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Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super-Yang–Mills theory

S.A. Frolov [a,1], R. Roiban [b,*], A.A. Tseytlin [c,2]

[a] *Max-Planck-Institute for Gravitational Physics, Albert-Einstein-Institute, Am Mühlenberg 1, D-14476 Golm, Germany*
[b] *Department of Physics, Princeton University, Princeton, NJ 08544, USA*
[c] *Department of Physics, The Ohio State University, Columbus, OH 43210, USA*

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Abstract

We consider a nonsupersymmetric example of the AdS/CFT duality which generalizes the supersymmetric exactly marginal deformation constructed in hep-th/0502086. The string theory background we use was found in hep-th/0503201 from the $AdS_5 \times S^5$ by a combination of T-dualities and shifts of angular coordinates. It depends on three real parameters γ_i which determine the shape of the deformed 5-sphere. The dual gauge theory has the same field content as $\mathcal{N} = 4$ SYM theory, but with scalar and Yukawa interactions "deformed" by γ_i-dependent phases. The special case of equal $\gamma_i = \gamma$ corresponds to the $\mathcal{N} = 1$ supersymmetric deformation. We compare the energies of semiclassical strings with three large angular momenta to the 1-loop anomalous dimensions of the corresponding gauge-theory scalar operators and find that they match as it was the case in the $SU(3)$ sector of the standard AdS/CFT duality. In the supersymmetric case of equal γ_i this extends the result of our previous work (hep-th/0503192) from the 2-spin to the 3-spin sector. This extension turns out to be quite nontrivial. To match the corresponding low-energy effective "Landau–Lifshitz" actions on the string theory and the gauge theory sides one is to make a special choice of the spin chain Hamiltonian representing the 1-loop gauge theory dilatation operator. This choice is adapted to low-energy approximation, i.e., it allows one to capture the right vacuum states and the "macroscopic spin wave" sector of states of the spin chain in the continuum coherent state effective action.

* Corresponding author.
E-mail addresses: frolovs@aei.mpg.de (S.A. Frolov), rroiban@princeton.edu (R. Roiban), tseytlin@mps.ohio-state.edu (A.A. Tseytlin).
[1] Also at SUNYIT, Utica, USA, and Steklov Mathematical Institute, Moscow.
[2] Also at Imperial College London and Lebedev Institute, Moscow.

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1. Introduction

Study of AdS/CFT duality in situations with reduced (or no) supersymmetry is of obvious interest and importance. Recently, a new example of such duality between an exactly marginal (in 4d sense) deformation of $\mathcal{N}=4$ super-Yang–Mills theory and an exactly marginal (in 2d sense) deformation of $AdS_5 \times S^5$ superstring theory was suggested in [1] and further explored in [2,3].

Here we shall be interested in generalizing the results of [2] about the correspondence between semiclassical string states and "long" gauge-theory operators to the case of 3-spin (J_1, J_2, J_3) string states dual to operators built out of the three holomorphic combinations of 6 real scalars (analog of $SU(3)$ in undeformed theory). The comparison between string and gauge theory in this sector turns out to be quite nontrivial.

We shall consider the case of real deformation parameter $\beta \equiv \gamma - i\boldsymbol{\sigma} = \gamma$. It turns out to be straightforward to generalize the discussion to the case of the more general nonsupersymmetric 3-parameter (γ_i) deformation of the $AdS_5 \times S^5$ geometry constructed in [3] using the same TsT (T-duality, shift, T-duality) transformation as in [1]. This deformation is quite natural as it treats all 3 isometric angles of S^5 on an equal footing. The corresponding type IIB supergravity background preserves $1/4$ of supersymmetries (8 supercharges) only in the "symmetric" LM [1] case

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma. \tag{1.1}$$

However, as we will see, this symmetric point is not special as far as the correspondence between string and gauge theory is concerned: the matching of leading-order semiclassical string energies and one-loop gauge theory anomalous dimensions we are going to establish below holds in the general γ_i case.

This appears to be one of the first nontrivial examples when implications of the AdS/CFT duality are observed at a quantitative level in a *nonsupersymmetric* case.[3] It provides a strong motivation for further study of this γ_i-dependent string theory and the conjectured dual non-supersymmetric large N gauge theory [3] is of obvious interest and importance. One particularly interesting aspect is the existence (for certain range of parameters) of closed-string tachyons and their reflection on the gauge theory side. This nonsupersymmetric theory is certainly stable in the nearly-flat and small γ_i limit and thus appears to be more under theoretical control than the type 0 example considered in [5].

We shall start in Section 2 with presenting the 3-parameter deformation of the $AdS_5 \times S^5$ background found by the direct generalization of the LM construction in [3]. We shall then discuss the BPS states and more general geodesics on γ_i-deformed S^5 representing semiclassical point-like string states. The geodesics happen to be described by a 1d integrable Neumann model which is the same as the system describing rotating [7] and pulsating [8] circular strings in S^5 part of $AdS_5 \times S^5$ [9]. The solutions are labeled in general by 3 conserved angular momenta (J_1, J_2, J_3) and one additional integral of motion ("oscillation number") and depend on deformation parameters γ_i through the combinations

$$\nu_i \equiv \epsilon_{ijk}\gamma_j J_k. \tag{1.2}$$

[3] The present case is obviously different from the examples of (non)supersymmetric orbifolds [4] of the $AdS_5 \times S^5$–$\mathcal{N}=4$ SYM duality where large N duality relations are "inherited" in untwisted sector. Same applies to the type 0 analog of the AdS/CFT duality [5] obtained by $(-1)^F$-type orbifolding; a discussion of matching of some of string energies and gauge theory anomalous dimensions in the BMN limit of type 0 theory appeared in [6].

These combinations are the twists that appear in the relations between the angle variables of S^5 and the γ_i-deformed five-sphere [3]. By using these relations one can show that in the special cases when ν_i are integer the circular pulsating and rotating strings of undeformed theory are, indeed, the images of the point-like strings in the deformed geometry, with ν_i being the counterparts of the circular string winding numbers m_i.[4] While in the standard $AdS_5 \times S^5$ (undeformed) case all geodesics were representing BPS states with energy E equal to the total angular momentum $\mathrm{J} = J_1 + J_2 + J_3$ here we shall find that only a few of them have this "vacuum state" property. These special "BPS" geodesics have energies that do not depend on the deformation parameters, i.e., are the same as in the undeformed case. They can be labeled by the angular momenta as: (i) $(\mathrm{J}, 0, 0)$, $(0, \mathrm{J}, 0)$, $(0, 0, \mathrm{J})$ and (ii) $(J_1, J_2, J_3)_{\mathrm{vac}}$ with $\nu_i = 0$, i.e.,

$$J_{i,\mathrm{vac}} = \frac{\gamma_i}{\boldsymbol{\gamma}}\mathrm{J}, \qquad \boldsymbol{\gamma} \equiv \gamma_1 + \gamma_2 + \gamma_3. \tag{1.3}$$

The $\nu_i = 0$ condition is satisfied for the (J, J, J) BPS state [1] in the symmetric LM case of $\gamma_i = \gamma$. In general, since J_i should take integer values in quantum theory, such states will exist only for special choices of γ_i. In addition to these special BPS states which are images of the corresponding point-like ($\nu_i = m_i = 0$) or BPS states of the undeformed theory, there is another simple subclass of geodesics for which radial directions are constant in time: these are (for integer ν_i) the TsT images of rigid rotating circular strings [7,9] in undeformed S^5. Their classical energy has nontrivial dependence on J_i and γ_i and receives also string α' corrections.

As in the undeformed case, it is straightforward to explore the fluctuation spectrum [10] near particular geodesics, i.e., quantum energies of semiclassical "small" (nearly point-like) string states in the limit of large total angular momentum J. The spectrum near the $(J, 0, 0)$ geodesic is similar to the standard BMN one [1,11]. In the case of the expansion near the $J_i \sim \gamma_i$ geodesic (1.3) the spectrum of small σ-dependent fluctuations turns out to be independent of the deformation parameters, i.e., to be the same as the BMN spectrum in the undeformed theory. The same conclusion was reached earlier in the symmetric LM $\gamma_i = \gamma$ case in [12,13]. This, in fact, is implied (to leading order in $1/\mathrm{J}$) by the TsT transformation of [3]. We shall discuss the spectrum of fluctuations on the gauge-theory side in Appendix A. The zero-mode part of the spectrum (corresponding to fluctuations depending only on time, i.e., within the space of geodesics of deformed theory) is, however, nontrivial [12]; we shall match it with the one-loop gauge theory prediction in Appendix B.

In Section 3, we shall turn to other semiclassical states represented by extended strings moving fast in deformed S^5. As in [2], they can be systematically described by reducing the classical string action to a kind of "Landau–Lifshitz" (LL) sigma model [14,15] for the "transverse" string degrees of freedom. In the present 3-spin case we shall obtain a deformed version of the $\mathbb{CP}^2$ LL model corresponding to the $su(3)$ sector of the $AdS_5 \times S^5$ string theory [15–17]. As in the deformed 2-spin case of [2], we shall find that the deformed 3-spin LL model contains a potential term which is responsible for lifting the energies of all of the string states apart from few BPS ones (the point-like states discussed above and some circular BPS strings existing as in [1] for special γ_i).

The challenge will then be to find the counterpart of this action on the gauge-theory side and to show that it coincides with the string expression; this would imply, in particular, the agreement

[4] If ν_i are not integer the formal images of geodesics of deformed geometry in $AdS_5 \times S^5$ theory do not satisfy closed-string periodicity conditions. These images are open strings subject to twisted boundary conditions.

between the leading correction to string energies and one-loop anomalous dimensions of the corresponding gauge-theory operators.

In Section 4 we shall present the direct generalization [18] (see also [19]) of the 1-loop dilatation operator for the exactly marginal β-deformation [20,21] of $\mathcal{N}=4$ SYM to the non-supersymmetric case of the three γ_i deformation parameters. As in the symmetric $\gamma_i=\gamma$ case, it can be identified with an integrable spin chain Hamiltonian (with 3 spin projections at each site corresponding to 3 chiral scalars Φ_i) which is a deformation of the $su(3)$ invariant XXX_1 Hamiltonian [22]. We shall then describe the corresponding generalization of the Bethe ansatz equations and apply them to show that the ground states of the 1-loop spin chain Hamiltonian are indeed the same as found on the string side. We shall also discuss the distinction between the $U(N)$ and $SU(N)$ gauge group cases which survives here the large N limit since the $U(1)$ parts of matter fields do not decouple.

In Section 5 we shall finally turn to the derivation of the effective coherent-state action for low-energy semiclassical states of the spin chain that should be dual to the semiclassical string states in the 3-spin sector. In general, there are many equivalent spin-chain Hamiltonians, corresponding to different choices of basis in the space of gauge-theory operators, that lead to the same anomalous dimensions. To establish the correspondence with string theory it turns out that one needs a special choice adapted to low-energy approximation. This is a subtlety not confronted in previous discussions of the coherent state approach in the undeformed [15] or deformed 2-spin [2] cases. We shall describe the choice of coherent states and the basis needed to capture the expected BPS states (1.3) in low-energy (slowly-changing coherent field) approximation in Sections 5.1 and 5.2. Then in Section 5.3 we shall find that this choice leads exactly to the same Landau–Lifshitz effective action as found in Section 3 on the string side. This provides a highly nontrivial check of the AdS/CFT duality not only in the supersymmetric LM deformation case [1] but also in the general nonsupersymmetric γ_i-deformed theory.

Section 6 will contain some concluding remarks.

In Appendix A we shall discuss fluctuations near the vacuum states of the one-loop spin chain and match their spectra with the string-theory results. In Appendix B we shall consider the spin-chain 0-mode fluctuations near the (J_1,J_2,J_3) vacuum and again demonstrate remarkable agreement with the string-theory predictions.

2. Three-parameter deformation of $AdS_5 \times S^5$ string theory

2.1. Background

We shall mostly follow the notation of [2]. The type IIB solution related by T-dualities and shifts transformation to the $AdS_5\times S^5$ background and which generalizes [3] the background of [1] to the case of unequal γ_i parameters can be represented as

$$ds^2_{\rm str} = R^2\left[ds^2_{AdS_5} + \sum_{i=1}^3\left(d\rho_i^2 + G\rho_i^2\,d\phi_i^2\right) + G\rho_1^2\rho_2^2\rho_3^2\left[d\left(\sum_{i=1}^3\tilde\gamma_i\phi_i\right)\right]^2\right], \tag{2.1}$$

$$B_2 = R^2 G w_2, \quad w_2 \equiv \tilde\gamma_3\rho_1^2\rho_2^2\,d\phi_1\,d\phi_2 + \tilde\gamma_1\rho_2^2\rho_3^2\,d\phi_2\,d\phi_3 + \tilde\gamma_2\rho_3^2\rho_1^2\,d\phi_3\,d\phi_1, \tag{2.2}$$

$$e^\phi = e^{\phi_0}G^{1/2}, \qquad \chi = 0, \tag{2.3}$$

$$G^{-1} \equiv 1 + \tilde\gamma_3^2\rho_1^2\rho_2^2 + \tilde\gamma_1^2\rho_2^2\rho_3^2 + \tilde\gamma_2^2\rho_1^2\rho_3^2, \qquad \sum_{i=1}^3\rho_i^2 = 1, \tag{2.4}$$

$$C_2 = -4R^2 e^{-\phi_0} w_1\, d\left(\sum_{i=1}^{3} \tilde{\gamma}_i \phi_i\right), \qquad dw_1 \equiv \cos\alpha \sin^3\alpha \sin\theta \cos\theta\, d\alpha\, d\theta, \tag{2.5}$$

$$F_5 = 4R^4 e^{-\phi_0} (\omega_{AdS_5} + G\omega_{S^5}), \quad \omega_{S^5} \equiv dw_1\, d\phi_1\, d\phi_2\, d\phi_3. \tag{2.6}$$

Here B_2 is the NSNS 2-form potential, ϕ is the dilaton and $d\chi$, dC_2 and F_5 are the RR field strengths. The angles θ, α appearing in dw_1 parametrize S^2 coordinates ρ_i as follows

$$\rho_1 = \sin\alpha \cos\theta, \qquad \rho_2 = \sin\alpha \sin\theta, \qquad \rho_3 = \cos\alpha. \tag{2.7}$$

Note also that

$$w_1 = \frac{1}{4}\rho_1^2\, d\big(\rho_2^2\big) - \frac{1}{8}\, d\big(\rho_1^2 \rho_2^2\big) = \frac{1}{8}\big(\rho_1^2 + \rho_2^2\big)^2 d\frac{\rho_2^2}{\rho_1^2 + \rho_2^2}. \tag{2.8}$$

The standard $AdS_5 \times S^5$ background is recovered after setting the deformation parameters $\tilde{\gamma}_i = R^2\gamma_i$ to zero. For equal $\tilde{\gamma}_i = \tilde{\gamma}$ this becomes the background of [1] ($\tilde{\gamma}_i$ were denoted as $\hat{\gamma}_i$ in [1,3]). We also assume that

$$g_s = e^{\phi_0} = \frac{g_{\mathrm{YM}}^2}{4\pi}, \qquad R^4 = 4\pi g_s N = N g_{\mathrm{YM}}^2 \equiv \lambda, \qquad \alpha' = 1, \tag{2.9}$$

$$\tilde{\gamma}_i = R^2 \gamma_i = \sqrt{\lambda}\gamma_i. \tag{2.10}$$

Here γ_i are the deformation parameters which appear on the gauge theory or spin chain side. In the symmetric case $\gamma_i = \gamma$ this parameter is the real part of the deformation parameter β in the superpotential $W = h\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)$. We shall consider only the case of real β where the duality appears to be much more under quantitative control (see [2]).

As discussed in [2], the parameters $\tilde{\gamma}_i$ which enter the supergravity background are assumed to be fixed in the semiclassical string limit. Since $\frac{R^2}{\alpha'} = \sqrt{\lambda}$ plays the role of the string tension, in this limit one also fixes other semiclassical parameters like $\mathcal{E}$ and $\mathcal{J}_i$ which determine the string energy and spins

$$E = \sqrt{\lambda}\mathcal{E}, \qquad J_i = \sqrt{\lambda}\mathcal{J}_i, \qquad \tilde{\lambda} \equiv \frac{\lambda}{\mathrm{J}^2} = \text{fixed}, \quad \mathrm{J} = \sum_{i=1}^{3} J_i \tag{2.11}$$

while $\sqrt{\lambda}$ and thus J are assumed to be large to suppress string α' corrections. That means that

$$\bar{\gamma}_i \equiv \gamma_i \mathrm{J} = \frac{\tilde{\gamma}_i}{\sqrt{\tilde{\lambda}}} \tag{2.12}$$

is also fixed in this limit, i.e., $\gamma_i \sim \frac{1}{\mathrm{J}}$. For definiteness, we shall assume that both J_i and γ_i are nonnegative.

On the gauge theory (spin chain) side, the limit which one takes is formally different [2,23]. Since one uses perturbative gauge theory, one first expands in λ and then takes J large. Here J plays the role of the length of the chain (or length of the operator), and we will be interested in extracting the dependence of the spin chain energies on the parameters $\tilde{\lambda}$ and $\gamma_i\mathrm{J}$ while looking at 1-loop (order λ) correction and taking large J limit. In all previously discussed examples of similar comparisons the leading order terms in the two expressions matched, and our aim will be to extend this matching to the present (nonsupersymmetric for unequal γ_i) case.

2.2. BPS states

By following the TsT transformation that relates the $AdS_5 \times S^5$ string theory to the γ_i-deformed string theory one can relate the angle variables $\tilde{\tilde{\phi}}_i$ of S^5 (in the notation of [3]) and the angle variables ϕ_i of the TsT-deformed geometry (2.1). The basic starting point is the equality between the $U(1)$ conserved current densities of strings on $AdS_5 \times S^5$ and on the γ_i-deformed background [3]:

$$\tilde{\tilde{\mathbf{J}}}_{i\,p} = \mathbf{J}_{i\,p}, \tag{2.13}$$

where $i = 1, 2, 3$ and $p = 0, 1$ are the world-sheet indices. Taking into account that the time components of the currents are the momentum densities conjugate to the angle variables, and expressing the time derivatives through the momenta, one can cast (2.13) in the following simple form

$$\tilde{\tilde{p}}_i = p_i = \mathbf{J}_{i\,0}, \tag{2.14}$$

$$\rho_i^2 \tilde{\tilde{\phi}}_i' = \rho_i^2 (\phi_i' - \epsilon_{ijk}\gamma_j p_k), \quad i = 1, 2, 3, \tag{2.15}$$

where in (2.15) we assume summation in j, k but no summation in i. If none of the “radii” ρ_i vanish on a string solution, one can cancel the ρ_i^2 factors in (2.15) to get

$$\tilde{\tilde{\phi}}_i' = \phi_i' - \epsilon_{ijk}\gamma_j p_k. \tag{2.16}$$

Integrating over σ and taking into account that ϕ_i are angle variables and the strings in the deformed background are assumed to be closed, i.e.,

$$\phi_i(2\pi) - \phi_i(0) = 2\pi n_i, \tag{2.17}$$

where n_i are integer winding numbers, we get the twisted boundary conditions for the angle variables $\tilde{\tilde{\phi}}_i$ of the original S^5 space

$$\tilde{\tilde{\phi}}_i(2\pi) - \tilde{\tilde{\phi}}_i(0) = 2\pi(n_i - \nu_i), \tag{2.18}$$

$$\nu_i \equiv \epsilon_{ijk}\gamma_j J_k, \qquad J_i = \int_0^{2\pi} \frac{d\sigma}{2\pi} p_i. \tag{2.19}$$

We see that if the twists ν_i (already mentioned in (1.2)) are not integer then the twisted strings in $AdS_5 \times S^5$ which are formal images of closed strings in the deformed geometry under the inverse of TsT transformation are open.

The relations (2.15) imply that if ϕ_i solve the equations of motion for a string in the γ_i-deformed background then $\tilde{\tilde{\phi}}_i$ solve those in $AdS_5 \times S^5$ with the twisted boundary conditions (2.18) imposed on the angle variables. It is easy to show [3] that the Virasoro constraints for both models also map to each other under the TsT-transformation; therefore, the energy of a twisted string in $AdS_5 \times S^5$ is equal to the energy of the corresponding closed string in the γ_i-deformed background. This observation allows one to readily determine all classically BPS states in the deformed model, i.e., the states that have minimal energy for the given charges,

$$E = \mathrm{J} \equiv J_1 + J_2 + J_3. \tag{2.20}$$

To this end we notice that a BPS state in the deformed background must be an image of a BPS state in $AdS_5 \times S^5$, that is an image of a point-like string or null geodesic in $AdS_5 \times S^5$. For such a string $\tilde{\tilde{\phi}}_i{}' = 0$, $\rho_i' = 0$; then $\tilde{\tilde{p}}_i = p_i = J_i$ do not depend on σ, i.e., all the charges are distributed uniformly along the string. Thus, for the BPS states the relation (2.16) takes the form

$$\phi_i' = \epsilon_{ijk}\gamma_j p_k = \nu_i, \qquad p_i = J_i, \tag{2.21}$$

where we also assume that all the charges J_i are not equal to 0. Since the string in the deformed background is closed, all the twists ν_i which play the role of the winding numbers then must be integer:

$$\nu_i = \epsilon_{ijk}\gamma_j J_k \quad \in \mathbb{Z}. \tag{2.22}$$

One is now to distinguish the case of nonzero ν_i when a solution is a circular string, and the case of $\nu_i = 0$ when the solution is a point-like string.

For $\nu_i \neq 0$ these equations can have a consistent (circular) string solution only if γ_i are rational (J_i take integer values in quantum theory) and the corresponding BPS state is a circular string similar to the ones studied in [7] (this generalizes the observation in [1] to the case of unequal γ_i).

For $\nu_i = 0$ the BPS state of deformed geometry is a point-like string. The general solution to $\nu_i = 0$ is

$$\nu_i = 0: \quad J_i = c\gamma_i, \tag{2.23}$$

where c is a proportionality coefficient which can be any real number. Since J_i must be integer in the quantum theory, such a solution exists only for special values of γ_i.[5] These (J_1, J_2, J_3) point-like BPS states generalize the (J, J, J) state [1] in the supersymmetric LM case $\gamma_i = \gamma$, $J_i = J$.

Note that any (J_1, J_2, J_3) solution in the deformed background for which (2.23) is satisfied can be obtained from a closed string solution in $AdS_5 \times S^5$, and the energies of these string states in the γ_i-deformed model and their images in the $AdS_5 \times S^5$ are equal to each other.[6]

If one of the 3 momenta is equal to zero, e.g., $J_3 = 0$, then the string states belong to the 2-spin sector which is the analog of the $su(2)$ sector of undeformed theory. It contains the obvious additional BPS state $(J, 0, 0)$ which is the direct TsT relative of the corresponding point-like state in $AdS_5 \times S^5$. Similarly, we have also $(0, J, 0)$ and $(0, 0, J)$ BPS states.

2.3. Point-like strings (geodesics) and near-by fluctuations

Let us now analyze some string solutions in the deformed geometry starting directly with (2.1), (2.2).

To find the classical point-like string states in the deformed geometry it is enough to concentrate on the string-frame metric (to study quantum corrections one will need of course the full Green–Schwarz fermionic action which will contain couplings to other background fields). We

[5] It is interesting that since for the classical strings there is no quantization condition, any solution satisfying $J_i \sim \gamma_i$ has the BPS energy $E = J_1 + J_2 + J_3$. It would be interesting to analyze the semiclassical expansion around such a solution, and to see how the quantization condition gets restored.

[6] Such (in general, non-BPS) states should be dual to the gauge-theory operators protected from the deformation at least to the leading order in γ_i.

should consider geodesics that wrap the "internal" S^5_γ part and that should be dual to special $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}+\cdots)$ operators on the gauge theory side.

The metric (2.1) has 3 isometries corresponding to shifts of the angles ϕ_i and thus the states should be characterized by 3 conserved angular momenta J_i. Starting with the string equations in conformal gauge with AdS_5 time $t=\mathcal{E}\tau$ it is straightforward to show that, while the metric looks rather complicated, the effective action that determines the time evolution of the S^2 coordinates ρ_i can be written simply as ($\sum_{i=1}^3\rho_i^2=1$)

$$S(\rho)=\frac{1}{2}\sqrt{\lambda}\int d\tau\,L,\qquad L(\rho)=\sum_{i=1}^{3}\bigl[\dot\rho_i^2-V_i(\rho_i)\bigr],\qquad V_i(\rho_i)=\frac{\mathcal{J}_i^2}{\rho_i^2}+\nu_i^2\rho_i^2,\tag{2.24}$$

$$J_i=\frac{\partial L(\rho,\phi)}{\partial\dot\phi_i}=\sqrt{\lambda}\mathcal{J}_i,\qquad \nu_i\equiv\epsilon_{ijk}\tilde\gamma_j\mathcal{J}_k=\epsilon_{ijk}\gamma_jJ_k.\tag{2.25}$$

Here $L(\rho,\phi)$ stands for the string Lagrangian before one solves for the derivatives of the angles. For $\nu_i=0$ this is the action of a particle moving on S^5. For general ν_i this is recognized as a Neumann–Rosochatius integrable system describing an oscillator on 2-sphere (or, equivalently, a special Neumann system describing an oscillator on 5-sphere, cf. [9]). The conformal gauge constraint implies that the corresponding Hamiltonian is equal to $\mathcal{E}^2$, i.e., $\sum_{i=1}^3[\dot\rho_i^2+V_i(\rho_i)]=\mathcal{E}^2$. In particular, in the LM case of $\tilde\gamma_i=\tilde\gamma$ we get explicitly for the particle Hamiltonian

$$\begin{aligned}H=\mathcal{E}^2&=\dot\rho_1^2+\dot\rho_2^2+\dot\rho_3^2+\frac{\mathcal{J}_1^2}{\rho_1^2}+\frac{\mathcal{J}_2^2}{\rho_2^2}+\frac{\mathcal{J}_3^2}{\rho_3^2}\\&\quad+\tilde\gamma^2\bigl[(\mathcal{J}_2-\mathcal{J}_3)^2\rho_1^2+(\mathcal{J}_3-\mathcal{J}_1)^2\rho_2^2+(\mathcal{J}_1-\mathcal{J}_2)^2\rho_3^2\bigr].\end{aligned}\tag{2.26}$$

This result is easy to find using the TsT relation [3] of the deformed theory to the $AdS_5\times S^5$ theory. The two string Hamiltonians are related by the TsT, so to get the particle Hamiltonian in the deformed theory all one has to do is to shift the σ-derivatives of the $AdS_5\times S^5$ angles by the momenta as in (2.16) and then to set all terms with σ-derivatives to zero.

The appearance of the Neumann system is not accidental: the same system was found in [9] to describe circular pulsating and rotating strings in undeformed S^5. These strings are, in fact, mapped (for integer ν_i) to point-like strings in S^5_γ under the TsT transformation of [3]: ν_i plays the role of the winding number m_i of the circular strings, and the conformal gauge constraint $m_iJ_i=0$ here is satisfied automatically.

Generic solution is labeled by (J_1,J_2,J_3) and one extra (in addition to the 1d energy or $H=\mathcal{E}^2$) integral of motion which may be interpreted as an "oscillation number" K for a (quasiperiodic) particle motion on S^2. The lower-energy solutions correspond to $K=0$ when $\rho_i=$ const. The form of the dependence of the energy on K and J_i will be the same as in the case of the pulsating strings in [15,24].

The special solutions that are the same as in the undeformed case and thus represent the lowest-energy ("BPS") states are found if (i) $\rho_1=1$, $\rho_2=\rho_3=0$ (and two other cases with interchange of $1,2,3$), representing $(J_1,0,0)$ state with $E=J_1$, and also if (ii) J_i are such that $\nu_i=0$, i.e., if $J_i\sim\tilde\gamma_i$ when $E=\mathrm{J}=J_1+J_2+J_3$. These are the same as already discussed in the previous subsection and they should be dual to vacuum (zero anomalous dimension) states on the spin chain side.

In the general nonsupersymmetric case of unequal γ_i there is an open question if such states are true vacua (i.e., states with $E = \mathrm{J}$ which is the absolute minimum of the energy), i.e., do not receive quantum corrections both on the string theory and on the gauge theory side.[7]

In addition, there are higher energy (non-BPS) states still having $\rho_i = \text{const}$, i.e., $K = 0$, which (for integer ν_i) are images of rigid (non-pulsating) circular rotating strings in S^5. As discussed in [7,9], the classical energy of the latter is a nontrivial function of J_i, the winding numbers m_i and the string tension; expanded in $\tilde{\lambda}$ it looks like $E = \mathrm{J} + \frac{\lambda}{\mathrm{J}} c_1(m_i, \frac{J_j}{J_k}) + \cdots$. The same expression is found for the point-like strings here with $m_i \to \nu_i$. The leading order corrections to $E = \mathrm{J}$ relation will scale as $\frac{\lambda}{\mathrm{J}}(\gamma_i J_n)^2 (\frac{J_j}{J_k})^2$ and may thus be compared to the gauge-theory side. We will do this automatically by matching the corresponding effective actions that describe such semiclassical states.

Next, let us follow [10,25] and study small semiclassical strings representing small fluctuations near the above geodesics.

2.3.1. $(\mathrm{J}, 0, 0)$ *case*

In the $\gamma_i = \gamma$ case the corresponding analog of the BMN spectrum of quadratic fluctuations near the $(\mathrm{J}, 0, 0)$ geodesic was found in [1,11]. Here we shall generalize it to the nonsupersymmetric γ_i-case. We shall first concentrate on the bosonic part of the fluctuation Lagrangian that follows from expanding the bosonic part of the string action which depends only on the string metric (2.1) and the 2-form field B_2 in (2.2)

$$I_B = -\frac{1}{2}\sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\sqrt{-g} g^{pq} \partial_p X^M \partial_q X^N G_{MN} - \epsilon^{pq} \partial_p X^M \partial_q X^N B_{MN}\right], \tag{2.27}$$

where $\epsilon^{01} = 1$ and in the conformal gauge which we shall use here $g_{pq} = \mathrm{diag}(-1, 1)$. Expanding the action near the solution $t = \phi_1 = \mathcal{J}\tau$, $\rho_1 = 1$, $\phi_{2,3} = \rho_{2,3} = 0$ we get for the part of the fluctuation Lagrangian which is different from the standard BMN $\gamma_i = 0$ case

$$L = \frac{1}{2}\left(\dot{y}_a^2 - y_a'^2 + \dot{z}_a^2 - z_a'^2\right) + \frac{1}{2}\mathcal{J}^2\left(1 + \tilde{\gamma}_3^2\right) y_a^2 + \frac{1}{2}\mathcal{J}^2\left(1 + \tilde{\gamma}_2^2\right) z_a^2 + \mathcal{J}\tilde{\gamma}_3 \epsilon_{ab} y_a y_b' + \mathcal{J}\tilde{\gamma}_2 \epsilon_{ab} z_a z_b'. \tag{2.28}$$

Here we assume summation over $a, b = 1, 2$ and y_a and z_a are $2 + 2$ fluctuations of Cartesian coordinates in the ρ_2, ϕ_2 and ρ_3, ϕ_3 planes. This is essentially the same Lagrangian as in the $\gamma_i = \gamma$ case [1,11] but with the parameters γ_2 and γ_3 in the each of the 2-planes transverse to the geodesic. The expansion near $(0, \mathrm{J}, 0)$ and $(0, 0, \mathrm{J})$ geodesics leads to similar expressions with the corresponding interchange of the parameters γ_i.

The corresponding characteristic frequencies that represent the analog of the BMN spectrum $E - \mathrm{J} = \frac{w_n}{\mathcal{J}} N_n$ are ($n = 0, \pm 1, \ldots$ labels string $e^{in\sigma}$ modes and is different for the two types for the excitations)

$$w_n^{(i)} = \sqrt{\mathcal{J}^2 + (n + \tilde{\gamma}_i \mathcal{J})^2} = \mathcal{J}\sqrt{1 + \tilde{\lambda}(n + \gamma_i \mathrm{J})^2}, \quad i = 2, 3. \tag{2.29}$$

[7] It is not a priori clear that string α' corrections are absent: while TsT transformation does not affect these special geodesics, it may (and, in fact, does) change the spectrum of fluctuations near them, and thus may alter the cancellation of the quantum correction to the vacuum energy.

As follows from the structure of the supergravity background, the quadratic fermionic action contains couplings only to the NSNS 3-form (with two parts proportional to γ_2 and γ_3 as reflected in (2.28)) and the standard RR 5-form flux. It thus has the structure as in Eq. (4.12) in [11] with γ_2 and γ_3 multiplying the corresponding fermionic projectors $\theta^A\Gamma^+(\tilde{\gamma}_2\Gamma_{y_1y_2}+\tilde{\gamma}_3\Gamma_{z_1z_2})\theta^A$ and a mass term coming from 5-form flux (see also [28] for a general structure of such actions). Then (as follows, e.g., from Eq. (4.21) in [11]) the corresponding fermionic spectrum is the same as the above bosonic one, implying that the quadratic fluctuation Lagrangian has 2d world-sheet supersymmetry. The latter is a consequence of space–time supersymmetry of the corresponding plane-wave background (for which (2.28) is the l.c. gauge fixed Lagrangian) present even though the original supergravity background is not supersymmetric for unequal γ_i. This has an important consequence that the contribution of the quadratic fluctuation energies to the $(\mathrm{J},0,0)$ ground state energy vanishes, i.e., (at least to the leading order in $1/\mathrm{J}$) this state is a true analog of the corresponding BPS state in the undeformed or in the supersymmetric deformed $\gamma_i=\gamma$ theory.

We shall see that these conclusions are corroborated by the analysis of the one-loop dilatation operator on the gauge theory side. In particular, the same fluctuation spectrum (for the relevant part of fluctuations) will appear from the coherent state action for the 3-spin or the holomorphic 3-scalar sector of the spin chain which is the analog of $su(3)$ sector in the undeformed theory.

2.3.2. $(J_1,J_2,J_3)_{\text{vac}}$ case

While the $(\mathrm{J},0,0)$ case is very similar to the standard BMN case, the expansion near the $(J_1,J_2,J_3)\sim(\gamma_1,\gamma_2,\gamma_3)$ geodesic is more involved. The bosonic part of the fluctuation Lagrangian follows from (2.27) expanded near the corresponding classical solution (again, we assume that $\gamma_i\geqslant 0$)

$$t=\phi_i=\mathcal{J}\tau,\qquad \rho_i^2=\frac{\mathcal{J}_i}{\mathcal{J}}=\frac{\gamma_i}{\boldsymbol{\gamma}},\quad \mathcal{J}\equiv\sum_{i=1}^{3}\mathcal{J}_i,\ \boldsymbol{\gamma}\equiv\sum_{i=1}^{3}\gamma_i. \tag{2.30}$$

The fluctuations in time and $\psi=\sum_{i=1}^{3}\tilde{\gamma}_i\phi_i$ directions decouple, i.e., are massless 2d fields, the fluctuations in the other 4 AdS_5 directions are the same as in the undeformed case (i.e., are described by massive 2d fields with mass $\mathcal{J}=\frac{\mathrm{J}}{\sqrt{\lambda}}$) while the remaining 4 nontrivial fluctuations in S^5_γ directions are found by setting ($a=1,2$)

$$\begin{aligned}&\phi_a=\mathcal{J}\tau+v_a,\qquad v_3=-\frac{\gamma_1}{\gamma_3}v_1-\frac{\gamma_2}{\gamma_3}v_2,\\&\rho_a=\sqrt{\frac{\gamma_a}{\boldsymbol{\gamma}}}(1+u_a),\qquad \rho_3=\sqrt{1-\rho_1^2-\rho_2^2},\end{aligned} \tag{2.31}$$

where v_1,v_2,u_1,u_2 are 4 independent 2d fluctuation fields. Computing the momenta for ϕ_i, i.e., the angular momenta J_i, from the Lagrangian (2.27) we get, to the leading order in fluctuations near the vacuum (2.31):

$$J_i=\frac{\gamma_i}{\boldsymbol{\gamma}}J+j_i,\quad j_i\equiv\sqrt{\lambda}\pi_i,\ \pi_i=\frac{\tilde{\gamma}_i}{\tilde{\boldsymbol{\gamma}}+\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}(\dot{v}_i+2\mathcal{J}u_i), \tag{2.32}$$

where u_3 is such that $\sum_{i=1}^{3}\gamma_iu_i=0$ and we ignored terms linear in σ-derivatives of the fluctuations v_i that integrate to zero. Thus with the assumption that $\sum_{i=1}^{3}\tilde{\gamma}_i\phi_i$ does not fluctuate we see that the value of the total momentum $J=\sum_{i=1}^{3}J_i$ is not, as required, changed by the fluctuations and thus the fluctuations of momenta satisfy $\sum_{i=1}^{3}j_i=0$.

To put the quadratic fluctuation Lagrangian into the canonical form it is useful to do further field redefinition to 4 fields z_a, y_a:

$$v_1 = \sqrt{\frac{\Delta}{2(\tilde\gamma_1+\tilde\gamma_3)}}\left(\sqrt{\frac{\tilde\gamma_3}{\tilde\gamma_1}}z_1 - \sqrt{\frac{\tilde\gamma_2}{\tilde{\boldsymbol\gamma}}}z_2\right), \qquad v_2 = \sqrt{\frac{\Delta(\tilde\gamma_1+\tilde\gamma_3)}{2\tilde\gamma_2\tilde{\boldsymbol\gamma}}}z_2, \tag{2.33}$$

$$u_1 = \sqrt{\frac{\tilde{\boldsymbol\gamma}}{2(\tilde\gamma_1+\tilde\gamma_3)}}\left(\sqrt{\frac{\tilde\gamma_3}{\tilde\gamma_1}}y_1 - \sqrt{\frac{\tilde\gamma_2}{\tilde{\boldsymbol\gamma}}}y_2\right), \qquad u_2 = \sqrt{\frac{\tilde\gamma_1+\tilde\gamma_3}{2\tilde\gamma_2}}y_2, \tag{2.34}$$

where

$$\Delta = \tilde{\boldsymbol\gamma} + \tilde\gamma_1\tilde\gamma_2\tilde\gamma_3, \quad \tilde{\boldsymbol\gamma} = \tilde\gamma_1+\tilde\gamma_2+\tilde\gamma_3. \tag{2.35}$$

Then the resulting fluctuation action is $I = \sqrt{\lambda}\int d\tau \int \frac{d\sigma}{2\pi} L_2$ where (we assume summation over $a, b = 1, 2$)

$$L_2 = \frac{1}{2}\left(\dot y_a^2 - y_a'^2 + \dot z_a^2 - z_a'^2\right) - \frac{1}{2}B^2 y_a^2 + A y_a \dot z_a + B\epsilon_{ab} y_b z_a', \tag{2.36}$$

and

$$A = 2\mathcal{J}\sqrt{\frac{\tilde{\boldsymbol\gamma}}{\tilde{\boldsymbol\gamma}+\tilde\gamma_1\tilde\gamma_2\tilde\gamma_3}}, \qquad B = 2\mathcal{J}\sqrt{\frac{\tilde\gamma_1\tilde\gamma_2\tilde\gamma_3}{\tilde{\boldsymbol\gamma}+\tilde\gamma_1\tilde\gamma_2\tilde\gamma_3}}, \qquad A^2+B^2 = 4\mathcal{J}^2. \tag{2.37}$$

This quadratic action can be interpreted also as an action for a string in a plane-wave background in the l.c. gauge $x^+ = \mathcal{J}\tau$ with y_a, z_a representing transverse coordinates. It has constant coefficients and can be readily quantized as discussed, e.g., in [27].

For $\gamma_i = \gamma$ (2.36) reduces to the fluctuation Lagrangian found near (J, J, J) geodesic in the LM case in [12,13]. In the case of $\gamma_i = 0$ (when $A = 2\mathcal{J}$) it reduces to the BMN Lagrangian in a rotated coordinate system corresponding to the expansion near the (J, J, J) geodesic.

By writing down the corresponding equations of motion and setting $y_a, z_a \sim \sum_n C_n e^{iw_n\tau + in\sigma}$ one finds that for $n \neq 0$ the corresponding characteristic frequencies are the same as in the BMN case, i.e., do not depend on γ_i (for the LM case of equal γ_i this was found in [12,13]):

$$w_n = \mathcal{J} \pm \sqrt{n^2+\mathcal{J}^2} = \mathcal{J}\left(1 \pm \sqrt{1+\tilde\lambda n^2}\right). \tag{2.38}$$

Since we assumed that $t = \mathcal{J}\tau$, the corresponding fluctuation energies are $E_n - \mathrm{J} = \frac{|w_n|}{\mathcal{J}}$.[8]

As was shown in [12] in the case of $\gamma_i = \gamma$ the fermionic part of the quadratic fluctuation Lagrangian (in this case fermions are coupled to both the NSNS and the RR 3-forms as well as the RR 5-form) leads to the same spectrum as the bosonic Lagrangian, implying again that there is a residual world-sheet supersymmetry (associated with supernumerary [26] target space supersymmetry). In particular, the correction to the ground state energy cancels out. This should be true also in the present unequal γ_i case.[9] The same bosonic spectrum will be found also on

8 One may argue that the conclusion that the spectrum of fluctuations near this vacuum $J_i \sim \gamma_i$ state does not depend on γ_i follows from the TsT construction of this string theory in [3]: the difference in fluctuation spectra should involve ν_i and is thus subleading in 1/J. However, this does not apply to the 0-modes, see below.

9 String theory TsT relation suggests this for integer ν_i in (2.25), but cancellation between the bosonic and fermionic contributions should not depend on whether ν_i is integer or not. This cancellation need not persist at subleading orders when non-linear interactions of fluctuation modes are to be included.

the gauge theory side from the analysis of the corresponding Bethe ansatz equations, and from the coherent state action.

The spectrum of the bosonic 0-modes (i.e., σ-independent fluctuations) is, however, non-trivial, as was already pointed out in the $\gamma_i = \gamma$ case in [12]. The 0-modes correspond to point-like strings, i.e., represent fluctuations within the set of geodesics. In the undeformed case all geodesics were BPS and thus had the same energy the spectrum of 0-modes was degenerate; here this is no longer the case (cf. (2.24)). The Lagrangian for τ-dependent 0-modes is $L_2 = \frac{1}{2}(\dot{y}_a^2 + \dot{z}_a^2) - \frac{1}{2}B^2 y_a^2 + A y_a \dot{z}_a$ and this system can be quantized by writing down the corresponding Schrödinger equation and separating the oscillator dynamics (corresponding to $n = 0$ case of the above w_n) from a free particle dynamics as discussed, e.g., in [27]. Same conclusion is reached also from the form of the corresponding Hamiltonian (the momenta are $p_i = \sqrt{\lambda}\pi_i$): $\mathcal{H} = \frac{1}{2}\pi_{y_a}^2 + \frac{1}{2}(\pi_{z_a} - A y_a)^2 + \frac{1}{2}B^2 y_a^2$ (we omit the overall factor of string tension $\sqrt{\lambda}$). Shifting y_a by $-\frac{A}{A^2+B^2}\pi_{z_a}$ to isolate the oscillator dynamics in y_a-directions we end up with

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\big[\pi_{y_a}^2 + \big(A^2 + B^2\big)\tilde{y}_a^2\big] + \frac{1}{2}\frac{B^2}{A^2 + B^2}\pi_{z_a}^2 \\ &= \frac{1}{2}\big(\pi_{y_a}^2 + 4\mathcal{J}^2\tilde{y}_a^2\big) + \frac{1}{2}\frac{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}{\tilde{\boldsymbol{\gamma}} + \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}\pi_{z_a}^2.\end{aligned} \tag{2.39}$$

To express π_{z_a} in terms of fluctuations of the angular momenta j_i in (2.32) one should note that since π_{z_a} is given by a linear combination of $\dot{z}_a$ and y_a (cf. (2.36)), redefining the latter by π_{z_a} changes also the relation between π_{z_a} and $\dot{z}_a$ (and thus commutation relations, etc.). Equivalently, the same result for H is found in a more transparent way by performing the fluctuation analysis directly in the Hamiltonian for the σ-independent modes, i.e., by expanding both the coordinates and the momenta. In terms of the fluctuations π_i of the momenta of the angular coordinates in (2.32), the required phase-space redefinition that separates the oscillator dynamics from the free particle dynamics is

$$\begin{aligned}u_1 &= \frac{1}{2\mathcal{J}}\left[\sqrt{\frac{\tilde{\gamma}_3\tilde{\boldsymbol{\gamma}}}{\tilde{\gamma}_1(\tilde{\gamma}_2 + \tilde{\gamma}_3)}}y_1 - \sqrt{\frac{\tilde{\gamma}_2}{\tilde{\gamma}_2 + \tilde{\gamma}_3}}y_2 + \frac{\tilde{\boldsymbol{\gamma}}}{\tilde{\gamma}_1}\pi_1\right], \\ u_2 &= \frac{1}{2\mathcal{J}}\left[\sqrt{\frac{\tilde{\gamma}_1 + \tilde{\gamma}_3}{\tilde{\gamma}_1}}y_2 + \frac{\tilde{\boldsymbol{\gamma}}}{\tilde{\gamma}_1}\pi_2\right].\end{aligned} \tag{2.40}$$

This leads to the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\big(\pi_{y_a}^2 + 4\mathcal{J}^2 y_a^2\big) + \frac{1}{2\mathcal{J}}\big[\tilde{\gamma}_2(\tilde{\gamma}_1 + \tilde{\gamma}_3)\pi_1^2 + 2\tilde{\gamma}_1\tilde{\gamma}_2\pi_1\pi_2 + \tilde{\gamma}_1(\tilde{\gamma}_2 + \tilde{\gamma}_3)\pi_2^2\big]. \tag{2.41}$$

Thus the 0-mode contribution to the fluctuation energy spectrum expressed in terms of the angular momenta of the fluctuation modes in (2.32)

$$j_i = J_i - \frac{\gamma_i}{\boldsymbol{\gamma}}\mathrm{J}, \qquad \sum_{i=1}^{3} j_i = 0 \tag{2.42}$$

takes the form ($E = \sqrt{\lambda}\mathcal{H}$)

$$E_{0\text{-mode}} = \frac{\lambda}{2\mathrm{J}}\big[\gamma_2(\gamma_1 + \gamma_3)j_1^2 + 2\gamma_1\gamma_2 j_1 j_2 + \gamma_1(\gamma_2 + \gamma_3)j_2^2\big]. \tag{2.43}$$

In the case when $\gamma_i = \gamma$ Eq. (2.43) becomes simply

$$E_{0\text{-mode}} = \frac{\lambda\gamma^2}{\mathrm{J}}\left[\left(J_1 - \frac{1}{3}\mathrm{J}\right)^2 + \left(J_2 - \frac{1}{3}\mathrm{J}\right)^2 + \left(J_1 - \frac{1}{3}\mathrm{J}\right)\left(J_2 - \frac{1}{3}\mathrm{J}\right)\right], \tag{2.44}$$

reproducing the expression in [12]

$$E_{0\text{-mode}} = \frac{\lambda\gamma^2}{3\mathrm{J}}\left[(J_1 - J_2)^2 + (J_1 - J_3)^2 - (J_1 - J_2)(J_1 - J_3)\right]. \tag{2.45}$$

Let us stress that the correct quantization of the zero mode sector should not be based on the expansion to quadratic order in fluctuations but should start directly with the (supersymmetric version of) the Neumann model (2.24), i.e., from the corresponding 0-mode truncation of the superstring action. In the undeformed $AdS_5 \times S^5$ case this amounts to quantizing the corresponding superparticle action leading to the spectrum of the BPS (supergravity) modes. The first attempt in this direction would be to keep only the bosonic fields (and thus ignore the "mixing" with the AdS_5 directions via fermions) and try to use the known information about quantum Neumann model (see, e.g., [30]). Such 0-mode sector quantization of this γ_i deformed string theory would be equivalent (for integer ν_i) to a "minisuperspace" quantization of the original $AdS_5 \times S^5$ string theory in the sector of rotating and pulsating circular strings.

Assuming the large J limit to suppress quantum corrections, the expression (2.43) can be found directly from the particle Hamiltonian (2.24), (2.26) by considering the semiclassical configurations with constant ρ_i (to minimize energy for given J_i). In the limit of large $\mathcal{J}_i \sim \mathcal{J}$ with fixed $\nu_i = \epsilon_{ijk}\tilde{\gamma}\gamma_j\mathcal{J}_k$ it is sufficient to evaluate the energy on the semiclassical configuration that extremises the dominant S^5 part of potential in (2.24):

$$\rho_i^2 = \frac{J_i}{\mathrm{J}}. \tag{2.46}$$

For zero γ_i, i.e., for the semiclassical particle states represented by geodesics on S^5 these are BPS states of undeformed theory. For $\gamma_i \neq 0$ if J_i happen to be equal to $\frac{\gamma_i}{\boldsymbol{\gamma}}\mathrm{J}$ these are the vacuum states of deformed theory having again $E = \mathrm{J}$. The energy of the configuration (2.46) is $E = \sqrt{\lambda}\sqrt{V}$, where $V = \sum_{i=1}^{3} V_i$ is the potential in (2.24) evaluated on ρ_i in (2.46), i.e.,

$$E = \sqrt{\mathrm{J}^2 + \lambda\left[\gamma_2\gamma_3 j_1^2 + \gamma_1\gamma_3 j_2^2 + \gamma_1\gamma_2 j_3^2 - \frac{1}{\mathrm{J}}(\gamma_1 + \gamma_2 + \gamma_3)^2 j_1 j_2 j_3\right]}, \quad j_i \equiv J_i - \frac{\gamma_i}{\boldsymbol{\gamma}}\mathrm{J}. \tag{2.47}$$

Here in general j_i or J_i are of the same order as $\mathrm{J} = J_1 + J_2 + J_3$. Expanding (2.47) in large J for fixed γj and $\tilde{\lambda} = \frac{\lambda}{\mathrm{J}^2}$ we get, as for semiclassical string states, to leading order in $\tilde{\lambda} = \frac{1}{\mathcal{J}^2}$

$$\begin{aligned} E = \mathrm{J} + \frac{\lambda}{2\mathrm{J}}\left[\left(\gamma_2\gamma_3 j_1^2 + \gamma_1\gamma_3 j_2^2 + \gamma_1\gamma_2 j_3^2\right) - \frac{1}{\mathrm{J}}(\gamma_1 + \gamma_2 + \gamma_3)^2 j_1 j_2 j_3\right] \\ + O\left(\frac{\lambda^2\gamma^4 j^4}{\mathrm{J}^3}\right). \end{aligned} \tag{2.48}$$

The j^2 term here is the same as in (2.43). We shall reproduce this expression from the fast string limit in Section 3 or from the Landau–Lifshitz model in Section 5. We shall also obtain the same spectrum of 0-mode fluctuations (2.48) on the gauge theory side directly from the Bethe ansatz in Appendix B.

Let us note that in the limit which we consider (J $\to \infty$ with $\tilde{\lambda} = \frac{\lambda}{\mathrm{J}^2}$ and $\mathrm{J}\gamma_i$ fixed) the j^2 contribution in (2.43) is finite provided $j_i \sim \mathrm{J}^{1/2}$ (or, for $\gamma_i = \gamma$, if $J_i - J_j \sim \mathrm{J}^{1/2}$ [12]). However, as already stressed above, the true condition of validity of (2.43) follows from the exact treatment of the 0-mode fluctuations and is only that $j_i \sim \mathrm{J}^{\mu}$, $\mu \leqslant 1$. The same condition will appear on the spin chain side in Appendix B. As for the quantum corrections, they are expected to modify (2.48) by terms with similar dependence on j_i but suppressed by extra powers of $1/\mathrm{J}$. Same structures should appear on the spin chain side as corrections to leading thermodynamic limit approximation.

3. Fast motion limit: Landau–Lifshitz action from the string action

Let us start with recalling the derivation of the reduced effective action that governs the dynamics of "slow" string degrees of freedom in the 3-spin ($su(3)$ invariant) sector of undeformed theory following [15]. We shall parametrize S^5 by 3 complex coordinates X_i such that

$$X_i = \rho_i e^{i\phi_i} \equiv U_i e^{i\psi}, \qquad \sum_{i=1}^{3} \rho_i^2 = 1, \tag{3.1}$$

where ρ_i and ϕ_i are real. We have isolated the common phase ψ that will be a collective coordinate representing fast string motion in the three planes. There is an obvious $U(1)$ gauge invariance $U_i \to e^{i\zeta} U_i$, $\psi \to \psi - \zeta$. The S^5 metric has then the form of the Hopf fibration of S^1 over $\mathbb{CP}^2$:

$$ds^2 = dX_i\, dX_i^* = \sum_{i=1}^{3} \left(d\rho_i^2 + \rho_i^2\, d\phi_i^2\right) = (d\psi + C)^2 + dU_i^*\, dU_i - C^2, \tag{3.2}$$

where $C = -iU_i^*\, dU_i$. The $\mathbb{CP}^2$ metric is $dU_i^*\, dU_i + (U_i^* dU_i)^2 = |DU_i|^2$ where $DU_i = dU_i - iCU_i$. We can then start with the general form of the bosonic part of the string action and apply the 2d duality (T-duality) in the ψ direction. The result is (including time direction of AdS_5 and assuming summation over i)

$$L = \epsilon^{pq} C_p \partial_q \tilde{\psi} - \frac{1}{2} \sqrt{-g} g^{pq} \left(-\partial_p t \partial_q t + \partial_p \tilde{\alpha} \partial_q \tilde{\alpha} + D_p U_i^* D_q U_i\right), \tag{3.3}$$

where the first term represents the 2-form coupling induced by off-diagonal form of the metric. The next step is solve for the 2d metric, replacing the second term in L by its Nambu counterpart, $\sqrt{-\det h}$, $h_{pq} = -\partial_p t \partial_q t + \partial_p \tilde{\alpha} \partial_q \tilde{\alpha} + D_p U_i^* D_q U_i$. The final step is to fix a static gauge: $t = \tau$, $\tilde{\alpha} = \mathcal{J}\sigma$, where the letter condition corresponds to fixing the angular momentum associated with the fast variable ψ, i.e., the total momentum $\mathrm{J} = J_1 + J_2 + J_3$, to be homogeneously distributed along σ (as this is the property of the spin chain description of the corresponding states). Finally, expanding in large $\mathcal{J}$ and assuming that time derivatives of the slow "transverse" variables U_i are small we end with the following $\mathbb{CP}^2$ analog of the Landau–Lifshitz action ($\tilde{\lambda} = \frac{\lambda}{\mathrm{J}^2}$):

$$I = \mathrm{J} \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\mathcal{L} + O\left(\tilde{\lambda}^2\right)\right], \qquad \mathcal{L} = -iU_i^* \partial_t U_i - \frac{1}{2} \tilde{\lambda} |D_\sigma U_i|^2. \tag{3.4}$$

Since this action has $U(1)$ gauge invariance (in addition to the global $SU(3)$ invariance), we may parametrize U_i in the same way as in (2.24), i.e., $U_i = \rho_i e^{i\phi_i}$ ($\sum_{i=1}^{3} \phi_i$ can be assumed to be gauge fixed to zero but will in any case decouple) getting explicitly

$$\mathcal{L} = \sum_{i=1}^{3} \rho_i^2 \dot{\phi}_i - \frac{1}{2}\tilde{\lambda}\left[\sum_{i=1}^{3} \rho_i'^2 + \sum_{i<j=1}^{3} (\phi_i' - \phi_j')^2 \rho_i^2 \rho_j^2\right]. \tag{3.5}$$

Note that since $\sum_{i=1}^3 \rho_i^2 = 1$ the WZ term depends (modulo a total derivative) only on $\phi_i - \phi_j$, i.e., $\sum_{i=1}^3 \phi_i$ indeed decouples. Other forms of this action were given in [16,17]. This $\mathbb{CP}^2$ action is integrable, i.e., the corresponding equations of motion admit a Lax pair representation. [10]

3.1. Deformed case

Let us now find a generalization of this $\mathbb{CP}^2$ action to the case of nonzero deformation parameters γ_i. We may choose $\psi = \sum_i \phi_i$ (e.g., set $\phi_1 = \psi - \varphi_2$, $\phi_2 = \psi + \varphi_1 + \varphi_2$, $\phi_3 = \psi - \varphi_1$) and start with the metric and B_2 background in (2.1), (2.2). After doing T-duality and the same gauge fixing as above we finish with the following generalization of (3.5)

$$\mathcal{L} = \sum_{i=1}^{3} \rho_i^2 \dot{\phi}_i - \frac{1}{2}\tilde{\lambda}\left[\sum_{i=1}^{3} \rho_i'^2 + \sum_{i<j=1}^{3} (\phi_i' - \phi_j' - \epsilon_{ijk}\bar{\gamma}_k)^2 \rho_i^2 \rho_j^2 - \bar{\boldsymbol{\gamma}}^2 \rho_1^2 \rho_2^2 \rho_3^2\right], \tag{3.6}$$

where

$$\bar{\gamma}_i \equiv \tilde{\gamma}_i \mathcal{J} = \gamma_i \mathrm{J}, \qquad \bar{\boldsymbol{\gamma}} \equiv \sum_{i=1}^{3} \bar{\gamma}_i. \tag{3.7}$$

Note the time-derivative (WZ) term does not get deformed.

Since the 3-parameter deformation of the full string model is integrable [3], this action should represent an integrable deformation of the $\mathbb{CP}^2$ LL model.[11]

The 2-spin case action is recovered by setting $\rho_3 = 0$; then $\rho_1^2 + \rho_2^2 = 1$ and the action depends only on γ_1 and reduces to the anisotropic version of the $\mathbb{CP}^2$ LL model found in [2].

We observe that the case of (3.6) with $\tilde{\boldsymbol{\gamma}} = 0$ is special: then the dependence on $\bar{\gamma}_i$ can be formally absorbed into a formal redefinition of ϕ_i (as was the case in the 2-spin sector in [2]), e.g., $\phi_1 \to \phi_1 + \bar{\gamma}_3\sigma$, $\phi_2 \to \phi_2$, $\phi_3 \to \phi_3 - \bar{\gamma}_1\sigma$; in terms of the shifted angles the action then becomes the same as (3.5).

Another special case is the symmetric one $\gamma_i = \gamma$ when we get explicitly

$$\mathcal{L} = \sum_{i=1}^{3} \rho_i^2 \dot{\phi}_i - \frac{1}{2}\tilde{\lambda}\mathcal{H}, \tag{3.8}$$

---

[10] To find the corresponding Lax representation it is useful to use the matrix form of the LL model [17] in which the LL equation takes the form (we rescale time to absorb $\tilde{\lambda}$): $\partial_t N = -\frac{i}{6}[N, \partial_\sigma^2 N]$. Here $N_{ij} = 3U_i^* U_j - \delta_{ij}$ satisfies $\operatorname{Tr} N = 0$, $N^\dagger = N^2 = N + 2$. This equation can be written as $\partial_t N = \partial_\sigma K$, where $K \equiv -\frac{i}{6}[N, \partial_\sigma N]$. We observe that N is "covariantly constant" $\partial_\sigma N = \frac{2i}{3}[N, K]$ and define the Lax connection (A_σ, A_t) as $A_\sigma = isN$, $A_t = isK + \frac{3i}{2}s^2 N$ (s is a spectral parameter). It then satisfies (as a consequence of the above two equations on N and K) $\partial_t A_\sigma - \partial_\sigma A_t - [A_t, A_\sigma] = 0$.

[11] Indeed, this action describes (a large $\mathcal{J}$) approximation to solutions of the original string action. It would be interesting to find explicitly the corresponding Lax pair.

$$\mathcal{H} = \sum_{i=1}^{3} \rho_i'^2 + (\phi_1' - \phi_2' - \bar{\gamma})^2 \rho_1^2 \rho_2^2 + (\phi_2' - \phi_3' - \bar{\gamma})^2 \rho_2^2 \rho_3^2 + (\phi_3' - \phi_1' - \bar{\gamma})^2 \rho_3^2 \rho_1^2 \\ - 9\bar{\gamma}^2 \rho_1^2 \rho_2^2 \rho_3^2. \tag{3.9}$$

Another generalization of (3.9) ("orthogonal" to the one in (3.6)) can be found by starting with more a general supergravity background in [1] dual to $\mathcal{N} = 4$ SYM deformation with complex parameter $\beta = \gamma - i\boldsymbol{\sigma}$. That background depends on both $\tilde{\gamma} = \sqrt{\lambda}\gamma$ and $\tilde{\boldsymbol{\sigma}} = \sqrt{\lambda}\boldsymbol{\sigma}$ and was obtained in [1] using S-duality transformations.[12] In this case the application of the above procedure leads to (3.9) with $\mathcal{H}$ replaced by[13]

$$\mathcal{H} = (\rho_1\rho_2' - \rho_2\rho_1' + \bar{\boldsymbol{\sigma}}\rho_1\rho_2)^2 + (\rho_3\rho_1' - \rho_1\rho_3' + \bar{\boldsymbol{\sigma}}\rho_3\rho_1)^2 + (\rho_2\rho_3' - \rho_3\rho_2' + \bar{\boldsymbol{\sigma}}\rho_2\rho_3)^2 \\ + (\phi_1' - \phi_2' - \bar{\gamma})^2 \rho_1^2 \rho_2^2 + (\phi_2' - \phi_3' - \bar{\gamma})^2 \rho_2^2 \rho_3^2 + (\phi_3' - \phi_1' - \bar{\gamma})^2 \rho_3^2 \rho_1^2 \\ - 9(\bar{\gamma}^2 + \bar{\boldsymbol{\sigma}}^2)\rho_1^2 \rho_2^2 \rho_3^2, \tag{3.10}$$

where $\bar{\boldsymbol{\sigma}} \equiv \tilde{\boldsymbol{\sigma}}\mathcal{J} = \boldsymbol{\sigma}\mathrm{J}$. This deformation of the $\mathbb{CP}^2$ action (3.5) is unlikely to be integrable. The action (3.6) admits a similar generalization to the case of the 3 different $\bar{\boldsymbol{\sigma}}_i$ parameters.

3.2. *Special solutions*

Let us now study some solutions of the action (3.6). The solutions are characterized by 3 conserved angular momenta

$$J_i = \mathrm{J}\int_0^{2\pi} \frac{d\sigma}{2\pi} \rho_i^2, \quad \mathrm{J} = J_1 + J_2 + J_3. \tag{3.11}$$

The main difference between (3.6) and its undeformed case (3.5) is the presence of the (non-negative) potential term:

$$V = \bar{\gamma}_3^2 \rho_1^2 \rho_2^2 + \bar{\gamma}_1^2 \rho_2^2 \rho_3^2 + \bar{\gamma}_2^2 \rho_3^2 \rho_1^2 - (\bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3)^2 \rho_1^2 \rho_2^2 \rho_3^2, \tag{3.12}$$

where ρ_i are subject to $\sum_{i=1}^3 \rho_i^2 = 1$. While in the case of $\gamma_i = 0$ the action (3.5) had σ-independent solutions $\phi_i = \text{const}$, $\rho_i = \text{const}$ (with ρ_i being arbitrary apart from $\sum_{i=1}^3 \rho_i^2 = 1$) which represented BPS geodesics with $E = \mathrm{J}$ now the potential selects only few of them that will minimize V and thus $E = \mathrm{J}(1 + \frac{1}{2}\tilde{\lambda}V)$, i.e., will have $V = 0$. These absolute minima of V correspond precisely to the BPS geodesics discussed above in Section 2, namely,

$$\text{(i)} \quad \rho_1 = 1, \quad \rho_2 = \rho_3 = 0; \\ \rho_2 = 1, \quad \rho_1 = \rho_3 = 0; \\ \rho_3 = 1, \quad \rho_2 = \rho_1 = 0, \tag{3.13}$$

$$\text{(ii)} \quad \rho_i = \sqrt{\frac{\bar{\gamma}_i}{\bar{\boldsymbol{\gamma}}}}, \quad \text{i.e.,} \quad J_i = \frac{\gamma_i}{\boldsymbol{\gamma}}\mathrm{J}. \tag{3.14}$$

[12] For comments on the corresponding string theory and AdS/CFT duality in this case see also [2].

[13] In this case B_2 has a term proportional to w_1 and one is to use (2.8).

Other geodesics described (in this large $\mathcal{J}$ approximation) by solutions with constant values of ρ_i (note that for σ-independent solutions $\phi_i - \phi_j$ play in (3.6) the role of the Lagrange multipliers imposing the condition of constancy of ρ_i) will have nonzero value of the energy. Explicitly, we find from (3.12) that for generic point-like solutions

$$E = \mathrm{J} + \frac{1}{2}\tilde{\lambda}\left[\frac{1}{\mathrm{J}}\left(\bar{\gamma}_3^2 J_1 J_2 + \bar{\gamma}_1^2 J_2 J_3 + \bar{\gamma}_2^2 J_3 J_1\right) - \frac{1}{\mathrm{J}^2}(\bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3)^2 J_1 J_2 J_3\right]. \tag{3.15}$$

For fixed $\bar{\gamma}_i$ such dependence of energy on spins J_i is characteristic of macroscopic string solutions [7]. An alternative representation making it clear that the energy vanishes for $J_i \sim \gamma_i$ states is (cf. (2.48))

$$E = \mathrm{J} + \frac{1}{2}\tilde{\lambda}\left[\frac{1}{\mathrm{J}}\left[\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_3) j_1^2 + 2\bar{\gamma}_1\bar{\gamma}_2 j_1 j_2 + \bar{\gamma}_1(\bar{\gamma}_2 + \bar{\gamma}_3) j_2^2\right] \right.$$
$$\left. - \frac{1}{\mathrm{J}^2}(\bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3)^2 j_1 j_2 j_3\right], \tag{3.16}$$

where as in (2.42) $j_i = J_i - \frac{\bar{\gamma}_i}{\bar{\gamma}_1+\bar{\gamma}_2+\bar{\gamma}_3}\mathrm{J}$. In the LM case of $\bar{\gamma}_i = \bar{\gamma}$ this can be written also as

$$E = \mathrm{J} + \tilde{\lambda}\bar{\gamma}^2\left[\frac{1}{\mathrm{J}}\left(j_1^2 + j_1 j_2 + j_2^2\right) - \frac{9}{2\mathrm{J}^2} j_1 j_2 j_3\right], \qquad j_i = J_i - \frac{1}{3}\mathrm{J}, \tag{3.17}$$

where the j^2 term is recognized to be equivalent to the 0-mode contribution in (2.44) or (2.45). Thus the zero-mode contributions are easily captured by the LL model.

If $\tilde{\lambda}$, $\bar{\gamma}$ and j^2/J are fixed as one may assume in the discussion of the quadratic zero-mode fluctuation contribution to the energy spectrum near the (J, J, J) geodesic, then j^3/J^2 term is subleading. As already mentioned in Section 2.3.2, the assumption that j^2/J is fixed is not needed in general, and in Appendix B we shall reproduce the whole expression (3.16) including the j^3 term from the Bethe ansatz on the gauge theory side.

It is also straightforward to study the non-zero part of the fluctuation spectrum near $(\mathrm{J}, 0, 0)$ or $J_i \sim \gamma_i$ geodesic and to show that it is agreement with the leading order $\tilde{\lambda}$ term in the corresponding part of spectrum found in Sections 2.3.1 and 2.3.2.[14]

Let us now comment on extended string solutions of the LL action. One observes that in general the circular string ansatz

$$\phi_i = m_i\sigma, \qquad \rho_i = \text{const}, \tag{3.18}$$

gives a solution of LL action (3.9) with $\gamma_i = \gamma$. Similar circular solutions exist in the full string equations and are analogs of the rigid circular strings in undeformed $AdS_5 \times S^5$ geometry [7,9].[15] Interestingly, as it is obvious from the comparison of (3.9) and (3.12), this case is formally equivalent to the case of *point-like* solutions in the γ_i-deformed theory with

$$\gamma_1 = \gamma + \frac{m_3 - m_2}{\mathrm{J}}, \qquad \gamma_2 = \gamma + \frac{m_1 - m_3}{\mathrm{J}}, \qquad \gamma_3 = \gamma + \frac{m_2 - m_1}{\mathrm{J}}, \qquad \sum_{i=1}^{3}\gamma_i = 3\gamma. \tag{3.19}$$

[14] One can also study fluctuations near more general geodesics, and here the spectrum will be similar to the one found in [9,29] near circular rotating strings in undeformed theory.

[15] Some string solutions in LM geometry were discussed in [34].

In this case we know that there are vacuum (J_1, J_2, J_3) states provided $J_i = \frac{\gamma_i}{\gamma_1+\gamma_2+\gamma_3}\mathrm{J}$. Here that leads to the conclusion that we can have ground-state solutions provided γ takes special rational values. These are, in fact, the string BPS states found in the supersymmetric deformed theory in [1]. They are TsT images of particular point-like BPS states in the undeformed $AdS_5 \times S^5$ theory (there perturbative large N BPS states are represented only by point-like strings).

Similar remark applies if we start with the generic γ_i case of LL action (3.6): specifying to the sector of circular strings with constant "radii" ρ_i is equivalent to studying point-like states in the theory with shifted γ_i parameters: $\gamma_1 \to \hat{\gamma}_1 = \gamma_1 + \frac{m_3-m_2}{\mathrm{J}}$, etc. One then finds additional ground states for special values of J_i and γ_i such that $\epsilon_{ijk} J_j \hat{\gamma}_k$ are zero.

4. Dilatation operator of deformed gauge theory

4.1. The spin chain Hamiltonian for the holomorphic 3-scalar sector

The one-loop planar dilatation operator of the $\mathcal{N}=4$ SYM theory in the holomorphic 3-scalar sector (i.e., the anomalous dimension matrix for the operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}+\cdots)$ built out of chiral scalars Φ_i, $i=1,2,3$) can be written as an $su(3)$ invariant nearest-neighbor ferromagnetic spin chain Hamiltonian [22]:

$$H = \sum_{k=1}^{L} H_{k,k+1}, \qquad H_{k,k+1} = \frac{\lambda}{8\pi^2}\mathcal{H}_{k,k+1}, \tag{4.1}$$

$$\mathcal{H}^{(0)}_{k,k+1} \equiv \mathbb{I}_{k,k+1} - \mathcal{P}_{k,k+1}. \tag{4.2}$$

Here H acts on products of 3-vectors at each site of the spin length L which is equal to the total momentum (in discussion of spin chains and Bethe ansatz we shall use the notation L instead of J)

$$L = \mathrm{J} \equiv J_1 + J_2 + J_3. \tag{4.3}$$

$\mathbb{I}$ is an identity and $\mathcal{P}$ is the permutation operator. In terms of the generators $(e_n^m)_i^j \equiv \delta_i^m \delta_n^j$ of the algebra $gl(3)$ we have

$$\mathbb{I}_{k,k+1} = \mathbb{I}_k \otimes \mathbb{I}_{k+1} = \sum_{m,n=1}^{3} e_m^m(k) e_n^n(k+1),$$
$$\mathcal{P}_{k,k+1} = \sum_{m,n=1}^{3} e_n^m(k) e_m^n(k+1). \tag{4.4}$$

The generalization of (4.1) to the case of the β-deformed $\mathcal{N}=4$ SYM theory was found in [18, 19]. It has a formal generalization to the case of 3 complex deformation parameters $q_i = e^{i\pi\beta_i}$ (here $e \otimes e \equiv e(k)e(k+1)$):

$$\begin{aligned}
\mathcal{H}_{k,k+1} = {} & |q_1|^{-2} e_2^2 \otimes e_3^3 + |q_1|^2 e_3^3 \otimes e_2^2 - \frac{q_1}{\bar{q}_1} e_2^3 \otimes e_3^2 - \frac{\bar{q}_1}{q_1} e_3^2 \otimes e_2^3 \\
& + |q_2|^{-2} e_3^3 \otimes e_1^1 + |q_2|^2 e_1^1 \otimes e_3^3 - \frac{q_2}{\bar{q}_2} e_3^1 \otimes e_1^3 - \frac{\bar{q}_2}{q_2} e_1^3 \otimes e_3^1 \\
& + |q_3|^{-2} e_1^1 \otimes e_2^2 + |q_3|^2 e_2^2 \otimes e_1^1 - \frac{q_3}{\bar{q}_3} e_1^2 \otimes e_2^1 - \frac{\bar{q}_3}{q_3} e_2^1 \otimes e_1^2.
\end{aligned} \tag{4.5}$$

This expression appeared in [18] as a step in the construction of the Hamiltonian for the supersymmetric deformation in the 3-spin sector which corresponds to equal parameters $q_i = q = e^{i\pi\beta}$.[16] It was noticed in [18] that the complex β deformation is not contained in the class of integrable deformations of $su(3)$-invariant Heisenberg chain described by a twisted R-matrix. It was later argued [19] that the spin chain describing the complex β case is not integrable.

In the case of real $\beta_i \equiv \gamma_i$ which we will be interested in here (4.5) becomes

$$\mathcal{H}_{k,k+1} = \mathbb{I}_{k,k+1} - \tilde{\mathcal{P}}_{k,k+1}, \quad \tilde{\mathcal{P}}_{k,k+1} = \sum_{m,n=1}^{3} e^{-2i\pi\alpha_{nm}} e_n^m(k) e_m^n(k+1), \tag{4.6}$$

$$\alpha_{mn} \equiv -\epsilon_{mni}\gamma_i. \tag{4.7}$$

This gives the 1-loop dilatation operator of the nonsupersymmetric deformation of $\mathcal{N}=4$ SYM theory [3] which should be dual to string theory defined by (2.1)–(2.8). This gauge theory has the following scalar potential [3]

$$V = \operatorname{Tr} \sum_{n>m=1}^{3} \left| e^{-i\pi\alpha_{mn}} \Phi_m \Phi_n - e^{i\pi\alpha_{mn}} \Phi_n \Phi_m \right|^2 + \operatorname{Tr} \sum_{m=1}^{3} [\Phi_m, \bar{\Phi}_m]^2 \tag{4.8}$$

and similarly deformed Yukawa couplings to ensure the marginality of the deformation as well as the cancellation of the self-energy corrections to the anomalous dimension matrix.

The terms displayed in (4.6) are determined by the interactions in the first sum in (4.8); for (4.6) to be indeed the dilatation operator it is necessary that, for general γ_i, the contribution of self-energy graphs, vector exchange graphs and the graphs containing $[\Phi_m, \bar{\Phi}_m]^2$ interaction vertices continue to cancel out, just like in the supersymmetric theory case. This cancellation is relatively easy to understand based on the similarity between the β deformation and noncommutative theories [1,33]: here we have a noncommutative structure related to the $U(1)$ symmetries inherited from the R-symmetry of the undeformed theory. In noncommutative theories, planar graphs in the deformed theory are equal [36] to those in the undeformed theory except that the external fields are multiplied with a *-product.[17] The cancellation mentioned above occurs as follows. The vector exchange graphs are independent of the deformation because the vector-scalar–scalar coupling is independent of γ_i. Similarly, the vertices analogous to those coming from the "D-term" $[\Phi_m, \bar{\Phi}_m]^2$ are undeformed because the charge vectors of Φ_m and $\bar{\Phi}_m$ are proportional. The deformation of the Yukawa couplings is done again with the *-product which now contains the fermion $U(1)$-charges (equal to their R-charges in the undeformed theory). The contribution of fermions to self-energy thus may have a nontrivial phase, but, based on the noncommutative structure, the planar self-energies would be the same as in the $\mathcal{N}=4$ theory except for a *-product between the external fields. Due to the $U(1)$-charge conservation, this phase is 1, i.e., there is no nontrivial contribution. As a result, there is the same cancellation as in the $\mathcal{N}=4$

[16] In the general complex β case one needs also a rescaling of the coefficient λ in front of $\mathcal{H}_{k,k+1}$ as discussed in [2].

[17] The noncommutativity due to the γ_i-deformation is certainly different from the one discussed in [36], so one may be inclined to question the applicability of the results of [36] to our case. Abstractly, the results of [36] are based on the fact that the fields carry certain additive charges and that the corresponding symmetry generators obey the chain rule. These are the properties of the $U(1)$ symmetries inherited from the R-symmetry of the $\mathcal{N}=4$ SYM theory in the present case as well as the momentum generators in the case of [36]. The difference between the two cases is that, while all the fields carry momentum, in our case of the γ_i-deformed theory some fields have trivial charges so are not affected by the *-product. We will return to this point at the end of this section.

theory between the vector exchange, self-energy and the contribution of the $[\Phi_m, \bar{\Phi}_m]^2$ vertices, as it should be for (4.6) to correspond to the one-loop dilatation operator of the gauge theory deformation suggested in [3].

Let us note that for the non-holomorphic sectors of the β deformed theory it is complicated to construct the dilatation operator by a direct computation, even in the case of real deformation. Using different techniques, it was shown in [32] that, for any sector, the Hamiltonian of the spin chain in the deformed theory H is related to the Hamiltonian in the undeformed theory $H^{(0)}$ by

$$H_{k,k+1} = \mathcal{U}_{k,k+1} H^{(0)}_{k,k+1} \mathcal{U}^{-1}_{k,k+1}, \quad \mathcal{U}_{k,k+1} = e^{i\pi \sum_{m,n=1}^{3} \alpha_{mn} \mathrm{h}_m(k) \mathrm{h}_n(k+1)}, \tag{4.9}$$

where $\mathrm{h}_n(k)$ are the Cartan generators of the symmetry group acting at site k. In the case of our present interest, i.e., the holomorphic 3-scalar sector

$$\mathcal{H}_{k,k+1} = \mathcal{U}_{k,k+1} (\mathbb{I}_{k,k+1} - \mathcal{P}_{k,k+1}) \mathcal{U}^{-1}_{k,k+1}, \tag{4.10}$$

$$\mathcal{U}_{k,k+1} = e^{i\pi \sum_{m,n=1}^{3} \alpha_{mn} e_m^m(k) e_n^n(k+1)} = \sum_{m,n=1}^{3} e^{i\pi\alpha_{mn}} e_m^m(k) e_n^n(k+1), \tag{4.11}$$

where we used (4.4) and that $e_n^n(k) e_m^m(k) = \delta_n^m e_m^m(k)$.

4.2. The Bethe ansatz

As usual, the diagonalization of a spin chain Hamiltonian with more than two states per site is done through the nested Bethe ansatz algorithm. From the details described in [18] it is straightforward though tedious to derive the Bethe equations for the 3-spin sector; one can also specialize the results of [32] to this case. The resulting Bethe equations are

$$e^{-2i\pi L\alpha_{21}} \left[\frac{u_{1,k} + \frac{i}{2}}{u_{1,k} - \frac{i}{2}} \right]^L = \prod_{\substack{i=1 \\ i \neq k}}^{J_2+J_3} \frac{u_{1,k} - u_{1,i} + i}{u_{1,k} - u_{1,i} - i} \left[\prod_{j=1}^{J_3} e^{-2i\pi(\alpha_{32}+\alpha_{21}+\alpha_{13})} \frac{u_{1,k} - u_{2,j} - \frac{i}{2}}{u_{1,k} - u_{2,j} + \frac{i}{2}} \right], \tag{4.12}$$

$$e^{2i\pi L(\alpha_{21}+\alpha_{13})} = \prod_{\substack{j=1 \\ j \neq l}}^{J_3} \frac{u_{2,l} - u_{2,j} + i}{u_{2,l} - u_{2,j} - i} \left[\prod_{i=1}^{J_2+J_3} e^{2i\pi(\alpha_{32}+\alpha_{21}+\alpha_{13})} \frac{u_{1,i} - u_{2,l} + \frac{i}{2}}{u_{1,i} - u_{2,l} - \frac{i}{2}} \right]. \tag{4.13}$$

Here $L = J_1 + J_2 + J_3$ and we should add also the condition that the eigenvectors are related to single-trace operators (the cyclicity condition):

$$e^{-2i\pi(J_2\alpha_{21}+J_3\alpha_{31})} \prod_{k=1}^{J_2+J_3} \frac{u_{1,k} + \frac{i}{2}}{u_{1,k} - \frac{i}{2}} = 1. \tag{4.14}$$

The contribution of a given Bethe root solution to the energy is

$$E = \sum_{k=1}^{J_2+J_3} \epsilon_k = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J_2+J_3} \frac{1}{u_{1,k}^2 + \frac{1}{4}}. \tag{4.15}$$

4.3. Ground states of the spin chain Hamiltonian

The vacua of the spin chain Hamiltonian should correspond to the "BPS" states of gauge theory that have zero anomalous dimensions. For the supersymmetric deformation with $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ there are at least two ways of finding them: (i) finding the generators of the chiral ring and (ii) directly finding the solutions of the Bethe equations which have zero energy. Since for general nonsupersymmetric deformations the first option is not available, we shall therefore concentrate on the second approach (in the supersymmetric limit we shall be able to compare the results with those of the chiral ring analysis).

First, let us note that since the dilatation operator is positive semidefinite, the contribution of any Bethe root distribution to the energy must be non-negative. Indeed, from the Bethe equations (4.12)–(4.13) one can see that, as in the undeformed case, the Bethe roots occur in complex conjugate pairs which give positive contributions to the energy (4.15). Thus, the vacua of the spin chain fall into the two categories:

(1) $J_2 + J_3 = 0$ case in which the energy (4.15) is obviously zero;
(2) configurations of Bethe roots for which $\epsilon_k = 0$ for all $k = 1, \ldots, J_2 + J_3$.

The first class corresponds to the classical vacuum of the spin chain ($\operatorname{Tr} \Phi_1^{J_1}$ operator) which was chosen to derive the Bethe equations, as well as to its obvious $\mathbb{Z}_3$ images.

For the second class the expression (4.15) clearly implies that all rapidities $u_{1,k}$ are to be infinite. This is similar to the case of other BPS states $\operatorname{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})_{\mathrm{symm}}$ in the undeformed $\mathcal{N} = 4$ SYM theory. As in the undeformed case, we will take the difference between any two unequal rapidities to also go to infinity. This is necessary in order to focus on solutions which exist regardless of whether $J_2 + J_3$ is even or odd. Unlike the undeformed case, however, due to the presence of the deformation parameters γ_i or α_{mn}, not any such rapidity configuration will be a solution of the Bethe equations and the cyclicity condition.

The cyclicity condition (4.14) implies that the angular momenta J_2 and J_3 must be chosen such that

$$J_2\gamma_3 - J_3\gamma_2 = 0. \tag{4.16}$$

Then, the main Bethe equation (4.12) further implies that

$$J_1\gamma_3 - J_3\gamma_1 = 0. \tag{4.17}$$

Finally, taking the product of the auxiliary Bethe equations (4.13) in the limit in which $u_{1,k} \to \infty$ implies that the third combination of the deformation parameters and the angular momenta must vanish as well:

$$J_1\gamma_2 - J_2\gamma_1 = 0. \tag{4.18}$$

This discussion however is insufficient because it implies a rather large degeneracy due to the fact that the auxiliary Bethe equation (4.13) is nontrivial. It is, however, easy to see that, if we focus on solutions which exist regardless of the parity properties of J_3, we must have $|u_{2,k} - u_{2,l}|$ for all $k \neq l$ (and therefore $u_{2,k}$ for any k) approach the infinity.

We conclude that for the general three real deformation parameters γ_i the spin chain Hamiltonian has three vacua corresponding to the operators $\operatorname{Tr} \Phi_i^J$, $i = 1, 2, 3$, as well as the fourth vacuum corresponding to an operator containing J_i copies of Φ_i with $i = 1, 2, 3$ provided

$$\epsilon_{ijk} J_j \gamma_k = 0, \quad \text{i.e.,} \quad J_i \sim \gamma_i. \tag{4.19}$$

Since J_i are integer, this forth vacuum can exist only for special values of γ_i. This matches the result of the string theory analysis in Section 2.2.

In the case of supersymmetric deformation $\gamma_i = \gamma$ the above condition (4.19) becomes $J_1 = J_2 = J_3$. The existence of such (J, J, J) BPS state can be derived from the construction of the chiral ring. The argument is the same as originally given for rational deformation parameter in [33], see also [1]. Indeed, the F-term constraints

$$\begin{aligned}
&e^{i\pi\beta} \Phi_1 \Phi_2 - e^{-i\pi\beta} \Phi_2 \Phi_1 = 0, \qquad e^{i\pi\beta} \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_3 \Phi_2 = 0, \\
&e^{i\pi\beta} \Phi_3 \Phi_1 - e^{-i\pi\beta} \Phi_1 \Phi_3 = 0
\end{aligned} \tag{4.20}$$

imply that, in the chiral ring, any single-trace operator can be brought to the form

$$\mathrm{Tr}\big(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3}\big). \tag{4.21}$$

Then, the same F-term constraints allow one to move any of the Φ_i fields around the trace. In general, this multiplies the initial operator by a phase whose argument is proportional to the difference between the number of fields of different types than the one which is transported around the trace. For the operator to be an element of the chiral ring it is necessary that this phase is unity which in turn implies that

$$J_1 = J_2 = J_3. \tag{4.22}$$

For rational γ there are also additional BPS states [1] corresponding to rotating circular strings; they are, in fact, images of certain BPS states in undeformed theory under the TsT transformation. We have described them explicitly in Section 3.2. They are visible also in the Bethe ansatz. Indeed, for rational γ_i it is possible that

$$\epsilon_{ijk} J_j \gamma_k \in \mathbb{Z}. \tag{4.23}$$

This is enough to eliminate completely the deformation from the Bethe equations. Thus, the energy of the states with such (J_1, J_2, J_3) quantum numbers are identical to those in the undeformed theory, i.e., they should be exact BPS states (despite the theory not being supersymmetric for unequal γ_i).

We shall discuss fluctuations near these vacua as implied by the Bethe ansatz equations in Appendices A and B.

4.4. *Comment on $U(N)$ vs. $SU(N)$ gauge theory*

It is worth emphasizing that the planar dilatation operator (4.5) and the corresponding Bethe equations (4.12)–(4.13) hold for the β_i-deformation of the $\mathcal{N} = 4$ SYM with $U(N)$ gauge group. The distinction between the $U(N)$ and the $SU(N)$ case is nontrivial here even in the large N limit. More precisely, it is immaterial for "long" single-trace operators we discuss in the main part of this paper but matters for some "short" operators. Indeed, in the presence of the deformation, the $U(1)$ factor no longer automatically decouples, and that has interesting consequences; in particular, the $U(N)$ theory is not automatically conformal [38], having running couplings of $U(1)$ matter fields.

It was recently observed in [38,39] that, while in the supersymmetric $U(N)$ β-deformed SYM theory the operators $\mathrm{Tr}(\Phi_i\Phi_j)$ $(i \neq j)$ have nonvanishing one-loop anomalous dimension, their anomalous dimension is zero in the supersymmetric β-deformed theory with $SU(N)$ gauge group. The nonzero (in the large N limit) contribution in the $U(N)$ case comes entirely from the nondecoupled $U(1)$ factor.

It is easy to see that the expression (4.5) for the spin chain Hamiltonian representing planar 1-loop $U(N)$ dilatation operator implies that the anomalous dimension of $\mathrm{Tr}(\Phi_i\Phi_j)$ is

$$\Delta_{(\mathrm{Tr}(\Phi_i\Phi_j))} = \frac{\lambda}{2\pi^2}\sin^2\pi\alpha_{ij}, \tag{4.24}$$

where α_{ij} is given by (4.7). In the supersymmetric limit of equal deformation parameters this reproduces the result of [38] for the $U(N)$ theory. The same result (4.24) may be obtained from the Bethe equations (4.12)–(4.13) (and $\mathbb{Z}_3$ symmetry). In the case of a single excitation above the $(2,0,0)$ vacuum the Bethe equations simplify considerably; since the result is determined by a single rapidity, it may, in fact, be obtained from the cyclicity condition which trivially leads to

$$\Delta_{(\mathrm{Tr}(\Phi_1\Phi_j))} = \frac{\lambda}{2\pi^2}\sin^2\pi\alpha_{1j}. \tag{4.25}$$

The other $\mathrm{Tr}(\Phi_i\Phi_j)$ anomalous dimensions may be obtained using $\mathbb{Z}_3$ transformations or by changing the vacuum of the spin chain. The expression (4.25) is, in fact, the $L \to 2$ limit of the anomalous dimension of $\mathrm{Tr}(\Phi_1^{L-1}\Phi_j)$.

In the case of the deformation of the $SU(N)$ SYM theory these anomalous dimensions vanish due to an "accidental cancellation". As was pointed out in [38] in the case of the supersymmetric deformation of the $SU(N)$ theory, the superpotential contribution to the potential can be written as

$$V = \sum_{a=1}^{N^2-1}\Big[\big|\mathrm{Tr}\big([\Phi_1,\Phi_2]_\beta T^a\big)\big|^2 + \big|\mathrm{Tr}\big([\Phi_2,\Phi_3]_\beta T^a\big)\big|^2 + \big|\mathrm{Tr}\big([\Phi_3,\Phi_1]_\beta T^a\big)\big|^2\Big], \tag{4.26}$$

where $[\Phi_1,\Phi_2]_\beta = \Phi_1 * \Phi_2 - \Phi_2 * \Phi_1 = e^{i\pi\beta}\Phi_1\Phi_2 - e^{-i\pi\beta}\Phi_2\Phi_1$ and T^a are the $SU(N)$ generators. As a result, the anomalous dimension of holomorphic 2-field operators is proportional to $\mathrm{Tr}\,T^a$ which vanishes. Clearly, such a cancellation does not occur in the deformed $U(N)$ theory. In the undeformed $U(N)$ $\mathcal{N}=4$ SYM theory, the dilatation operator (4.1), (4.2) combined with the cyclicity of the trace still leads to the vanishing anomalous dimension for $\mathrm{Tr}(\Phi_i\Phi_j)$.[18]

It is not hard to see that, in the case of the nonsupersymmetric $SU(N)$ gauge theory with unequal $\beta_i = \gamma_i$ the cancellation due to the tracelessness of gauge group generators also takes place. As was mentioned above, the nonsupersymmetric γ_i-deformed theory is obtained by replacing in the component Lagrangian of the $\mathcal{N}=4$ SYM the ordinary product of fields with the

[18] The reason why there is a difference between the $SU(N)$ and $U(N)$ cases even in the large N limit has to do with non-decoupling of $U(1)$ part of scalar multiplets (in the pure gauge field sector $U(1)$ part of $U(N)$ always decouples at large N); it is also special to the case of length-2 operators. A quick way to see why the double-trace quartic scalar vertex present in the $SU(N)$ case ($(T^a)^i_j(T^a)^k_l = \delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l$) does contribute to the anomalous dimension of the $\mathrm{Tr}(\Phi_i\Phi_j)$ operator in the same way as the single-trace vertex is to consider the generating functional $Z(k)$ for the correlators of the $\mathrm{Tr}(\Phi^2)$ operators (suppressing all indices). The corresponding action will look like $S = \frac{N}{\lambda}\int[\cdots + \mathrm{Tr}(\Phi^2)^2 + \frac{1}{N}(1 + k(x))\,\mathrm{Tr}(\Phi^2)]$, where we do not make distinction between the structure of Φ^2 term in the vertex and in the operators for which $k(x)$ is a source. Then it is clear that derivatives over $k(x)$ will scale as N^0, which is the same cylinder-diagram scaling as for the 2-point function of $\mathrm{Tr}(\Phi^2)$ with insertions of $\mathrm{Tr}\,\Phi^4$ vertex. We thank K. Zarembo for a discussion of this point.

noncommutative product

$$\Phi_i * \Phi_j(x) \mapsto \lim_{y\to x} e^{-i\pi\alpha_{mn}\mathrm{h}_m(x)\mathrm{h}_n(y)}\Phi_i(x)\Phi_j(y), \tag{4.27}$$

where again $\alpha_{mn} = -\epsilon_{mnk}\gamma_k$, h_n are the three global $U(1)$ symmetry generators (i.e., $\mathrm{h}_n\Phi_i = \delta_{ni}\Phi_i$), and summation over m, n is assumed. Thus the potential relevant for the calculation of anomalous dimensions of scalar operators may be written as

$$V = \sum_{a=1}^{N^2-1}\Big[\big|\mathrm{Tr}\big([\Phi_1,\Phi_2]_{\gamma_3}T^a\big)\big|^2 + \big|\mathrm{Tr}\big([\Phi_2,\Phi_3]_{\gamma_1}T^a\big)\big|^2 + \big|\mathrm{Tr}\big([\Phi_3,\Phi_1]_{\gamma_2}T^a\big)\big|^2\Big]. \tag{4.28}$$

Using this form of V for the calculation of anomalous dimensions of the same $\mathrm{Tr}(\Phi_i\Phi_j)$ operators it is easy to see that their anomalous dimensions are proportional to

$$\big(e^{2i\pi\gamma_k\epsilon_{ijk}} - e^{-2i\pi\gamma_k\epsilon_{ijk}}\big)\,\mathrm{Tr}\,T^a, \tag{4.29}$$

and thus vanish again in the $SU(N)$ gauge group case regardless the values of the deformation parameters γ_i.

Such cancellations appear not to exist for longer operators, even in the supersymmetric $\gamma_i = \gamma$ case. Indeed, in that case the chiral ring argument appears to imply that the only chiral operators are in the representations (J, 0, 0), (0, J, 0), (0, 0, J) and (J, J, J).

Given that it is possible to break supersymmetry by an arbitrarily small amount (the deformation parameters γ_i are continuous) and that the spectrum of 1-loop anomalous dimensions has a gap, it is reasonable to search for protected operators in the nonsupersymmetric theory among the protected operators in its supersymmetric limiting case. Since we have argued that we already know all such operators with vanishing anomalous dimensions, we do not expect additional operators with vanishing anomalous dimensions for sufficiently small $\gamma_1, \gamma_2, \gamma_3$.

An interesting question is what is the dual string theory prediction for the anomalous dimensions of operators $\mathrm{Tr}(\Phi_i\Phi_j)$ or, alternatively, which theory does the dual string theory describe: the $U(N)$ or the $SU(N)$ gauge theory.[19] The answer is nontrivial here and appears to be $SU(N)$: even though the diagonal $U(1)$ gauge fields still decouple in the presence of the deformation, the $U(1)$ scalars and fermions do not. Due to the absence of coupling with the gauge fields, it is expected that the RG beta-functions of the couplings of these $U(1)$ fields is positive and they flow to zero at low energies.[20] For the supersymmetric deformation this is indeed the case so

[19] There are several notable differences between the γ_i-deformed theory and "standard" noncommutative field theories where one finds nondecoupling of the $U(1)$ gauge fields. As was mentioned above, the *-product describing the γ_i deformation can be thought of as a Moyal product based on the Cartan generators of the remnant of the $\mathcal{N}=4$ SYM R-symmetry group. It acts nontrivially only on the fields carrying nonzero charges under these generators, i.e., the fermions and the scalar fields, but *not* the gauge fields. Therefore, one is allowed to truncate away the $U(1)$ gauge field, but not the $U(1)$ scalars and fermions. An analog of this non-decoupling in matter sector may be observed on the string theory side as well. Realizing the gauge theory on a collection of coincident D3-branes, the decoupling of the diagonal $U(1)$ degrees of freedom is associated to the translational invariance of the collection of branes. The form of the string background (2.1)–(2.7) dual to the deformed gauge theory suggests that in a similar set-up there should exist nontrivial fluxes in the space transverse to the branes. These fluxes will break translational invariance and lead to the nondecoupling of the fields carrying charges under the flat space "transverse", i.e., internal, symmetry group. Translational invariance if, however, maintained along the branes and thus the gauge fields continue to decouple. This is also reflected in the fact that here we have the standard AdS_5 factor in the geometry while in the noncommutative case the solution of [37] does not have an AdS_5 asymptotics.

[20] We thank O. Aharony, J. Maldacena and E. Sokachev for discussions on this point.

only the $SU(N)$ theory is conformal [35,38]. For the nonsupersymmetric deformation new effects may appear. For example, the three terms $|[\Phi_i, \Phi_j]_{\beta_k}|^2$ are related by a $\mathbb{Z}_3$ symmetry which also acts on the deformation parameters. Thus, it is possible that their coefficients undergo different renormalization and, while equal at one-loop level, at higher loops they may have different values at the fixed point. Similarly to the case of the supersymmetric deformation, further new phenomena may appear for rational values of β_i related to the appearance of additional operators with vanishing anomalous dimensions.

The observation that the coupling of the $U(1)$ matter fields runs suggests that the string theory in the deformed background (2.1)–(2.7) describes the deformation of the conformal $SU(N)$ gauge theory. The anomalous dimensions of holomorphic operators $\mathrm{Tr}(\Phi_i\Phi_j)$ with $i \neq j$ were computed to two loops in the supersymmetric case in [39] and found to be subleading in the $1/N$ expansion. It may be possible that these operators remain marginal to all orders in the planar limit. It would be interesting to check this explicitly, by analyzing in supergravity the corrections to the masses of the fields in the **20** of $SO(6)$ once the deformation is turned on.

5. Coherent state effective action

In the case of undeformed $\mathcal{N} = 4$ SYM – $AdS_5 \times S^5$ string duality the matching of predictions for energies of states with large quantum numbers can be done in a universal way by comparing the effective action for the long wave length spin chain excitations with the effective action for the "slow" world sheet modes obtained as a limit of the classical string action after separating the "fast" collective string modes [14].

The relevant spin chain degrees of freedom can be described by the spin coherent states $|n\rangle\rangle$ with the action

$$S_{\mathrm{coh}} = i\,\langle\langle n|\partial_t|n\rangle\rangle - \langle\langle n|H|n\rangle\rangle, \tag{5.1}$$

which appears in the exponent in the coherent state path integral. The limit one is interested in, i.e., $\mathrm{J} \to \infty$, $\tilde{\lambda} = \frac{\lambda}{\mathrm{J}^2}$, is the semiclassical limit for the spin chain path integral in which one can take the continuum limit keeping only the leading 2-derivative terms in S_{coh}.

In the case of the 2-spin sector, i.e., operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2} + \cdots)$, this strategy was successfully applied to the supersymmetric deformed theory [2], demonstrating the equivalence of the two effective actions. The case of the 3-spin sector (and larger nonholomorphic sectors) is, however, somewhat different. In the context of the coherent state continuum limit the problem arises in that a naive derivation that follows the same strategy as in the undeformed 3-spin case [16,17] or the deformed 2-spin case [2] leads to an effective action that does not properly describe all expected vacuum states as seen in the Bethe ansatz and also on the string theory side.

Indeed, as in the case of the 2-spin sector in [2], the fact that the "Wess–Zumino" term in the string action (3.6) is independent of the deformation parameters suggests that we may use the same coherent state as in the $su(3)$ sector in the undeformed theory. Then the resulting effective action as found in the continuum limit from (5.1) with H given by (4.1), (4.6) happens to contain the potential

$$V_{\mathrm{naive}} = \gamma_1^2\rho_2^2\rho_3^2 + \gamma_2^2\rho_3^2\rho_1^2 + \gamma_3^2\rho_1^2\rho_2^2, \qquad \sum_{i=1}^{3}\rho_i^2 = 1, \tag{5.2}$$

where we used the same notation ρ_i as in the string-theory potential (3.12) for the corresponding coherent state parameter. Compared to (3.12) this potential misses the last ρ^6 term. As a result, it captures the vacua (J, 0, 0), etc., but misses the nontrivial one with $(J_1, J_2, J_3) \sim (\gamma_1, \gamma_2, \gamma_3)$ or (J, J, J) in the equal γ_i case.

To understand the source of the problem it is useful to recall the story of BPS vacua in the $su(3)$ sector of the undeformed theory. There the spin chain Hamiltonian (4.2) containing permutation operator has vacua represented by all totally symmetrized products of the three chiral fields $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})_{\mathrm{symm}}$. Apart from $\mathrm{Tr}\,\Phi_1^J$, $\mathrm{Tr}\,\Phi_2^J$ and $\mathrm{Tr}\,\Phi_3^J$, i.e., (J, 0, 0), (0, J, 0) and (0, 0, J) vacua these are *not* "slow" modes of the spin chain: the field component Φ_i in general changes rapidly from site to site. The coherent state operators that are mapped onto semiclassical string states (in this case geodesics or point-like strings all of which here are BPS) are particular linear combinations of these quantum spin chain vacua:[21] if the generic coherent state operator is $\mathrm{Tr}(\prod_{k=1}^{\mathrm{J}}[\sum_{i=1}^3 n_i(k)\Phi_i])$ then the vacua correspond to constant n_i, i.e., to $\mathrm{Tr}(\sum_{i=1}^3 n_i\Phi_i)^{\mathrm{J}}$, which are indeed linear combinations of symmetrized products.

In the present deformed case we do not have all possible (J_1, J_2, J_3) BPS quantum vacua to built a coherent linear superposition, and, moreover, the nontrivial vacuum $(J_1, J_2, J_3) \sim (\gamma_1, \gamma_2, \gamma_3)$ is not a "slow" state. Yet, the fact that it is naturally found also on the string theory side suggests that there should be a way to capture it in the coherent state action.

The source of the problem thus appears to be in the choice of a description of the relevant spin chain modes by coherent states. One is either to generalize the definition of coherent states, or, alternatively, to use the "undeformed" coherent states but choose a different representative in the class of equivalent spin chain Hamiltonians with the same spectrum.

The latter option is equivalent to changing the basis. The spin chain Hamiltonian represents the gauge-theory anomalous dimension matrix in the basis of single-trace single-term operators. As we shall show below, there is a way to choose a more suitable basis so that the resulting coherent state action (5.1) adequately describes the "low-energy" approximation with all vacua included, and, moreover, matches its string-theory counterpart (3.6).

We shall start with reviewing the choice of the coherent states which will be the same as in the undeformed $su(3)$ case. We shall then describe a change of basis leading to an equivalent (but more appropriate for the low-energy description with standard set of coherent states) Hamiltonian $\tilde{H} = U^{-1}HU$. Finally, we shall use $\tilde{H}$ to compute S_{coh} in (5.1) and find its continuum limit.

5.1. The coherent state

While the standard definition of coherent states based on global symmetry of the Hamiltonian[22] does not formally apply in the present deformed case, we can still use the $SU(3)$-invariant coherent state which is a tensor product over the spin 1 (3-component) chain sites of a state obtained by a 3×3 rotation which keeps fixed some specified 3-vector:

$$R = R(h)R(k),$$

$$R(h) = \mathrm{diag}\big(e^{ih_1}, e^{ih_2}, e^{ih_3}\big), \qquad \sum_{i=1}^3 h_i = 0,$$

[21] All quantum BPS vacua correspond to Kaluza–Klein modes (spherical harmonics), while their particular coherent combinations have semiclassical interpretation as point-particles moving along geodesics of S^5. All such geodesics are related by $SO(6)$ rotations.

[22] In general, it is given by an G/G_0 transformation applied to a ground state, where G is a symmetry of H and G_0 is a symmetry of the ground state.

$$R(k) = \mathbb{I} - 2|k\rangle\langle k|, \quad \langle k|k\rangle = 1. \tag{5.3}$$

This state can be parametrized by an element of $\mathbb{CP}^2$. With appropriate redefinitions ($k_1 = \frac{1}{2}\sqrt{1-n_1},\ k_2 = -\frac{n_2}{2\sqrt{1-n_1}},\ k_3 = -\frac{n_3}{2\sqrt{1-n_1}}$) the coherent state can be written as:

$$|n\rangle = n_1|1\rangle + n_2|2\rangle + n_3|3\rangle, \tag{5.4}$$

where e_n^m in (4.4) acts on $|i\rangle$ as

$$e_i^m|j\rangle = \delta_{ij}|m\rangle, \tag{5.5}$$

and

$$n_i = m_i e^{ih_i}, \quad \sum_{i=1}^{3} m_i^2 = 1, \ \sum_{i=1}^{3} h_i = 0. \tag{5.6}$$

On the string theory side (cf. (3.6)) m_i will correspond to ρ_i, and h_i to ϕ_i but for generality we shall use this separate notation.

The total spin-chain state coherent state is then

$$|n\rangle\rangle = |n\rangle_1 \otimes |n\rangle_2 \otimes \cdots \otimes |n\rangle_L, \quad |n\rangle_k = \sum_{i=1}^{3} n_i(k)|i\rangle. \tag{5.7}$$

It thus corresponds to the operator

$$\mathrm{Tr}\left[\left(\sum_{i=1}^{3} n_i(1)\Phi_i\right)\cdots\left(\sum_{j=1}^{3} n_j(L)\Phi_j\right)\right], \tag{5.8}$$

up to the cyclicity of the trace composed with cyclic permutations of the site labels $1, 2, \ldots, L$.[23]

With this choice of the coherent state the first WZ term in the continuum effective action (5.1) has the same standard form as in the undeformed case:

$$S_{\mathrm{WZ}} = \mathrm{J}\int_0^{2\pi} \frac{d\sigma}{2\pi} \sum_{i=1}^{3} m_i^2 \dot{h}_i. \tag{5.9}$$

5.2. The change of basis: choice of an equivalent Hamiltonian

Let us first recall the meaning of the change of basis for the spin chain Hamiltonian. The precise statement about the relation between string energies and the gauge-theory anomalous dimensions is that the string energies are equal to the eigenvalues of the dilatation operator. The latter are computed by finding the anomalous dimension matrix and then diagonalizing it. The spin chain Hamiltonian is the anomalous dimension matrix, that is the dilatation operator in the

[23] One may also include the factor of $\frac{1}{\sqrt{L}}$ that makes the state (and the operator) unit normalized. This factor ends up playing no role; it cancels because there are always L identical terms contributing to the expectation value of any operator.

basis of operators used to compute the anomalous dimension matrix,

$$\Delta O_A = H_A^B O_B, \tag{5.10}$$

where A, B are multi-indices. The spin chain Hamiltonian (4.1), (4.5), (4.6) was computed in the "standard" basis, that is the basis of single-term single-trace operators $\mathrm{Tr}(\Phi_{i_1} \cdots \Phi_{i_L})$. Changing this basis leads to a change of the expression for the spin chain Hamiltonian. From (5.10) we see that a general change of basis $O_A = U_A^B \tilde{O}_B$ acts on the Hamiltonian by the transformation

$$H \to \tilde{H} = U^{-1} H U. \tag{5.11}$$

Since the original H in (4.6) contains only nearest-neighbor interactions it is clear that the operator U which we need should be nontrivial since it should be able to generate in the continuum limit of S_{coh} higher than 4th powers of the "radii" m_i in order to get an effective potential that will have more than just three obvious vacua $m_i = (1, 0, 0)$, etc.

Consequently, U cannot be a site-wise tensor product. It is natural to try the next simple possibility, i.e., a product of operators overlapping only on one site

$$U = \prod_{k=1}^{L} U_{k,k+1}, \tag{5.12}$$

with the additional assumption that $U_{k-1,k}$ commutes with $U_{k,k+1}$. Without this additional assumption $U^{-1} H U$ would be a double sum over the spin chain sites and, therefore, would lead to a nonlocal effective action. Combined with the observation that the original spin chain Hamiltonian can be written in the form (4.10), i.e.,

$$H = \sum_{k=1}^{L} H_{k,k+1} = \sum_{k=1}^{L} \mathcal{U}_{k,k+1}^{-1} H_{k,k+1}^{(0)} \mathcal{U}_{k,k+1}, \tag{5.13}$$

this suggest the following natural ansatz:

$$U_{k,k+1}(\xi) = (\mathcal{U}_{k,k+1})^{\xi} = e^{i\xi\pi \sum_{m,n=1}^{3} \alpha_{mn} e_m^m(k) e_n^n(k+1)} = \sum_{m,n=1}^{3} e^{i\xi\pi\alpha_{mn}} e_m^m(k) e_n^n(k+1), \tag{5.14}$$

$$U_{k,k+1}^{-1}(\xi) = U_{k,k+1}(-\xi), \qquad U_{k,k+1}(1) = \mathcal{U}_{k,k+1}. \tag{5.15}$$

Here α_{mn} are the same phases as in (4.6) and ξ is a parameter. This ansatz is, in fact, quite unique. For example, allowing for off-diagonal generators in the exponent would violate the locality requirement.[24]

It is relatively easy to find the transformed Hamiltonian. The main observation is that in each term in the sum defining $\tilde{H}$ in (5.13) all factors of U cancel out except for those which have

[24] For generic complex ξ the transformation above acts on the basis operators by adding a phase and a rescaling depending on the order of fields in a monomial, so this is a rather simple change of the basis.

nontrivial overlap with $H_{k,k+1}$. We find

$$\begin{aligned}\tilde{H} &= U^{-1}(\xi)HU(\xi)\\ &= \sum_{k=1}^{L} U_{k-1,k}(-\xi)U_{k,k+1}(-\xi+1)U_{k+1,k+2}(-\xi)H^{(0)}_{k,k+1}\\ &\quad\times U_{k-1,k}(\xi)U_{k,k+1}(\xi-1)U_{k+1,k+2}(\xi)\\ &\equiv \sum_{k=1}^{L}\tilde{H}_{[k]},\end{aligned}\tag{5.16}$$

where $[k]$ is used to indicate the dependence on the sites $k-1, k, k+1, k+2$. Using trivial identities following from the properties of e^n_m, $\tilde{H}_{[k]}$ can be simplified to:

$$\begin{aligned}\tilde{H}_{[k]} &= \frac{\lambda}{8\pi^2}\tilde{\mathcal{H}}_{[k]}\\ &= \frac{\lambda}{8\pi^2}\sum_{m,n,p,r,q,t=1}^{3} e^{i\pi\xi(\alpha_{mr}-\alpha_{mn})}e^{i\pi\xi(\alpha_{qt}-\alpha_{pt})}e^{i\pi(\xi-1)(\alpha_{rq}-\alpha_{np})}\\ &\quad\times e^m_m(k-1)\big[e^n_n(k)e^p_p(k+1)H^{(0)}_{k,k+1}e^r_r(k)e^q_q(k+1)\big]e^t_t(k+2).\end{aligned}\tag{5.17}$$

5.3. Continuum limit and the effective action

The important piece of information in constructing the effective action is the cyclicity property of the states described by it. In the initial form (5.13) of H the states the Hamiltonian acted on were periodic. An arbitrary change of the basis may affect this and lead to nonperiodic states. The transformation (5.14) has the crucial property that it commutes with the shift operator. Therefore, the states the transformed Hamiltonian acts on continue to be cyclically symmetric. This implies that we are allowed to use the coherent state (5.7) to construct the effective action.

Using the expression (4.2), (4.4) for the undeformed Hamiltonian in the $su(3)$ sector or $[H^{(0)}_{k,k+1}]^{np}_{rq} = \delta^n_r\delta^p_q - \delta^p_r\delta^n_q$ it follows that the expectation value of $\tilde{\mathcal{H}}_{[k]}$ in the above coherent state $|n\rangle\rangle$ (5.7) is

$$\begin{aligned}&\langle\langle n|\tilde{\mathcal{H}}_{[k]}|n\rangle\rangle\\ &\quad= \sum_{n,p=1}^{3}\Bigg[\big(m_n(k)\big)^2\big(m_p(k+1)\big)^2\\ &\qquad- \sum_{q=1}^{3}\big(m_q(k-1)\big)^2 e^{i\pi\xi(\alpha_{qp}-\alpha_{qn})}e^{-2i\pi(\xi-1)\alpha_{np}}m_n(k)m_p(k)m_p(k+1)m_n(k+1)\\ &\qquad\times e^{i(h_n(k)-h_n(k+1)-h_p(k)+h_p(k+1))}\sum_{r=1}^{3}\big(m_r(k+2)\big)^2 e^{i\pi\xi(\alpha_{nr}-\alpha_{pr})}\Bigg].\end{aligned}\tag{5.18}$$

Expanding this expression in α_{mn} (i.e., in the deformation parameters $\gamma_i = \frac{1}{2}\epsilon_{inm}\alpha_{mn}$) and in the spin chain spacing a up to the second order and suitably combining the resulting terms we find

$(\partial m(k) \equiv \frac{m(k+1)-m(k)}{a})$

$$\langle\langle n|\tilde{\mathcal{H}}_{[k]}|n\rangle\rangle \simeq \sum_{i=1}^{3}\big(\partial m_i(k)\big)^2 + \sum_{p<n=1}^{3}\left[\partial h_p(k) - \partial h_n(k) + \frac{2\pi}{a}\alpha_{pn}\right]^2 \big(m_p(k)m_n(k)\big)^2$$
$$- \ 2\xi(1-\xi)\left(\frac{2\pi}{a}\right)^2\left(\sum_{p<n=1}^{3}\alpha_{pn}\right)^2\big[m_1(k)m_2(k)m_3(k)\big]^2. \tag{5.19}$$

As usual, the sum over sites is replaced by an integral over $\sigma \in [0, 2\pi]$, and using the relation between the lattice spacing a and the length of the chain ($L \equiv \mathrm{J} = \sum_{i=1}^{3} J_i$)

$$a = \frac{2\pi}{\mathrm{J}} \tag{5.20}$$

we get for the continuum limit of the coherent state expectation value of the transformed Hamiltonian (here $\tilde{\lambda} = \frac{\lambda}{\mathrm{J}^2}$, $' = \partial_\sigma$)

$$\langle\langle n|\tilde{H}|n\rangle\rangle = \frac{1}{2}\mathrm{J}\tilde{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\big[(m_1m_2' - m_2m_1')^2 + (h_1' - h_2' + \mathrm{J}\alpha_{12})^2(m_1m_2)^2$$
$$+ (m_2m_3' - m_3m_2')^2 + (h_2' - h_3' + \mathrm{J}\alpha_{23})^2(m_2m_3)^2$$
$$+ (m_3m_1' - m_1m_3')^2 + (h_3' - h_1' + \mathrm{J}\alpha_{31})^2(m_3m_1)^2$$
$$- 2\xi(1-\xi)(\mathrm{J}\alpha_{12} + \mathrm{J}\alpha_{23} + \mathrm{J}\alpha_{31})^2(m_1m_2m_3)^2\big]. \tag{5.21}$$

As required in our scaling limit, the action is finite (modulo the overall factor of J) for fixed $\tilde{\lambda} = \frac{\lambda}{\mathrm{J}^2}$ and $\mathrm{J}\alpha_{mn} = -\epsilon_{mni}\mathrm{J}\gamma_i$. It thus describes a particular sector of low-energy excitations of the spin chain ("macroscopic spin waves") which correspond to semiclassical fast-moving strings in the 3-spin sector.

We are now in position to determine the free parameter ξ in the definition of $U(\xi)$ by requiring that the effective action reflects the correct vacuum structure. Intuitively, the existence of the vacua $(\mathrm{J}, 0, 0)$, $(0, \mathrm{J}, 0)$ and $(0, 0, \mathrm{J})$ should not impose any constraints on ξ because U acts trivially on these states. This can indeed be verified by the explicit calculation (the corresponding critical points are $m_1 = 1, m_2 = 0, m_3 = 0$, etc.). The existence of the (J_1, J_2, J_3) vacua with $J_i \sim \gamma_i$ does, however, require the specific value of ξ. For example, in the case of $\gamma_i = \gamma$ the corresponding critical point of the potential term in (5.21) is $m_i = \pm\frac{1}{\sqrt{3}}$ and the value of the potential at this point is $V = \frac{1}{2}\mathrm{J}\tilde{\lambda}(\mathrm{J}\gamma)^2[\frac{1}{3} - \frac{2}{3}\xi(1-\xi)]$. By requiring that it vanishes we get

$$\xi = \frac{1}{2}(1 \pm i), \qquad 2\xi(1-\xi) = 1. \tag{5.22}$$

One can directly verify that in the general γ_i case the effective potential with these values of ξ is nonnegative and vanishes only at the required four critical points.

The full coherent state action which correctly reproduces the spin chain vacuum structure is thus given by the difference of (5.9) and (5.21) with the coefficient of the last term being $2\xi(1-\xi) = 1$. Remarkably, it then also reproduces the fast string action (3.4), (3.6) with the identification $\rho_i = m_i$, $\phi_i = h_i$.

Let us note that the complex value of ξ implies that the transformation in (5.12), (5.14) is not unitary. This manifests itself at higher orders in the γ_i expansion and therefore implies that at higher loops a further change of basis is necessary. While the unitarity of the basis change is not a required condition, we suspect that there may exist also a unitary change of basis that leads to the same real coherent state effective action.

6. Concluding remarks

In this paper we have studied an example of large N AdS/CFT duality in a nonsupersymmetric context.

The string theory we considered is obtained from the $AdS_5 \times S^5$ string theory by a combination of T-dualities and shifts of angular coordinates and is parametrized in addition to the radius $R = \lambda^{1/4}$ $(\alpha' = 1)$ of the AdS_5 space by the three real parameters $\tilde{\gamma}_i = R^2\gamma_i$ which determine the shape of the deformed $S^5_{\gamma_i}$ space. The special case of equal $\gamma_i = \gamma$ corresponds to the supersymmetric deformation of $AdS_5 \times S^5$ string theory introduced in [1] and further studied in [2,3].

The dual gauge theory has the same field content as the $\mathcal{N} = 4$ SYM theory, but with scalar quartic interactions and Yukawa couplings being "*-deformed" using γ_i as phase multiplying the $U(1)$-charges of the fields [3]. The three $U(1)$ symmetries and the corresponding charges are inherited from the $SU(4)$ R-symmetry of $\mathcal{N} = 4$ SYM theory. In the case of $\gamma_i = \gamma$ the gauge theory becomes the exactly marginal $\mathcal{N} = 1$ supersymmetric deformation of the $\mathcal{N} = 4$ SYM theory with real deformation parameter $\beta = \gamma$.

We have compared the energies of the semiclassical strings in $AdS_5 \times S^5_{\gamma_i}$ geometry having three large angular momenta in $S^5_{\gamma_i}$ to the 1-loop anomalous dimensions of the corresponding gauge-theory scalar operators and found that they match just as it was the case in the $SU(3)$ sector of the standard $AdS_5 \times S^5$ duality [7,16,17,24].

In particular, in the supersymmetric special case of $\gamma_i = \gamma$ this extends the result of [2] from the 2-spin sector to the 3-spin sector. This extension turns out to be quite nontrivial. To match the corresponding low-energy effective actions on the string theory and the gauge theory side one is to make a special choice of the spin chain Hamiltonian representing the 1-loop gauge theory dilatation operator. This choice is "adapted" to the low-energy or semiclassical approximation, i.e., it allows one to capture the right vacuum states and the "macroscopic spin wave" sector of states of the spin chain in the continuum coherent state effective action.

Our results suggest that some quantitative aspects of the AdS/CFT correspondence may be less sensitive to the presence of supersymmetry than it was previously expected. There are, of course, many ways to break supersymmetry of the original maximally supersymmetric AdS/CFT set-up. The important observation of [1] extended in [3] to the nonsupersymmetric case is that the TsT duality preserves the regularity of the geometry and thus leads to tractable examples of the duality. Also, the present theory has continuous tunable parameters which is an advantage over the orbifold [31] models.[25]

One of the by-products of our investigation of the spectrum of fluctuations near nontrivial (J_1, J_2, J_3) vacuum of deformed theory on the gauge theory side is the discovery of a new type of solutions of the Bethe equations for the 3-spin sector of deformed theory (see Appendices A and B). Switching on the deformation parameters lifts the degeneracy of the spectrum of conformal dimensions of the $\mathcal{N} = 4$ SYM theory and leads to new nontrivial relations

[25] Other $\mathcal{N} = 1$ or $\mathcal{N} = 0$ models based on replacing S^5 by less-symmetric spaces of different topology (like $T^{1,1}$ [43] or $S^2 \times S^3$) and corresponding to non-perturbative isolated conformal fixed points on the gauge theory side, appear to be under less theoretical control.

between the structure of solutions of the Bethe equations and dimensions of gauge-theory operators with given $U(1)$ charges. A very interesting related question is the construction of the string Bethe equations describing states which in the undeformed case belong to the same irreducible representation of $PSU(2,2|4)$. The zero-mode states corresponding to fluctuations around the $(J_1, J_2, J_3)_{\rm vac}$ are only one of the simplest examples; in general, the Bethe solutions describing extended multi-spin string states in the same multiplet as the highest-weight states will exhibit similar subtleties. Their proper description appears to require a modification of the standard thermodynamic limit arguments which should also be reflected in the construction of the string Bethe equations.

There remain many interesting questions and directions for future research.

It would be important to study this nonsupersymmetric γ_i-deformed SYM theory in detail, finding out, in particular, if it remains conformal (for properly adjusted coupling and deformation parameters) even for finite N. In the large N limit this follows (to all orders in perturbation theory) from the noncommutative nature of the deformation, the result of [36] and the fact that γ_i cannot be renormalized.[26] At finite N it is, in principle, straightforward to check the conformal invariance (for properly adjusted parameters of the deformed Lagrangian) to the first two loop orders using existing general relations for the β-functions of generic nonsupersymmetric gauge theories [42] (note that here, compared to orbifold models, all the fields are in the same adjoint representation of $U(N)$). The existence of exactly marginal nonsupersymmetric deformations of $\mathcal{N}=4$ SYM theory implied by the AdS/CFT duality seems an interesting subject worth detailed study.[27]

On the string theory side, it remains to construct the explicit form of the Green–Schwarz action describing the γ_i-deformed theory. To do that one may apply the TsT transformation to the superstring action on $AdS_5 \times S^5$ [46] to using the world-sheet rules of T-duality formulated for the Green–Schwarz superstring in [47].[28] The approach used in [3] should then lead to a local and periodic Lax representation for the complete Green–Schwarz sigma model on the γ_i-deformed background, related to the Lax pair for the $AdS_5 \times S^5$ string [48] by the TsT transformations.[29] Having found the Lax representation one may then analyze the properties of the monodromy matrix and derive the string Bethe equations for the γ_i-deformed model analogous to those found for superstring on $AdS_5 \times S^5$ in [49,50]. The string Bethe equations could then be compared to the thermodynamic limit of the Bethe equations for the γ_i-deformed SYM theory (this was already done for the simplest $su(2)_\gamma$ case in [2]). One may also hope that the analysis of the string Bethe equations will shed light on the structure of the dressing factor that appears in the Bethe ansatz

[26] To see this we note that the operator (4.27) deforming the ordinary product must have definite total dimension. Since the $U(1)$ generators have vanishing anomalous dimensions, it follows that the same must hold for γ_i. The same argument implies that, in noncommutative field theories, the θ-parameter is not renormalized. Explicit calculations show that this is indeed correct to two loops and general analysis of the renormalization of such theories suggests that this is generally true.

[27] Supersymmetric exactly marginal deformations of $\mathcal{N}=4$ SYM theory which can be obtained by orbifolding were discussed in [40].

[28] Incidentally, that would give the first nontrivial example of the GS action in a nonsupersymmetric background. To find the explicit form of this action it may be useful to start with the $AdS_5 \times S^5$ action in a particular κ-symmetry gauge where its fermionic structure is explicit. One candidate for such a gauge is $(\Gamma_t + \Gamma_\phi)\theta = 0$ where ϕ is the direction which is T-dualized.

[29] Again, a more direct way to derive the Lax pair for the GS string may be to start from the $AdS_5 \times S^5$ action and do the transformations in the sigma model rather than start with the sigma model in the supergravity background constructed using the T-duality rules.

for quantum strings [51] (in the deformed case the dressing factor may depend on γ_i and that may lead to an additional consistency condition for it).

It would be of much interest to study the stability of this string theory, i.e., the presence of tachyons in its spectrum. The tachyons should be absent for small enough γ_i (as well as at the supersymmetric point of equal γ_i).[30]

This deformed model may thus be useful for understanding aspects of closed-string tachyon physics in the AdS/CFT context (complementing orbifold model examples like type 0 one [5] with the advantage of having a tunable deformation parameter; an interesting possibility is that in the nonsupersymmetric γ_i-deformed theory double-trace operators are not generated in perturbation theory, cf. [44]). Some particular questions are if tachyons are present in the supergravity approximation for generic values of γ_i and how to identify the corresponding operators on the gauge-theory side.

The present work gives also another illustration of the utility of the approach based on the low-energy effective actions of Landau–Lifshitz type. Another interesting problem (already mentioned in [2]) is to try to use the LL action found on the gauge theory side to reconstruct the dual geometry. This remains a challenge for the second $\mathcal{N}=1$ exactly marginal deformation of [20] which preserves only one $U(1)$ isometry.[31]

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Appendix A. Fluctuations near ground states of spin chain

In Section 2 we discussed fluctuations around vacua from the string theory perspective. The fluctuations are found by expanding near the corresponding null geodesics representing point-like string states with lowest energy. Here we shall attempt to analyze these excitations from the gauge-theory standpoint, using the one-loop Bethe ansatz.

There is a qualitative difference between the vacua of the type $(\mathrm{J}, 0, 0)$ and those of the type (J_1, J_2, J_3): the latter are quantum states, corresponding to a nontrivial condensate of roots.

[30] Since the masses should be smooth functions of the deformation parameters γ_i, it seems that it is the lightest modes of the undeformed background, i.e., the supergravity modes, that may become tachyonic first. In general, there may be a mixing between "momentum" and "winding" modes under the TsT transformation (cf. the discussion of geodesics in Section 2). There is some analogy with the case of the Melvin twist of the flat-space theory [41], where tachyons appear in the winding sector for large enough twist parameter $\gamma > \gamma_{\mathrm{crit}} = \frac{1}{2} w \frac{R}{\alpha'}$; tachyons are present for generic twist parameters, but are absent at special supersymmetric points. These winding tachyons can be seen at the supergravity level if one applies T-duality to the flat Melvin background [41].

[31] For an attempt in this direction in the BMN limit see [11]; a perturbative supergravity approach was developed in [45].

A.1. *Fluctuations near the* (J, 0, 0) *vacuum*

Let us first consider the small fluctuations around the obvious classical vacuum $(L,0,0)$ $(L \equiv \mathrm{J})$ of the spin chain, i.e., consider the states with J_2 and J_3 being small.[32]

The rapidities of type u_1 in fact are divided into two groups. The first group of rapidities which we denote $u^{(2)}_{1,k}$ $(k = 1, \dots, J_2)$ corresponds to fluctuations changing the charge J_2, and the second group of rapidities $u^{(3)}_{1,k}$ $(k = 1, \dots, J_3)$ corresponds to fluctuations changing the charge J_3. The auxiliary rapidities $u_{2,k} \equiv u^{(3)}_{2,k}$ $(k = 1, \dots, J_3)$ are associated to rapidities $u^{(3)}_{1,k}$.

Following the intuition from the undeformed theory, we conclude that if the number of excitations is small, their momenta are also small and therefore their rapidities are large. To make this explicit we introduce, as usual, the rescaled rapidities

$$x^{(2)}_{1,k} = \frac{u^{(2)}_{1,k}}{L}, \qquad x^{(3)}_{1,k} = \frac{u^{(3)}_{1,k}}{L}. \tag{A.1}$$

To get a consistent system of equations in the large L limit we also have to assume that the auxiliary rapidities $u_{2,k}$ have the following scaling behavior

$$u^{(3)}_{2,k} = u^{(3)}_{1,k} + w^{(3)}_{2,k} = L x^{(3)}_{1,k} + w^{(3)}_{2,k}, \tag{A.2}$$

where $w^{(3)}_{2,k}$ do not depend on L. That means that in the large L limit an auxiliary rapidity $u^{(3)}_{2,k}$ may differ from $u^{(3)}_{1,k}$ only by a constant.

Then, in terms of the rescaled rapidities the logarithm of the Bethe equations (4.12)–(4.13) and of the momentum constraint (4.14) expanded for large L become

$$-2\pi L\alpha_{21} + 2\pi J_3(\alpha_{32} + \alpha_{21} + \alpha_{13}) - 2\pi n^{(2)}_{1,k} + \frac{1}{x^{(2)}_{1,k}}$$
$$= \sum_{\substack{i=1\\ i\neq k}}^{J_2} \frac{2/L}{x^{(2)}_{1,k} - x^{(2)}_{1,i}} + \sum_{i=1}^{J_3} \frac{1/L}{x^{(2)}_{1,k} - x^{(3)}_{1,i}},$$

$$-2\pi L\alpha_{21} + 2\pi J_3(\alpha_{32} + \alpha_{21} + \alpha_{13}) + 2\pi n^{(3)}_{1,k} + \frac{1}{x^{(3)}_{1,k}}$$
$$= \sum_{\substack{i=1\\ i\neq k}}^{J_3} \frac{1/L}{x^{(3)}_{1,k} - x^{(3)}_{1,i}} + \sum_{i=1}^{J_2} \frac{2/L}{x^{(3)}_{1,k} - x^{(2)}_{1,i}} - i\ln\frac{w^{(3)}_{2,k} + i/2}{w^{(3)}_{2,i} - i/2},$$

$$2\pi L(\alpha_{21} + \alpha_{13}) - 2\pi(J_2 + J_3)(\alpha_{32} + \alpha_{21} + \alpha_{13}) + 2\pi n^{(3)}_{2,k}$$
$$= \sum_{\substack{i=1\\ i\neq k}}^{J_3} \frac{1/L}{x^{(3)}_{1,k} - x^{(3)}_{1,i}} + \sum_{i=1}^{J_2} \frac{1/L}{x^{(2)}_{1,i} - x^{(3)}_{1,k}} + i\ln\frac{w^{(3)}_{2,k} + i/2}{w^{(3)}_{2,k} - i/2},$$

[32] The fluctuations around the other two similar vacua, $(0, L, 0)$ and $(0, 0, L)$, are related to those around the $(L, 0, 0)$ by simple relabeling.

$$2\pi L(J_2\alpha_{21} - J_3\alpha_{13}) - 2\pi m = \sum_{k=1}^{J_2} \frac{1}{x_{1,k}^{(2)}} + \sum_{k=1}^{J_3} \frac{1}{x_{1,k}^{(3)}}, \tag{A.3}$$

where, as in (4.7), $\alpha_{12} = -\gamma_3,\ \alpha_{23} = -\gamma_1,\ \alpha_{31} = -\gamma_2$ and $L\alpha_{ij}$ is assumed to be fixed in the scaling limit. These equations hold regardless of any assumptions on the size of the quantum numbers J_i. In the case of the (J_1, J_2, J_3) fluctuations around the vacuum $(L, 0, 0)$ we further require that $J_1 \sim \mathcal{O}(L)$ while $J_2, J_3 \sim \mathcal{O}(1)$. This assumption implies that most of the terms in the equations can be safely neglected, and the system takes the following simple form

$$\begin{aligned}
&-2\pi L\alpha_{21} - 2\pi n_{1,k}^{(2)} + \frac{1}{x_{1,k}^{(2)}} = 0,\\
&-2\pi L\alpha_{21} + 2\pi n_{1,k}^{(3)} + \frac{1}{x_{1,k}^{(3)}} = -i \ln \frac{w_{2,k}^{(3)} + i/2}{w_{2,k}^{(3)} - i/2},\\
&2\pi L(\alpha_{21} + \alpha_{13}) + 2\pi n_{2,k}^{(3)} = i \ln \frac{w_{2,k}^{(3)} + i/2}{w_{2,k}^{(3)} - i/2},\\
&2\pi L(J_2\alpha_{21} - J_3\alpha_{13}) - 2\pi m = \sum_{k=1}^{J_2} \frac{1}{x_{1,k}^{(2)}} + \sum_{k=1}^{J_3} \frac{1}{x_{1,k}^{(3)}},
\end{aligned} \tag{A.4}$$

where we took into account that $\alpha_{ij} \sim 1/L$.

To find the energy spectrum we need to know only the rapidities $x_1^{(2)}$ and $x_1^{(3)}$ which can be easily determined from these equations

$$\begin{aligned}
\frac{1}{x_{1,k}^{(2)}} &= -2\pi\left(L\alpha_{21} + n_{1,k}^{(2)}\right) = -2\pi\left(n_{1,k}^{(2)} + \gamma_3 L\right),\\
\frac{1}{x_{1,k}^{(3)}} &= -2\pi\left(L\alpha_{13} + n_{1,k}^{(3)} + n_{2,k}^{(3)}\right) = -2\pi\left(n_{1,k}^{(3)} + n_{2,k}^{(3)} + \gamma_2 L\right).
\end{aligned} \tag{A.5}$$

Shifting $n_1^{(3)} \to n_1^{(3)} - n_2^{(3)}$, we find the energy spectrum

$$E = \frac{\lambda}{2L^2}\left[\sum_{k=1}^{J_2}\left(n_{1,k}^{(2)} + \gamma_3 L\right)^2 + \sum_{k=1}^{J_3}\left(n_{1,k}^{(3)} + \gamma_2 L\right)^2\right], \tag{A.6}$$

which agrees precisely with the leading term in the expansion of the string theory result in (2.29). Furthermore, the number of such states is also correct, being equal to the number of states in the undeformed theory.

The discussion above can be thought of as an explicit implementation of the general arguments of [52] regarding the structures appearing in the thermodynamic limit of the $\mathcal{N} = 4$ SYM spin chain. Adapting their analysis to our context it follows that, for an arbitrary number of excitations, the relevant equations in the thermodynamic limit are the first two equations (A.3) corresponding to the 1-stacks, and the sum of the second and third equations (A.3), corresponding to the 2-stacks.[33] Indeed, we have seen that these combinations led to the solutions (A.5).

[33] The scaling (A.2) implements the fact that the separation of roots inside a stack is $\mathcal{O}(1)$.

A.2. Fluctuations near the (J_1, J_2, J_3) *vacuum*

Let us now turn to the analysis of the fluctuations around the quantum vacuum (J_1, J_2, J_3) (4.19), i.e., now we will assume that $J_i \sim \mathcal{O}(L)$ for all $i = 1, 2, 3$. Since such states are built out of large numbers of excitations above the classical vacuum $(L, 0, 0)$ of the spin chain, it is convenient to take the thermodynamic limit, i.e., $L \to \infty$. From the previous discussion and Ref. [52] the relevant equations are a subset of (A.3)

$$2\pi(J_3\alpha_{32} - J_1\alpha_{21}) + 2\pi(J_3\alpha_{13} - J_2\alpha_{21}) - 2\pi n^{(2)}_{1,k} + \frac{1}{x^{(2)}_{1,k}}$$

$$= \sum_{\substack{i=1\\ i\neq k}}^{J_2} \frac{2/L}{x^{(2)}_{1,k} - x^{(2)}_{1,i}} + \sum_{i=1}^{J_3} \frac{1/L}{x^{(2)}_{1,k} - x^{(3)}_{1,i}},$$

$$2\pi(J_3\alpha_{32} - J_1\alpha_{21}) + 2\pi(J_3\alpha_{13} - J_2\alpha_{21}) + 2\pi n^{(3)}_{1,k} + \frac{1}{x^{(3)}_{1,k}}$$

$$= \sum_{\substack{i=1\\ i\neq k}}^{J_3} \frac{1/L}{x^{(3)}_{1,k} - x^{(3)}_{1,i}} + \sum_{i=1}^{J_2} \frac{2/L}{x^{(3)}_{1,k} - x^{(2)}_{1,i}} - i\ln\frac{w^{(3)}_{2,k} + i/2}{w^{(3)}_{2,i} - i/2},$$

$$2\pi(J_1\alpha_{13} - J_2\alpha_{32}) + 2\pi(J_3\alpha_{13} - J_2\alpha_{21}) + 2\pi n^{(3)}_{k} + \frac{1}{x^{(3)}_{1,k}}$$

$$= \sum_{\substack{i=1\\ i\neq k}}^{J_3} \frac{2/L}{x^{(3)}_{1,k} - x^{(3)}_{1,i}} + \sum_{i=1}^{J_2} \frac{1/L}{x^{(3)}_{1,k} - x^{(2)}_{1,i}},$$

$$2\pi L(J_2\alpha_{21} - J_3\alpha_{13}) - 2\pi mL = \sum_{k=1}^{J_2} \frac{1}{x^{(2)}_{1,k}} + \sum_{k=1}^{J_3} \frac{1}{x^{(3)}_{1,k}}, \tag{A.7}$$

with $n^{(3)}_k = n^{(3)}_{1,k} + n^{(3)}_{2,k}$. As we have discussed in Section 4, the vacuum $(J_1, J_2, J_3)_{\text{vac}}$ exists whenever the angular momentum vector $J_{i,\text{vac}}$ is a zero eigenvector of the deformation matrix α_{mn} (i.e., $\alpha_{12}J_{2,\text{vac}} + \alpha_{13}J_{3,\text{vac}} = \alpha_{12}J_{2,\text{vac}} - \alpha_{31}J_{3,\text{vac}} = 0$, etc.). Fluctuations around this vacuum have $j_i = J_i - J_{i,\text{vac}} \sim L^{\mu}$ with $\mu < 1$ and therefore the deformation-dependent terms on the left-hand side of the equations above are of order $1/L^{1-\mu}$ (since $\alpha_{ij}L$ is fixed). Such a source term appears in the equations determining the vacuum rapidities as well and is an illustration of the usual fact that excitations around any quantum vacuum back-react on the vacuum condensate. In this case, the deviation of the angular momentum vector from being a 0-eigenvector of the deformation matrix acts as a source in the equations for the vacuum rapidities and renders them finite (albeit larger than the other ones by a factor of L).

From the discussion in Section 4 it is clear that not all mode numbers are free parameters. Because of the fact that the rapidities building the vacuum state are infinite in the absence of additional excitations, Eqs. (A.7) imply that the corresponding mode numbers vanish. We therefore have the following structure:

$$n_{1,k} = 0, \quad k = 1, \ldots, J_{2,\text{vac}} + J_{3,\text{vac}},$$

$$n_{2,k} = 0, \quad k = 1, \ldots, J_{3,\text{vac}},$$

$$n_{1,k} = \text{free}, \quad k = J_{2,\text{vac}} + J_{3,\text{vac}} + 1, \ldots, J_2 + J_3,$$
$$n_{2,k} = \text{free}, \quad k = J_{3,\text{vac}} + 1, \ldots, J_3. \tag{A.8}$$

Further analyzing (A.8) requires making a distinction between the case in which all the mode numbers which are free parameters are nonzero and the case in which at least one of the mode numbers vanish. We will analyze here the first case and defer to Appendix B the case of all vanishing mode numbers.

If all the free mode numbers are nonvanishing, it follows that the corresponding rapidities $x_{1,k}$ are of order unity as well as that in the corresponding equations the deformation-dependent terms are subleading compared to the mode numbers. Then, using the fact that all vacuum rapidities are large, it follows that in the equations with nonvanishing mode number only very few terms survive on the right-hand side, insufficient to compensate for the explicit $1/L$ suppression. This implies that $x_{1,k}^{(2)}$ and $x_{1,k}^{(3)}$ are given by

$$\frac{1}{x_{1,k}^{(2)}} = 2\pi n_{1,k}^{(2)}, \qquad \frac{1}{x_{1,k}^{(3)}} = 2\pi n_k^{(3)}, \tag{A.9}$$

i.e., are the same as in the undeformed theory.

The consistency of all other equations is also guaranteed by the fact that the deformation enters at higher orders in the $1/L$ expansion. If mode numbers of the auxiliary Bethe equations are nonzero, they lead to quite complicated expressions for the corresponding rapidities u_2. Fortunately, we do not need them since they do not enter the expressions for the energy or the momentum constraint given by

$$E = \frac{\lambda}{2L^2} \sum_{k=J_{2,\text{vac}}+J_{3,\text{vac}}+1}^{J_2+J_3} n_{1,k}^2, \qquad \sum_{k=J_{2,\text{vac}}+J_{3,\text{vac}}+1}^{J_2+J_3} n_{1,k} = mL = 0. \tag{A.10}$$

The vanishing of the momentum number m is implied by the fact that we considered only few excitations above the vacuum.

The conclusion is that the small fluctuations around the (J_1, J_2, J_3) vacuum having nonzero mode numbers are identical to those in the undeformed theory. This is the same result as was found on the string-theory side in Section 2.3.2.

Appendix B. The anomalous dimensions of operators dual to lowest energy pointlike strings

The special case in which all mode numbers in (A.7) vanish is quite interesting and nontrivial. The corresponding operators are BPS in the absence of the deformation and thus their anomalous dimensions are solely due to the presence of the deformation. In Section 2.3.2 we have seen that the zero-mode fluctuations around the $(J_1, J_2, J_3)_{\text{vac}}$ geodesics are part of a larger class of pointlike string configurations which in the large angular momentum limit become (approximate) solutions. Their energies in this limit are given by (2.48). In this appendix we will go beyond the zero-mode approximation $J_i/L \simeq J_{i,\text{vac}}/L$ and find the anomalous dimensions of the gauge theory operators corresponding to all such pointlike strings captured by the deformed $su(3)$ sector.

There exists a conceptual issue related to the $(J_1, J_2, J_3)_{\text{vac}}$ state being a *quantum rather than classical vacuum*. As mentioned before, the Bethe equations (4.12)–(4.13) employ the state

$(L, 0, 0)$ as the vacuum ("classical" vacuum, i.e., a state with no excitations); the quantum vacuum appears as a nontrivial configuration of Bethe roots. From this perspective, fluctuations around this quantum state are on the one hand similar to a generic state with large angular momenta and on the other hand special because they are accidentally close to a zero energy state. While an analog of the Bethe equations having $(J_1, J_2, J_3)_{\rm vac}$ as its "classical" vacuum would be a desirable starting point for studying the fluctuations near that state, deriving such equations remains an interesting open problem.

In the following we will discuss the excitations with vanishing mode numbers n_k close to the $(J_1, J_2, J_3)_{\rm vac}$ state, using the Bethe equations (4.12)–(4.13). Remarkably, we will find that the results agree with the exact string theory predictions (2.48).

Depending on the departure from the $(J_1, J_2, J_3)_{\rm vac}$ state, it is easy to see what is the scaling of the Bethe roots with the length of the chain. The relevant equations follow from (A.7)

$$2\pi j_3\boldsymbol{\gamma} + \frac{1}{x^{(2)}_{1,k}} = \sum_{\substack{i=1\\ i\neq k}}^{J_2} \frac{2/L}{x^{(2)}_{1,k} - x^{(2)}_{1,i}} + \sum_{i=1}^{J_3} \frac{1/L}{x^{(2)}_{1,k} - x^{(3)}_{1,i}}, \tag{B.1}$$

$$-2\pi j_2\boldsymbol{\gamma} + \frac{1}{x^{(3)}_{1,k}} = \sum_{\substack{i=1\\ i\neq k}}^{J_3} \frac{2/L}{x^{(3)}_{1,k} - x^{(3)}_{1,i}} + \sum_{i=1}^{J_2} \frac{1/L}{x^{(3)}_{1,k} - x^{(2)}_{1,i}}, \tag{B.2}$$

$$2\pi(j_2\gamma_3 - j_3\gamma_2), = \sum_{k=1}^{J_2} \frac{1/L}{x^{(2)}_{1,k}} + \sum_{k=1}^{J_3} \frac{1/L}{x^{(3)}_{1,k}}, \tag{B.3}$$

where we explicitly used the expression of the vacuum quantum numbers $(J_1, J_2, J_3)_{\rm vac}$ and, as before,

$$j_i = J_i - \frac{\gamma_i}{\boldsymbol{\gamma}} L, \qquad \boldsymbol{\gamma} = \gamma_1 + \gamma_2 + \gamma_3. \tag{B.4}$$

The parameters j_i describe the deviation of the state (J_1, J_2, J_3) from the vacuum; therefore, their scaling with the length of the chain is $j \sim L^{\mu}$ with $0 \leqslant \mu \leqslant 1$. Then, from (B.3) it trivially follows that the constant source terms scale like $j\boldsymbol{\gamma} \sim L^{\mu-1}$ which leads to rapidities $x^{(2)}_{1,k}, x^{(3)}_{1,k} \sim L^{1-\mu}$. Still, to leading order, the expression for the energy does not involve any fractional powers of L. Indeed, the energy is an even function of the constant source in the Bethe equations, which vanishes in its absence. Thus, schematically and to the leading order, the energy behaves as

$$\begin{aligned} E &= \frac{\lambda}{8\pi^2} \sum_{k=1}^{J_2+J_3} \frac{1}{u^2_{1,k} + \frac{1}{4}} = \frac{\lambda}{8\pi^2 L^{2(2-\mu)}} F\big(x^{(2)}_1, x^{(3)}_1\big) \\ &\sim \frac{\lambda}{8\pi^2 L^{2(2-\mu)}} \big(\gamma j L^{1-\mu}\big)^2 \big[\mathcal{O}(J_2 + J_3) + \cdots\big] \sim \frac{\tilde\lambda}{8\pi^2} (\gamma L)^2 \frac{j^2}{L} \big[\mathcal{O}(1) + \cdots\big]. \end{aligned} \tag{B.5}$$

This L-dependence is similar to the one derived on the string theory side in (2.43). The existence of a rescaling of the rapidities which makes all terms in the Bethe equations of the same order also implies that we can safely neglect terms of the type $1/(Lx)$. This observation will be useful shortly.

The Eqs. (B.1)–(B.3) are similar to those in [53]. The differences are the nonintegrality of the constant term on their left-hand side and potential term on the left hand side of (B.2). The solution is, however, similar to that of [53]. To analyze them it is useful to proceed in the standard way and introduce the resolvents

$$G_i(x) = \frac{1}{L}\sum_{k=1}^{J_i}\frac{1}{x - x_{1,k}^{(i)}}, \quad i = 2, 3, \tag{B.6}$$

in terms of which the anomalous dimensions are

$$E = -\frac{\lambda}{8\pi^2 L}\big(G_2'(0) + G_3'(0)\big). \tag{B.7}$$

To find $G_2'(0)$ and $G_3'(0)$ we begin by multiplying the first equation in (B.1)–(B.3) by $\frac{1}{x - x_{1,k}^{(2)}}$ and the second by $\frac{1}{x - x_{1,k}^{(3)}}$ and summing all the equations. This leads to

$$\begin{aligned}
&2\pi j_3\boldsymbol{\gamma} G_2(x) + \frac{1}{x}\big(G_2(x) - G_2(0)\big) \\
&\quad = G_2(x)^2 - \frac{1}{L}G_2'(x) + \frac{1}{L^2}\sum_{k=1}^{J_2}\sum_{i=1}^{J_3}\frac{1}{(x - x_{1,k}^{(2)})(x_{1,k}^{(2)} - x_{1,i}^{(3)})}, \\
&-2\pi j_2\boldsymbol{\gamma} G_3(x) + \frac{1}{x}\big(G_3(x) - G_3(0)\big) \\
&\quad = G_3(x)^2 - \frac{1}{L}G_3'(x) + \frac{1}{L^2}\sum_{k=1}^{J_3}\sum_{i=1}^{J_2}\frac{1}{(x - x_{1,k}^{(3)})(x_{1,k}^{(3)} - x_{1,i}^{(2)})}, \\
&-2\pi(j_2\gamma_3 - j_3\gamma_2) = G_2(0) + G_3(0).
\end{aligned} \tag{B.8}$$

Further summing the first two equations and neglecting subleading terms we find

$$\begin{aligned}
&(2\pi j_3\boldsymbol{\gamma})G_2(x) + (-2\pi j_2\boldsymbol{\gamma})G_3(x) + \frac{1}{x}\big[G_2(x) + G_3(x) - G_2(0) - G_3(0)\big] \\
&\quad = \big[G_2(x) + G_3(x)\big]^2 - G_2(x)G_3(x), \\
&-2\pi(j_2\gamma_3 - j_3\gamma_2) = G_2(0) + G_3(0).
\end{aligned} \tag{B.9}$$

The limit $x \to 0$ expresses the derivative of the sum of the resolvents evaluated at the origin in terms of the values of G_2 and G_3 at $x = 0$:

$$G_2'(0) + G_3'(0) = C^2 - G_2(0)G_3(0) - AG_2(0) - BG_3(0), \tag{B.10}$$

$$G_2(0) + G_3(0) = -C, \tag{B.11}$$

where, to shorten later equations, we introduced the notation

$$A = 2\pi j_3\boldsymbol{\gamma}, \qquad B = -2\pi j_2\boldsymbol{\gamma}, \qquad C = 2\pi(j_2\gamma_3 - j_3\gamma_2). \tag{B.12}$$

To find $G_2(0)$ and $G_3(0)$ we need another equation in addition to (B.11); it can be obtained by first multiplying (B.1) by the factor $\sum_{m=1}^{J_3}\frac{1/L^2}{x_{1,m}^{(3)} - x_{1,k}^{(2)}}$ and summing over k, and by multiplying

(B.2) by $\sum_{m=1}^{J_2} \frac{1/L^2}{x_{1,m}^{(2)}-x_{1,k}^{(3)}}$ and summing over k:

$$\begin{aligned}
&A \sum_{m=1}^{J_3} \sum_{k=1}^{J_2} \frac{1}{x_{1,m}^{(3)} - x_{1,k}^{(2)}} + \sum_{m=1}^{J_3} \sum_{k=1}^{J_2} \frac{1}{x_{1,k}^{(2)}(x_{1,m}^{(3)} - x_{1,k}^{(2)})} \\
&\quad = \sum_{m=1}^{J_3} \sum_{k\neq i=1}^{J_2} \frac{1/L}{(x_{1,m}^{(3)} - x_{1,k}^{(2)})(x_{1,m}^{(3)} - x_{1,i}^{(2)})} + \sum_{m,i=1}^{J_3} \sum_{k=1}^{J_2} \frac{1/L}{(x_{1,m}^{(3)} - x_{1,k}^{(2)})(x_{1,k}^{(2)} - x_{1,i}^{(3)})}, \\
&B \sum_{m=1}^{J_2} \sum_{k=1}^{J_3} \frac{1}{x_{1,m}^{(2)} - x_{1,k}^{(3)}} - \sum_{m=1}^{J_2} \sum_{k=1}^{J_3} \frac{1}{x_{1,k}^{(3)}(x_{1,k}^{(3)} - x_{1,m}^{(2)})} \\
&\quad = -\sum_{m=1}^{J_2} \sum_{k\neq i=1}^{J_3} \frac{1/L}{(x_{1,k}^{(3)} - x_{1,m}^{(2)})(x_{1,m}^{(2)} - x_{1,i}^{(3)})} - \sum_{m,i=1}^{J_2} \sum_{k=1}^{J_3} \frac{1/L}{(x_{1,k}^{(3)} - x_{1,m}^{(2)})(x_{1,k}^{(3)} - x_{1,i}^{(2)})}.
\end{aligned} \tag{B.13}$$

Then, summing these two equations and dividing by L^2 leads to

$$(A-B) \sum_{m=1}^{J_3} \sum_{k=1}^{J_2} \frac{1/L^2}{x_{1,m}^{(3)} - x_{1,k}^{(2)}} + G_2(0)G_3(0) = 0. \tag{B.14}$$

The unknown sum can be determined by summing Eqs. (B.1) or Eqs. (B.2):

$$\begin{aligned}
\sum_{m=1}^{J_3} \sum_{k=1}^{J_2} \frac{1/L^2}{x_{1,m}^{(3)} - x_{1,k}^{(2)}} &= -G_3(0) + B\boldsymbol{\alpha}_2 = G_2(0) - A\boldsymbol{\alpha}_3 \\
&= -G_3(0) + B\left(\frac{\gamma_2}{\boldsymbol{\gamma}} + \frac{j_2}{L}\right) = G_2(0) - A\left(\frac{\gamma_3}{\boldsymbol{\gamma}} + \frac{j_3}{L}\right),
\end{aligned} \tag{B.15}$$

where we used the notation

$$\boldsymbol{\alpha}_i \equiv \frac{J_i}{L} = \frac{\gamma_i}{\boldsymbol{\gamma}} + \frac{j_i}{L}$$

for the filling fractions. Thus, the $G_2(0)$ and $G_3(0)$ are determined by

$$\begin{aligned}
&(A-B)\big[B\boldsymbol{\alpha}_3 - G_3(0)\big] + G_2(0)G_3(0) = 0, \\
&G_2(0) + G_3(0) = -C.
\end{aligned} \tag{B.16}$$

From the definition (B.6) it follows that resolvents G_2 and G_3 identically vanish if $J_2 = J_3 = 0$; we will therefore pick the solution for $G_2(0)$ and $G_3(0)$ which also vanishes in this limit:

$$\begin{aligned}
G_2(0) &= \frac{1}{2}\Big[A - B - C - \sqrt{4\boldsymbol{\alpha}_3 B(A-B) + (A-B+C)^2}\Big], \\
G_3(0) &= \frac{1}{2}\Big[-A + B - C + \sqrt{4\boldsymbol{\alpha}_3 B(A-B) + (A-B+C)^2}\Big].
\end{aligned} \tag{B.17}$$

Using (B.10), it is then easy to find what $G_2'(0) + G_3'(0)$ is

$$G_2'(0) + G_3'(0) = \boldsymbol{\alpha}_3 B(A-B) + C(A+C). \tag{B.18}$$

Finally, using the definitions (B.12) to express (B.18) in terms of the deformation parameters γ_i and the deviations of the angular momenta from the vacuum values j_i, the anomalous dimensions are found to be

$$\begin{aligned} E &= -\frac{\lambda}{8\pi^2 L}\big(G_2'(0) + G_3'(0)\big) \\ &= \frac{\lambda}{2L}\left[\gamma_1\gamma_2 j_3^2 + \gamma_2\gamma_3 j_1^2 + \gamma_3\gamma_1 j_2^2 - \frac{1}{L}(\gamma_1+\gamma_2+\gamma_3)^2 j_1 j_2 j_3\right]. \end{aligned} \tag{B.19}$$

As promised, this reproduces the exact string theory result (2.48).

The calculation above shows that there exists a configuration of Bethe roots whose energy matches that of the string theory zero modes. Even though we have not found explicitly the rapidity distribution (since we only needed the values of the resolvents and their first derivative at the origin) we may comment on some of its features. In the undeformed theory the (J_1, J_2, J_3) BPS states are described by infinite rapidities which are also infinitely separated. As we turn on the deformation, the Bethe roots $u_{1,2}$ descend to finite distance, of the order of $L/(j\gamma)$. The fact that initially their differences were also infinite suggests that in the presence of the deformation they will also be of the order of $L/(j\gamma)$. The distance between them is still large, and that suggests that the Bethe roots describing the zero modes do not condense.

Besides the solution described above, Eqs. (B.1)–(B.3) have additional ones. For example, if $j_2 = j_3$ and $\gamma_2 = \gamma_3$ it is possible to construct a solution satisfying $G_2(x) - G_3(x) = \mathcal{C}x(G_2(x) + G_3(x))$ where $\mathcal{C}$ is a constant which may be determined from the asymptotic behavior of the resolvents. It turns out that $G_2(x) + G_3(x)$ has no cut, so it also does not describe a root condensate. Rather, it has two poles, at $\pm i(\sqrt{3}\mathcal{C})^{-1}$. Its energy has the same scaling with the length of the chain as in (B.5), but it is a nonanalytic function of the deformation parameters γ_i. This feature might tempt one to discard it, based on the fact that the undeformed theory should be reached smoothly in the limit $\gamma_i \to 0$. The physical interpretation of this solution is not clear at the moment.

It is worth pointing out that, in the calculation and the matching described above, the value of the power $0 \leqslant \mu \leqslant 1$ in the scaling $j \sim L^\mu$ was unimportant. This is in agreement with the string theory discussion in Section 2.3.2. From the perspective of the Bethe ansatz we expect finite size corrections to the energies (B.19). Since Eqs. (B.1)–(B.3) include all terms up to $O(1)$ in the $1/L$ expansion of the logarithm of the Bethe equations, these corrections should be suppressed by additional powers of $1/L$. It would be interesting, though appears to be quite challenging, to compare these corrections to the α' corrections on the string theory side. Techniques developed in [54] may be useful in this respect.

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