carry over to the fermionic case quite easily. They will therefore only be mentioned shortly in this paper. What poses a problem is to establish the existence of a fermionic analogue of the Hot-Bang state. Proving the positivity of this state requires significantly more effort in the fermionic case than in the bosonic one. This proof will cover the main part of this work.

The existence of such a Hot-Bang state is desirable, since it provides a nontrivial example of a local equilibrium state. Furthermore, since the region in which it is thermal is the interior of a lightcone, it explicitly shows the implementation of the thermodynamical arrow of time. Apart from that it is the only local thermal equilibrium state that has locally sharply defined temperature.

2. Massless Free Fermions

The setting for the analysis will be the CAR Algebra of massless free fermions [2, 3]. In the massless case, the Dirac equation decomposes into two independent equations, called Weyl equations. They describe the left-handed and the right-handed parts of the fermions separately. We will consider one of these parts described by the smeared out fields $\psi_r(f^r)$ and $\bar{\psi}_{\dot{r}}(g^{\dot{r}})$, where f and g are smooth functions with compact support and takes values in $\mathbb{C}^2$. As common in the literature, r, $\dot{r}$ range from 1 to 2 and denote the components of f and g. The dots over the indices shall indicate the different transformation behavior under Lorentz transformations.

These $\psi_r(f^r)$ and $\bar{\psi}_{\dot{r}}(g^{\dot{r}})$ create a C^*-algebra $\mathcal{F}$ subject to the relations:

$$\left\{\psi_r(f^r), \bar{\psi}_{\dot{s}}(g^{\dot{s}})\right\} = 2\pi \int \mathrm{d}p\, \delta(p^2)\varepsilon(p_0)\tilde{f}^r(p)\, p_{r\dot{s}}\, \tilde{g}^{\dot{s}}(-p)\cdot 1 \tag{2.1}$$

$$\left\{\psi_r(f^r), \psi_s(g^s)\right\} = \left\{\bar{\psi}_{\dot{r}}(f^{\dot{r}}), \bar{\psi}_{\dot{s}}(g^{\dot{s}})\right\} = 0, \tag{2.2}$$

where ε is the sign-function, $\tilde{}$ denotes the Fourier transform, and

$$\psi_r\left(-i\partial^{\dot{s}r} g_{\dot{s}}\right) = \bar{\psi}_{\dot{r}}\left(i\partial^{\dot{r}s} f_s\right) = 0. \tag{2.3}$$

Here, for a vector $a \in \mathbb{R}^4$ the 2×2 matrices $a_{r\dot{s}}$ and $a^{\dot{r}s}$ are defined by

$$a_{r\dot{s}} \doteq \begin{pmatrix} a^0+a^3 & a^1-ia^2 \\ a^1+ia^2 & a^0-a^3 \end{pmatrix}_{r\dot{s}} \quad \text{and} \quad a^{\dot{r}s} \doteq \begin{pmatrix} a^0-a^3 & -a^1+ia^2 \\ -a^1-ia^2 & a^0+a^3 \end{pmatrix}^{\dot{r}s}.$$

Furthermore, the $*$-relation is given by

$$\psi_r(f^r)^* = \bar{\psi}_{\dot{s}}(\bar{f}^{\dot{s}}), \qquad \bar{\psi}_{\dot{r}}(g^{\dot{r}})^* = \psi_s(\bar{g}^s), \tag{2.4}$$

where $\bar{f}$ is the componentwise complex conjugate function of f. The double covering of the Poincaré group, the elements of which consist of pairs (A, a) with $A \in SL(2,\mathbb{C})$, $a \in \mathbb{R}^4$, acts on $\mathcal{F}$ via

$$\alpha_{(A,a)}\psi_r(f^r) = \psi_r\left(A^r{}_s f^s_{(\Lambda,a)}\right) \tag{2.5}$$

$$\alpha_{(A,a)}\bar{\psi}_{\dot r}(g^{\dot r}) = \bar{\psi}_{\dot r}\left(A^{\dot r}{}_{\dot s} g^{\dot s}_{(\Lambda,a)}\right) \tag{2.6}$$

with $f_{(\Lambda,a)}(x) = f(\Lambda^{-1}(x-a))$, the Lorentz transform Λ being the corresponding one to $A \in SL(2,\mathbb{C})$. As in the standard literature, $A^r{}_s$ denotes the rth line and sth column of $(A^{\mathrm{T}})^{-1}$, similarly $A^{\dot r}{}_{\dot s}$ the $\dot r$th line and $\dot s$th column of $(A^\dagger)^{-1}$.

The global gauge group $U(1)$ acts on $\mathcal{F}$ by

$$\alpha_\varphi \psi_r(f^r) \doteq \mathrm{e}^{i\varphi}\psi_r(f^r), \quad \alpha_\varphi \bar{\psi}_{\dot r}(g^{\dot r}) \doteq \mathrm{e}^{-i\varphi}\bar{\psi}_{\dot r}(g^{\dot r}). \tag{2.7}$$

In the case of $\mathcal{F}$, the KMS states and their properties are known [3]. Since the theory is massless and free, the global equilibrium situations need to be characterized by inverse temperature $|\beta| > 0$, but not by chemical potential. Furthermore, every KMS state determines the rest system with respect to which it is in equilibrium, due to the fact that Lorentz symmetry is spontaneously broken in KMS states [10]. A rest system is uniquely defined by a future directed, timelike unit vector e. We combine these two parameters to a vector in the forward lightcone $\beta = |\beta| e \in V^+$. We will consider gauge-invariant KMS states only, and for each temperature vector $\beta \in V^+$ there is a unique gauge-invariant KMS-state ω_β. All these states are quasifree and thus completely determined by their two-point function, which is given by

$$\omega_\beta\left(\bar{\psi}_{\dot r}(f^{\dot r})\,\psi_s(g^s)\right) = 2\pi \int_{\mathbb{R}^4} \mathrm{d}p\,\delta(p^2)\,\varepsilon(p_0)\,\frac{\tilde{g}^s(p)\,p_{s\dot r}\,\tilde{f}^{\dot r}(-p)}{1+\mathrm{e}^{-(\beta,p)}} \tag{2.8}$$

$$\omega_\beta\left(\psi_r(f^r)\,\psi_s(g^s)\right) = \omega_\beta\left(\bar{\psi}_{\dot r}(f^{\dot r})\,\bar{\psi}_{\dot s}(g^{\dot s})\right) = 0. \tag{2.9}$$

A special case of this is the vacuum state ω_∞. It is given by (2.8) and (2.9), where β tends to timelike infinity, i.e. one has:

$$\omega_\infty\left(\bar{\psi}_{\dot r}(f^{\dot r})\,\psi_s(g^s)\right) = 2\pi \int_{\mathbb{R}^4} \mathrm{d}p\,\delta(p^2)\,\theta(p_0)\,\tilde{g}^s(p)\,p_{s\dot r}\,\tilde{f}^{\dot r}(-p). \tag{2.10}$$

3. Local Equilibrium States

We briefly review the method to characterize local themal equilibrium states by Buchholz et al. [4, 5].

Let B be a compact subset of the forward lightcone V^+ and $\mathrm{d}\rho$ be a normalized measure on B. Due to (2.8) the function $\beta \mapsto \omega_\beta(A)$ is continuous for any polynomial A in the smeared fields. Thus one can build the statistical mixtures of KMS states:

$$\omega_B = \int_B \mathrm{d}\rho(\beta)\,\omega_\beta. \tag{3.11}$$

The mixtures for all B and all $\mathrm{d}\rho$ form the set $\mathcal{C}$ of thermal reference states. Within this setting we are able to compare a given state ω with the reference states ω_B and thus can analyze the thermal properties of ω at any point in spacetime. We do this by using a set of observables that correspond to measurements of thermal properties at a single point. It is obvious that the observables in $\mathcal{F}$ are not suitable for this because of their extended localization. To proceed, we need to go over to idealized observables at a point, that exist only in the sense of forms.

Let $\boldsymbol{\mu} = (\mu_1 \ldots \mu_m)$ be a multi-index. We define the gauge invariant point fields (forms):

$$\begin{aligned}\lambda^{\boldsymbol{\mu}\nu}(x) &\doteq \eth^{\boldsymbol{\mu}} : \bar{\psi}_{\dot r}(x)\sigma^{\nu,\dot r s}\psi_s(x) : \\ &\doteq \lim_{\substack{\zeta\to 0\\ \zeta^2<0}} \partial_\zeta^{\boldsymbol{\mu}} : \bar{\psi}_{\dot r}(x+\zeta)\sigma^{\nu,\dot r s}\psi_s(x-\zeta) :\end{aligned} \tag{3.12}$$

where the normal ordering is performed with respect to the vacuum state ω_∞. The $\lambda^{\boldsymbol{\mu}\nu}(x)$ are called thermal observables at $x \in \mathbb{R}^4$ and correspond to idealized measurements at x. We will only consider states in which the limit (3.12) exists.

For $x \in \mathbb{R}^4$, the linear span of the $\lambda^{\boldsymbol{\mu}\nu}(x)$ (for all $\boldsymbol{\mu}$, ν) will be denoted by $\mathcal{S}_x$. The thermal observables transform under Poincaré transformations according to

$$\alpha_{(A,a)}\lambda^{\boldsymbol{\mu}\nu}(x) = \Lambda_{\mu_1'}{}^{\mu_1} \ldots \Lambda_{\mu_m'}{}^{\mu_m}\Lambda_{\nu'}{}^{\nu}\lambda^{\boldsymbol{\mu}'\nu'}(\Lambda x + a).$$

In particular, the spaces $\mathcal{S}_x$ are transformed into each other by the action of the translations, $\alpha_{y-x}\mathcal{S}_x = \mathcal{S}_y$. These thermal observables and the reference states are used to characterize local equilibrium states:

DEFINITION 3.1. *Let ω be a state over $\mathcal{F}$ and $\mathcal{O} \subset \mathbb{R}^4$ be open. The state ω is called $\mathcal{S}_{\mathcal{O}}$-thermal, if the following conditions hold:*

(i) *For every $x \in \mathcal{O}$ there is a reference state $\omega_{B_x} \in \mathcal{C}$ such that $\omega\big(\lambda^{\boldsymbol{\mu}\nu}(x)\big) = \omega_{B_x}\big(\lambda^{\boldsymbol{\mu}\nu}(x)\big)$ for all $\lambda^{\boldsymbol{\mu}\nu}(x) \in \mathcal{S}_x$.*
(ii) *For every compact subset $U \subset \mathcal{O}$ there is a compact subset $B \subset V^+$ such that $B_x \subset B$ for $x \in U$.*

Thus an $\mathcal{S}_{\mathcal{O}}$-thermal state coincides with some global equilibrium situation at every point $x \in \mathcal{O}$ with regard to the chosen observables $\lambda^{\boldsymbol{\mu}\nu}(x)$. In this sense it is locally close to equilibrium.
For an element $\lambda^{\boldsymbol{\mu}\nu}(x)$ the corresponding function

$$V^+ \ni \beta \longmapsto \omega_\beta\big(\lambda^{\boldsymbol{\mu}\nu}(x)\big) \doteq L^{\boldsymbol{\mu}\nu}(\beta) \tag{3.13}$$

is called thermal function. By straightforward calculation one shows that

$$L^{\boldsymbol{\mu}\nu}(\beta)=\omega_\beta(\lambda^{\boldsymbol{\mu}\nu}(x))=c_m\left(\partial_\beta^{\boldsymbol{\mu}\nu}\frac{1}{(\beta,\beta)}\right), \tag{3.14}$$

with $m=\deg\boldsymbol{\mu}$ and

$$c_m=\left\{\begin{array}{ll}\frac{i\,\pi^{m+1}\,(2^{2m+2}-2^{m+1})}{(m+3)!}\,(-1)^{(m+3)/2}\,B_{(m+3)/2} & \text{for odd } m\\ 0 & \text{for even } m\end{array}\right\}, \tag{3.15}$$

where the B_n are the Bernoulli numbers. Since the KMS states ω_β are translation-invariant, the expectation values $L^{\boldsymbol{\mu}\nu}(\beta)$ do not depend on x. This allows for a thermal interpretation of the local observables $\lambda^{\boldsymbol{\mu}\nu}(x)$.

The choice (3.12) of observables provides significant information about thermal properties. By macroscopic thermodynamic considerations [5,6] one knows what value intensive thermal properties such as energy density, entropy current density and particle phase space density should have in global equilibrium states. The thermal energy-momentum tensor in a system of massless, free fermions being in a state of constant temperature $\beta\in V^+$, for example, has the form:

$$E^{\mu\nu}(\beta)=\frac{\pi^2}{60}\left(\frac{4\beta^\mu\beta^\nu}{(\beta,\beta)^3}-\frac{\eta^{\mu\nu}}{(\beta,\beta)^2}\right) \tag{3.16}$$

at every point $x\in\mathbb{R}^4$. From the microscopic point of view, the thermal observable

$$:\theta^{\mu\nu}(x):\doteq\frac{1}{2i}(\lambda^{\mu\nu}(x)+\lambda^{\nu\mu}(x)) \tag{3.17}$$

is the normal-ordered, symmetrized energy-momentum tensor of the free, massless Dirac field, and one finds $\omega_\beta(:\theta^{\mu\nu}(x):)=E^{\mu\nu}(\beta)$ by (3.14). So in $\mathcal{S}_x$ there is an observable for the thermal energy at $x\in\mathbb{R}^4$. In fact, the $\mathcal{S}_x$ contain enough elements to determine most important thermal properties of a system, as will be shown in the following.

4. Macroobservables and Thermodynamic Interpretation

The thermal observables $\lambda^{\boldsymbol{\mu}\nu}(x)$ correspond to measurements of thermodynamic properties of a system at a point $x\in\mathbb{R}^4$. Let now $f\in\mathscr{D}(\mathbb{R}^4)$ be a test function that integrates to one and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}^4$ going to spacelike infinity sufficiently fast. Then one can show [5] that the limit

$$L^{\boldsymbol{\mu}\nu}\doteq\lim_{n\to\infty}n^{-4}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\lambda^{\boldsymbol{\mu}\nu}(x)\,f(n^{-1}x-x_n) \tag{4.18}$$

exists and determines an observable that commutes with every element in $\mathcal{F}$. It serves as an observable measuring the average of $\lambda^{\boldsymbol{\mu}\nu}(x)$ over spacetime, thus implementing a notion of macroscopic properties of the system. The $L^{\mu\nu}$ are called macroobservables.

Since the reference states are translation invariant, we have

$$\omega_B\big(\lambda^{\mu\nu}(x)\big) = \omega_B\big(L^{\mu\nu}\big) = \int_B \mathrm{d}\rho(\beta)\, L^{\mu\nu}(\beta), \tag{4.19}$$

where $L^{\mu\nu}(\beta)$ is the thermal function of $\lambda^{\mu\nu}(x)$ (3.14). Equation (4.19) shows that the reference states cannot distinguish between microscopic and macroscopic scales.

The mean particle phase space density of a system of massless fermions being in equilibrium at temperature $\beta \in V^+$ is given by

$$N_p(\beta) = (2\pi)^{-3} \frac{1}{1+\mathrm{e}^{(p,\beta)}} \tag{4.20}$$

with $p = (|\vec{p}|,\, \vec{p})$. One can see that $\Box_\beta N_p(\beta) = 0$, i.e. N_p satisfies the wave equation w.r.t. β. From (3.14) one sees that all thermal functions do so, too. Choosing a set of suitable seminorms one can show [1,4] that the linear span of thermal functions is dense in the space of all smooth solutions of the wave equation on V^+. Every such solution Ξ determines a macroobservable in the following way. Condition (ii) in Definition 3.1 guarantees the $\mathcal{S}_\mathcal{O}$-thermal states to be continuous with respect to these seminorms. Hence one can extend a $\mathcal{S}_\mathcal{O}$-thermal state ω to Ξ pointwise via

$$\omega\big(\Xi\big)(x) \doteq \omega_{B_x}\big(\Xi\big) = \int_{B_x} \mathrm{d}\rho_x(\beta)\, \Xi(\beta),$$

where ω_{B_x} is a reference state ω coincides with on all elements in $\mathcal{S}_x$, $x \in \mathcal{O}$.

By the above considerations we see that the function (4.20) determines a macroobservable N_p that corresponds to the mean phase space density. Furthermore, it is measurable in every $\mathcal{S}_\mathcal{O}$-thermal state and for any such state ω it determines a function

$$(x, p) \longmapsto \omega\big(N_p\big)(x) \tag{4.21}$$

that serves as the particle phase space density of the system being in the state ω. Corresponding functions can be defined for thermal properties like entropy current density or Gibbs potential [1,5]. So the thermal observables $\lambda^{\mu\nu}(x)$ can be used to approximate important thermodynamic properties, which allows for a local thermodynamic interpretation of $\mathcal{S}_\mathcal{O}$-thermal states.

5. The Hot-Bang State

In the last section we demonstrated, how each local thermal equilibrium state ω gives rise to a well-defined mean phase space density $N(x, p) = \omega\big(N_p\big)(x)$. On the other hand, one can show that each function $N(x, p)$ satisfying certain properties, determines a quasifree linear functional on a subalgebra of $\mathcal{F}$ [1] by

$$\omega_{\mathrm{hb}}\Big(\bar{\psi}_{\dot{r}}(x)\psi_s(y)\Big) = \int_{\mathbb{R}^4} \mathrm{d}p\, \delta(p^2)\, \varepsilon(p_0)\, p_{s\dot{r}}\, e^{i(p,x-y)}\, N(x+y, p) \tag{5.22}$$

$$\omega_{\mathrm{hb}}\Big(\psi_r(x)\psi_s(y)\Big) = \omega_{\mathrm{hb}}\Big(\bar{\psi}_{\dot{r}}(x)\bar{\psi}_{\dot{s}}(y)\Big) = 0. \tag{5.23}$$

It is a priori not clear whether this functional is positive, i.e. can be extended to a state on $\mathcal{F}$. Also, one cannot be sure whether the potential state has any thermal properties at all. In the following, though, we will provide an example, for which not only positivity can be proven, but one is also able to show that the resulting state is $\mathcal{S}_{V^+}$-thermal, i.e. is locally in thermal equilibrium in the lightcone. Furthermore, the resulting phase-space density is exactly the function one has started with. This indicates that the procedure mentioned above may provide a way to generate large families of $\mathcal{S}_{\mathcal{O}}$-thermal states [1,4].

To an $\mathcal{S}_{\mathcal{O}}$-thermal state ω, there is, by definition, a thermal reference state ω_{B_x} at each point $x \in \mathcal{O}$, such that ω and ω_{B_x} agree on the elements of $\mathcal{S}_x$. It is easy to show that the only $\mathcal{S}_{\mathcal{O}}$-thermal states over $\mathcal{F}$ that have a KMS-state $\omega_{\beta(x)}$ as reference state at every point, must have $\beta(x)$ being linear in x. [1] Thus, the resulting phase space density has to be (up to reflections and translations) of the form:

$$N(x, p) = (2\pi)^{-3} \frac{1}{1 + \mathrm{e}^{\lambda(x,p)}} \tag{5.24}$$

for $\lambda > 0$. So the condition of locally sharp temperature severely restricts the possibilities of $\mathcal{S}_{\mathcal{O}}$-thermal states.

In the following, we will show that there in fact is a $\mathcal{S}_{V^+}$-thermal state ω_{hb} with phase-space density $\omega_{\mathrm{hb}}\big(N_p\big)(x)$ equal to (5.24). In particular, we will show that the quasifree linear functional (5.22), (5.23) with (5.24)

$$\omega_{\mathrm{hb}}\Big(\bar{\psi}_{\dot{r}}(x)\psi_s(y)\Big) = (2\pi)^{-3} \int_{\mathbb{R}^4} \mathrm{d}p\, \delta(p^2)\, \varepsilon(p_0)\, p_{s\dot{r}}\, \frac{\mathrm{e}^{i(p,x-y)}}{1 + \mathrm{e}^{\lambda(x+y,p)}} \tag{5.25}$$

$$\omega_{\mathrm{hb}}\Big(\psi_r(x)\psi_s(y)\Big) = \omega_{\mathrm{hb}}\Big(\bar{\psi}_{\dot{r}}(x)\bar{\psi}_{\dot{s}}(y)\Big) = 0, \tag{5.26}$$

with $\lambda > 0$ is positive on a subalgebra of $\mathcal{F}$. To show that ω_{hb} is in fact $\mathcal{S}_{V^+}$-thermal is then just a straightforward use of (5.25) and the definition of the local thermal observables (3.12). The expectation values of thermal macroobservables then imply that ω_{hb} describes the fate of a temperature singularity at the tip of the lightcone V^+, which justifies the name "Hot Bang state". [1]

We will now show that (5.25), (5.26) defines a positive linear functional on $\mathcal{F}(V^+)$, that is, the algebra generated by $\psi_r(f^r)$ and $\bar{\psi}_{\dot{s}}(g^{\dot{s}})$ with $f, g \in \mathscr{D}(V^+, \mathbb{C}^2)$.

Quasifreedom implies [3] that proving the positivity for the two-point function is sufficient for proving the positivity of (5.25):

$$\omega_{\text{hb}}\Big(\bar{\psi}_{\dot{r}}(\bar{f}^{\dot{r}})\psi_s(f^s)\Big) \geqslant 0 \tag{5.27}$$

$$\omega_{\text{hb}}\Big(\psi_r(f^r)\bar{\psi}_{\dot{r}}(\bar{f}^{\dot{r}})\Big) \geqslant 0. \tag{5.28}$$

To show (5.27) and (5.28), we proceed as follows: first, we rewrite both two-point functions by expanding the exponential in the denominator in (5.25) into an alternating series. Afterwards, we will identify the terms of the series as values of a certain logarithmically convex function L. The convexity of L will then allow to compare certain terms in the series to each other, proving the positivity of (5.27) and (5.28).

We start by observing with the help of (5.25) that for $f \in \mathscr{D}(V^+, \mathbb{C}^2)$:

$$\begin{aligned}
&\omega_{\text{hb}}\Big(\bar{\psi}_{\dot{s}}(\bar{f}^{\dot{s}})\psi_r(f^r)\Big) \\
&\quad = 2\pi \sum_{n=0}^{\infty} (-1)^n \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|}\, \tilde{f}^r\Big((1+i\lambda n)p'\Big) p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}\Big((1+i\lambda n)p'\Big)} + \\
&\qquad + 2\pi \sum_{n=1}^{\infty} (-1)^{n-1} \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|}\, \tilde{f}^r\Big((-1+i\lambda n)p'\Big) p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}\Big((-1+i\lambda n)p'\Big)}
\end{aligned} \tag{5.29}$$

with $p' = (|\vec{p}|, \vec{p})$. Each of the integrals in the sum is nonnegative, since $p'_{r\dot{s}}$ is a positive semidefinite matrix. So the sum is alternating, and it is this fact that makes the proof of positivity of (5.29) difficult. In the proof of positivity for the corresponding bosonic state, the alternating sign is missing, which simplifies matters tremendously. In our case, we need to investigate the integrals in (5.29) further:

Let $z = \rho e^{i\phi} \in \overline{\mathbb{C}_+}\backslash\{0\}$. Then by scaling we have

$$\int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|}\, \tilde{f}^r(zp') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(zp')} = \frac{1}{\rho^3} L(\phi). \tag{5.30}$$

with

$$[0,\pi] \ni \phi \longmapsto L(\phi) = \int_{\mathbb{R}^3} \frac{d^3p}{2|\vec{p}|}\, \tilde{f}^r(e^{i\phi}p')\, p'_{r\dot{s}}\, \overline{\tilde{f}^{\dot{s}}(e^{i\phi}p')} \in \mathbb{R}^+. \tag{5.31}$$

With this, we obtain

$$\omega_{\text{hb}}\Big(\bar{\psi}_{\dot{s}}(\bar{f}^{\dot{s}})\psi_r(f^r)\Big) = 2\pi\sum_{n=0}^{\infty}(-1)^n\,|\cos^3(\phi_n)|\,L(\phi_n) + \\ + 2\pi\sum_{n=1}^{\infty}(-1)^{n-1}\,|\cos^3(\pi-\phi_n)|\,L(\pi-\phi_n) \tag{5.32}$$

with $\phi_n = \arctan\lambda n$.

By making use of the anticommutation relations, we also get

$$\omega_{\text{hb}}\Big(\psi_r(f^r)\bar{\psi}_{\dot{s}}(\bar{f}^{\dot{s}})\Big) = 2\pi\sum_{n=1}^{\infty}(-1)^{n-1}\,|\cos^3(\phi_n)|\,L(\phi_n) + \\ + 2\pi\sum_{n=0}^{\infty}(-1)^{n}\,|\cos^3(\pi-\phi_n)|\,L(\pi-\phi_n). \tag{5.33}$$

The next crucial step will be the proof of convexity of L for any $f\in\mathscr{D}(V^+,\mathbb{C}^2)$. This will allow for a comparison of certain terms in the alternating sums in (5.32) and (5.33), respectively. By this we will be able to show that both sums are non-negative. To tackle the properties of L, we need to investigate some functional analytic properties of the integral in (5.31).

LEMMA 5.1. *Let* $f\in\mathscr{D}(V^+,\mathbb{C}^2)$. *Let* $\mathbb{C}_+ \doteq \{z\in\mathbb{C}\,|\,Im\ z>0\}$, *then*

$$F(z) \doteq \int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{2|\vec{p}|}\,\tilde{f}^r(zp')\,p'_{r\dot{s}}\,\overline{\tilde{f}^{\dot{s}}(\bar{z}^{-1}p')}, \tag{5.34}$$

where $p'=(|\vec{p}|,\vec{p})$, *exists for* $z\in\overline{\mathbb{C}_+}\setminus\{0\}$ *and is continuous in* z. *Furthermore,* F *is holomorphic on* $\mathbb{C}_+$.

Proof. First consider the complex Fourier transform of f, with $\zeta\in\mathbb{C}^4$, which is an entire analytic function on $\mathbb{C}^4$:

$$\tilde{f}^r(\zeta) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\mathrm{d}x\,\mathrm{e}^{i(\zeta,x)}f^r(x).$$

Because of supp $f\subset V^+$, the theorem of Paley–Wiener and the fact that f is a test function imply

$$\big|\tilde{f}^r(zp')\big| \leqslant C_N\frac{\mathrm{e}^{-\delta|\vec{p}|\,\mathrm{Im}\ z}}{(1+|z|\,|\vec{p}|)^N} \qquad \text{for all } z\in\overline{\mathbb{C}_+}. \tag{5.35}$$

For fixed $p' = (|\vec{p}|, \vec{p})$ the integrand

$$z \longmapsto \frac{1}{2|\vec{p}|} \tilde{f}^r(zp')\, p'_{r\dot{s}}\, \overline{\tilde{f}^{\dot{s}}(\overline{z}^{-1}p')} = \frac{1}{2|\vec{p}|} \int_{\mathbb{R}^8} \mathrm{d}x\, \mathrm{d}y\, f^r(x)\, p'_{r\dot{s}}\, \overline{f^{\dot{s}}(y)}\, \mathrm{e}^{i(p', zx - z^{-1}y)}.$$

is holomorphic on $\mathbb{C}\backslash\{0\}$, since f has compact support. As $(p'_{r\dot{s}}/2|\vec{p}|)$ is uniformly bounded, it thus follows from (5.35) that

$$\left| \frac{1}{2|\vec{p}|} \tilde{f}^r(zp')\, p'_{r\dot{s}}\, \overline{\tilde{f}^{\dot{s}}(\overline{z}^{-1}p')} \right| \leqslant \frac{2C_N^2}{(1+|\vec{p}|^2)^N} \tag{5.36}$$

for $z \in \overline{\mathbb{C}_+}\backslash\{0\}$. Thus, for z varying in $\mathbb{C}_+$, the integrand is uniformly bounded by an integrable function of $\vec{p}$. Hence the integral exists and is holomorphic in $\mathbb{C}_+$. Furthermore, if $\{z_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}_+$ converging to $\rho \in \mathbb{R}\backslash\{0\}$, we may interchange integration and limit and get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2|\vec{p}|} \tilde{f}^r(z_n p')\, p'_{r\dot{s}}\, \overline{\tilde{f}^{\dot{s}}(\overline{z_n}^{-1}p')} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2|\vec{p}|} \tilde{f}^r(\rho p')\, p'_{r\dot{s}}\, \overline{\tilde{f}^{\dot{s}}(\rho^{-1}p')},$$

which completes the proof of the statement. □

PROPOSITION 5.1. *Let* $f \in \mathscr{D}(V^+, \mathbb{C}^2)$. *Then the function* (5.31)

$$[0,\pi] \ni \phi \longmapsto L(\phi) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2|\vec{p}|} \tilde{f}^r(\mathrm{e}^{i\phi}p')\, p'_{r\dot{s}}\, \overline{\tilde{f}^{\dot{s}}(\mathrm{e}^{i\phi}p')} \in \mathbb{R}$$

is either identically zero or logarithmically convex and positive. Furthermore it is continuous on $[0,\pi]$ *and smooth on* $(0,\pi)$.

Proof. The claim about the continuity and smoothness is evident from Lemma 5.1 and $L(\phi) = F(\mathrm{e}^{i\phi})$. By the Cauchy–Schwartz inequality and scaling we get for $z = \rho\mathrm{e}^{i\phi} \in \overline{\mathbb{C}_+}\backslash\{0\}$:

$$|F(z)|^2 \leqslant \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2|\vec{p}|} \tilde{f}^r(\rho\mathrm{e}^{i\phi}p')p'_{r\dot{s}}\overline{\tilde{f}^{\dot{s}}(\rho\mathrm{e}^{i\phi}p')} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2|\vec{p}|} \tilde{f}^r(\rho^{-1}\mathrm{e}^{i\phi}p')p'_{r\dot{s}}\overline{\tilde{f}^{\dot{s}}(\rho^{-1}\mathrm{e}^{i\phi}p')}$$

$$= \left(\int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2|\vec{p}|} \tilde{f}^r(\mathrm{e}^{i\phi}p')p'_{r\dot{s}}\overline{\tilde{f}^{\dot{s}}(\mathrm{e}^{i\phi}p')} \right)^2$$

$$= L(\phi)^2.$$

Thus, if L is zero for some $\phi \in [0, \pi]$, then F is zero on a ray emerging from the origin through $e^{i\phi}$. If ϕ is in $(0, \pi)$, then F is a holomorphic function that is zero on a set with accumulation points in its domain of analyticity and hence must be zero. If $\phi = 0$ or $\phi = \pi$, then F has zero boundary values on a set that is open in the boundary and hence must be zero by the Schwartz reflection principle. As L is nonnegative by definition, either it is zero everywhere or strictly positive.

It remains to show that in the latter case L is logarithmically convex. Let $\alpha \in (0, 1)$ and $\mathbb{C}_{+,\alpha} \doteq \{z \in \mathbb{C}_+ \mid 0 < \arg z < \pi/(1+\alpha)\}$. Consider the function

$$\mathbb{C}_{+,\alpha} \ni z \longmapsto F_\alpha(z) \doteq \int_{\mathbb{R}^3} \frac{d^3 p}{2|\vec{p}|} \tilde{f}^r(z^{1+\alpha} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\bar{z}^{\alpha-1} p')} \in \mathbb{C}. \tag{5.37}$$

The integrand is holomorphic, as $z \mapsto z^{1+\alpha}$ is on $\mathbb{C}_{+,\alpha}$. Furthermore, if $z \in \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\}$, then $\bar{z}^{\alpha-1}$, $z^{\alpha+1} \in \overline{\mathbb{C}_+} \setminus \{0\}$. So by (5.35) we have:

$$\left| \frac{1}{2|\vec{p}|} \tilde{f}^r(z^{1+\alpha} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\bar{z}^{\alpha-1} p')} \right| \leqslant \frac{2C_N^2}{(1+|z|^{2\alpha}|\vec{p}|^2)^N}$$

for all $z \in \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\}$ and $\vec{p} \in \mathbb{R}^3$. Therefore, the integral exists for all $z \in \overline{\mathbb{C}_{+,\alpha}} \setminus \{0\}$. The integrand is uniformly bounded by an integrable function if z varies in some compact subset of $\mathbb{C}_{+,\alpha}$. So, F_α is holomorphic on $\mathbb{C}_{+,\alpha}$ and has continuous boundary values for $\rho \in \mathbb{R}^+$ given by

$$\lim_{z \to \rho} \int_{\mathbb{R}^3} \frac{d^3 p}{2|\vec{p}|} \tilde{f}^r(z^{1+\alpha} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\bar{z}^{\alpha-1} p')} = \int_{\mathbb{R}^3} \frac{d^3 p}{2|\vec{p}|} \tilde{f}^r(\rho^{1+\alpha} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\rho^{\alpha-1} p')}$$

$$= \rho^{-3\alpha} \int_{\mathbb{R}^3} \frac{d^3 p}{2|\vec{p}|} \tilde{f}^r(\rho p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\rho^{-1} p')} = \rho^{-3\alpha} F(\rho).$$

Hence, the two functions $z \mapsto F_\alpha(z)$ and $z \mapsto z^{3\alpha} F(z)$ are both holomorphic on $\mathbb{C}_{+,\alpha}$ and have the same continuous boundary values on $\mathbb{R}^+$. So, by an application of the Schwartz reflection principle, they have to be equal:

$$F(z) = z^{-3\alpha} F_\alpha(z) \tag{5.38}$$

on $\overline{\mathbb{C}_{+,\alpha}} \setminus \{0\}$. So, for every $0 < \phi < \pi/(1+\alpha)$ we have

$$L(\phi)^2 = |\mathrm{e}^{-3i\alpha} F_\alpha(\mathrm{e}^{i\phi})|^2 = \left| \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{2|\vec{p}|} \tilde{f}^r(\mathrm{e}^{i(1+\alpha)\phi} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\mathrm{e}^{i(1-\alpha)\phi} p')} \right|^2$$

$$\leqslant \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{2|\vec{p}|} \tilde{f}^r(\mathrm{e}^{i(1+\alpha)\phi} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\mathrm{e}^{i(1+\alpha)\phi} p')} \times$$

$$\times \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{2|\vec{p}|} \tilde{f}^r(\mathrm{e}^{i(1-\alpha)\phi} p') p'_{r\dot{s}} \overline{\tilde{f}^{\dot{s}}(\mathrm{e}^{i(1-\alpha)\phi} p')} = L(\phi(1+\alpha)) L((1-\alpha)\phi).$$

This means that for every $\phi \in (0, \pi)$ there is a $\delta > 0$ such that

$$L(\phi)^2 \leqslant L(\phi + \varepsilon) L(\phi - \varepsilon)$$

for all $\varepsilon < \delta$. So L is logarithmically convex, and thus the proposition is proven. □

The convexity of L now allows to compare the terms in the series (5.32) and (5.32), respectively. The following proposition will assure the positivity of both series. For this, the convexity of L plays a crucial part, since it severely restricts the intervals on which L has a certain monotony type.

PROPOSITION 5.2. *Let $L : [0, \pi] \to \mathbb{R}^+$ be a continuous, convex function that is smooth on $(0, \pi)$. Define $\rho : [0, \pi] \to \mathbb{R}$ by $\rho(\phi) = |\cos^3 \phi|$ and $g(\phi) = \rho(\phi) L(\phi)$. Let, furthermore, $\{\phi_n\}_{n\in\mathbb{N}}$ be a monotonically increasing sequence in $[0, \pi/2)$ converging to $\pi/2$, such that*

$$\sum_{n=0}^{\infty} g(\phi_n) < \infty. \tag{5.39}$$

Then the two series

$$A_L \doteq \sum_{n=0}^{\infty} (-1)^n \Big[g(\phi_n) + g(\pi - \phi_{n+1}) \Big] \tag{5.40}$$

$$B_L \doteq \sum_{n=0}^{\infty} (-1)^n \Big[g(\pi - \phi_n) + g(\phi_{n+1}) \Big] \tag{5.41}$$

converge absolutely and are both nonnegative.

Proof. Because L is continuous at $\phi = \pi/2$, it follows from (5.39) that $\sum_n g(\pi - \phi_n)$ converges, too. Since $L > 0$ we have $g \geqslant 0$, and therefore the series (5.40) and (5.41) converge absolutely.

To establish positivity of the two series, we first show that the function g is either monotone on $[0, \pi/2]$ or on $[\pi/2, \pi]$: Assume g not to be monotone on $[0, \pi/2]$.

Then there is a $\phi_{\text{Null}} \in (0, \pi/2)$ with $g'(\phi_{\text{Null}}) = 0$. Since $L > 0$ and $\rho'(\phi_{\text{Null}}) < 0$, we then have that

$$L'(\phi_{\text{Null}}) = \frac{-\rho'(\phi_{\text{Null}}) L(\phi_{\text{Null}})}{\rho(\phi_{\text{Null}})} > 0,$$

and thus, since L is convex, $L' > 0$ on $[\phi_{\text{Null}}, \pi)$. Therefore, for all $\phi \in [\pi/2, \pi)$, we have that

$$g'(\phi) = \rho(\phi) L'(\phi) + \rho'(\phi) L(\phi) > 0,$$

since ρ and ρ' are nonnegative on $[\pi/2, \pi)$. So g is monotone on $[\pi/2, \pi]$.

Now assume g not to be monotone on $[\pi/2, \pi]$. Replace L by $\overline{L}$ given by

$$\overline{L}(\phi) \doteq L(\pi - \phi). \tag{5.42}$$

The function $\overline{L}$ is convex, too, and $\overline{g} = \rho \cdot \overline{L}$ is not monotone on $[0, \pi/2]$. Thus, the above argument can be applied to $\overline{g}$ instead of g and shows that g is monotone on $[0, \pi/2]$.

Since $g(0)$, $g(\pi) > 0$ and $g(\pi/2) = 0$, we know that either g is monotonically decreasing on $[0, \pi/2]$ or monotonically increasing on $[\pi/2, \pi]$. Without loss of generality, we can assume the latter to be the case. Otherwise, we could replace L by $\overline{L}$ as in (5.42), since by (5.40) and (5.41) we see that $A_L = B_{\overline{L}}$ and $B_L = A_{\overline{L}}$. So by this replacement both series are just interchanged.

Thus, from now on, g will be assumed to be monotonically increasing on $[\pi/2, \pi]$. There are two possibilities: L may or may not be monotone on $[0, \pi/2]$.

Case 1: L is monotone on $[0, \pi/2]$:

Let L be monotonically decreasing on $[0, \pi/2]$, then g is, too. This means that g is monotone on $[0, \pi/2]$ and $[\pi/2, \pi]$. By reordering of (5.40) and (5.41), we get:

$$A_L = \sum_{n=0}^{\infty} \Big[g(\phi_{2n}) - g(\phi_{2n+1}) \Big] + \sum_{n=1}^{\infty} \Big[g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n}) \Big]$$

$$B_L = \sum_{n=0}^{\infty} \Big[g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1}) \Big] + \sum_{n=1}^{\infty} \Big[g(\phi_{2n-1}) - g(\phi_{2n}) \Big].$$

Since $\phi_m \leqslant \phi_{m+1}$ for all m, we see that every expression in square brackets is non-negative, and so are A_L and B_L.

Let L be monotonically increasing on $[0, \pi/2]$. Thus, since $L'' > 0$, we have, for all $\phi \in (0, \pi/2)$ that $0 < L(\phi) < L(\pi - \phi)$ and $0 < L'(\phi) < L'(\pi - \phi)$. So we have

$$\begin{aligned}|g'(\phi)| &\leqslant |\rho'(\phi)|\cdot|L(\phi)| + |\rho(\phi)|\cdot|L'(\phi)|\\ &\leqslant \rho'(\pi-\phi)L(\pi-\phi) + \rho(\pi-\phi)L'(\pi-\phi)\\ &= g'(\pi-\phi).\end{aligned}$$

Thus, for $0 \leqslant a \leqslant b \leqslant \pi/2$ we have

$$|g(a) - g(b)| \leqslant \int_a^b |g'(\phi)|\mathrm{d}\phi \leqslant \int_a^b g'(\pi-\phi)\mathrm{d}\phi = g(\pi-a) - g(\pi-b). \tag{5.43}$$

We rewrite (5.40) and (5.41) as follows:

$$A_L = g(\phi_0) + \sum_{n=1}^{\infty}\Big[g(\pi-\phi_{2n-1}) - g(\pi-\phi_{2n}) + g(\phi_{2n}) - g(\phi_{2n-1})\Big]$$

$$B_L = g(\phi_0) + \sum_{n=0}^{\infty}\Big[g(\pi-\phi_{2n}) - g(\pi-\phi_{2n+1}) + g(\phi_{2n+1}) - g(\phi_{2n})\Big].$$

By $\phi_n \leqslant \phi_{n+1}$ for all $n \in \mathbb{N}$ and (5.43), the expressions in square brackets are nonnegative for all $n \in \mathbb{N}$, and since g is nonnegative, both expressions are nonnegative as well.

Case 2: L is not monotone on $[0, \pi/2]$

Since L is convex, there is a $\phi_{\mathrm{Null}} \in (0, \pi/2)$ such that L is monotonically decreasing on $[0, \phi_{\mathrm{Null}}]$ and monotonically increasing on $[\phi_{\mathrm{Null}}, \pi/2]$. So $|g'(\phi)| \leqslant g'(\pi-\phi)$ for all $\phi \in [\phi_{\mathrm{Null}}, \pi/2]$, by the same argument as above. Thus, relation (5.43) is valid for all $\phi_{\mathrm{Null}} \leqslant a \leqslant b \leqslant \pi/2$. This means that for ϕ_0 such that $\phi_{\mathrm{Null}} \leqslant \phi_0$ we are done. If $\phi_0 < \phi_{\mathrm{Null}}$, there is $p \in \mathbb{N}$ such that $\phi_p \leqslant \phi_{\mathrm{Null}} \leqslant \phi_{p+1}$. Now consider the sequence $\{\tilde{\phi}\}_{n\in\mathbb{N}}$, which is given by

$$\begin{aligned}&\tilde{\phi}_n = \phi_n, &&\text{for } n \leqslant p\\ &\tilde{\phi}_{p+1} = \tilde{\phi}_{p+2} = \phi_{\mathrm{Null}}\\ &\tilde{\phi}_{n+3} = \phi_{n+1} &&\text{for } n \geqslant p.\end{aligned}$$

One easily sees by (5.40) and (5.41) that A_L and B_L evaluated with the sequence $\{\tilde{\phi}_n\}_{n\in\mathbb{N}}$ have the same values as evaluated with the sequence $\{\phi_n\}_{n\in\mathbb{N}}$. So, without loss of generality, we may assume ϕ_{Null} to be a member of $\{\phi_n\}_{n\in\mathbb{N}}$, i.e. $\phi_{\mathrm{Null}} = \phi_{2m+1} = \phi_{2m}$ for some $m \in \mathbb{N}$. Again, we reorder the series (5.40) and (5.41) and get:

$$A_L = \sum_{n=0}^{m-1} \Big[g(\phi_{2n}) - g(\phi_{2n+1}) \Big] + g(\phi_{2m}) + \sum_{n=m+1}^{\infty} \Big[g(\phi_{2n}) - g(\phi_{2n-1}) \Big] + \\ + \sum_{n=1}^{m} \Big[g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n}) \Big] + \sum_{n=m+1}^{\infty} \Big[g(\pi - \phi_{2n-1}) - g(\pi - \phi_{2n}) \Big].$$

Since L is monotonically decreasing on $[0, \phi_{2m}]$, so is g. Thus, the first sum is nonnegative. L is increasing on $[\phi_{2m}, \pi/2]$ and therefore relation (5.43) is valid for all $\phi_{2m} \leqslant a \leqslant b \leqslant \pi/2$. So, the second sum could be negative, but the fourth sum dominates it, so the sum of both is nonnegative. That the third sum is nonnegative is a consequence of the fact that g is monotonically increasing on $[\pi - \phi_{2m}, \pi]$. So, A_L is nonnegative. Similarly, we have:

$$B_L = \sum_{n=1}^{m} \Big[g(\phi_{2n-1}) - g(\phi_{2n}) \Big] + g(\phi_{2m+1}) + \sum_{n=m+1}^{\infty} \Big[g(\phi_{2n+1}) - g(\phi_{2n}) \Big] + \\ + \sum_{n=0}^{m} \Big[g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1}) \Big] + \sum_{n=m+1}^{\infty} \Big[g(\pi - \phi_{2n}) - g(\pi - \phi_{2n+1}) \Big].$$

Again, L is monotonically decreasing on $[0, \phi_{2m+1}]$ and monotonically increasing on $[\phi_{2m+1}, \pi/2]$. So, by analogous arguments as above, B_L is nonnegative, too, which completes the proof of the proposition. □

Collecting these results, we now arrive at the desired statement. Propositions 5.1 and 5.2, together with Equations (5.32) and (5.33) prove the following:

THEOREM 5.1. *Let ω_{hb} be the gauge-invariant, quasifree functional given by (5.25) and (5.26). Then*

$$\omega_{\mathrm{hb}}\Big(\bar{\psi}_{\dot{r}}(\bar{f}^{\dot{r}}) \psi_s(f^s) \Big) \geqslant 0$$

$$\omega_{\mathrm{hb}}\Big(\psi_r(f^r) \bar{\psi}_{\dot{r}}(\bar{f}^{\dot{r}}) \Big) \geqslant 0$$

for all $f \in \mathscr{D}(V^+, \mathbb{C}^2)$. Thus ω_{hb} is a state on $\mathcal{F}(V^+)$.

So, we have shown that ω_{hb} defines a state on $\mathcal{F}(V^+)$. This is a closed sub-C^*-algebra of $\mathcal{F}$, and hence we can extend ω_{hb} to all of $\mathcal{F}$. Since due to the anticommutation relations (2.1) positivity is equivalent to boundedness by one, the theorem of Hahn–Banach guarantees that such a (nonunique) extension can be chosen to be positive and hence to be a state, too.

Showing that ω_{hb} is in fact $\mathcal{S}_{V^+}$-thermal is straightforward by its definition (5.25) and (3.12). At each point $x \in V^+$, it coincides with the KMS-state $\omega_{2\lambda x}$ on $\mathcal{S}_x$ [1]. Its thermodynamic interpretation as the future of a temperature singularity is thus justified, as is its name “Hot-Bang state”.

6. Conclusion and Outlook

In this paper, we reviewed the definition of a local thermal equilibrium state over the algebra of the free, massless Dirac field. A specific example for such a state was given, and its positivity was proven. In comparison with the bosonic case, where an analogous Hot Bang state exists, this proof required significantly more effort.

The general properties of local thermal equilibrium sates of massless fermions were investigated in [1]. All the important results of the free bosons [4], such as transport equations and thermodynamic arrow of time, could be established for the fermionic case as well. In future work, it would be of interest to look at the influence of the size of the space of thermal reference states $\mathcal{S}_x$ on these properties. Also, one should apply the characterization scheme for local thermal equilibrium states to other models, as has happened for massive bosons [8] or conformal fields [9].

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