

ON THE RELATION BETWEEN MATHEMATICAL AND NUMERICAL RELATIVITY

LARS ANDERSSON

ABSTRACT. The large scale binary black hole effort in numerical relativity has led to an increasing distinction between numerical and mathematical relativity. This note discusses this situation and gives some examples of successful interactions between numerical and mathematical methods in general relativity.

1. INTRODUCTION

After a lengthy period of fighting various “monsters” [48], such as spurious radiation, constraint instabilities, boundary effects, collapse of the lapse, etc., the effort in numerical general relativity directed at modelling mergers of binary black holes is now rapidly entering a phase of “normal science”. Although not all of the monsters have been tamed, a number of groups are reporting multiple orbit evolutions and the goal of providing reliable wave forms in sufficient numbers and of sufficient accuracy for use in gravitational wave data analysis is in sight.

The conceptual framework for the numerical work on the binary black hole (BBH) problem, which arguably has played an absolutely necessary role as foundation and stimulus for this work, has been provided by the global picture of spacetimes, including singularity theorems, ideas of cosmic censorship, post-Newtonian and other analytical approximations of the 2-body problem in general relativity, which have been arrived at purely by analytical and geometric techniques. Further, the theoretical analysis of numerical approximations to solutions of systems of PDE’s, the analysis of the Cauchy problem for the Einstein equations, developments in computer science concerning parallel processing, all provided essential stepping stones on the path towards successful BBH simulations.

Due to the large scale of the effort that goes into the BBH work, the division of the general relativity community into “numerical” and “theoretical/mathematical” groups has become pronounced. With this in mind, it seems natural to ask oneself what grounds there are for future interactions between these two communities. On the one hand, one may take the point of view that the “strong field” regime of general relativity is going to be the essentially exclusive domain of numerical general relativity, the phenomena one is likely to encounter being too complex to be amenable to mathematical analysis; a consequence of this point of view is the recommendation to mathematical relativists interested in these aspects of general relativity to devote themselves to becoming numerical relativists. On the other hand, one may take the point of view that the strong field regime of general relativity likely contains new phenomena of interest both for our understanding of

Date: July 2, 2006.

Supported in part by the NSF, under contract no. DMS 0104402 with the University of Miami.

the analytical nature of the Einstein equations, as well as for our understanding of physical reality.

In the latter point of view, which I am proposing in this note, the relation between numerical and mathematical general relativity is similar to that of experimental mathematics to mathematics, i.e. as a tool for discovering new phenomena, testing conjectures, and developing a heuristic framework which can be used in a precise mathematical analysis. In either case, there is a clear need for an effort to bridge the emerging gap between the two communities.

1.1. Numerical experiments and mathematics. Mathematics has a long history of interaction between computer simulations and analytical work. Areas where this interaction has been prominent are number theory, dynamical systems, and fluid mechanics. The interaction has provided both the discovery of new phenomena, as well as proofs of theorems conjectured on the basis of numerical experiments.

A few examples where the interaction between mathematics and computer simulations has played an important role are provided by the accidental discovery in 1963 by Lorentz [51] of chaotic behavior in a system of equations derived from atmospheric models¹, the discovery by Feigenbaum [33] of universality in period doubling bifurcations, the discovery and study of strange attractors in dynamical systems, and the analysis of fractals including the Mandelbrot set [52].

The proof of the existence of solutions to the Feigenbaum functional equation was computer based, using rigorous numerical computer techniques [49]. The existence of strange attractors for the Hénon map [40] was proved by Benedicks and Carleson [11], using analytic techniques. The proof was preceded by a lengthy period of theoretical work as well as very detailed computer simulations which gave strong support to the conjectured picture of the attractor and the dynamics of the Hénon map. The Hénon map was derived as a model for the Poincaré map of the Lorentz system. It was recently proved that the Lorentz system contains a strange attractor [67], thus providing a solution to Smale's 14th problem. The proof of this fact was again computer based.

1.2. Overview of this paper. Below, in section 2, I shall discuss three examples from general relativity. The first is the Bianchi IX, or Mixmaster system, an anisotropic homogenous cosmological model, and in particular modelled by a system of ODE's, see section 2.1. The second is the Gowdy T^2 -symmetric cosmological model, which is modelled by a 1+1 dimensional system of wave equations, see section 2.2. Third, I will discuss critical collapse, see section 2.3 which was first discovered during numerical simulations of the collapse of a self-gravitating scalar field.

In section 3, I mention some open problems where it seems likely that the interaction of numerical and analytical techniques will play an important role. In section 3.1, I discuss general T^2 symmetric cosmologies, which provide a simple model for the full BKL type behavior followed by a few remarks on generic singularities in section 3.2. The next section 3.3, introduces the problem of self-gravitating wave maps and the $U(1)$ model. Finally, the stability of the Kerr black hole is discussed in section 3.4. Concluding remarks are given in section 4.

¹The famous Fermi-Pasta-Ulam experiment of 1955 is perhaps the non-chaotic counterpart of the Lorentz experiment.

2. SUCCESS STORIES

2.1. The Mixmaster spatially homogenous cosmology. The Bianchi IX or Mixmaster model is given by restricting the vacuum Einstein equations to the spatially homogenous case with S^3 spatial topology. The dynamics of this system was first discussed in some detail by Misner, see [53] and references therein, see also [69]. Misner gave a Hamiltonian analysis which indicated that the system exhibits “bounces” interspersed with periods of “coasting”. The thesis of Chitre [21] gave an approximation of the dynamics as a hyperbolic billiard. It was quickly realized that the billiard system is chaotic in a certain sense, namely it projects to the Gauss map, $x \mapsto \{\frac{1}{x}\}$, see [8], which has been well studied.

The heuristic picture of the oscillatory, and chaotic, asymptotic behavior of the Mixmaster model played a central role in the proposal of Belinskiĭ, Khalatnikov, and Lifshitz (BKL) concerning the structure of generic singularities for the gravitational field [9, 10]. An essential aspect of the BKL proposal is that the dynamics near typical spatial points is asymptotically “Mixmaster”-like. In the case of spacetimes containing stiff matter on the other hand, the asymptotics is “Kasner”-like, and quiescent. The quiescent behavior also occurs under certain symmetry conditions, an important example being the Gowdy spacetimes to be discussed below. Apart from the intrinsic beauty of the Mixmaster system, the BKL proposal provides one of the main motivations for studying the Mixmaster system in detail.

It should be remarked that the chaotic nature of the Mixmaster dynamics was used by Misner as a basis for the so-called “chaotic cosmology” proposal, in which it was argued that the dynamics of Mixmaster gave a way around the horizon problem which plagued cosmology during this period (i.e. pre-inflation).

The full Mixmaster model resisted analysis for a long time, in spite of a large number of papers devoted to this subject. Numerical experiments indicated that the model has sensitive dependence on initial data, but also revealed that evolving the system of ODE’s describing the Mixmaster dynamics for sufficiently long times to give useful insights, and with sufficient accuracy to give reliable results, presented a difficult challenge. It was only with the work of Berger, Garfinkle and Strasser [14] which made use of symplectic integration techniques and an analytic approximation that it was possible to overcome the extremely stiff nature of the system of ODE’s for the Mixmaster model. This numerical work gave strong support for the basic conjectures concerning the Mixmaster system, and led to a renewed interest within the mathematical general relativity community in the analysis of the Mixmaster dynamics. The volumes [41] as well as the paper [54] played an important role in spreading the word about this problem.

The main conjectures concerning the mixmaster model, including proof of cosmic censorship in the Bianchi class A models, the oscillatory nature of the Bianchi IX singularity, as well as the existence of an attractor for the Bianchi IX system were proved by Ringström in a series of papers [57, 58]. However, in spite of these very important results, many basic and important questions concerning both the full Mixmaster system, as well as the billiard approximation, remain open.

Recent work of Damour, Henneaux, Nicolai and others [30] have shown that the BKL conjecture extends in a very interesting way to higher dimensional theories of gravitation inspired by supergravity theories in D=11 spacetime dimensions. A formal argument indicates that these models have asymptotic Mixmaster like behavior, governed by a hyperbolic billiard, determined by the Weyl chamber of

a certain Kac-Moody Lie algebra. Applying this analysis to 3+1 vacuum gravity reproduces the Chitre model. These domains occurring in these hyperbolic billiards are arithmetic, which has interesting consequences for the length spectrum of the billiard.

An important open problem is to understand the relation between the ‘‘Hamiltonian’’ approach, developed by Misner-Chitre, and which also is used in the work of Damour-Henneaux with the scale invariant variables approach developed by Ellis-Wainwright-Hsu, and which was used in the work of Ringstrom on Mixmaster. The scale invariant variables formalism has been generalized to inhomogenous models by Uggla et. al, [68], and applied to formal and numerical analysis of inhomogenous cosmological models [4, 37].

2.2. The Gowdy T^2 -symmetric cosmologies. The cosmological models on $T^3 \times \mathbb{R}$, with T^2 symmetry, and with hypersurface orthogonal Killing fields, the so-called Gowdy model, is one of the simplest inhomogenous cosmological models. The Einstein equations reduce to a system of PDE’s on $S^1 \times \mathbb{R}$, consisting of a pair of nonlinear wave equations of wave maps type, and a pair of constraint equations. Eardley, Liang and Sachs [31] introduced the notion of asymptotically velocity dominated singularities, to describe the asymptotically locally Kasner like, non-oscillating behavior of certain cosmological models. We will refer to this behavior as quiescent. In particular, analysis showed that one could expect the Gowdy model to exhibit quiescent behavior at the singularity.

A programme to study the Gowdy model analytically, with a view towards proving strong cosmic censorship for this class of models, was initiated by Moncrief. The methods used included a Hamiltonian analysis, and formal power series expansions around the singularity. The formal power series expansions of Grubisic and Moncrief [38] supported the idea that a family of Gowdy spacetimes with ‘‘full degrees of freedom’’, i.e. roughly speaking parametrized by four functions, exhibited quiescent behavior at the singularity. In the course of this work, an obstruction to the convergence of the formal power series was discovered. The condition for the consistency of the formal power series expansions was that the ‘‘asymptotic velocity’’ k of the Gowdy spacetime, satisfies $0 < k < 1$. The term asymptotic velocity has its origin in the fact that the evolution of a Gowdy spacetime corresponds to the motion of a loop in the hyperbolic plane. The asymptotic velocity $k(x)$ for $x \in S^1$ is defined as the asymptotic hyperbolic velocity of the point with parameter x on the evolving loop in the hyperbolic plane. It is a highly nontrivial fact that this limiting value exists, see [62].

Numerical studies carried out by Berger and Moncrief [16] and later by Berger and Garfinkle [13] gave rise to a good heuristic picture of the asymptotic dynamics of Gowdy models. In particular, the numerical work showed that Gowdy spacetimes exhibit sharp features (spikes), which formed and appeared to persist until the singularity. The spatial scale of the spikes turned out to be shrinking exponentially fast, and it was therefore impossible to resolve these features numerically for more than a limited time. Kichenassamy and Rendall [43] showed, using Fuchsian techniques, that the picture developed in the work on formal expansions could be made rigorous, and in particular that full parameter families of ‘‘low velocity’’ Gowdy spacetimes could be constructed with quiescent singularities. Further, Rendall and Weaver [56] used a combination of Fuchsian and solution generating techniques to construct Gowdy spacetimes containing spikes with arbitrary prescribed velocity.

This allowed one to gain detailed understanding of the nature of the spikes, and in particular of the nature of the discontinuity of the asymptotic velocity at spikes.

These developments gave through the numerical work, a vivid graphical picture of the dynamics of Gowdy spacetimes, but also established with rigor some of the fundamental conjectured aspects of Gowdy spacetimes. Based on these developments, Ringstrom [59, 61, 60] was able to analyze the nature of the singularity of generic Gowdy spacetimes, and in particular give a proof of strong cosmic censorship for this class of spacetimes.

2.3. Critical collapse. Critical behavior in singularity formation was discovered by Matt Choptuik [22], during numerical studies of the collapse of self-gravitating scalar fields. He found that for one-parameter families of initial data, interpolating between data leading to dispersion and data leading to collapse, data on the borderline between dispersion and collapse exhibited, for a period depending on the parameter, a discrete self-similar behavior before dispersing or collapsing. Further, Choptuik found that the rate of divergence from the self-similar behavior exhibited a “universal” behavior, analogous to the universality discovered by Feigenbaum in connection with period doubling bifurcations. This work opened up a very rich field of investigation which is still active.

The basic principle is now well established through a large number of numerical experiments and investigations, see the review paper [39]. A formal analysis indicates that the “universal” behavior mentioned above may be explained in terms of a linearized analysis around the self-similar critical solution [32, 47]. It turns out that the detailed behavior, in particular the rate, depends on the details of the non-linearity, or in the case of general relativity, the matter model under consideration, but the basic idea of universality within a matter model, and the above mentioned mechanism for the critical behavior is well established over a wide range of models. Depending on the matter model, the self-similar behavior may be discrete, continuous, or even in some cases a mixture of the two types. Virtually all numerical work on critical behavior has so far been in the spherically symmetric case. The reason is the extreme demands on numerical precision presented by the problem.

By generalizing the notion of critical behavior from general relativity to semilinear wave equations, Yang-Mills equations, and wave maps equations, Bizon and others [19], have been able in some cases to find the explicit form of the first unstable (self-similar) mode, and thus give an analytic description of the blowup solutions. They find good agreement with numerical data. However, beyond the linearized stability analysis mentioned above, not many rigorous results are known for critical behavior for the hyperbolic equations mentioned above, including the case of general relativity. This state of affairs should be contrasted with the asymptotic analysis of singular solutions of semilinear parabolic equations.

In the 2+1 dimensional case, a new phenomenon arises. For wave maps on (2+1)-dimensional Minkowski space, with spherical target, there is no self-similar solution. Instead there is a one-parameter family of static solutions, and numerical work in the equivariant case shows that this family mediates the blowup [20]. This is borne out by the proof due to Struwe [65] that in the equivariant case, a rescaling limit of a blowup solution converges to a harmonic map from \mathbb{R}^2 to S^2 . Due to this fact, the nature of the critical blowup in 2+1 dimensions is fundamentally different from the 3+1 dimensional case, and the analysis of the asymptotic rate of concentration of

blowup solutions is much more delicate. Recent work of Rodnianski and Sterbenz [63] sheds light on this question in the general case without symmetries.

The further study of the asymptotic behavior of blowup solutions of semilinear wave equations as well as the gravitational field, in the non-spherically symmetric case is one of the important challenges for the near future. Here it appears likely that a lot of the technology developed during the course of the BBH work, such as adaptive mesh-refinement, etc. will play a decisive role. Indeed, the original work by Choptuik on critical collapse used a version of adaptive mesh refinement in the spherically symmetric situation.

3. OPEN PROBLEMS

In this section, I shall briefly indicate some problems which I consider to be of interest from the point of view of the interaction between numerical and mathematical work in general relativity and related fields. The survey papers [1, 55] provide general references for many of the problems mentioned below.

The problems I will mention are not exactly coincident with the “forefront” of numerical relativity, and may by some workers in that field be considered as simple problems, not worthy of their attention. There are several reasons for this. One is that the development of the mathematical theory of relativity is to a large extent lagging behind the exploratory and goal oriented work being performed within numerical relativity. Further, in order to provide reliable insights into the nonlinear problems under consideration, the numerical experiments must necessarily be carried out to a high degree of accuracy. This level of precision is so far not available in general in the 3+1 or even 2+1 dimensional numerical evolutions, in particular not in the strong field regime where many of the phenomena of interest take place. In particular, a serious numerical study of the asymptotic behavior at cosmological and other types of singularities, with a view to better understanding the BKL proposal in general relativity, as well as the asymptotic behavior of blowup solutions of geometric wave equations, without symmetry assumptions, is likely to be at least as challenging as the BBH problem.

3.1. General T^2 -symmetric cosmologies. The full T^2 symmetric model on $T^3 \times \mathbb{R}$, without the condition the the Killing fields be surface orthogonal exhibits oscillatory behavior at the singularity. While this has not been rigorously established, this is indicated by formal and numerical studies. The formal work includes the analysis of the silent boundary due to Uggla et al [68]. There are formal and numerical studies using both the metric formulation by Berger et al. [15] and the scale invariant formulation by Andersson et al. [4]. The last mentioned work gives numerical support to the silent boundary picture for the case of T^2 -symmetric cosmologies. The review by Berger [12] provides a general reference on the numerical investigation of spacetime singularities. The numerical studies lend support to the BKL proposal on the nature of generic cosmological singularities, and also indicate some new dynamical features of the T^2 singularity. These new features can be interpreted as spikes. However in contrast to the Gowdy case, where the asymptotic velocity at the spike is a constant, the spikes in T^2 are dynamical features, which according to the numerical experiments [4] exhibit a simple dynamics, closely related to the asymptotic billiard for the silent boundary system for T^2 .

3.2. Singularities in generic cosmologies. The $U(1)$ model has been studied in the spatially compact case by Choquet-Bruhat and Moncrief [26, 25]. They proved global existence in the expanding direction for small data on spacetimes with topology $\Sigma \times \mathbb{R}$ with $\text{genus}(\Sigma) > 1$. For the polarized case, one has a self-gravitating scalar field. In the polarized case, one expects to have quiescent behavior at the singularity, a full parameter family of such solutions was constructed by Choquet-Bruhat, Isenberg and Moncrief [42, 23]. For the full $U(1)$ model, one expects an oscillatory singularity, as in the full T^2 case. Numerical and analytical work of Berger and Moncrief [18, 17] give support to this picture. For this model, as for the T^2 model, the BKL proposal, and in particular, the silent boundary proposal of Uggla et al [68] provides a heuristic picture of the dynamical behavior that one expects to see.

In fact, the scale invariant variables introduced by Uggla et al provides, with for example CMC time gauge, a well posed elliptic-hyperbolic system, which can be used to model the dynamical behavior at the singularity. Some preliminary numerical experiments using a CMC code have been carried out. Even for the polarized case in the 2+1 dimensions the need for adaptive codes is apparent. The oscillatory nature of the singularity means that spatial structure is created at small scales. This will make it impossible to produce even somewhat realistic evolutions of the full $U(1)$ model without using an adaptive code. See however the recent work by Garfinkle [37] for some numerical experiments in the 3+1 case. For these, even though they reproduce the heuristic picture derived from the silent boundary conjecture, the accuracy is too low to provide reliable information. Earlier work by Garfinkle [36] on cosmological singularities in self-gravitating scalar field model in 3+1 dimensions made use of spacetime harmonic (or wave) coordinates. The fact that this experiment, which for essentially the first time made use of spacetime harmonic coordinates for a numerical relativity code was successful, has later had a significant influence on current work on the BBH problem. The self-gravitating scalar field model is known to have large families of data which give rise to quiescent singularities [3].

3.3. Self-gravitating wave maps and $U(1)$. Vacuum 3+1 dimensional gravity with a spatial $U(1)$ action gives, after a Kaluza-Klein reduction, a self-gravitating wave-maps model in 2+1 dimensions, with hyperbolic target space. Further imposing on this model a rotational symmetry, i.e. another spatial $U(1)$ action, which acts equivariantly, results in an equivariant self-gravitating 2+1 dimensional wave map with hyperbolic target. The equivariant $U(1)$ action does not correspond to a Killing field in the 3+1 dimensional picture. In this case it is natural to impose asymptotic flatness for the 2+1 dimensional spacetime [7]. If this is done, turning off the gravitational interaction gives a flat space wave maps model with hyperbolic target.

According to standard conjectures, the flat space 2+1 dimensional wave map with hyperbolic target is expected to be well posed in energy norm, see [66], and it is therefore reasonable to expect that also the self-gravitating version of this model is globally well-posed. For the 2+1 dimensional wave maps model with spherical target, on the other hand, numerical experiments indicate that one has blowups for large data, see section 2.3 above.

Based on the idea that the blowup in the wave maps model with spherical target is mediated by static solutions, one may argue that in the self-gravitating case,

one cannot have blowups for sufficiently large values of the coupling constant. The reason for this is that the energy balance between the gravitational field and the wave maps field does not leave enough energy for the wave maps field to produce the static solutions which mediate blowup. Some numerical experiments have been carried out which support this picture. It would be of great interest to have detailed numerical experiments in this situation. The full 2+1 dimensional version of the self-gravitating wave maps problem is very challenging both numerically and theoretically.

3.4. Stability of Kerr. As mentioned above, the understanding of the structure of full 3+1 dimensional cosmological singularities and the strong cosmic censorship represents a major challenge to the numerical and mathematical relativity community. However, the stability of the Kerr black hole is perhaps closer to the type of problems which occupy most of the attention of current work in numerical relativity. As is well known, according to the cosmic censorship picture, the end state of the evolution of an asymptotically flat data set is a single Kerr black hole. A proof of the nonlinear stability of Kerr would provide an important step towards a proof of this far-reaching conjecture.

The nonlinear stability of Minkowski space was proved by Christodoulou and Klainerman, see [28], see also [35] for an earlier partial result, using a conformally regular form of the Einstein equations. For quasilinear wave equations which satisfy the so-called null condition of Christodoulou [27], global existence for sufficiently small data is known to hold in dimension $n + 1$, $n \geq 3$. It is a very important fact that the Einstein equations do not satisfy the classical null condition [24].

The proof of Christodoulou and Klainerman relied upon detailed and rather delicate estimates of higher order Bel-Robinson energies, using a combination of techniques. The geometry of certain null foliations was studied, exploiting the transport equations for geometric data along null rays. Further, a variant of the vector fields method of Klainerman [44] was used.

The vector fields method was developed to prove decay estimates for solutions of wave equations, and requires at least approximate symmetries of the background solution. The method has been used by Klainerman and Rodnianski in a micro-local setting in order to prove well-posedness for the Einstein equations with rough data. The method of Christodoulou and Klainerman has later, in a series of papers by Nicolo and Klainerman [45, 46] been shown to yield the correct peeling behavior at null infinity, which is expected from the Penrose picture.

Recently, a substantially simpler proof of the nonlinear stability of Minkowski space was given by Lindblad and Rodnianski [50]. Their proof relies upon the so-called weak null condition.

Neither of the above mentioned techniques generalize easily to the case of a non-flat background solution. One serious problem is that the light cones in a black hole spacetime differ by a logarithmic term from those in Minkowski space. Further, due to the presence of the horizon, and in particular the ergo region, one has different types of decay behavior in the region close to the black hole and in the asymptotic region.

Several natural problems arise in this context. The decay of scalar field on Schwarzschild and Kerr backgrounds, in particular the behavior at the horizon (Price law) is a natural starting point. Several recent papers have studied this

problem [29, 34] and for the case of a Schwarzschild background the estimates agree with the conjectured Price law behavior.

While some mathematical results on for example the decay of scalar fields on Schwarzschild and Kerr backgrounds are available, the techniques used to prove these are intimately tied to the symmetries of the background, and make heavy use of spherical harmonics expansions. Therefore these proofs do not directly generalize to spacetimes which are close in a suitable sense to Kerr. Further, one could even say that we don't have a good notion of what "close to Kerr" actually means.

Thus, the problem of stability of Kerr opens up a natural arena for the interaction of numerical and mathematical relativity. The aspects of this problem where numerical experiments may be able to provide crucial insights include the asymptotic decay behavior of the gravitational field and matter fields near the horizon. A question closely related to this, and of direct relevance for numerical work, is the asymptotic behavior of dynamical horizons [6, 5, 2, 64]. It is not unlikely that a good understanding of the asymptotic geometry of dynamical horizons near timelike infinity will play a crucial role in the global analysis of black hole spacetimes.

In the far region and intermediate region, one expects linear effects to dominate and here there is a lot of information available from systematic post-Newtonian calculations. It is of interest to compare this to the results of numerical simulations, and a great deal of work in this direction is already being carried out in the context of the BBH programme.

4. CONCLUDING REMARK

This note represents a personal view and the rather incomplete discussion here leaves out very large areas of numerical relativity and numerical geometric analysis, including Ricci flow, heat flow, higher dimensional general relativity models, including black strings and other areas which are being worked on intensely. Further, the asymptotic behavior of cosmologies in the expanding direction, which has not been discussed here, provides interesting open questions, which can be fruitfully studied using numerical techniques.

Acknowledgements. This note is loosely based on a talk given during the Newton Institute programme on Global Problems in Mathematical Relativity, fall of 2005. I am grateful to the organizers of the GMR programme for inviting me to the Newton Institute and to the Newton institute for the excellent working environment. I would also like to thank the editors of the special issue on numerical general relativity, for the invitation to contribute an article.

REFERENCES

- [1] Lars Andersson, *The global existence problem in general relativity*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 71–120.
- [2] Lars Andersson, Marc Mars, and Walter Simon, *Local existence of dynamical and trapping horizons*, Physical Review Letters **95** (2005), 111102.
- [3] Lars Andersson and Alan D. Rendall, *Quiescent cosmological singularities*, Comm. Math. Phys. **218** (2001), no. 3, 479–511.
- [4] Lars Andersson, Henk van Elst, Woei Chet Lim, and Claes Uggla, *Asymptotic silence of generic cosmological singularities*, Physical Review Letters **94** (2005), 051101.
- [5] Abhay Ashtekar and Gregory J. Galloway, *Some uniqueness results for dynamical horizons*, Adv. Theor. Math. Phys. **9** (2005), no. 1, 1–30.

- [6] Abhay Ashtekar and Badri Krishnan, *Dynamical horizons: energy, angular momentum, fluxes, and balance laws*, Phys. Rev. Lett. **89** (2002), no. 26, 261101, 4.
- [7] Abhay Ashtekar and Madhavan Varadarajan, *A striking property of the gravitational hamiltonian*, Physical Review D **50** (1994), 4944.
- [8] John D. Barrow, *Chaotic behaviour in general relativity*, Phys. Rep. **85** (1982), no. 1, 1–49.
- [9] Vladimir A. Belinskiĭ, Isaac M. Khalatnikov, and Evgeny M. Lifshitz, *Oscillatory approach to a singular point in the relativistic cosmology*, Adv. Phys. **19** (1970), 525–573.
- [10] ———, *A general solution of the Einstein equations with a time singularity*, Adv. Phys. **31** (1982), 639–667.
- [11] Michael Benedicks and Lennart Carleson, *The dynamics of the Hénon map*, Ann. of Math. (2) **133** (1991), no. 1, 73–169.
- [12] Beverly K. Berger, *Numerical approaches to spacetime singularities*, Living Rev. Relativ. **5** (2002), 2002–1, 58 pp. (electronic).
- [13] Beverly K. Berger and David Garfinkle, *Phenomenology of the Gowdy universe on $T^3 \times R$* , Physical Review D **57** (1998), 4767.
- [14] Beverly K. Berger, David Garfinkle, and Eugene Strasser, *New algorithm for Mixmaster dynamics*, Classical Quantum Gravity **14** (1997), no. 2, L29–L36.
- [15] Beverly K. Berger, James Isenberg, and Marsha Weaver, *Oscillatory approach to the singularity in vacuum spacetimes with T^2 isometry*, Phys. Rev. D (3) **64** (2001), no. 8, 084006, 20.
- [16] Beverly K. Berger and Vincent Moncrief, *Numerical investigation of cosmological singularities*, Phys. Rev. D (3) **48** (1993), no. 10, 4676–4687.
- [17] Beverly K. Berger and Vincent Moncrief, *Exact $U(1)$ symmetric cosmologies with local Mixmaster dynamics*, Physical Review D **62** (2000), 023509.
- [18] ———, *Signature for local Mixmaster dynamics in $U(1)$ symmetric cosmologies*, Physical Review D **62** (2000), 123501.
- [19] Piotr Bizoń, Tadeusz Chmaj, and Zbysław Tabor, *Dispersion and collapse of wave maps*, Nonlinearity **13** (2000), no. 4, 1411–1423.
- [20] ———, *Formation of singularities for equivariant $(2+1)$ -dimensional wave maps into the 2-sphere*, Nonlinearity **14** (2001), no. 5, 1041–1053.
- [21] D. M. Chitre, *Investigations of Vanishing of a Horizon for Bianchy Type X (the Mixmaster) Universe.*, Ph.D. Thesis (1972).
- [22] M. W. Choptuik, *“Critical” behaviour in massless scalar field collapse*, Approaches to numerical relativity (Southampton, 1991), Cambridge Univ. Press, Cambridge, 1992, pp. 202–222.
- [23] Y. Choquet-Bruhat, J. Isenberg, and V. Moncrief, *Topologically general $U(1)$ symmetric vacuum space-times with AVTD behavior*, Nuovo Cimento Soc. Ital. Fis. B **119** (2004), no. 7–9, 625–638.
- [24] Yvonne Choquet-Bruhat, *Asymptotic solutions of non linear wave equations and polarized null conditions*, Actes des Journées Mathématiques à la Mémoire de Jean Leray, Sémin. Congr., vol. 9, Soc. Math. France, Paris, 2004, pp. 125–141.
- [25] ———, *Future complete $U(1)$ symmetric Einsteinian spacetimes, the unpolarized case*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 251–298.
- [26] Yvonne Choquet-Bruhat and Vincent Moncrief, *Nonlinear stability of an expanding universe with the S^1 isometry group*, Partial differential equations and mathematical physics (Tokyo, 2001), Progr. Nonlinear Differential Equations Appl., vol. 52, Birkhäuser Boston, Boston, MA, 2003, pp. 57–71.
- [27] Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), no. 2, 267–282.
- [28] Demetrios Christodoulou and Sergiu Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, vol. 41, Princeton University Press, Princeton, NJ, 1993.
- [29] Mihalis Dafermos and Igor Rodnianski, *The red-shift effect and radiation decay on black hole spacetimes*, 2005.
- [30] T. Damour, M. Henneaux, and H. Nicolai, *Cosmological billiards*, Classical Quantum Gravity **20** (2003), no. 9, R145–R200.
- [31] D. Eardley, E. Liang, and R.K. Sachs, *Velocity-dominated singularities in irrotational dust cosmologies*, J. Math. Phys. **13** (1972), no. 1, 99–107.

- [32] Charles R. Evans and Jason S. Coleman, *Critical phenomena and self-similarity in the gravitational collapses of radiation fluid*, Phys. Rev. Lett. **72** (1994), 1782–1785.
- [33] Mitchell J. Feigenbaum, *Quantitative universality for a class of nonlinear transformations*, J. Statist. Phys. **19** (1978), no. 1, 25–52.
- [34] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau, *Decay of solutions of the wave equation in the Kerr geometry*, Comm. Math. Phys. **264** (2006), no. 2, 465–503.
- [35] Helmut Friedrich, *On the existence of n -geodesically complete or future complete solutions of Einstein’s field equations with smooth asymptotic structure*, Comm. Math. Phys. **107** (1986), no. 4, 587–609.
- [36] David Garfinkle, *Harmonic coordinate method for simulating generic singularities*, Phys. Rev. D (3) **65** (2002), no. 4, 044029, 6.
- [37] David Garfinkle, *Numerical simulations of generic singularities*, Physical Review Letters **93** (2004), 161101.
- [38] Boro Grubišić and Vincent Moncrief, *Asymptotic behavior of the $T^3 \times \mathbf{R}$ Gowdy space-times*, Phys. Rev. D (3) **47** (1993), no. 6, 2371–2382.
- [39] Carsten Gundlach, *Critical phenomena in gravitational collapse*, Living Rev. Relativ. **2** (1999), 1999–4, 58 pp. (electronic).
- [40] M. Hénon, *A two-dimensional mapping with a strange attractor*, Comm. Math. Phys. **50** (1976), no. 1, 69–77.
- [41] David Hobill, Adrian Burd, and Alan Coley (eds.), *Deterministic chaos in general relativity*, NATO Advanced Science Institutes Series B: Physics, vol. 332, New York, Plenum Press, 1994.
- [42] James Isenberg and Vincent Moncrief, *Asymptotic behaviour in polarized and half-polarized $U(1)$ symmetric vacuum spacetimes*, Classical Quantum Gravity **19** (2002), no. 21, 5361–5386.
- [43] Satyanad Kichenassamy and Alan D. Rendall, *Analytic description of singularities in Gowdy spacetimes*, Classical Quantum Gravity **15** (1998), no. 5, 1339–1355.
- [44] Sergiu Klainerman, *Weighted L^∞ and L^1 estimates for solutions to the classical wave equation in three space dimensions*, Comm. Pure Appl. Math. **37** (1984), no. 2, 269–288.
- [45] Sergiu Klainerman and Francesco Nicolò, *The evolution problem in general relativity*, Progress in Mathematical Physics, vol. 25, Birkhäuser Boston Inc., Boston, MA, 2003.
- [46] ———, *Peeling properties of asymptotically flat solutions to the Einstein vacuum equations*, Classical Quantum Gravity **20** (2003), no. 14, 3215–3257.
- [47] Tatsuhiro Koike, Takashi Hara, and Satoshi Adachi, *Critical behavior in gravitational collapse of radiation fluid: A renormalization group (linear perturbation) analysis*, Phys. Rev. Lett. **74** (1995), 5170–5173.
- [48] Imre Lakatos, *Proofs and refutations*, Cambridge University Press, Cambridge, 1976, The logic of mathematical discovery, Edited by John Worrall and Elie Zahar.
- [49] Oscar E. Lanford, III, *A computer-assisted proof of the Feigenbaum conjectures*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 427–434.
- [50] Hans Lindblad and Igor Rodnianski, *Global existence for the Einstein vacuum equations in wave coordinates*, Comm. Math. Phys. **256** (2005), no. 1, 43–110.
- [51] E. N. Lorentz, *Deterministic nonperiodic flow*, J. Atmos. Sci. **20** (1963), 130–141.
- [52] Benoit Mandelbrot, *Fractal aspects of the iteration of $z \mapsto \lambda z(1 - z)$, for complex λ, z* , Nonlinear dynamics (R. G. H. Helleman, ed.), Annals of the New York Academy of Sciences, vol. 357, 1980, pp. 249–259.
- [53] Charles W. Misner, *The mixmaster cosmological metrics*, Deterministic chaos in general relativity (Kananaskis, AB, 1993), NATO Adv. Sci. Inst. Ser. B Phys., vol. 332, Plenum, New York, 1994, pp. 317–328.
- [54] Alan D. Rendall, *Global dynamics of the mixmaster model*, Classical Quantum Gravity **14** (1997), no. 8, 2341–2356.
- [55] ———, *Theorems on existence and global dynamics for the Einstein equations*, Living Rev. Relativ. **5** (2002), 2002–6, 62 pp. (electronic). MR **MR1932418 (2003h:83016)**
- [56] Alan D. Rendall and Marsha Weaver, *Manufacture of Gowdy spacetimes with spikes*, Classical Quantum Gravity **18** (2001), no. 15, 2959–2975.
- [57] H. Ringström, *Curvature blow up in Bianchi VIII and IX vacuum spacetimes*, Classical Quantum Gravity **17** (2000), no. 4, 713–731.
- [58] Hans Ringström, *The Bianchi IX attractor*, Ann. Henri Poincaré **2** (2001), no. 3, 405–500.

- [59] ———, *Asymptotic expansions close to the singularity in Gowdy spacetimes*, Classical Quantum Gravity **21** (2004), no. 3, S305–S322, A spacetime safari: essays in honour of Vincent Moncrief.
- [60] ———, *On a wave map equation arising in general relativity*, Comm. Pure Appl. Math. **57** (2004), no. 5, 657–703.
- [61] ———, *On Gowdy vacuum spacetimes*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 2, 485–512.
- [62] Hans Ringström, *Existence of an asymptotic velocity and implications for the asymptotic behavior in the direction of the singularity in T^3 -gowdy*, Comm. Pure Appl. Math. **59** (2006), 977–1041.
- [63] Igor Rodnianski and Jacob Sterbenz, *On the formation of singularities in the critical $o(3)$ sigma-model*, 2006.
- [64] Erik Schnetter, Badri Krishnan, and Florian Beyer, *Introduction to dynamical horizons in numerical relativity*, 2006.
- [65] Michael Struwe, *Equivariant wave maps in two space dimensions*, Comm. Pure Appl. Math. **56** (2003), no. 7, 815–823, Dedicated to the memory of Jürgen K. Moser.
- [66] Terence Tao, *Geometric renormalization of large energy wave maps*, 2004, submitted, Forges les Eaux conference proceedings.
- [67] Warwick Tucker, *A rigorous ODE solver and Smale's 14th problem*, Found. Comput. Math. **2** (2002), no. 1, 53–117.
- [68] Claes Uggla, Henk van Elst, John Wainwright, and George F. R. Ellis, *The past attractor in inhomogeneous cosmology*, Phys. Rev. D **68** (2003), no. 10, 103502–22.
- [69] J. Wainwright and G. F. R. Ellis (eds.), *Dynamical systems in cosmology*, Cambridge University Press, Cambridge, 1997, Papers from the workshop held in Cape Town, June 27–July 2, 1994.

E-mail address: `larsa@math.miami.edu`

ALBERT EINSTEIN INSTITUTE, AM MÜHLENBERG 1, D-14476 POTSDAM, GERMANY AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FL 33124, USA