Active mass under pressure

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After a historical introduction to Poisson's equation in Newtonian gravity, we review its analog for static gravitational fields in Einstein's theory. The source of the potential, which we call the active mass density, comprises not only all possible sources of energy, but also the pressure term $3P/c^2$. In the Hamburg seminar on relativity in the 1950s we discussed whether this term due to Fermi pressure in different atomic nuclei could be detected in Cavendish-type experiments. Our reasoning contained an instructive mistake that we are now able to resolve. We conclude that this term should not lead to discrepancies for different materials in a Cavendish-type experiment, although it is important in the early universe and collapsing stellar cores.

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I. HISTORICAL INTRODUCTION

Active mass is a term used for mass as the source of a gravitational field. This mass can be distinguished from *passive* mass, which is a measure of the response to a gravitational field and of *inertial* mass which is the resistance against acceleration by any force, gravitational or otherwise. In Newton's and Einstein's theories these distinctions are not necessary, in agreement with available physical and astronomical evidence and post-Newtonian approximations. However, these distinctions are useful for considering non-Newtonian or non-Einsteinian theories of gravity or to assess which aspects of mass are tested in experiments.

In 1773 Lagrange[1] introduced the function V through the relation

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' . \tag{1}$$

He recognized that it is easier to first compute V and then the force by differentiation than to calculate the force directly. The function $\rho(\mathbf{x})$ denotes the density of the active mass, which is given by

$$M = \int \rho(\mathbf{x}') d^3x' . \tag{2}$$

We recognize Lagrange's function V as the negative of the gravitational potential ϕ. Lagrange did not use vector notation and did not exhibit Newton's gravitational constant G because astronomers set it equal to unity as we do in this paper. The use of the negative of the potential was customary before the principle of the conservation of energy began to dominate physics and astronomy.

Lagrange did not give a name for his function V. Gauss called it the potential in his 1839 paper.[2] He had not been aware of Green's 1828 article[3] that called V the potential function. Green had published his paper privately and rarely referred to it in his later papers. It was only several years after Green's death in 1841 that William Thomson (Lord Kelvin) discovered Green's paper and arranged for the publication of its results.

The acceleration vector $\ddot{\mathbf{x}}$ for a massive particle was given by Lagrange in an inertial system by

$$\ddot{\mathbf{x}} = \nabla V. \tag{3}$$

This relation implies that the inertial mass may be identified with the passive gravitational mass.

In 1782 Laplace[4] introduced the equation

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{4}$$

which is now known as the Laplace equation. Laplace did not express it in Cartesian coordinates, but used spherical polar coordinates. He published the equation in Cartesian coordinates in 1787.[5]

It took another 26 years until Poisson pointed out that the Laplace equation does not hold within the substance of the attracting body[6] and has to be replaced there by

$$\nabla^2 V = -4\pi\rho. \tag{5}$$

Equation (5) is now known as Poisson's equation. A proof was first given by Gauss in 1839.[2] If we write Eq. (5) in terms of the potential ϕ, we have

$$\nabla^2 \phi = 4\pi\rho, \tag{6}$$

which is the relation in Newton's theory between the gravitational potential and the density of the active mass in an inertial system. If ϕ is required to vanish at infinity, then Eq. (6) implies Eq. (1).

II. THE POISSON EQUATION FOR STATIC FIELDS IN EINSTEIN'S THEORY OF GRAVITATION

In general relativity[7] the ten components $g_{\mu\nu}$ of the metric tensor

$$ds^2 = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu \tag{7}$$

replace the potential of Lagrange. Moreover, not just the mass-density, but all ten components of the energy-momentum-stress tensor $T_{\mu\nu}$ become contributing sources to the gravitational field. For neutral matter $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu + P_{\mu\nu}, \tag{8}$$

with the energy density ρ, pressure tensor $P_{\mu\nu}$, and four-velocity u^μ subject to the normalization

$$u^\mu u_\mu = 1, \tag{9}$$

and

$$P_{\mu\nu}u^\nu = 0. \tag{10}$$

For a perfect fluid

$$P_{\mu\nu} = P(u_\mu u_\nu - g_{\mu\nu}). \tag{11}$$

Instead of Poisson's equation relating the gravitational potential to the active mass density ρ, we have the Einstein field equations,

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = -8\pi T_{\mu\nu}, \tag{12}$$

connecting $g_{\mu\nu}$ and its derivatives up to second order with the components of $T_{\mu\nu}$. The Riemann tensor is defined through the interchange of the order in the second covariant derivatives of a covariant vector field ξ_μ,

$$\xi_{\mu;\alpha;\nu} - \xi_{\mu;\nu;\alpha} = -\xi_\gamma R^\gamma_{\mu\alpha\nu}. \tag{13}$$

The covariant derivative of ξ_μ with respect to the coordinate x^α is denoted by

$$\xi_{\mu;\alpha} = \frac{\partial \xi_\mu}{\partial x^\alpha} - \xi_\beta \Gamma^\beta_{\mu\alpha}. \tag{14}$$

The Christoffel symbols $\Gamma^\beta_{\mu\alpha}$ are defined by

$$\Gamma^\beta_{\mu\alpha} = \frac{1}{2}g^{\beta\gamma}\left(\frac{\partial g_{\mu\gamma}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\gamma}\right). \tag{15}$$

The Ricci tensor is obtained by contraction from the Riemann tensor $R^\gamma_{\mu\alpha\nu}$,

$$R_{\mu\nu} = R^\gamma_{\mu\gamma\nu}. \tag{16}$$

The Ricci scalar R is found by contracting the Ricci tensor R^μ_ν. We use units such that the speed of light $c=1$.

For a static gravitational field in which matter is at rest we can find an analog of Lagrange's potential and its Poisson equation. We give a derivation in Appendix A. In this derivation we restrict ourselves to discussing this relation for a static gravitational field in Einstein's theory. As we show in Appendix A such a static gravitational field can be described by the metric

$$ds^2 = g_{00}(x^l)(dx^0)^2 + g_{jk}(x^l)dx^j dx^k \tag{17}$$

with no g_{0i} terms. Latin indices run from 1 to 3. The components of the metric tensor do not depend on the time coordinate $t=x^0$, and the four-velocity u^μ of the matter is given by

$$u^\mu = \frac{1}{\sqrt{g_{00}}}\delta^\mu_0. \tag{18}$$

We have used the Kronecker δ-symbol $\delta^\mu_\nu=1$ for $\mu=\nu$ and $\delta^\mu_\nu=0$ for $\mu\neq\nu$. The three-dimensional proper volume element is given by

$$\sqrt{-\hat{g}}d^3x \tag{19}$$

with $\hat{g}=\det\|g_{jk}\|$.

In a perfect fluid the static potential $\sqrt{g_{00}}$ is given by

$$-\frac{1}{\sqrt{-\hat{g}}}(\sqrt{-\hat{g}}g^{jk}(\sqrt{g_{00}})_{,k})_{,j} \equiv \nabla^2\sqrt{g_{00}} = 4\pi\sqrt{g_{00}}\left(\rho + 3P - \frac{\lambda}{4\pi}\right). \tag{20}$$

The spatial metric differs from the flat metric only by terms of order ϕ. On the left-hand side of the relativistic Poisson equation, Eq. (20), the Laplacian operates on the spaces where $t=$constant, where it is applied to the function $\sqrt{g_{00}}$, which we identify as the static relativistic potential. This expression for the left-hand side of Eq. (20) is the Laplace operator in Riemannian geometry. It was derived by Beltrami in 1868 and is now known as the second Beltrami parameter.[8]

The cosmological λ term in Eq. (12) can be written on the right-hand side of this equation as the tensor $T_{\mu\nu}(\lambda)$,

$$-\lambda g_{\mu\nu} \equiv -8\pi T_{\mu\nu}(\lambda). \tag{21}$$

We then interpret $T_{\mu\nu}(\lambda)$ as the energy-momentum tensor of a perfect fluid with energy density $\rho(\lambda)$ and pressure $P(\lambda)$,

$$T_{\mu\nu}(\lambda) = [\rho(\lambda) + P(\lambda)]u_\mu u_\nu - P(\lambda)g_{\mu\nu}, \tag{22}$$

and

$$\rho(\lambda) = -P(\lambda), \quad P(\lambda) = -\frac{\lambda}{8\pi}. \tag{23}$$

The contribution to the active mass density ("dark energy") becomes

$$\rho(\lambda) + 3P(\lambda) = -\frac{\lambda}{4\pi}. \tag{24}$$

Poisson's $4\pi\rho$ term corresponds to the right-hand side of Eq. (20). Apart from Einstein's lambda term that acts—if it is positive—as a negative mass density contributing to the active mass, there are two differences in comparison with Poisson's equation. One is the factor $\sqrt{g_{00}}$ on the right-hand side of the Eq. (20). Because of energy conservation, the potential is defined as the work needed to displace a unit mass (charge) from infinity and is usually normalized to vanish at spatial infinity. In Einstein's theory of gravitation the component g_{00} for a finite mass distribution in a static gravitational field is usually taken as c^2 at infinity. The reason is that we can then include the rest energy of a particle as part of the potential energy and assume that the metric of an isolated system becomes Euclidean at large distances. At large distances r from the center of mass we have

$$\sqrt{g_{00}(r)} \approx 1 - \frac{M}{r}, \tag{25}$$

where M is the total active mass. Thus, for weak fields the factor $\sqrt{g_{00}}$ in the relativistic Poisson equation differs only slightly from unity.

The justification for identifying $\sqrt{g_{00}}$ with the relativistic global potential rests on its definition as the specific potential energy, that is, the potential energy per unit mass. We shall show in Appendix A that a test particle of unit mass at rest at **x** has potential energy (apart from its rest energy)

$$\phi = \sqrt{g_{00}} - 1. \tag{26}$$

The Poisson equation in Newtonian gravity refers to a global inertial frame. The frame used in Eq. (20) was derived invariantly, that is, independent of coordinate transformations. It is the closest analog to such a system in Newtonian theory. This analogy, which identifies ϕ with $\sqrt{g_{00}}-1$, is not perfect because the acceleration of a particle in the field of the potential is no longer exactly given by its gradient. In general relativity we instead have for the acceleration of a particle at rest in a static gravitational field

$$\dot{u}_k = -\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial x^k}\sqrt{g_{00}} = -\frac{\phi_{,k}}{(1+\phi)}. \tag{27}$$

III. THE PRESSURE CONTRIBUTION TO THE ACTIVE MASS DENSITY

The most surprising correction to the relativistic Poisson equation is the $3P/c^2$ term. This term was first noted explicitly as a consequence of Einstein's field equations by Levi-Civita in 1917.[9]

In the statistical mechanics of an ideal gas of particles with mass m and momentum **p**, velocity **v**, and constant number density n, the pressure P is given by Bernoulli's formula,[10]

$$P = \tfrac{1}{3} n \overline{\mathbf{p}\cdot\mathbf{v}}, \tag{28}$$

where the bar indicates an average over the momentum distribution. In Eq. (28) P is the kinetic contribution to the pressure. In general, P or P_{jk} contains contributions from short-range interactions, which must be added to Eq. (28). Attractive interactions contribute negative contributions to P. Equation (28) holds also for a relativistic ideal gas.[11] In the high energy limit of photons with energy ϵ, Eq. (28) becomes $\epsilon = \mathbf{p}\cdot\mathbf{v}$ and

$$P = \tfrac{1}{3} n \bar{\epsilon} = \tfrac{1}{3}\rho. \tag{29}$$

IV. THE HAMBURG PARADOX

We were members of a seminar on relativity that regularly met at Hamburg University in the 1950s. When we learned about the $3P$ term in the Poisson equation, the following test was suggested: Because nucleons move in atomic nuclei with about two-tenths of the speed of light, the $3P$ term might significantly contribute to the active mass density of all nuclei except the proton and the neutron. A simple calculation for the pressure in an ideal Fermi gas of nucleons at zero temperature (see Appendix B) gives a pressure contribution to the active mass density of 4.3% for nuclear matter. Therefore, a ball of hydrogen should have an active mass that is about 4% smaller then a ball of lead of the same inertial and passive gravitational mass. This difference could be checked by weighing them on a scale.

The only way such an effect might be seen in the laboratory is by the Cavendish experiment for the determination of the gravitational constant where the active mass comes into play. Although it would be forbidding to work with a ball of ultra-cold solid hydrogen, we might consider a material with a high hydrogen content that is solid at room temperature such as polyethylene (CH_2) or lithium hydride (LiH). Lithium consists of 92.5% of the isotope $^7_3$Li. The rest is $^6_3$Li. Thus, lithium hydride has a molecular weight of essentially 8 and the hydrogen contribution is 1/8 by mass.

For lithium hydride the active mass should be less by $0.043/8 = 5\times10^{-3}$ and for polyethylene $2\times0.043/14 = 6\times10^{-3}$ compared with the active mass of matter containing no hydrogen. If we were to use balls of materials containing hydrogen to determine the gravitational constant through the Cavendish experiment, we should obtain lower values for the gravitational constant G. In the 1950s the value of G was uncertain by about 0.1%[12] and so the effect might just be measurable. In 1958, one of us (ELS) spoke to Robert Dicke of Princeton University about a possible experimental test. Dicke had his doubts about whether it could be done because machining homogeneous spheres in these materials might be forbidding. However, a year later it became clear that the value for G should be the same for balls of hydrogen. Dieter Brill, who had joined the Hamburg seminar, alerted us to a paper by Misner and Putnam[13] about active mass. (See John Wheeler's Memoir[14] to learn about Peter Putnam who died in 1987.)

Misner and Putnam showed, assuming gravity to be negligible, that the $3P$ term for a gas in a container is canceled by negative contributions to the mass from the stresses in the walls of the container that kept the gas together. It had not been clear to us that negative surface contributions to the energy would exactly cancel the positive $3P$-volume contribution to the total mass for the model of a bubble that we had in mind.

V. THE SPHERICAL BUBBLE

We imagine a gas of constant energy density ρ and pressure P enclosed in a spherical shell of radius r with surface mass density τ and surface tension σ. The surface tension should be just strong enough to keep the bubble in equilib-

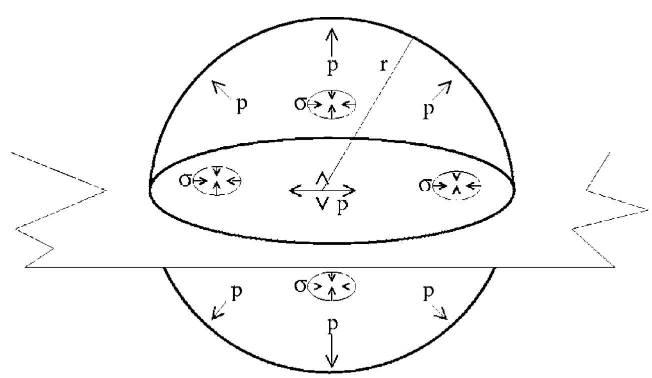

Fig. 1. A spherical bubble of radius r is filled with a gas of pressure P. The bubble is kept in equilibrium by a surface tension σ with dimensions of force times length.

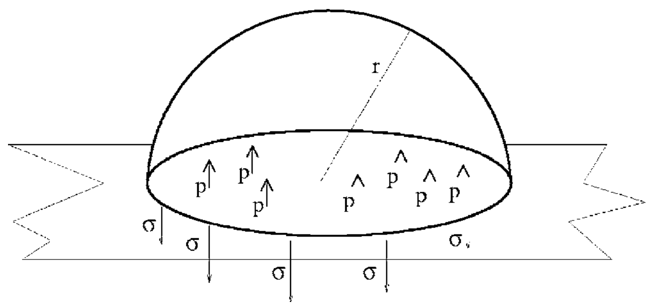

Fig. 2. The lower hemisphere of the bubble (Fig. 1) is removed and replaced by its forces on the upper hemisphere.

rium. The gravitational binding energy of the bubble is assumed to be negligible compared to its mass. To find the relation between the surface tension and the pressure at equilibrium, we imagine a plane cut through the bubble that removes the southern hemisphere. To keep the northern hemisphere in equilibrium, we have to balance the upward pressure over the equatorial disc of area πr^2 against the surface tension pulling down along the equator over the length $2\pi r$. This balance gives the relation

$$P\pi r^2 = \sigma 2\pi r, \tag{30a}$$

$$\sigma = \tfrac{1}{2}Pr. \tag{30b}$$

The total active mass M is equal to the sum of the surface contribution $4\pi r^2(\tau-\sigma)$ and the volume contribution $4\pi r^3(\rho+3P)/3$:

$$M = 4\pi r^2(\tau-\sigma) + \frac{4\pi}{3}r^3(\rho+3P) = 4\pi r^2\tau + \frac{4\pi}{3}r^3\rho + 2\pi r^3 P, \tag{31}$$

leaving us with half the volume contribution of the $3P$-term to the active mass. Something is incorrect. It was only last summer when we discussed this problem again in a nostalgic moment that we saw the solution.

If the active mass density of a three-dimensional distribution needs to be complemented by a $3P$ term, then a two-dimensional shell needs a $2P$ term and a one-dimensional disk a P term (corresponding to the trace of a two- and one-dimensional isotropic stress tensor). If these terms were stresses instead of pressures, they would enter with a negative sign.

Now all was clear: the surface contribution to the active mass of the bubble is $4\pi r^2(\tau-2\sigma)$ and we obtain instead of Eq. (31) the result

$$M = 4\pi r^2(\tau-2\sigma) + \frac{4\pi}{3}r^3(\rho+3P) = 4\pi r^2\tau + \frac{4\pi}{3}r^3\rho. \tag{32}$$

This simple remark also settles the result for the active mass of a circular disk.

VI. THE ACTIVE MASS OF A CIRCULAR DISK

We consider a circular disk of radius r with mass density τ and pressure P. The disk is kept in equilibrium by a one-dimensional string around its circumference of linear mass density μ and stress λ. To find the relation between the pres-

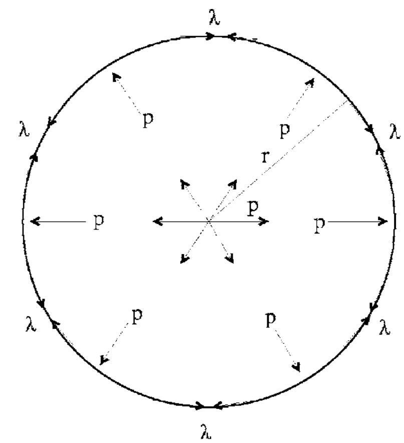

Fig. 3. Circular disk of radius r carries a surface pressure P. The pressure is balanced by the tension λ along its perimeter.

sure P and the stress λ, we imagine a linear cut through the center of the disk that removes the lower half. The pressure over the diameter $2r$ must now be balanced by the stress λ at the left and the right end of the semicircle. This consideration gives

$$2rP = 2\lambda, \tag{33a}$$

$$\lambda = rP. \tag{33b}$$

The active mass M of the disk is obtained by taking the active mass density $\tau+2P$ over the area πr^2 and adding the active mass density $\mu-\lambda$ of the bounding string along the circumference $2\pi r$. We obtain

$$M = (\tau+2P)\pi r^2 + (\mu-\lambda)2\pi r = \tau\pi r^2 + \mu 2\pi r, \tag{34}$$

the promised result: The total mass M of the disc is equal to the sum of the mass of the interior and the mass of the circumference. The pressure and stress contributions of interior and circumference cancel each other.

VII. CONCLUSION

If we extend Newtonian theory by adding $3P/c^2$ to the density of the active mass, we should not expect to see changes in the total active mass of atomic nuclei. The bubble model suggests that the integrated contribution of the pressure term is compensated by the negative energy contribution of the membrane if we also postulate a $2P/c^2$ term for the active mass of the membrane. We have also extended Newton's theory by assuming the equivalence of energy and mass. In a model that goes beyond the Misner-Putnam

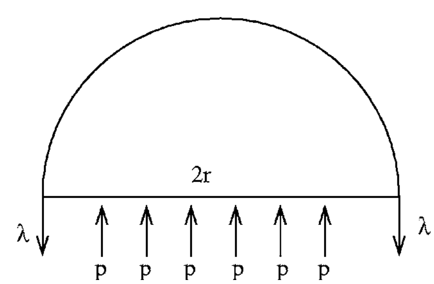

Fig. 4. The lower half of the disk (Fig. 3) has been removed and replaced by the forces acting on the upper part.

calculation[13] and takes gravitational binding energy and Einstein's field equation into account, we also find complete compensation of the pressure term.[15] However, the compensation of the pressure term pertains to the static case. If we wish to find the effect of the $3P$ term, we have to look at non-equilibrium situations in the early universe or in the late stages of a type II supernova core.[15]

Because of Newton's third law, the active and passive masses are equal in Newtonian theory. If this equality did not hold, the center of mass of a system of two passive unit masses but different active masses would accelerate, thus violating Newton's third law. In Einstein's theory of gravitation the equality of active and passive mass is not obvious because Einstein's cosmological constant can give rise to self-acceleration for the center of mass of a double star. Although we have no reason to doubt the equality of the three kinds of masses, tests involving the active mass would be desirable. Such tests are especially called for situations where the gravitational binding energy significantly contributes to the mass.

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APPENDIX A: THE ANALOG OF THE POISSON EQUATIONS IN EINSTEIN'S GRAVITATION

In the following we give an invariant definition of the Poisson equation for static gravitational fields in general relativity. We take the gravitational potential of Newton's theory as the analog of the electrostatic potential in Maxwell's theory. In its relativistic version it appears as the zero component of the 4-vector potential. As the analog of the 4-vector potential for gravitation in Einstein's theory, we take a time-like Killing vector field. Its existence assumes that we are dealing with a stationary metric.

Killing's equation. We demonstrate the analogy to the equations of Maxwell's electromagnetism. A stationary gravitational field is characterized by the existence of a time-like Killing field that generates an infinitesimal transformation which leaves the metric unchanged. The field is called static if the Killing vector field is hypersurface-orthogonal, which means that the covariant vector is a product of a gradient by a scalar function. A Killing vector field ξ_μ satisfies

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{A1}$$

If we write Eq. (13) three times with a cyclic permutation of the indices

$$\xi_{\mu;\alpha;\nu} - \xi_{\mu;\nu;\alpha} = -\xi_\gamma R^\gamma_{\mu\alpha\nu}, \tag{A2a}$$

$$\xi_{\alpha;\nu;\mu} - \xi_{\alpha;\mu;\nu} = -\xi_\gamma R^\gamma_{\alpha\nu\mu}, \tag{A2b}$$

$$\xi_{\nu;\mu;\alpha} - \xi_{\nu\alpha;\mu} = -\xi_\gamma R^\gamma_{\nu\mu\alpha}, \tag{A2c}$$

add Eqs. (A2a) and (A2b) and subtract Eq. (A2c), we obtain that

$$2\xi_{\mu;\alpha;\nu} = 2\xi_\gamma R^\gamma_{\nu\mu\alpha}, \tag{A3}$$

where we also have used the cyclic symmetry of the Riemann tensor, $R^\gamma_{\mu\alpha\nu}+R^\gamma_{\alpha\nu\mu}+R^\gamma_{\nu\mu\alpha}=0$. Equation (A3) is known as the integrability condition for the Killing field. If we define

$$\xi_{\mu;\alpha} - \xi_{\alpha;\mu} = 2\xi_{\mu;\alpha} \equiv F_{\mu\alpha}, \quad 2\xi_\gamma R^\gamma_\mu \equiv j_\mu, \tag{A4}$$

we obtain from Eq. (A3) by anti-symmetrization and contraction Maxwell's equations for the field tensor $F_{\mu\nu}$ and four-current j_μ,

$$F_{\mu\alpha;\nu} + F_{\alpha\nu;\mu} + F_{\nu\mu;\alpha} = 0, \quad F^{\mu\nu}_{\ ;\nu} = j^\mu. \tag{A5}$$

Because $\xi_{\mu;\alpha}-\xi_{\alpha;\mu}=\xi_{\mu,\alpha}-\xi_{\alpha,\mu}=F_{\mu\alpha}$, the Killing vector for a stationary gravitational field plays the role of an electromagnetic four-potential.

Adapted coordinates. The next step makes the time-independence evident by the choice of adapted coordinates. To get from the stationary setting to the static one where space and time are neatly separated, we have to do the analog of excluding magnetic fields in electrodynamics. This analog appears here as the condition that the covariant Killing vector field is the multiple of the four-dimensional gradient of a scalar field. The hypersurfaces on which this field is constant are orthogonal to the time-like Killing vector field and define the time coordinate x^0. We choose coordinates so that

$$\xi^0 = 1, \quad \xi^j = 0 \quad (j = 1,2,3), \tag{A6}$$

which can be done for any contravariant vector field in a finite region. This special choice of coordinates still allows a gauge transformation with an arbitrary function $\chi(x^k)$:

$$\bar{x}^0 = x^0 + \chi(x^k), \quad \bar{x}^j = \bar{x}^j(x^k). \tag{A7}$$

The Killing equation (A1) can be written as

$$g_{\mu\nu,\lambda}\xi^\lambda + g_{\lambda\nu}\xi^\lambda_{,\mu} + g_{\mu\lambda}\xi^\lambda_{,\nu} = 0. \tag{A8}$$

This equation gives with the normalization in Eq. (A6),

$$g_{\mu\nu,0} = 0, \tag{A9}$$

which is independent of the coordinate x^0. Because we required that the Killing vector ξ^μ be time-like, we have that x^0 is a distinguished time coordinate with $\xi_\mu=g_{0\mu}$ and $\xi^\mu\xi_\mu=g_{00}>0$. Its distinction is that the metric tensor $g_{\mu\nu}$ is independent of time.

We now use the condition that the Killing vector is hypersurface-orthogonal (static field),

$$\xi_\mu(\xi_{\nu,\lambda} - \xi_{\lambda,\nu}) + \xi_\nu(\xi_{\lambda,\mu} - \xi_{\mu,\lambda}) + \xi_\lambda(\xi_{\mu,\nu} - \xi_{\nu,\mu}) = 0. \tag{A10}$$

After contraction with ξ^μ we have

$$g_{00}(g_{0\nu,\lambda} - g_{0\lambda,\nu}) - g_{0\nu}g_{00,\lambda} + g_{0\lambda}g_{00,\nu} = 0. \tag{A11}$$

This equation gives after division by $(g_{00})^2$,

$$(g_{0\nu}/g_{00})_{,\lambda} - (g_{0\lambda}/g_{00})_{,\nu} = 0. \tag{A12}$$

Therefore, we have that $g_{0\nu}/g_{00}$ is a gradient of a scalar function ψ:

$$g_{0\nu} = g_{00}\psi_{,\nu}, \quad \psi_{,0} = 1. \tag{A13}$$

We can write

$$\psi = x^0 + \chi(x^k). \qquad (A14)$$

A comparison of Eq. (A14) with Eq. (A7) shows that we can choose the gauge transformation χ such that $\psi=\bar{x}^0$. By dropping the bar on $\bar{x}^0$, we then obtain the form for the metric given in Eq. (17):

$$ds^2 = g_{00}(x^l)(dx^0)^2 + g_{jk}(x^l)dx^j dx^k. \qquad (A15)$$

The purely spatial coordinate transformations are still free. The time-like hypersurface-orthogonal Killing vector is unique up to a constant factor. The square of this factor multiplies g_{00}.

Poisson equation. We now define the Laplace operator for a scalar function that is the square root of the length of the static time-like Killing vector $\sqrt{g_{00}}$. A state of rest is then described by a time-like unit vector

$$u^\mu = \frac{1}{\sqrt{g_{00}}}\xi^\mu, \quad u^\mu u_\mu = 1. \qquad (A16)$$

We study the second set of Maxwell's equations (A5) in the above coordinates. We have because of time independence

$$F^{\mu\nu}_{;\nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu k})_{,k}. \qquad (A17)$$

From Eq. (A4) we have $F_{\mu\nu}=(g_{00}\delta^0_\mu)_{,\nu}-(g_{00}\delta^0_\nu)_{,\mu}$, and the only non-vanishing covariant components are

$$F_{0j} = g_{00,j} = -F_{j0}. \qquad (A18)$$

If we substitute this result into Eq. (A17), we find

$$F^{\mu\nu}_{;\nu} = \delta^\mu_0 \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g_{00,j}\frac{1}{g_{00}}g^{jk}\right)_{,k}, \qquad (A19)$$

with

$$\sqrt{-g} = \sqrt{g_{00}}\sqrt{-\hat{g}}, \quad \hat{g} = \det\|g_{jk}\|. \qquad (A20)$$

We obtain

$$F^{\mu\nu}_{;\nu} = \delta^\mu_0 \frac{2}{\sqrt{-g}}(\sqrt{-\hat{g}}g^{jk}(\sqrt{g_{00}})_{,j})_{,k} = -\frac{2}{\sqrt{g_{00}}}\delta^\mu_0 \nabla^2\sqrt{g_{00}}. \qquad (A21)$$

Here ∇^2 is the Laplace operator as in Eq. (20) for a positive definite metric. The minus sign on the right-hand side of Eq. (A21) occurs because we use the Lorentz signature $+---$ of the four-dimensional metric.

The Laplace operator is applied to the scalar potential function $\sqrt{g_{00}}$ which appears here as the zero component of the four-vector $g_{0\mu}/\sqrt{g_{00}}$ whose 3-vector part vanishes. In special relativity the Poisson equation of electrostatics is obtained similarly for a four-potential A_μ whose space components A_j vanish.

The four-current j_μ from Eq. (A4) becomes with the Einstein field equations (12)

$$j_\mu = 2\xi_\gamma R^\gamma_\mu = 2\left(-\kappa T^\gamma_\mu + \tfrac{1}{2}\delta^\gamma_\mu \kappa T + \lambda\delta^\gamma_\mu\right)\xi_\gamma, \qquad (A22)$$

or

$$j^\mu = 2\left(-\kappa T^\mu_0 + \tfrac{1}{2}\delta^\mu_0(\kappa T + 2\lambda)\right). \qquad (A23)$$

The relativistic gravitational constant κ is given by $8\pi G/c^2$ in terms of of Newton's gravitational constant G and the vacuum speed of light. In the units used here it is simply 8π. Here T is the trace (contraction) of the tensor T^μ_ν. The second set of Maxwell's equations states that

$$T^j_0 = 0, \qquad (A24)$$

which implies that the density of momentum and energy flux vanish in this static situation. If we call the trace of the pressure tensor $3P$,

$$T = \rho - 3P, \quad \rho \equiv T^0_0, \quad 3P \equiv -T^a_a, \qquad (A25)$$

we have

$$j^0 = -\kappa(\rho + 3P) + 2\lambda. \qquad (A26)$$

If we use Eqs. (A5) and (A21), Eq. (20) gives the relativistic Poisson equation for a static gravitational field:

$$\nabla^2\sqrt{g_{00}} = \sqrt{g_{00}}(4\pi(\rho + 3P) - \lambda). \qquad (A27)$$

The energy of a particle in a static gravitational field. To justify the identification of $\sqrt{g_{00}}$ with the gravitational potential, we show that it agrees with the definition of the specific energy necessary to move a mass to infinity. A particle of constant mass m and four-velocity v^μ moving on a geodesic in a static gravitational field obeys

$$v^\mu v_\mu = 1, \quad \dot{v}^\mu \equiv v^\mu_{;\nu}v^\nu = 0. \qquad (A28)$$

The energy integral for unit mass is given by

$$E = v^\mu \xi_\mu, \qquad (A29)$$

because the derivative $\dot{E}$ of E along the four-velocity v^μ vanishes:

$$\dot{E} = (mv^\mu\xi_\mu)^{\cdot} = mv^\mu_{;\nu}v^\nu\xi_\mu + m\xi_{\mu;\nu}v^\mu v^\nu = 0. \qquad (A30)$$

The first term on the right-hand side of Eq. (A30) vanishes because of the geodesic equation (A28), and the second term is zero due to the Killing equation (A1).

In terms of the components of the local rest frame we have

$$E = \frac{m}{\sqrt{1-\beta^2}}\sqrt{g_{00}}, \qquad (A31)$$

where β is the local velocity in terms of the speed of light.

Acceleration of a particle at rest. We demonstrate here that for weak fields the accelerations are obtained from the negative gradient of the potential bringing us back to Eq. (3) of Lagrange.

A particle at rest is characterized by its four-velocity

$$u_0 = \sqrt{g_{00}}, \quad u_j = 0, \quad j = 1,2,3. \qquad (A32)$$

The acceleration of the particle is given by

$$\dot{u}_\mu \equiv u_{\mu;\nu}u^\nu = u_{\mu,\nu}u^\nu - \Gamma^\lambda_{\mu\nu}u_\lambda u^\nu. \qquad (A33)$$

Because g_{00} is independent of time, the first term on the right-hand side of Eq. (A33) vanishes. We then obtain using Eq. (A32)

$$\dot{u}_\mu = -\Gamma^\lambda_{\mu\nu}u_\lambda u^\nu = -\frac{1}{g_{00}}\Gamma^0_{\mu 0} = -\frac{1}{2g_{00}}g_{00,\mu}, \qquad (A34)$$

or

$$\dot{u}_k = -\frac{1}{2g_{00}}g_{00,k} = -\frac{(\sqrt{g_{00}})_{,k}}{\sqrt{g_{00}}}. \qquad (A35)$$

APPENDIX B: THE ATOMIC NUCLEUS AS A DEGENERATE FERMI GAS

We wish to calculate the pressure inside atomic nuclei. The simplest model assumes that the nucleons move freely in a potential well subject to the Pauli exclusion principle. For the ground state of the nucleus only the lowest energy levels are occupied.

The pressure P and the ratio of $3P$ to the energy density ρc^2 for an ideal Fermi gas at zero temperature was derived by Fermi.[16] He obtained

$$\frac{3P}{\rho c^2} = \frac{3}{5}\left(\frac{h}{mc}\right)^2\left(\frac{3n}{4\pi g}\right)^{2/3}, \tag{B1a}$$

or in terms of $\hbar$,

$$\frac{3P}{\rho c^2} = \frac{3}{5}\left(\frac{\hbar}{mc}\right)^2\left(\frac{6\pi^2 n}{g}\right)^{2/3}. \tag{B1b}$$

Here m is the mass of the fermions, n their nuclear density, and g the statistical weight. The generalization to arbitrary g is due to Pauli.[17]

Heisenberg considered an atomic nucleus as a Fermi gas of nucleons.[18] The interactions among nucleons that keep them confined are taken into account by a potential well in which they move freely. For a rough estimate we disregard the mass difference between protons and neutrons and take the statistical weight g to be $g=4$. For a nucleus with N nucleons the density of nuclear matter is

$$n = \frac{N}{N(4\pi/3)r_0^3} = \frac{3}{4\pi r_0^3}. \tag{B2}$$

The constant r_0 is given by[19] $r_0=1.2$ fm. We have from Eq. (B1b)

$$\frac{3P}{\rho c^2} = \frac{3}{5}\left(\frac{\hbar}{mcr_0}\right)^2\left(\frac{9\pi}{8}\right)^{2/3}, \tag{B3}$$

which gives

$$\frac{3P}{\rho c^2} = 1.39\left(\frac{\hbar}{mcr_0}\right)^2 = 0.043. \tag{B4}$$

With the known mass density of nuclear matter we found the relative contribution of the $3P/c^2$ term to the active mass. In this model it is the same for all nuclei except those with only one nucleon.

APPENDIX C: SUGGESTED PROBLEMS

Problem 1. For the three-dimensional metric

$$ds^2 = g_{jk}dx^j dx^k = A^2(x,y,z)dx^2 + B^2(x,y,z)dy^2 + C^2(x,y,z)dz^2 \tag{C1}$$

calculate the Laplace operator for a scalar function $f(x,y,z)$,

$$\nabla^2 f = \frac{1}{\sqrt{g}}[\sqrt{g}g^{jk}f_{,k}]_{,j}, \quad g = \det g_{jk} \tag{C2}$$

and specialize to the case of spherical polar coordinates and cylindrical coordinates of an Euclidean space.

Problem 2. Show that the variation of the three-dimensional integral I extended over the volume V,

$$I = \int\int\int g^{jk}(x^l)f_{,j}f_{,k}\sqrt{g}d^3x, \tag{C3}$$

for a function f fixed at the boundary of the volume V gives

$$\delta I = -2\int\int\int \nabla^2 f\delta f\sqrt{g}d^3x. \tag{C4}$$

Useful problems in relativity with solutions can be found in Ref. 20.

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