

Partition function of dyonic black holes in $N = 4$ string theory

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The dominant contribution to the semicanonical partition function of dyonic black holes of $N = 4$ string theory is computed for generic charges, generalizing recent results of Shih and Yin. The result is compared to the black hole free energy obtained from the conjectured relation to topological strings. If certain perturbative corrections are included agreement is found to subleading order. These corrections modify the conjectured relation and implement covariance with respect to electric-magnetic duality transformations.

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1 Introduction

The conjecture of Ooguri, Strominger, and Vafa [1] relates the black hole partition function to that of topological strings. In their proposal, the relevant black hole ensemble is the one in which the magnetic black hole charges p^I are treated microcanonically while the electric charges q_I are treated canonically. This semicanonical partition sum is related to the microcanonical partition sum $d(p, q)$ by a Laplace transform,

$$Z(p, \phi) = \sum_{q_I} d(p, q) e^{q_I \phi^I}, \quad (1.1)$$

where the continuous variables ϕ^I are the electrostatic potentials conjugate to the quantized electric charges. When viewing $Z(p, \phi)$ as a holomorphic function in ϕ^I , the black hole degeneracies $d(p, q)$ can be retrieved by performing contour integrations as will be reviewed. The conjecture amounts to comparing the black hole partition function $Z(p, \phi)$ with the square of the topological partition sum: $e^{\mathcal{F}(p, \phi)} = |e^{F_{\text{top}}}|^2$. Many encouraging results have been presented to this extent [2–6]. The conjecture is, however, still lacking a precise formulation. Modifications of the original conjecture are needed to implement electric-magnetic duality covariance and duality symmetries such as S- or T-duality. A comprehensive discussion appears in [7]. Preliminary accounts of this work have been presented at many occasions.¹

In this paper, the partition function of 1/4-BPS states is studied that arises in $N = 4$ compactifications of type-II string theory on $K3 \times T^2$. These models have a dual description in terms of heterotic strings on T^6 . For 1/4-BPS states a formula for the exact state degeneracy was proposed by Dijkgraaf, Verlinde, and Verlinde [8] and recently rederived by Shih, Strominger, and Yin [9]. It involves the automorphic form $\Phi_{10}(\rho, \sigma, v)$ that transforms with weight 10 under the modular group $\text{Sp}(2, \mathbb{Z})$. The arguments ρ , σ , and v form the period matrix of a Riemann surface of genus 2. The state degeneracy of dyons depends only on the $\text{SO}(6, 22)$ -duality invariant products of the electric and magnetic charge vectors,

$$Q = 2q_0 p^1 + q^2, \quad P = -2q_1 p^0 + p^2, \quad R = q_0 p^0 - q_1 p^1 + p \cdot q. \quad (1.2)$$

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¹ See, for instance, <http://www.fields.utoronto.ca/audio/05-06/strings/wit/index.html>.

Here, p^2 and q^2 are the contractions of p^M and q_M (with $M = 2, \dots, 27$) and involve a metric C_{MN} , which is related to the intersection matrix on $K3$, and its inverse C^{MN} . Their precise form will play no role. The charges can be identified with those of D-branes wrapping the various cycles of $K3 \times T^2$ and with the quanta of winding and momentum of wrapped NS5-branes and F-strings (see, for instance, [9] for details). The dyon degeneracies $d(p, q)$ depend only on the invariants Q, P , and R and are given by the coefficients of the formal Fourier expansion of $1/\Phi_{10}(\rho, \sigma, v)$. They can be extracted by performing the contour integrals

$$d(p, q) = \int d\rho d\sigma dv \frac{e^{i\pi(Q\sigma + P\rho + (2v-1)R)}}{\Phi_{10}(\rho, \sigma, v)}. \tag{1.3}$$

In the limit of large charges, the logarithmic degeneracy agrees with the entropy of the corresponding dyonic black holes, as was first observed in [8]. Subsequently, it was shown in [10] that the degeneracy formula precisely captures both perturbative and non-perturbative corrections to the Bekenstein-Hawking area law for black hole entropy, the origin of which can be traced back to the presence of certain higher-derivative curvature and non-holomorphic interaction terms in the effective action. For the present set-up, the supergravity description was first discussed in [11]. As expected, the gravity description reproduces only the semiclassical behavior of (1.3). The analysis in [10] shows there are two type of corrections to the asymptotic density of states: there are contributions that are exponentially suppressed in the limit of large charges as well as perturbative corrections that are subleading in this limit. In the following, the dominant contribution to the black hole partition is evaluated. Here too, both type of corrections will appear.

In [12], Shih and Yin calculated the leading contribution to $Z(p, \phi)$ for vanishing D6-brane charge p^0 and determined the perturbative corrections in this limit. In this paper, this computation is repeated for generic charges. While the presence of a D6-brane charge does not lead to substantial technical difficulties, it does uncover certain subtleties concerning subleading terms of the measure. For large charges, the dominant contribution to $Z(p, \phi)$ is of the form

$$Z(p, \phi) \sim \sum_{k^I} \left[\sqrt{\Delta(p, \phi + 2\pi i k)} e^{\mathcal{F}(p, \phi + 2\pi i k)} + \dots \right]. \tag{1.4}$$

Here, $\mathcal{F}(p, \phi)$ is precisely the non-holomorphic generalization of the free energy function given in [10]. As discussed above, the ellipsis indicates that microscopically one has exponentially suppressed corrections to the leading contribution $\sqrt{\Delta} e^{\mathcal{F}}$. These originate from other rational quadratic divisors of Φ_{10} and form the non-perturbative completion of the result.

The relevant contributions to the measure are accounted for by the factor $\sqrt{\Delta}$. From electric-magnetic duality covariance one expects that this factor is constructed from the determinant of a generalized period matrix, and an argument is presented to this extent in Sect. 3. The microscopic analysis shows that beyond the subleading order the microscopic partition function (1.1) differs from (1.4). A extensive discussion of these subtle issues is given in [7].

Recently, Jatkar and Sen [13] generalized the dyonic degeneracy formula to a class of CHL-models and showed that, asymptotically, it reproduces the entropy of the corresponding black holes. The present set-up is a simple special case of these more general models. The findings of this note can be generalized to that class of CHL-models, as is discussed in [7].

2 Microscopic black hole partition function

In this section, the dominant contribution to the partition function (1.1) is calculated, neglecting contributions that are exponentially suppressed in the limit of large generic charges. Following [12], the sum over q_0 and q_1 is converted into a sum over invariants. From (1.2) it is clear that only the combination Q and P can be

used as independent summation variables. The result is

$$Z(p, \phi) = \frac{1}{p^1 p^0} \sum_{\phi^{0,1} \rightarrow \phi^{0,1} + 2\pi i k^{0,1}}^{p^{1,0}-1} \sum_{q_M} \sum_{Q,P} d(p, q) e^{\frac{\phi^0}{2p^1}(Q-q^2) - \frac{\phi^1}{2p^0}(P-p^2) + q \cdot \phi}, \quad (2.1)$$

where R is given by

$$R = \frac{p^0}{2p^1}(Q - q^2) + \frac{p^1}{2p^0}(P - p^2) + p \cdot q. \quad (2.2)$$

There is a summation over imaginary shifts of ϕ^0 and ϕ^1 which is implemented by replacing $\phi^0 \rightarrow \phi^0 + 2\pi i k^0$ and $\phi^1 \rightarrow \phi^1 + 2\pi i k^1$ in each summand and, subsequently, by summing over the integers $k^{0,1} = 0, \dots, p^{1,0} - 1$. These shift sums enforce that only those summands contribute for which $(Q - q^2)/2p^1$ and $(P - p^2)/2p^0$ are integers. Furthermore, they implement the required shift invariance of $Z(p, \phi)$ under $\phi^{0,1} \rightarrow \phi^{0,1} + 2\pi i$. Using the integral expression for the degeneracies (1.3), one performs the sums over Q and P . This yields the sums over delta-functions $\sum_{n \in \mathbb{Z}} \delta(\sigma - \sigma(v) + n)$ and $\sum_{m \in \mathbb{Z}} \delta(\rho - \rho(v) + m)$, where

$$\begin{aligned} \sigma(v) &= -\frac{\phi^0}{2\pi i p^1} - (2v - 1) \frac{p^0}{2p^1}, \\ \rho(v) &= \frac{\phi^1}{2\pi i p^0} - (2v - 1) \frac{p^1}{2p^0}. \end{aligned} \quad (2.3)$$

These sums can be integrated against the contour integrals of σ and ρ , which run in the strip $\sigma \sim \sigma + 1$ and $\rho \sim \rho + 1$, with the result

$$Z(p, \phi) = \frac{1}{p^1 p^0} \sum_{\phi^{0,1} \rightarrow \phi^{0,1} + 2\pi i k^{0,1}}^{p^{1,0}-1} \sum_{q_M} \int \frac{dv}{\Phi_{10}(\rho(v), \sigma(v), v)} e^{i\pi\sigma(v)q^2 + i\pi\rho(v)p^2 + q_M(\phi^M + i\pi(2v-1)p^M)}. \quad (2.4)$$

Note that in view of (2.3) the integrand is invariant under $\phi^{0,1} \rightarrow \phi^{0,1} + 2\pi i p^{1,0}$ as desired. As pointed out by [12], an extra phase factor $\exp[-i\pi R]$ is included in (1.3) relative to the degeneracy formulae that appear in [8, 13]. In order to compare with the macroscopic results it is useful to Poisson-resum with respect to q_M . The result is

$$Z(p, \phi) = \frac{1}{p^1 p^0} \sum_{\phi^I \rightarrow \phi^I + 2\pi i k^I} \sqrt{\det iC_{MN}} \int \frac{dv e^{\frac{i(\phi + i\pi(2v-1)p)^2}{4\pi\sigma(v)} + i\pi\rho(v)p^2}}{\sigma(v)^{(n-1)/2} \Phi_{10}(\rho(v), \sigma(v), v)}, \quad (2.5)$$

where $n = 27$ for the present example, and the sum over k^M is over all integers. The shift-symmetry in $\phi^{0,1}$ is no longer obvious.

In a last step, the contour integral over v is performed. The contour runs horizontally in the strip defined by $v \sim v + 1$ and is confined to $\text{Im}\rho \text{Im}\sigma > \text{Im}v^2$, which for (2.3) is given by $(a + b)\text{Im}v < ab$ with $2\pi a = \text{Re}\phi^0/p^0$ and $2\pi b = -\text{Re}\phi^1/p^1$. One can show, using an $\text{Sp}(2, \mathbb{Z})$ transformation, that $\Phi_{10}(\rho, \sigma, v)$ is an even function in v . Using this and the periodicity $v \sim v + 1$, the v -contour can be closed thereby picking up the encircled residues of $1/\Phi_{10}$. The result is twice the desired integral. In general, the discussion of the various contours in the definition (1.3) is subtle, since Φ_{10} has zeroes even in the interior of the Siegel upper half plane. Fortunately, when focusing on the leading contribution to $Z(p, \phi)$, these subtleties do not play a role as long as the dominant residues are picked up. As discussed in [8, 10, 12], the leading contribution to the partition function comes from points that lie on the rational quadratic divisor

$$D = \rho\sigma - v^2 + v = 0. \quad (2.6)$$

Around these points, Φ_{10} has the expansion (see [10] for details)

$$\Phi_{10}(\rho, \sigma, v) = \frac{\eta(\sigma')^{24} \eta(\gamma')^{24}}{\sigma^{12}} D^2 + \mathcal{O}[D^4], \quad (2.7)$$

where σ' and γ' are defined by

$$\sigma' = -\frac{\rho}{\rho\sigma - v^2}, \quad \gamma' = \frac{\rho\sigma - v^2}{\sigma}. \quad (2.8)$$

Inserting $\rho(v)$ and $\sigma(v)$ given in (2.3) into these expressions one finds

$$D = (2v - 1) \frac{\phi^0 p^1 - p^0 \phi^1}{4\pi i p^0 p^1} + \frac{\phi^0 \phi^1 + \pi^2 p^1 p^0}{4\pi^2 p^1 p^0}. \quad (2.9)$$

The piece in D quadratic in v has canceled, and the critical value v_* is given by

$$(2v_* - 1) = -i \frac{\phi^0 \phi^1 + \pi^2 p^1 p^0}{\pi (\phi^0 p^1 - \phi^1 p^0)}. \quad (2.10)$$

Therefore, the contour integral over v is given by the residue

$$Z(p, \phi) = \sum_{\phi^I \rightarrow \phi^I + 2\pi i k^I} \sqrt{\det iC_{MN}} \frac{(-8)\pi^3 i p^0 p^1}{(\phi^0 p^1 - \phi^1 p^0)^2} \times \frac{d}{dv} \left[\frac{\sigma(v)^{12-(n-1)/2} e^{\frac{i}{4\pi\sigma(v)}\phi^2 + i\pi \left[\rho(v) - \frac{(2v-1)^2}{4\sigma(v)} \right] p^2 - \frac{2v-1}{2\sigma(v)} \phi \cdot p}}{\eta(\sigma'(v))^{24} \eta(\gamma'(v))^{24}} \right]_{v_*}, \quad (2.11)$$

where other exponentially suppressed contributions that come from other divisors have been suppressed. The result takes the form

$$Z(p, \phi) = \sum_{k^I} \mathcal{M}(p, \phi + 2\pi i k) e^{\mathcal{F}(p, \phi + 2\pi i k)} + \dots \quad (2.12)$$

It is now shown that $\mathcal{F}(p, \phi)$ is exactly the non-holomorphic generalization of the free energy given in [10]. In addition, there is a measure factor $\mathcal{M}(p, \phi)$, which is discussed below. To this extent the following definitions are adopted:

$$Y^I = \frac{\phi^I}{2\pi} + \frac{i}{2} p^I, \quad \bar{Y}^I = \frac{\phi^I}{2\pi} - \frac{i}{2} p^I, \quad (2.13)$$

which define the moduli $S = -iY^1/Y^0$, $\bar{S} = i\bar{Y}^1/\bar{Y}^0$, and $T^M = -iY^M/Y^0$, $\bar{T}^M = i\bar{Y}^M/\bar{Y}^0$. These relations are to be understood as defining the quantities such as S and \bar{S} as functions of the complex variables ϕ^I . In particular, S and \bar{S} , for instance, are related by complex conjugation only if the ϕ^I are real. In this sense one finds that on the divisor

$$4\rho_* - \frac{(2v_* - 1)^2}{\sigma_*} = -\frac{1}{\sigma_*} = i(S + \bar{S}), \quad \frac{2v_* - 1}{2\sigma_*} = -\frac{i}{2}(S - \bar{S}), \quad (2.14)$$

where $\rho_* = \rho(v_*)$ and $\sigma_* = \sigma(v_*)$, and that $\gamma'(v_*) = iS$ and $\sigma'(v_*) = i\bar{S}$. For $\mathcal{F}(p, \phi)$ in (2.12) these substitutions lead to

$$\mathcal{F}(p, \phi) = (S + \bar{S}) \left[\frac{\phi^2}{4\pi} - \frac{\pi p^2}{4} \right] + \frac{i}{2} (S - \bar{S}) \phi \cdot p - \log \left[(S + \bar{S})^{12} \eta(iS)^{24} \eta(i\bar{S})^{24} \right]. \quad (2.15)$$

To arrive to this result, the factor σ_*^{12} that arises in (2.7) is absorbed into the exponent, while the factor $\sigma_*^{-(n-1)/2}$ is a necessary part of the measure. The measure factor \mathcal{M} is given by

$$\mathcal{M} = 4\pi^2 \sqrt{\det C_{MN}} (S + \bar{S})^{(n-1)/2} \left[-\frac{(T + \bar{T})^2}{2} + 2\pi (S + \bar{S}) \left(12 - \frac{n-1}{2}\right) \frac{(p^0)^2}{(\phi^0 p^1 - \phi^1 p^0)^2} \right. \\ \left. - \frac{12}{\pi (Y^0)^2} \partial_S \log \eta(iS) - \frac{12}{\pi (\bar{Y}^0)^2} \partial_{\bar{S}} \log \eta(i\bar{S}) \right]. \quad (2.16)$$

Using that

$$2\pi (S + \bar{S}) \frac{(p^0)^2}{(\phi^0 p^1 - \phi^1 p^0)^2} = -\frac{1}{2\pi (S + \bar{S})} \frac{(Y^0 - \bar{Y}^0)^2}{|Y^0|^4}, \quad (2.17)$$

the measure can be rewritten as

$$\mathcal{M} = 4\pi^2 \sqrt{\det C_{MN}} (S + \bar{S})^{(n-1)/2} \\ \times \left[-\frac{(T + \bar{T})^2}{2} + \mathcal{D}\Omega + \frac{(n-1)}{4\pi (S + \bar{S})} \frac{(Y^0 - \bar{Y}^0)^2}{|Y^0|^4} + \frac{36}{2\pi} \frac{1}{(S + \bar{S}) |Y^0|^2} \right], \quad (2.18)$$

where the operator \mathcal{D} is given by

$$\mathcal{D} = \frac{2}{(Y^0)^2} \partial_S + \frac{2}{(\bar{Y}^0)^2} \partial_{\bar{S}} - \frac{2(S + \bar{S})}{|Y^0|^2} \partial_S \partial_{\bar{S}} \quad (2.19)$$

and Ω is the same function that appeared in [10]:

$$\Omega = -\frac{6}{\pi} \log \eta(iS) - \frac{6}{\pi} \log \eta(i\bar{S}) - \frac{3}{\pi} \log [S + \bar{S}]. \quad (2.20)$$

This completes the computation of the dominant contribution to the semiclassical black hole partition function. The expressions $\exp \mathcal{F}(p, \phi)$ and $\mathcal{M}(p, \phi)$ coincide with the expressions found in [12] in the limit $p^0 \rightarrow 0$, while, not surprisingly, the ranges of the sums over k^0 and k^1 are different.

In the next section it is argued that the first two terms in the bracket of (2.18) are to be treated as the leading terms and that this gives rise to a precise agreement with the leading perturbative corrections induced by the measure factor $\sqrt{\Delta}$ of (1.4). The third, n -dependent term in (2.18) is not T-duality invariant and its presence reflects the fact that for the set-up discussed here (1.1) breaks T-duality invariance. Both this and the fourth term in (2.18) are not captured directly by the approach discussed in the following.

3 Semiclassical black hole partition function and duality

The Ooguri-Strominger-Vafa proposal [1] must be modified in order to ensure the covariance with respect to electric-magnetic duality transformations and, in particular, to obtain S - and T -duality invariant results [7]. These modifications should account for the leading perturbative corrections calculated in the previous section. One way to derive these modifications is to start from a symplectically covariant expression for the black hole partition sum,

$$Z(\chi, \phi) = \sum_{p, q} d(p, q) e^{q_I \phi^I - p^I \chi_I}. \quad (3.1)$$

The additional sum over the magnetic charges is weighted by the magnetostatic potentials χ_I . The electro- and magnetostatic potentials (ϕ^I, χ_I) transform as a vector under electric-magnetic duality transformations. Assuming that the microscopic degeneracies transform as a function [as is expected to be the case for (1.3)], the left-hand side of (3.1) is invariant under symplectic transformations. By definition, one has $Z(\chi + 2\pi i, \phi) = Z(\chi, \phi + 2\pi i) = Z(\chi, \phi)$. Viewing $Z(\chi, \phi)$ as a holomorphic function in χ_I and ϕ^I , the degeneracies $d(p, q)$ or $Z(p, \phi)$ can be retrieved by an inverse Laplace transform. For example,

$$d(p, q) = \prod_{I,J} \frac{1}{(2\pi i)^2} \int d\chi_I d\phi^J Z(\chi, \phi) e^{-q_K \phi^K + p^K \chi_K}, \tag{3.2}$$

where contours run in the strips $\chi_I \sim \chi_I + 2\pi i$ and $\phi^J \sim \phi^J + 2\pi i$. Of course, it would be desirable to derive $Z(\chi, \phi)$ directly from a degeneracy formula such as (1.3), but this seems difficult.

Inspired by [1], a symplectically covariant function $Z(\chi, \phi)$ is suggested in [7] that reproduces, using (3.2), the expected black hole entropy in the semiclassical regime of large charges. The existence of such a function is intimately related to existence of a variational principle for black hole attractors and black hole entropy. The leading contribution to $Z(\chi, \phi)$ is of the form

$$Z(\chi, \phi) \sim \sum_{l_I, k^J} e^{2\pi \mathcal{H}(\chi + 2\pi i l, \phi + 2\pi i k)} + \dots, \tag{3.3}$$

where $\mathcal{H}(\chi, \phi)$ is a generalized version of the Hesse potential and includes the effects of higher-derivative curvature interactions and possibly of non-holomorphic corrections. The sums over l_I and k^J are expected to be present and reflect the fact that $\mathcal{H}(\chi, \phi)$ generically does not have any periodicity properties in χ and ϕ . The example of the previous section shows that the ranges of some of the summations can be restricted even though the corresponding periodicities might become apparent only after resummation. The ellipsis indicates that, similar to (1.4), one expects exponentially suppressed contributions that form the full non-perturbative completion of the expression. The assumption (3.3) is rather compelling: inserting (3.3) into (3.2) and performing a saddle-point approximation with respect to both the electro- and magnetostatic potentials, one finds that this semiclassical result is in precise agreement with the general black hole entropy formula. Clearly, when comparing with microscopic entropy formulae, corrections to this semiclassical black hole entropy arise [10, 13]. Such effects lead to additional subleading contributions to (3.3) and are discussed in [7].

In order to make a connection with (1.4) one can now use (3.3) as the starting point and perform an inverse Laplace transform with respect to the magnetostatic potentials only. For generic directions χ_I , the sums over the shifts combine with the integrals along the strips $\chi^I \sim \chi^I + 2\pi i$ to give contours running parallel to the whole imaginary axis. When performing these integrals in saddle-point approximation one recovers precisely (1.4), where Δ is given by the determinant of the period matrix that is suitably generalized to include certain higher-derivative curvature interactions and non-holomorphic corrections. For the set-up discussed in the previous section, the result is given, up to a numerical factor, by the sum of two squares:

$$\Delta(p, \phi) \sim \det C_{MN} (S + \bar{S})^{n-1} \left[\left(-\frac{1}{2} (T + \bar{T})^2 + \mathcal{D}\Omega \right)^2 - \frac{4 (S + \bar{S})^2}{|Y^0|^4} |D_S \partial_S \Omega|^2 \right], \tag{3.4}$$

where

$$D_S \partial_S \Omega = \left(\partial_S^2 + \frac{2}{S + \bar{S}} \partial_S \right) \Omega. \tag{3.5}$$

Up to an overall rescaling by $|Y^0|^4 (S + \bar{S})^2$, the two terms in the bracket (3.4) are each invariant under S- and T-duality transformations. In order to relate this to the microscopic result, one compares $\log \sqrt{\Delta}$ with $\log \mathcal{M}$ given by (2.18). Thereby, one treats the first term in the bracket of (3.4),

$$(S + \bar{S}) |Y^0|^2 \left(-\frac{1}{2} (T + \bar{T})^2 + \mathcal{D}\Omega \right), \tag{3.6}$$

as the leading, duality invariant part and expands $\log \sqrt{\Delta}$ in inverse powers of this quantity. The same is done for the expression $\log \mathcal{M}$ given in (2.18) and one finds precise agreement to leading order in these expressions. The two partition functions therefore agree to subleading order. Beyond this order there are, not unexpectedly, certain deviations. A discussion of the origins of these effects is given in [7].

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