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Dynamics of compact object clusters: a post-Newtonian study

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ABSTRACT

Compact object clusters are likely to exist in the centres of some galaxies because of mass segregation. The high densities and velocities reached in them need a better understanding. The formation of binaries and their subsequent merging by gravitational radiation emission are important to the evolution of such clusters. We address the evolution of such a system in a relativistic regime. The recurrent mergers at high velocities create an object with a mass much larger than the average. For this purpose we modified the direct NBODY6++ code to include post-Newtonian effects on the force during two-body encounters. We adjusted the equations of motion to include for the first time the effects of both periastron shift and energy loss by emission of gravitational waves, and so to study the eventual decay and merger of radiating binaries. The method employed allows us to give here an accurate post-Newtonian description of the formation of a runaway compact object by successive mergers with surrounding particles, as well as the distribution of characteristic eccentricities in the events. This study should be envisaged as a first step towards a detailed, accurate study of possible gravitational wave sources, thanks to the combination of the direct NBODY numerical tool with the implementation of post-Newtonian terms.

Key words: black hole physics – gravitational waves – methods: N-body simulations – galaxies: star clusters.

1 INTRODUCTION

It is nowadays well established that most, if not all, galaxies harbour a supermassive black hole (SMBH) in their centre with a mass of some 10^{6-9} $M_\odot$ (see e.g. the recent reviews by Ferrarese et al. 2001; Kormendy & Gebhardt 2001; Ferrarese & Ford 2005). There is also evidence for masses of 10^6 $M_\odot$ (Greene & Ho 2005). In the case of our Galaxy this is even more established: an SMBH with a mass of about $\sim$3–4 $\times$ 10^6 $M_\odot$ (Ghez et al. 2000; Eckart et al. 2002; Schödel et al. 2003; Ghez et al. 2003) must be ensconced in its centre. If one extends the correlation between the SMBH mass and the stellar velocity dispersion of the bulge of the host galaxy (the $\mathcal{M}_\bullet$–σ correlation) observed for galactic nuclei (Gebhardt et al. 2000; Ferrarese & Merritt 2000) to smaller systems, like globular clusters, one should expect intermediate-mass black holes (IMBHs) with masses of between 10^3 and 10^4 $M_\odot$ to be lurking in the centres of such stellar clusters. There are observations of M15 in the Milky Way or G1 in M31 (Gerssen et al. 2002; Gebhardt, Rich & Ho 2002; van der Marel et al. 2002) which are compatible with this possibility, but N-body models of these clusters have been made which do not require the presence of an IMBH (Baumgardt et al. 2003b).

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The densities observed in the central regions of galaxies, where these very massive objects are located, are very high and may even exceed the core density of globular clusters by a factor of 100 (about 10^7–10^8 $M_\odot$ pc^{-3} for the Galactic Centre, for instance), thus making them very special laboratories for stellar dynamics.

On the other hand, it is not strictly excluded that the central mass concentrations are not massive black holes (MBHs). Mass segregation creates a flow of compact objects like neutron stars or stellar black holes to the central parts of the cluster (Lee 1987; Miralda-Escudé & Gould 2000), where they may constitute a subcluster. This could mimic the effect of the MBH, and thus give an alternative explanation of the properties of clusters that have undergone core collapse, like M15 and G1 (Gebhardt et al. 2002; van der Marel et al. 2002; Baumgardt et al. 2003a,b). On the other hand, MBHs are favoured in the case of galaxies, in particular the Milky Way (Maoz 1998; Miller 2006).

For the case of a globular cluster it has been deduced that stellar black holes are probably ejected from the system. Stellar black holes should form three-body binaries and kick each other out of the cluster (Phinney & Sigurdsson 1991; Kulkarni, Hut & McMillan 1993; Sigurdsson & Hernquist 1993; Portegies Zwart & McMillan 2000). None the less, if the velocity dispersion is high enough, then binaries will not be created because of three-body encounters, as in the classical case considered before, but their potential energy is converted to gravitational waves during two-body encounters.

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A simple way to understand this is that the components of a binary merge before a third particle or a second binary comes in sufficiently close to interact with them so as to eject the binary or one of its components. Thus ejections cannot happen in such a scenario. As a matter of fact, for velocity dispersions of $\gtrsim$300 km s^{-1} the merging time in clusters with two mass components is already shorter than the required time between interactions before a third particle or a second binary can bring about an ejection (Lee 1995).

Relativistic stellar dynamics are of paramount importance for the study of a number of subjects. For instance, if we want to have a better understanding of what the constraints on alternatives to SMBHs are, in order to investigate the possibility of ruling out stellar clusters, one must do detailed analysis of the dynamics of relativistic clusters and determine in particular the core collapse time (Miller 2006). Also, when we want to investigate more competently the formation of MBHs, learn how the dynamics around them are, for instance to estimate captures of compact objects on a central SMBH via extreme mass ratio in-spiralling, or study a system of many SMBHs, etc., the inclusion of relativistic effects is essential.

Our current work includes the study of stars on relativistic orbits around a SMBH, so as to be able to estimate captures of compact objects on a central SMBH via extreme mass ratio in-spiralling and binary evolution of two SMBHs.

Efforts to understand the dynamical evolution of a stellar cluster in which relativistic effects may be important have already been made by Lee (1987), Quinlan & Shapiro (1989, 1990) and Lee (1993). In his work, Lee (1993, hereafter MHL93) addressed the problem of the dynamical evolution of a cluster composed of compact objects by, with some approximations, adding an estimate of the gravitational wave emission term correction to NBODY5 (see Section 3). Nevertheless, he neglected the first two post-Newtonian terms, $1\mathcal{PN}$ and $2\mathcal{PN}$ (see the next section), and made use of the formalism introduced by Peters (1964), possibly because of computational concerns. The computation of the $\mathcal{PN}$ corrections is CPU-consuming, for we have to compute both the accelerations and their time-derivatives (see next section). Also, NBODY5 is not suitable for supercomputers or special-purpose GRAPE (GRAvity Pip E) hardware; here either NBODY6++ or NBODY4 has to be used (Spurzem 1999; Aarseth 1999).

In this work we describe a new tool that allows us to address this problem in a much more rigorous way than has been done in the existing literature, including deviations from the Newtonian formalism of the standard direct NBODY6++ code (Spurzem 1999), based on Aarseth's direct NBODY codes (Aarseth 1999). We have modified the code in order to allow for post-Newtonian ($\mathcal{PN}$) effects, implementing in it the $1\mathcal{PN}$, $2\mathcal{PN}$ and $2.5\mathcal{PN}$ corrections without any further approximation than those inherent to the calculation of the $\mathcal{PN}$ terms themselves (Soffel 1989).

In Section 2 we give a brief description of the method and of the implementation of the $\mathcal{PN}$ terms into a standard NBODY code. An analysis of the formation and evolution of a particle that gains more and more mass from successive mergers in the system (the 'runaway particle') is made in Section 3 and, to conclude, in Section 4 we present a summary and discussion of the main results obtained.

2 METHOD: DIRECT SUMMATION NBODY WITH POST-NEWTONIAN CORRECTIONS

The version of the direct summation NBODY method that we have employed for the calculations, NBODY6++, includes 'Kustaanheimo–Stiefel (KS) regularization'. This means that when two particles are tightly bound to each other or the separation between them becomes smaller during a hyperbolic encounter, the couple becomes a candidate for a 'KS pair' in order to avoid problematically small individual time-steps (Kustaanheimo & Stiefel 1965a,b). We modified this scheme to allow for relativistic corrections to the Newtonian forces by expanding the acceleration in a series of powers of $1/c$ (where c is the speed of light) in the following way (Damour & Dervelle 1981; Soffel 1989):

$$\boldsymbol{a} = \underbrace{\boldsymbol{a}_0}_{\text{Newt.}} + \underbrace{\underbrace{c^{-2}\boldsymbol{a}_2}_{1\mathcal{PN}} + \underbrace{c^{-4}\boldsymbol{a}_4}_{2\mathcal{PN}}}_{\text{periastron shift}} + \underbrace{\underbrace{c^{-5}\boldsymbol{a}_5}_{2.5\mathcal{PN}}}_{\text{grav. rad.}} + \mathcal{O}(c^{-6}), \tag{1}$$

where $\boldsymbol{a}$ is the acceleration of particle 1, $\boldsymbol{a}_0 = -Gm_2\boldsymbol{n}/r^2$ is the Newtonian acceleration, G is the gravitational constant, m_1 and m_2 are the masses of the two particles, r is the distance between the particles, $\boldsymbol{n}$ is the unit vector pointing from particle 2 to 1, and $1\mathcal{PN}$, $2\mathcal{PN}$ and $2.5\mathcal{PN}$ are post-Newtonian corrections to the Newtonian acceleration, responsible for the pericentre shift ($1\mathcal{PN}$, $2\mathcal{PN}$) and the quadrupole gravitational radiation ($2.5\mathcal{PN}$), respectively, as shown in equation (1). The expressions for the accelerations are

$$\begin{aligned}\boldsymbol{a}_2 = \frac{Gm_2}{r^2}\Bigg\{\boldsymbol{n}\Bigg[&-v_1^2 - 2v_2^2 + 4v_1v_2 + \frac{3}{2}(nv_2)^2 \\ &+ 5\left(\frac{Gm_1}{r}\right) + 4\left(\frac{Gm_2}{r}\right)\Bigg] + (\boldsymbol{v}_1 - \boldsymbol{v}_2)[4nv_1 - 3nv_2]\Bigg\},\end{aligned} \tag{2}$$

$$\begin{aligned}\boldsymbol{a}_4 = \frac{Gm_2}{r^2}\Bigg\{\boldsymbol{n}\Bigg[&-2v_2^4 + 4v_2^2(v_1v_2) - 2(v_1v_2)^2 \\ &+ \frac{3}{2}v_1^2(nv_2)^2 + \frac{9}{2}v_2^2(nv_2)^2 - 6(v_1v_2)(nv_2)^2 \\ &- \frac{15}{8}(nv_2)^4 + \left(\frac{Gm_1}{r}\right)\left(-\frac{15}{4}v_1^2 + \frac{5}{4}v_2^2 - \frac{5}{2}v_1v_2\right. \\ &\left.+ \frac{39}{2}(nv_1)^2 - 39(nv_1)(nv_2) + \frac{17}{2}(nv_2)^2\right) \\ &+ \left(\frac{Gm_2}{r}\right)\Big(4v_2^2 - 8v_1v_2 + 2(nv_1)^2 \\ &- 4(nv_1)(nv_2) - 6(nv_2)^2\Big)\Bigg] \\ &+ (\boldsymbol{v}_1 - \boldsymbol{v}_2)\Bigg[v_1^2(nv_2) + 4v_2^2(nv_1) - 5v_2^2(nv_2) \\ &- 4(v_1v_2)(nv_1) + 4(v_1v_2)(nv_2) - 6(nv_1)(nv_2)^2 \\ &+ \frac{9}{2}(nv_2)^3 + \left(\frac{Gm_1}{r}\right)\left(-\frac{63}{4}nv_1 + \frac{55}{4}nv_2\right) \\ &+ \left(\frac{Gm_2}{r}\right)(-2nv_1 - 2nv_2)\Bigg]\Bigg\} \\ &+ \frac{G^3m_2}{r^4}\boldsymbol{n}\left(-\frac{57}{4}m_1^2 - 9m_2^2 - \frac{69}{2}m_1m_2\right),\end{aligned} \tag{3}$$

$$\begin{aligned}\boldsymbol{a}_5 = &\frac{4}{5}\frac{G^2m_1m_2}{r^3} \\ &\times \Bigg\{(\boldsymbol{v}_1 - \boldsymbol{v}_2)\left[-(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 + 2\left(\frac{Gm_1}{r}\right) - 8\left(\frac{Gm_2}{r}\right)\right] \\ &+ \boldsymbol{n}(nv_1 - nv_2)\left[3(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - 6\left(\frac{Gm_1}{r}\right) + \frac{52}{3}\left(\frac{Gm_2}{r}\right)\right]\Bigg\}.\end{aligned} \tag{4}$$

In the last expressions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the velocities of the particles. For simplification, we have denoted the dot product of two vectors, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, as $x_1\, x_2$. The basis of direct NBODY4 and NBODY6++ codes relies on an improved Hermit integrator scheme (Makino & Aarseth 1992; Aarseth 1999) for which we need not only the accelerations but also their time derivatives. These derivatives are not included in these pages for succinctness. We integrated our correction terms into the KS regularization scheme as perturbations, similarly to what is done to account for passing stars influencing a KS pair. Note that formally the perturbation force in the KS formalism does not need to be small compared with the two-body force (Mikkola 1997). If the internal KS time-step is properly adjusted, the method will work even for relativistic terms becoming comparable to the Newtonian force component.

3 DYNAMICAL EVOLUTION OF A CLUSTER OF COMPACT OBJECTS

3.1 The initial system and units

The units used in our models correspond to the so-called N-body unit system, in which $G = 1$, the total initial mass of the stellar cluster is 1 and its initial total energy is $-1/4$ (Hénon 1971; Heggie & Mathieu 1986). The system was chosen to be initially identical to that employed by MHL93: i.e. a spherical cluster with a number of compact stars $\mathcal{N}_\star = 10^3$ of identical mass m. These were distributed in an isotropic Plummer sphere, which means that the phase-space distribution function is proportional to $|E|^{7/2}$, where E is the energy per unit mass of one star. The density profile is thus $\rho(r) = \rho_0[1 + (r/R_{\rm Pl})^2]^{-5/2}$, where $R_{\rm Pl}$ is the Plummer scaling length. For such a model the N-body length unit is $\mathcal{U}_{\rm l} = (16/3\pi)R_{\rm Pl}$.

In the situations considered here, the evolution of the cluster is driven by two-body relaxation. A natural time-scale is the (initial) half-mass relaxation time. We use the definition of Spitzer (1987),

$$T_{\rm rh}(0) = \frac{0.138N}{\ln\Lambda}\left(\frac{R_{1/2}^3}{G\mathcal{M}_{\rm cl}}\right)^{1/2}. \tag{5}$$

For instance, for a Plummer model, the half-mass radius is $R_{1/2} = 0.769\mathcal{U}_{\rm l} = 1.305R_{\rm Pl}$. $\mathcal{M}_{\rm cl}$ is the total stellar mass and $\ln\ \Lambda = \ln(\gamma N)$ is the Coulomb logarithm.

For the situation considered in this work, the square ratio of the central velocity dispersion $\sigma_{\rm cen}$ to the speed of light c,

$$\left(\frac{\sigma_{\rm cen}}{c}\right)^2 \approx \frac{G\mathcal{M}_{\rm cl}}{\mathcal{R}_{\rm cl}c^2} \approx \frac{\mathcal{R}_{\rm Schw}^{\rm cl}}{\mathcal{R}_{\rm cl}}, \tag{6}$$

is big enough, so that we can expect that relativistic effects play a noticeable role in the evolution of the system. For this aim, we chose $\sigma_{\rm cen}$ to be $\sim$4300 km s^{-1}. G is the gravitational constant, $\mathcal{R}_{\rm cl}$ is the radius of the cluster and $\mathcal{R}_{\rm Schw}^{\rm cl} = 2G\mathcal{M}^{\rm cl}/c^2$ is the Schwarzschild radius of the cluster.

In our calculations the $\mathcal{PN}$ terms are acting all the time during the calculations but obviously become important only when velocities are high. Our criterion for particles to merge is that they reach their common Schwarzschild radius $\mathcal{R}_{\rm Schw}$ – i.e. the sum of their Schwarzschild radii. This is of course approximate because the $\mathcal{PN}$ treatment breaks down when particles are that close (and $v \sim c$), but this should not matter, for the merging phase is *much* shorter than any stellar dynamical time. The gravitational recoil, the expected loss of linear momentum in asymmetric systems in which the merger remnant receives a kick from the gravitational wave emission, obviously does not show up in our models, because we truncate the series at 2.5 $\mathcal{PN}$ and it is only to be treated as an effect of higher-order terms.

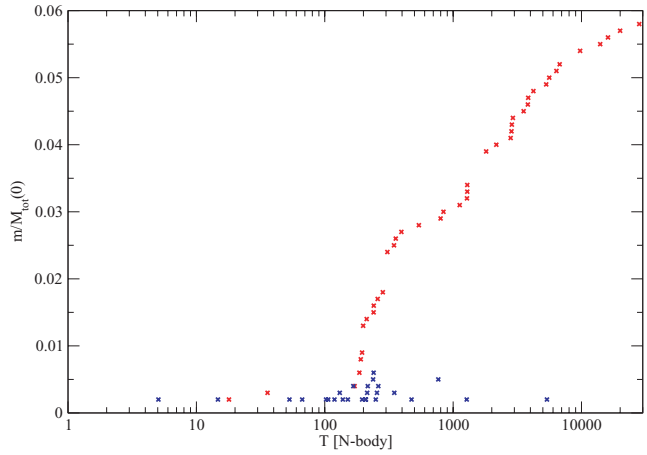

Figure 1. Time evolution of merging masses. The formation of the runaway particle is about the time of the cluster core collapse. Fore more details, see text.

3.2 Formation of a runaway body

Even though we started with a single-mass stellar system, the masses of some objects in the cluster increased by relativistic mergers. In Fig. 1 we survey the time evolution of the mass increase. We find a number of mergers that lead to the variation of the initial single-mass situation. The particle masses increase after the relativistic merging events, since we are assuming that the particles merge perfectly when they reach the distance of their $\mathcal{R}_{\rm Schw}$ (see above). We find the formation of a runaway particle that reaches almost 6 per cent of the initial total mass by the end of the simulation (see Fig. 1). We have denoted the mass of the runaway body by red crosses and the mass of other mergers by blue crosses.

One can observe that the runaway body dominates the system after its fast-growing phase around 300 time-units, which is approximately the moment at which the core collapse of the system happens, as we can see in Fig. 2. Only some merger events which are independent of the runaway body can occur after this phase. This fast-growing phase occurs at the core collapse of the system (Meylan & Heggie 1997). In Fig. 2 we follow the evolution of the

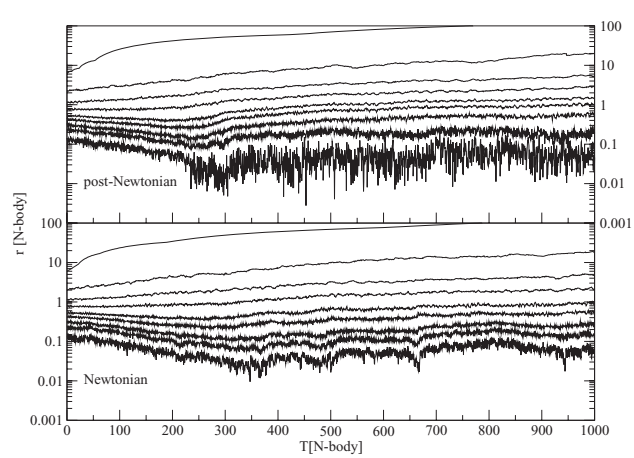

Figure 2. Evolution of the Lagrangian radii corresponding, from bottom to top, to 1, 5, 10, 20, 30, 50, 70, 90 and 100 per cent of the total mass.

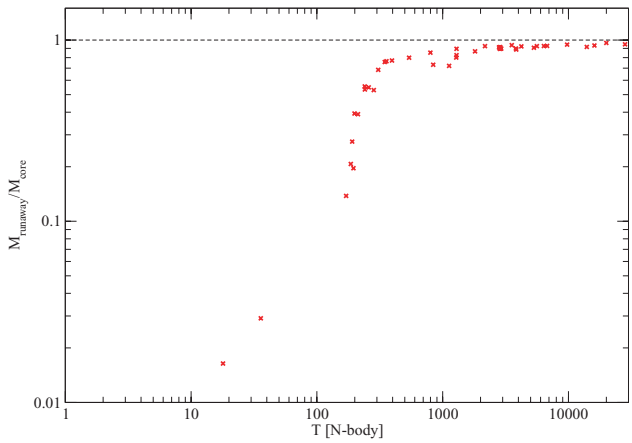

Figure 3. Evolution of the runaway particle mass in units of the core mass (at the same time).

so-called Lagrangian radii of the system, spheres containing 1, 5, 10, 20, 30, 50, 70, 90 and 100 per cent of the total mass of the cluster; the centre of the cluster is defined to be the centre of the mass density. Since the runaway particle is included, and in the end its mass reaches 5 per cent of the total initial mass of the cluster, the curves corresponding to 1 and 5 per cent roughly correspond to its evolution. We observe that the runaway stops the core collapse and allows for an expansion.

The process of mergers translates directly into a production of energy in the central regions of the cluster. The centre adapts to supply the cluster with the same amount of energy that it can obtain via relaxation, and this amount is determined by the large-scale structure.

According to, for instance, the table given in Freitag & Benz (2001), the standard value for the core collapse time is roughly ∼15–20 times the half-mass relaxation time $T_{\rm rh}$. We find nevertheless that the core collapse time is $t_{\rm cc} \sim 11T_{\rm rh}$, with a value of $\gamma = 0.11$ in the Coulomb logarithm (Giersz & Heggie 1994), which clearly suggests that the $\mathcal{PN}$ terms accelerate the collapse. This can be seen more clearly in Fig. 2, which corresponds to the same simulation but without making use of relativistic corrections. There we can see that $t_{\rm cc} \sim 380 \sim 14T_{\rm rh}$.

In Fig. 3 we show the evolution of the runway particle mass normalized to the mass contained in the core of the cluster, defined as in Casertano & Hut (1985). The mass of the runway particle can grow only up to the core mass. The core mass continuously decreases as the core collapse proceeds. We see this in the figure, where the runaway particle grows and saturates to the core mass after ∼1200 time units.

The evolution and formation of the runaway particle mass are not as fast as they were in MHL93, as we can see in fig. 5 of that paper. For our simulation the sudden jump in the growth of the mass comes in slightly later and is smoother, reaching final values for the runaway particle mass about three times smaller than in MHL93. The differences can be attributed to the following: MHL93 calculated the influence of the $2.5\mathcal{PN}$ term on the orbits in an unperturbed pair and made them merge after a decay time-scale, following the Peters (1964) formalism. This requires the assumption that particles move along their orbits on an ellipse, only valid when they are very far from the relativistic regime. On the other hand, we implemented the $2.5\mathcal{PN}$ term in the code itself, so that the relativistic corrections are a natural feature the influence of which on the evolution of the system comes in when the velocities of the stars become high enough. The influence of the $1\mathcal{PN}$ and $2\mathcal{PN}$ terms corresponds to the conservative-phase evolution of the orbit and cannot be relevant because they do not change its energy and angular momentum.

4 CONCLUSIONS

In this work we have presented a study of the formation and evolution of a runaway particle in a dense cluster of compact objects – which initially had the same mass – as a result of relativistic mergers. We have employed a modified version of the direct summation NBODY6 code in which we have implemented the $1\mathcal{PN}$, $2\mathcal{PN}$ and $2.5\mathcal{PN}$ terms to take into account post-Newtonian corrections to the standard NBODY Newtonian acceleration.

The runaway particle reaches at the end of our simulations ∼6 per cent of the initial total stellar mass of the cluster. We have also compared our work with a previous result based on a more approximate scheme, the approach described by Peters (1964), and we have found that the net result is that the growth of the runaway particle in the study of MHL93 is ∼3 times larger. Since the $1\mathcal{PN}$, $2\mathcal{PN}$ terms modify the extrinsic features of the orbits (e.g. the orientation) but do not affect their intrinsic parameters (like frequency), we therefore can expect their effect to be averaged out during the evolution of the system and not influence the merger rates. One should thus attribute the differences to the approach that Man Hoi Lee made, somehow inadequate for the velocity regime considered.

This study should be envisaged as successful test of the code, which has been shown to be robust. This tool can be applied to other astrophysical scenarios that require a post-Newtonian treatment. This includes on-going work, such as e.g. captures of compact objects by a SMBH in a galactic centre, also known as extreme mass ratio in-spirals. One of the fundamental aims is to explore the parameter space rigorously, so that we can provide the *LISA* data analysis community with realistic estimates of, for instance, the eccentricity, mass ratio, etc., at the beginning of the final merger, when the smaller compact object enters the *LISA* band. An assumption for the initial parameter space is necessary in order to develop waveform 'banks' for this kind of event. One must note here that the inclusion of the $1\mathcal{PN}$ and $2\mathcal{PN}$ terms is very relevant, for resonant relaxation (or Kozai) effects, which may increase the rate of in-spiral significantly, may be strongly affected by relativistic precession and may thus have an impact on the number of captures (Kozai 1962; Hopman & Alexander 2006). The inclusion of higher order $\mathcal{PN}$ terms is also part of the current studies, and will also shed light on other aspects of this subject (spin–spin coupling, spin–orbit interaction and radiation recoil).

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