



## Renormalization of Tensorial Group Field Theories

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30/10/2012

International Loop Quantum Gravity Seminar

Joint work with Daniele Oriti and Vincent Rivasseau: [arXiv:1207.6734](https://arxiv.org/abs/1207.6734) [hep-th] and more.

TGFTs are an approach to quantum gravity, which can be justified by two complementary logical paths:

- **The Tensor track** [Rivasseau '12]: matrix models, tensor models [Sasakura '91, Ambjorn et al. '91, Gross '92],  $1/N$  expansion [Gurau, Rivasseau '10 '11], universality [Gurau '12], renormalization of tensor *field* theories... [Ben Geloun, Rivasseau '11 '12]
- **The Group Field Theory approach to Spin Foams** [Rovelli, Reisenberger '00, ...]
  - Quantization of simplicial geometry.
  - No triangulation independence  $\Rightarrow$  lattice gauge theory limit [Dittrich et al.] or sum over foams.
  - GFT provides a prescription for performing the sum: simplicial gravity path integral = Feynman amplitude of a QFT.
  - Amplitudes are generically divergent  $\Rightarrow$  renormalization?
  - Need for a **continuum limit**  $\Rightarrow$  many degrees of freedom  $\Rightarrow$  renormalization (phase transition along the renormalization group flow?)

## Big question

Can we find a renormalizable TGFT exhibiting a phase transition from discrete geometries to the continuum, and recover GR in the classical limit?

- State of the art: several renormalizable TGFTs with nice topological content:
  - $U(1)$  model in 4d: just renormalizable up to  $\varphi^6$  interactions, asymptotically free [Ben Geloun, Rivasseau '11, Ben Geloun '12]
  - $U(1)$  model in 3d: just renormalizable up to  $\varphi^4$  interactions, asymptotically free [Ben Geloun, Samary '12]
  - even more renormalizable models [Ben Geloun, Livine '12]
- Question: what happens if we start adding geometrical data (discrete connection)?

## Main message of this talk

Introducing holonomy degrees of freedom is possible, and generically improves renormalizability. It implies a generalization of key QFT notions, including: **connectedness**, **locality** and **contraction** of (high) subgraphs.

Example I:  $U(1)$  super-renormalizable models in  $4d$ , for any order of interaction.

Example II: a just-renormalizable Boulatov-type model for  $SU(2)$  in  $d = 3!$

- 1 A class of dynamical models with gauge symmetry
- 2 Multi-scale analysis
- 3  $U(1)$  4d models
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- Dynamical variable: rank- $d$  complex field

$$\varphi : (\mathbf{g}_1, \dots, \mathbf{g}_d) \ni G^d \mapsto \mathbb{C},$$

with  $G$  a (compact) Lie group.

- Partition function:

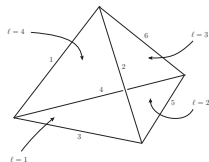
$$\mathcal{Z} = \int d\mu_C(\varphi, \bar{\varphi}) e^{-S(\varphi, \bar{\varphi})}.$$

- $S(\varphi, \bar{\varphi})$  is the interaction part of the action, and should be a sum of **local** terms.
- Dynamics + geometrical constraints contained in the **Gaussian measure**  $d\mu_C$  with covariance  $C$  (i.e. 2nd moment):

$$\int d\mu_C(\varphi, \bar{\varphi}) \varphi(\mathbf{g}_\ell) \bar{\varphi}(\mathbf{g}'_\ell) = C(\mathbf{g}_\ell; \mathbf{g}'_\ell)$$

- Natural assumption in  $d$  dimensional Spin Foams: elementary building block of space-time =  $(d + 1)$ -simplex.  
In GFT, translates into a  $\varphi^{d+1}$  interaction, e.g. in 3d:

$$S(\varphi, \bar{\varphi}) \propto \int [dg]^6 \varphi(g_1, g_2, g_3) \varphi(g_3, g_5, g_4) \varphi(g_5, g_2, g_6) \varphi(g_4, g_6, g_1) + \text{c.c.}$$



Problems:

- Full topology of the simplicial complex not encoded in the 2-complex [Bonzom, Girelli, Oriti ']; [Bonzom, Smerlak '12];
- (Very) degenerate topologies.

- A way out: add **colors** [Gurau '09]

$$S(\varphi, \bar{\varphi}) \propto \int [dg]^6 \varphi_1(g_1, g_2, g_3) \varphi_2(g_3, g_5, g_4) \varphi_3(g_5, g_2, g_6) \varphi_4(g_4, g_6, g_1) + \text{c.c.}$$

... then **uncolor** [Gurau '11; Bonzom, Gurau, Rivasseau '12] i.e.  $d$  auxiliary fields and 1 true dynamical field  $\Rightarrow$  infinite set of **tensor invariant effective interactions**.

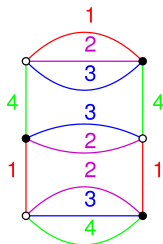
- Instead, **start** from **tensor invariant interactions**. They provide:
  - a good combinatorial control over topologies: full homology, pseudo-manifolds only etc.
  - analytical tools:  $1/N$  expansion, universality theorems etc.
- $S$  is a (finite) sum of **connected** tensor invariants, indexed by  **$d$ -colored graphs** ( $d$ -bubbles):

$$S(\varphi, \bar{\varphi}) = \sum_{b \in \mathcal{B}} t_b I_b(\varphi, \bar{\varphi}).$$

- $d$ -colored graphs are regular (valency  $d$ ), bipartite, edge-colored graphs.
- Correspondence with tensor invariants:
  - white (resp. black) dot  $\leftrightarrow$  field (resp. complex conjugate field);
  - edge of color  $l \leftrightarrow$  convolution of  $l$ -th indices of  $\varphi$  and  $\bar{\varphi}$ .

$$\int [dg_i]^{12} \varphi(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4) \bar{\varphi}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_5) \varphi(\mathbf{g}_8, \mathbf{g}_7, \mathbf{g}_6, \mathbf{g}_5)$$

$$\bar{\varphi}(\mathbf{g}_8, \mathbf{g}_9, \mathbf{g}_{10}, \mathbf{g}_{11}) \varphi(\mathbf{g}_{12}, \mathbf{g}_9, \mathbf{g}_{10}, \mathbf{g}_{11}) \bar{\varphi}(\mathbf{g}_{12}, \mathbf{g}_7, \mathbf{g}_6, \mathbf{g}_4)$$





- In general, the Gaussian measure has to implement the geometrical constraints:

- gauge symmetry

$$\forall h \in G, \quad \varphi(hg_1, \dots, hg_d) = \varphi(g_1, \dots, g_d); \quad (1)$$

- simplicity constraints.

⇒  $C$  expected to be a projector, for instance

$$C(g_1, g_2, g_3; g'_1, g'_2, g'_3) = \int dh \prod_{\ell=1}^3 \delta(g_\ell h g'_\ell{}^{-1}) \quad (2)$$

in 3d gravity (Ponzano-Regge amplitudes).

- But: not always possible in practice...

- In 4d, with Barbero-Immirzi parameter: simplicity and gauge constraints don't commute →  $C$  not necessarily a projector.
- Even when  $C$  is a projector, its cut-off version is not ⇒ differential operators in radiative corrections e.g. Laplacian in the Boulatov-Ooguri model [Ben Geloun, Bonzom '11].

- Advantage: **built-in notion of scale** from  $C$  with non-trivial spectrum.

We would like to have a TGFT with:

- a built-in notion of scale i.e. a non-trivial propagator spectrum;
- a notion of discrete connection at the level of the amplitudes.

Particular realization that we consider:

- Gauge constraint:

$$\forall h \in G, \quad \varphi(hg_1, \dots, hg_d) = \varphi(g_1, \dots, g_d), \quad (3)$$

- supplemented by the non-trivial kernel (conservative choice, also justified by [Ben Geloun, Bonzom '11])

$$\left( m^2 - \sum_{\ell=1}^d \Delta_{\ell} \right)^{-1}. \quad (4)$$

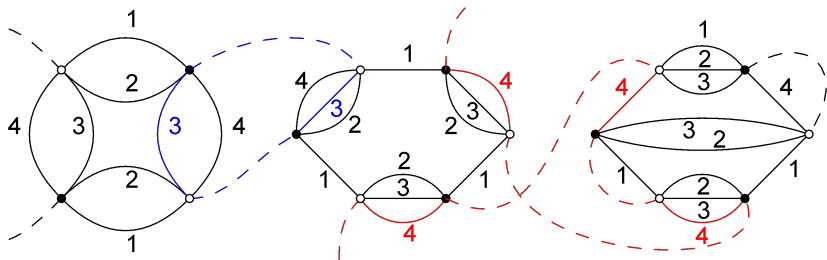
This defines the measure  $d\mu_C$ :

$$\int d\mu_C(\varphi, \bar{\varphi}) \varphi(g_{\ell}) \bar{\varphi}(g'_{\ell}) = C(g_{\ell}; g'_{\ell}) = \int_0^{+\infty} d\alpha e^{-\alpha m^2} \int dh \prod_{\ell=1}^d K_{\alpha}(g_{\ell} h g'_{\ell}{}^{-1}), \quad (5)$$

where  $K_{\alpha}$  is the heat kernel on  $G$  at time  $\alpha$ .

- The amplitudes are indexed by  $(d + 1)$ -colored graphs, obtained by connecting  $d$ -bubble vertices through propagators (dotted, color-0 lines).

Example: 4-point graph with 3 vertices and 6 (internal) lines.



## Nomenclature:

- $L(\mathcal{G})$  = set of (dotted) lines of a graph  $\mathcal{G}$ .
- Face of color  $\ell$**  = connected set of (alternating) color-0 and color- $\ell$  lines.
- $F(\mathcal{G})$  (resp.  $F_{\text{ext}}(\mathcal{G})$ ) = set of internal (resp. external) i.e. closed (resp. open) faces of  $\mathcal{G}$ .

- The amplitude of  $\mathcal{G}$  depends on oriented products of group elements along its faces:

$$\begin{aligned}
 \mathcal{A}_{\mathcal{G}} &= \left[ \prod_{e \in L(\mathcal{G})} \int d\alpha_e e^{-m^2 \alpha_e} \int dh_e \right] \left( \prod_{f \in F(\mathcal{G})} K_{\alpha(f)} \left( \overrightarrow{\prod}_{e \in \partial f} h_e^{\epsilon_{ef}} \right) \right) \\
 &\quad \left( \prod_{f \in F_{\text{ext}}(\mathcal{G})} K_{\alpha(f)} \left( g_{s(f)} \left[ \overrightarrow{\prod}_{e \in \partial f} h_e^{\epsilon_{ef}} \right] g_{t(f)}^{-1} \right) \right), \\
 &= \left[ \prod_{e \in L(\mathcal{G})} \int d\alpha_e e^{-m^2 \alpha_e} \right] \{ \textit{Regularized Boulatov-like amplitudes} \}
 \end{aligned}$$

where  $\alpha(f) = \sum_{e \in \partial f} \alpha_e$ ,  $g_{s(f)}$  and  $g_{t(f)}$  are boundary variables, and  $\epsilon_{ef} = \pm 1$  when  $e \in \partial f$  is the incidence matrix between oriented lines and faces.

- A **gauge symmetry** associated to vertices ( $h_e \mapsto g_{t(e)} h_e g_{s(e)}^{-1}$ ) allows to impose  $h_e = \mathbf{1}$  along a maximal tree of (dotted) lines.

# New notion of connectedness

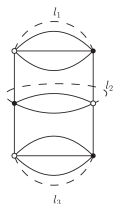
Spin Foam wisdom: lines  $\rightarrow$  faces; faces  $\rightarrow$  bubbles.

Amplitudes depend on holonomies along faces, built from group elements associated to lines  $\Rightarrow$  new notion of connectedness: incidence relations between lines and faces instead of incidence relations between vertices and lines.

## Definition

- A **subgraph**  $\mathcal{H} \subset \mathcal{G}$  is a subset of (dotted) lines of  $\mathcal{G}$ .
- **Connected components** of  $\mathcal{H}$  are the subsets of lines of the maximal factorized rectangular blocks of its  $\epsilon_{ef}$  incidence matrix.

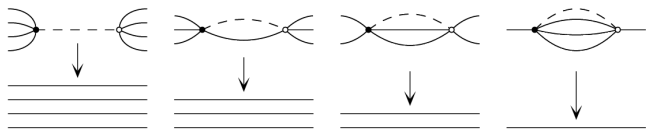
Equivalently, two lines of  $\mathcal{H}$  are elementarily connected if they have a common internal face in  $\mathcal{H}$ , and we require transitivity.



- $\mathcal{H}_1 = \{l_1\}$ ,  $\mathcal{H}_{12} = \{l_1, l_2\}$  are connected;
- $\mathcal{H}_{13} = \{l_1, l_3\}$  has two connected components (despite the fact that there is a single vertex!).

# Contraction of a subgraph

- The **contraction** of a line is implemented by so-called **dipole moves**, which in  $d = 4$  are:



Definition:  $k$ -dipole = line appearing in exactly  $k$  closed faces of length 1.

- The contraction of a subgraph  $\mathcal{H} \subset \mathcal{G}$  is obtained by successive contractions of its lines.

## Net result

The contraction of a subgraph  $\mathcal{H} \in \mathcal{G}$  amounts to delete all the internal faces of  $\mathcal{H}$  and reconnect its external legs according to the pattern of its external faces.

$\Rightarrow$  well-suited for coarse-graining / renormalization steps!

**Remark** Would be interesting to analyse these moves in a coarse-graining context [Dittrich et al.].

- 1 A class of dynamical models with gauge symmetry
- 2 Multi-scale analysis
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- 1) Decompose amplitudes according to slices of "momenta" (Schwinger parameter);
- 2) Replace high divergent subgraphs by effective local vertices;
- 3) Iterate.

⇒ Effective multi-series (1 effective coupling per interaction at each scale).

Can be reshuffled into a renormalized series (1 renormalized coupling per interaction).

Advantages of the effective series:

- Physically transparent, in particular for overlapping divergencies;
- No "renormalons":  $|\mathcal{A}_G| \leq K^n$ .



- The Schwinger parameter  $\alpha$  determines a momentum scale, which can be sliced in a geometric way. One fixes  $M > 1$  and decomposes the propagators as

$$C = \sum_i C_i, \quad (6)$$

$$C_0(g_\ell; g'_\ell) = \int_1^{+\infty} d\alpha e^{-\alpha m^2} \int dh \prod_{\ell=1}^d K_\alpha(g_\ell h g'_\ell{}^{-1}) \quad (7)$$

$$C_i(g_\ell; g'_\ell) = \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha e^{-\alpha m^2} \int dh \prod_{\ell=1}^d K_\alpha(g_\ell h g'_\ell{}^{-1}). \quad (8)$$

- A natural regularization is provided by a cut-off on  $i$ :  $i \leq \rho$ . To be removed by renormalization.
- The amplitude of a connected graph  $\mathcal{G}$  is decomposed over scale attributions  $\mu = \{i_e\}$  where  $i_e$  runs over all integers (smaller than  $\rho$ ) for every line  $e$ :

$$\mathcal{A}_{\mathcal{G}} = \sum_{\mu} \mathcal{A}_{\mathcal{G},\mu}.$$

## Strategy

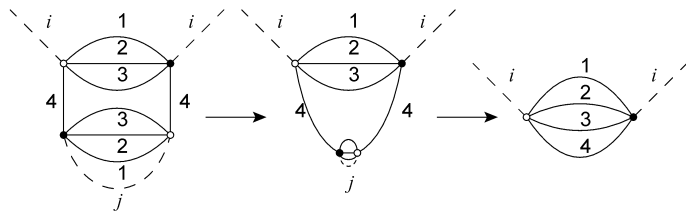
Find optimal bounds on each  $\mathcal{A}_{\mathcal{G},\mu}$ , in terms of the scales  $\mu$ .

## High subgraphs

To a couple  $(\mathcal{G}, \mu)$  is associated a set of **high subgraphs**  $\mathcal{G}_i^{(k)}$ : for each  $i$ , one defines  $\mathcal{G}_i$  as the subgraph made of all lines with scale higher or equal to  $i$ , and  $\{\mathcal{G}_i^{(k)}\}$  its connected components.

**Necessary condition:** **divergent high subgraphs must be quasi-local, i.e. look like (connected) tensor invariants.**

Example:  $i < j$



2 sources of loss of locality:

- When  $i \rightarrow +\infty$ ,  $H_f(\{h_e\}) \rightarrow \mathbf{1}$  in  $\mathcal{G}_i^{(k)}$ , but not necessarily  $h_e \rightarrow \mathbf{1}$ ;
- Combinatorial loss of connectedness when contracting a  $\mathcal{G}_i^{(k)}$ .

We therefore define

## Definition

- A connected subgraph  $\mathcal{H} \subset \mathcal{G}$  is called **contractible** if there exists a maximal tree of lines  $\mathcal{T} \subset L(\mathcal{H})$  such that

$$\left( \forall f \in F_{int}(\mathcal{H}), \prod_{e \in \partial f} \overrightarrow{h_e}^{\epsilon_{ef}} = \mathbf{1} \right) \Rightarrow (\forall e \in L(\mathcal{H}), h_e = \mathbf{1})$$

for any assignment of group elements  $(h_e)_{e \in L(\mathcal{H})}$  that verifies  $h_e = \mathbf{1}$  for any  $e \in \mathcal{T}$ .  
(approximate invariance)

- A connected subgraph  $\mathcal{H} \subset \mathcal{G}$  is called **tracial** if it is contractible and its contraction in  $\mathcal{G}$  conserves its connectedness. (approximate connected invariance)

## Theorem

- (i) If  $G$  has dimension  $D$ , there exists a constant  $K$  such that the following bound holds:

$$|\mathcal{A}_{\mathcal{G},\mu}| \leq K^{L(\mathcal{G})} \prod_{(i,k)} M^{\omega[\mathcal{G}_i^{(k)}]}, \quad (9)$$

where the **degree of divergence**  $\omega$  is given by

$$\omega(\mathcal{H}) = -2L(\mathcal{H}) + D(F_{int}(\mathcal{H}) - r(\mathcal{H})) \quad (10)$$

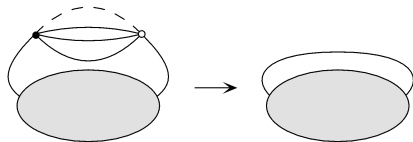
and  $r(\mathcal{H})$  is the rank of the  $\epsilon_{ef}$  incidence matrix of  $\mathcal{H}$ .

- (ii) These bounds are optimal when  $G$  is Abelian, or when  $\mathcal{H}$  is contractible.

- Subgraphs with  $\omega < 0$  are **convergent** i.e. have finite contributions when  $\rho \rightarrow \infty$ .
- Subgraphs with  $\omega \geq 0$  are **divergent** and need to be renormalized. Traciality (or at the very least contractibility) of divergent subgraphs is therefore needed for renormalizability to hold.

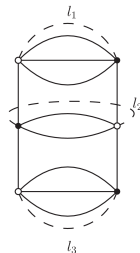
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The renormalization of such models is triggered by so-called **melopoles**. They are the tadpole connected subgraphs that can be reduced to a single line by successive 4-dipole contractions.



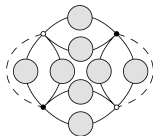
Example:

- $\mathcal{H} = \{l_1\}$ ,  $\mathcal{H} = \{l_1, l_2\}$  or  $\mathcal{H} = \{l_1, l_2, l_3\}$  are melopoles;
- $\mathcal{H} = \{l_2\}$  and  $\mathcal{H} = \{l_1, l_3\}$  are not (the last one because it is not connected).



## Theorem

- If  $\omega(\mathcal{H}) = 1$ , then  $\mathcal{H}$  is a **vacuum melopole**.
- If  $\omega(\mathcal{H}) = 0$ , then  $\mathcal{H}$  is either a **non-vacuum melopole**, or a **submelonic vacuum graph**.
- Otherwise,  $\omega(\mathcal{H}) \leq -1$  and  $\omega(\mathcal{H}) \leq -\frac{N(\mathcal{H})}{4}$ ,  $N(\mathcal{H})$  being the number of external legs of  $\mathcal{H}$ .



**Submelonic vacuum graph:** grey blobs represent melopole insertions.

## Corollary

For a given finite set of non-zero couplings, the theory has a finite set of divergent subgraphs.

## Lemma

Melopoles are tracial.

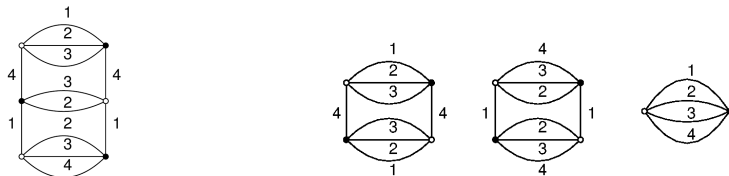
Renormalization is therefore possible in the realm of connected tensor invariants.

- One can use a **Wick ordering** procedure to remove divergencies. It is given by a linear map:

$$\Omega_\rho : \{\text{invariants}\} \rightarrow \{\text{invariants}\}$$

depending on the cut-off  $\rho$ .

- Precise expression of  $\Omega_\rho(I_b)$  given as a sum over all possible contractions of melopoles in  $b$ .





One defines the renormalized theory through melordering:

$$\begin{aligned} \mathcal{Z}_{\Omega_\rho} &= \int d\mu_{C_\rho}(\varphi, \bar{\varphi}) e^{-S_{\Omega_\rho}(\varphi, \bar{\varphi})}, \\ S_{\Omega_\rho}(\varphi, \bar{\varphi}) &= \sum_{b \in \mathcal{B}} t_b^R \Omega_\rho(l_b)(\varphi, \bar{\varphi}). \end{aligned}$$

### Theorem

For any finite set of non-zero renormalized couplings  $\{t_b^R\}$ , the amplitudes are convergent when  $\rho \rightarrow +\infty$ .

**Conclusion:**  $U(1)$  4d models with gauge symmetry are super-renormalizable at any order of perturbation theory.

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## Hypotheses:

- rank- $d$  tensors;
- $G$  of dimension  $D$ ;
- $v_{max}$  = maximal order of interactions.

**Question:** **necessary conditions** on  $d$ ,  $D$  and  $v_{max}$  in order to construct **just-renormalizable models** (i.e. with infinite sets of divergent graphs) ?

## Notations:

- $n_{2k}(\mathcal{H})$  = number of vertices with valency  $2k$  in  $\mathcal{H}$ ;
- $N(\mathcal{H})$  = number of external legs attached to vertices of  $\mathcal{H}$ ;
- $\mathcal{H}/\mathcal{T}$  = contraction of  $\mathcal{H}$  along a tree of lines (gauge-fixing).

## Proposition

Let  $\mathcal{H}$  be a non-vacuum subgraph. Then:

$$\omega(\mathcal{H}) = D(d-2) - \frac{D(d-2) - 2}{2} N \quad (11)$$

$$- \sum_{k=1}^{v_{max}/2-1} [D(d-2) - (D(d-2) - 2)k] n_{2k} \quad (12)$$

$$+ D\rho(\mathcal{H}/\mathcal{T}), \quad (13)$$

with

$$\rho(\mathcal{G}) \leq 0 \quad \text{and} \quad \rho(\mathcal{G}) = 0 \Leftrightarrow \mathcal{G} \text{ is a melopole.} \quad (14)$$

Type	$d$	$D$	$v_{max}$	$\omega$
A	3	3	6	$3 - N/2 - 2n_2 - n_4 + 3\rho$
B	3	4	4	$4 - N - 2n_2 + 4\rho$
C	4	2	4	$4 - N - 2n_2 + 2\rho$
D	5	1	6	$3 - N/2 - 2n_2 - n_4 + \rho$
E	6	1	4	$4 - N - 2n_2 + \rho$

**Table:** Classification of potentially just-renormalizable models.

$$\omega(\mathcal{H}) = 3 - \frac{N}{2} - 2n_2 - n_4 + 3\rho(\mathcal{H}/\mathcal{T}) \quad (15)$$

$N$	$n_2$	$n_4$	$\rho$	$\omega$
6	0	0	0	0
4	0	0	0	1
4	0	1	0	0
2	0	0	0	2
2	0	1	0	1
2	0	2	0	0
2	1	0	0	0

**Table:** Classification of non-vacuum divergent graphs for  $d = D = 3$ . All of them are melonic.

## Theorem

The  $\varphi^6$   $SU(2)$  model in  $3d$  is renormalizable. Divergencies generate coupling constants, mass and wave-function counter-terms.

## Summary:

- Introducing connection degrees of freedom is possible in renormalizable TGFTs.
- Generically improves renormalizability.
- $U(1)$  4d models with any finite number of interactions are super-renormalizable.
- 5 types of just-renormalizable models, including a  $SU(2)$  model in  $d = 3$ .

## What's next?

- Flow of the  $SU(2)$  model in 3d [wip]: asymptotic freedom? relation to Ponzano-Regge?
- Constructibility (of  $U(1)$  models first) [Gurau wip].
- Generalization to 4d gravity models [wip]: EPRL, FK, BO, etc.
  - geometry: interplay between simplicity constraints and tensor invariance?
  - with or without Laplacian (or other differential operator)?

**Thank you for your attention**