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Scale invariant hairy black holes

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Scalar fields coupled to three-dimensional gravity are considered. We uncover a scaling symmetry present in the black hole reduced action, and use it to prove a Smarr formula valid for any potential. We also prove that nonrotating hairy black holes exists only for positive total energy. The extension to higher dimensions is also considered.

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I. INTRODUCTION AND DISCUSSION

The system of gravity coupled to scalar fields has recently been under considerable scrutiny. Asymptotically AdS hairy black holes have been shown to exist in [1–4]. The issue of defining meaningful conserved charges has been considered in [1–3,5–7]. Earlier references include [8–11]. Within the AdS/CFT correspondence, the coupling of scalar matter was considered in [12,13].

Our aim in this paper is to make some general remarks on the structure of hairy black holes in three dimensions. Our key ingredient is the existence of a scale symmetry in the reduced action governing the black hole ansatz. This symmetry exists for any potential $V(\phi)$ and provides, via Noether's theorem, a radially conserved charge. We use this charge to find a relationship between the black hole parameters at infinity with those at the horizon.

Our main result is the following. Let M, J, and S be the total energy, angular momentum, and entropy of a black hole solution with some nonzero scalar field ϕ. Let T and Ω be the black hole's temperature and angular velocity. Assuming that the matter field is finite at the horizon and vanishes at infinity, it follows that these parameters must satisfy the three-dimensional Smarr [14] relation,

$$M = \frac{1}{2} TS - \Omega J. \tag{1}$$

The remarkable aspect of this result is its universality. In fact the scalar field and its potential play no role. The only condition on the matter field is that it must be finite everywhere, and zero asymptotically. Of course this imposes nontrivial constraints on the class of potentials being considered, which must elude the no-hair theorems. But, if the black hole exists, then it must satisfy (1).

The first law of black hole thermodynamics,

$$\delta M = T\delta S - \Omega \delta J, \tag{2}$$

is also valid in this theory. Inverting the Smarr relation (1) one finds that $S(M, J)$ must be a homogeneous[1] function of degree 1/2 of its arguments, $S(\sigma M, \sigma J) = \sigma^{1/2} S(M, J)$. This is certainly true for the vacuum Bañados-Teitelboim-Zanelli (BTZ) black hole. Our result implies that hairy black holes, regardless of the potential chosen, satisfy the same scaling relation.

A remark is in order here: the homogeneity property of $S(M, J)$ is not a consequence of simple dimensional analysis and scaling arguments as is the case e.g. for the Kerr-Newman metrics, cf. [15]. This is due to the presence of an additional dimensionful parameter, the curvature radius of the AdS space-time or, equivalently, the cosmological constant.[2] The reason why (1) holds nevertheless, even in the presence of scalar hair, is the scaling symmetry and the associated radially conserved charge.

In the nonrotating case, $J = 0$, we can use (1) and (2) to find the general expression for the temperature of nonrotating black holes,

$$T = \kappa M^{1/2}, \tag{3}$$

where κ is a constant with no variation. This means, in particular, that for any potential $V(\phi)$ the specific heat of the black hole is positive.

It is interesting to compare (3) with the result reported in [1]. In three dimensions, [1] considered the potential,

$$V = -\frac{1}{8}(\cosh^6\phi + \nu \sinh^6\phi), \tag{4}$$

where ν is a real parameter. An explicit black hole configuration was displayed, whose temperature as a function of the total energy follows the general law (3), and κ becomes a complicated function of the parameter ν.

The three-dimensional structure can be generalized to higher-dimensional black holes with toroidal topology [19,20], as well as to black holes on flat branes [21,22]. This is analyzed in Sec. VII.

[1]In terms of the total mass, the homogeneity property reads $M(\sigma S, \sigma^2 J) = \sigma^2 M(S, J)$.

[2]Generalized Smarr relations have been considered in [16–18].

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After this work was completed we became aware of [23] in which the Smarr relation for hairy black holes in three dimensions was also found. The parametrization of the reduced action used in this reference is very different from ours, and the relevant symmetry is a $SL(2, R)$ group rather than the scaling symmetry which we have employed.

II. REDUCED ACTION IN $D = 3$ AND SCALING SYMMETRY

Consider the action describing three-dimensional gravity coupled to a scalar field ϕ,

$$I = \frac{1}{16\pi G}\int (R - 8g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 16V(\phi))\sqrt{-g}d^3x. \tag{5}$$

We assume that $V(\phi)$ has a nonzero negative value at $\phi = 0$, such that the gravitational background is anti-de Sitter space.

We shall first consider nonrotating solutions. The generalization can be done straightforwardly and will be indicated in Sec. VI. Consider static, spherically symmetric solutions of the form

$$ds^2 = -\gamma(r)^2 h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\varphi^2, \qquad \phi = \phi(r). \tag{6}$$

Solutions of this form include, for example, black holes and soliton solutions. The solitons are relevant for AdS/CFT applications, as recently considered in [24]. We shall concentrate in this paper on black holes. One can write a reduced action for this problem,

$$I[h, \gamma, \phi] = -\frac{(t_2 - t_1)}{8G}\int dr\gamma(h' + 8rh\phi'^2 + 16rV(\phi)) + B. \tag{7}$$

where B is a boundary term that we shall consider below. The equations of motion are

$$h' + 8rh\phi'^2 + 16rV = 0, \qquad -\gamma' + 8r\gamma\phi'^2 = 0, \qquad -(r\gamma h\phi')' + r\gamma V_{,\phi} = 0. \tag{8}$$

They can be shown to be consistent with the original Einstein equations.

The key observation is that the action (7) is invariant under the scale transformations[3]

$$\tilde{r} = \sigma r, \tag{9}$$

$$\tilde{h}(\tilde{r}) = \sigma^2 h(r), \tag{10}$$

$$\tilde{\gamma}(\tilde{r}) = \sigma^{-2}\gamma(r), \tag{11}$$

$$\tilde{\phi}(\tilde{r}) = \phi(r), \tag{12}$$

with σ a (positive) constant.

By direct application of Noether's theorem to the above symmetry one finds that the combination

$$C = \gamma\left(-h + \frac{1}{2}rh' + 8r^2h\phi'^2\right) \tag{13}$$

is conserved, $C' = 0$. One can in fact prove this directly from the Eqs. (8). A crucial property of this conservation law is that it holds for any potential $V(\phi)$. This will allow us to make general statements about the nature of 3d black holes coupled to scalar fields.

Our strategy is now the following. Since C does not depend on r we can use it to find a relationship between the asymptotic parameters M, β and the horizon r_+. As we shall see, this relation is precisely the Smarr relation (1). But before we can state this result, we need to find an expression for the energy of this system.

III. ENERGY AND ENTROPY

The analysis in this section assumes a generic potential. For some specific cases, as masses saturating the Breitenlohner-Freedman bound, a separate analysis may be needed.

The boundary term B that appears in (7) is fixed by the condition that, upon varying the action, all boundary terms cancel for a set of given boundary conditions. At this point we shall switch to the Euclidean formalism, and interpret the on-shell action as the free energy of the thermodynamical system [26]. The Euclidean action I_E is the same as (7), except that $(t_2 - t_1) = 1$ and an overall sign, such that the weight in the functional integral is e^{-I_E}.

In the Euclidean Hamiltonian formalism, the boundary consist of two disconnected pieces, one in the asymptotic region $r \to \infty$ and the other at the horizon. The boundary term B is specified by the condition,

$$\delta B = -\frac{1}{8G}\gamma(\delta h + 16rh\phi'\delta\phi)|_{r=\infty} + \frac{1}{8G}\gamma\delta h|_{r=r_+}, \tag{14}$$

where the horizon is defined by the equation $h(r_+) = 0$. We assume that all fields are regular there.

Assuming that the matter field vanishes at infinity, Eqs. (8) imply that, asymptotically, $\gamma' = 0$. We write $\gamma(\infty) = \beta$, where β is a constant equal to the Euclidean period at infinity.[4]

The boundary term now has the form $\delta B = \beta\delta M - \delta S$, where the variation of mass and entropy are given by

$$\delta M = -\frac{1}{8G}(\delta h + 16rh\phi'\delta\phi)|_{r=\infty}, \tag{15}$$

[3]In the matter-free case this scale symmetry was already observed in [25].

[4]Note that solutions of the form $\gamma \sim \log(r)$ will not occur for a generic potential.

$$\delta S = -\frac{1}{8G}\gamma \delta h|_{r=r_+}. \tag{16}$$

As usual in the Hamiltonian formalism the entropy comes from the variation of the action at the horizon[27–29]. Our task now is to identify the actual values of S and M.

The boundary term at the horizon gives the usual Bekenstein-Hawking entropy without any modifications. In fact, from $h(r_+) = 0$ and $(h + \delta h)(r_+ + \delta r_+) = 0$ it follows that $\delta h(r_+) = -h'(r_+)\delta r_+$, as long as $h'(r_+) \neq 0$, which is satisfied for nonextreme black holes. In addition, the value of γ at the horizon cannot be arbitrary. To avoid conical singularities at $r = r_+$ one must impose [26]

$$\gamma(r_+)h'(r_+) = 4\pi. \tag{17}$$

These two conditions allow us to identify S as

$$S = \frac{2\pi r_+}{4G}, \tag{18}$$

just as in the matter-free system.

We now turn to the problem of integrating (15) to extract the value of M. This problem is more subtle because we have not specified the potential. We shall integrate (15) by using again the scale invariance discussed above, which maps solutions to solutions.

The idea is the following. The functions h and ϕ have scaling dimensions 2 and 0, respectively. From (15) we conclude that M must have scaling dimension 2. This means that under the scale variations of h and ϕ,

$$\delta h = \delta\sigma(-rh' + 2h), \tag{19}$$

$$\delta\phi = -r\delta\sigma\phi', \tag{20}$$

the corresponding variation of M satisfies

$$\delta M = 2\delta\sigma M. \tag{21}$$

We now replace (19) and (20) in (15) and, comparing with (21), we obtain the desired formula for M,[5]

$$M = \frac{1}{8G}\left(-h + \frac{1}{2}rh' + 8r^2 h\phi'^2\right). \tag{22}$$

Before explaining and discussing the validity of this formula let us check that it gives the right results in known cases. For a BTZ black hole, $h = r^2 - 8Gm$ and $\phi = 0$. One finds $M = m$, as expected. A less trivial example is the exact hairy black hole solution found in [1] with $h = r^2 + 4Br - 3(1 + \nu)B^2 + \mathcal{O}(1/r)$ and $\phi = (B/r)^{1/2} - 2/3(B/r)^{3/2} + \mathcal{O}(1/r^{5/2})$. Replacing this field in (22) one obtains

$$M = \frac{3(1 + \nu)B^2}{8G}, \tag{23}$$

in full agreement with [1].

Now, some comments on the derivation and validity of (22) are necessary. The variations (19) and (20) do not explore the full set of asymptotic solutions. In fact, (19) and (20) represent a 1-parameter (σ) set of variations. On the other hand, the equations are of first order for $h(r)$, and second order for $\phi(r)$ and the full space of solutions has three parameters. The key step is that since M is a "function of state" (exact differential), its value does not depend on the path chosen and in this sense the formula (22) is the correct one. However, we must now make sure that δM, as given in (15), is actually an exact differential. An equivalent way of stating this is that the existence of a well-defined variational principle requires B in (7) to exist, not just δB.

We do not need to worry about the first term in (15), δh, which is exact. The second piece, $rh\phi'\delta\phi$, needs a separate analysis. For a generic potential the asymptotic form of the scalar field on AdS is

$$\phi = \frac{a}{r^{\triangle_-}} + \frac{b}{r^{\triangle_+}} + \cdots, \tag{24}$$

where, for static black holes, a and b are arbitrary constants and represent the 2 degrees of freedom associated to ϕ. The exponents $\triangle_\pm$ are the solutions to a quadratic equation and satisfy $\triangle_+ + \triangle_- = 2$. We assume that both are positive.

Plugging (24) into (15) one finds finite terms of the form $f(a, b)\delta a + g(a, b)\delta b$. In order to write these terms as total variations (to achieve path independence) one needs to assume a relationship between a and b. This restriction on the space of solutions is generic and was also found in [1–3,30].

The particular choice considered in these references (generalized here to arbitrary $\triangle_\pm$) is

$$b = \eta a^{\triangle_+/\triangle_-}, \qquad \delta\eta = 0, \tag{25}$$

where η is held fixed. This choice is consistent with the full anti-de Sitter asymptotic group, although this will not be relevant for our discussion.[6]

For our purposes, the choice (25) is singled out by demanding scale invariance of the asymptotic solution. In fact, once a relationship between a and b is assumed, the only function $b = b(a)$ consistent with (20) is precisely

[5]The relationship between asymptotic functional variations and scale transformations can be checked explicitly in some examples. For the BTZ black hole with $h(r) = r^2 + h_0$, one has $\delta h = \delta h_0$. The constant h_0 has scaling dimension 2, $\delta h_0 = 2\delta\sigma h_0$. One can check that in fact $\delta\sigma(-rh' + 2h) = \delta h_0$. In the system studied in [1], the asymptotic solution is $h \simeq r^2 + 4Br - 3(1 + \nu)B^2$. It is direct to check that $\delta h = \delta\sigma(-rh' + 2h)$ with $\delta B = \delta\sigma B$, as claimed. Note finally that this correspondence fails in higher-dimensional gravity. See Sec. VII for details on this case.

[6]Note that this particular choice is by no means the most general. For solitonic solutions, as in [24], a and b become related in a different way. We shall consider solitons in this theory elsewhere.

(25). We conclude that on the space of solutions satisfying the boundary conditions (25), the formula (22) for M is correct.

Finally, we point out that the remarkable cancellations of divergent pieces in the total mass M, discovered in [1–3], can be seen in this case from a different perspective. Note that, up to the factor $\gamma(r)$ which becomes a constant at infinity, M is exactly equal to the scale charge C displayed in (13). Since C does not depend on r, it cannot diverge; the total mass is then finite.

IV. THERMODYNAMICS OF THE HAIRY BLACK HOLE

A. The first law

The first law for our class of black hole solutions can be checked by standard Hamiltonian arguments. The form of the action, derived in the previous section, after all boundary terms have been included, is

$$I[\beta] = \int dr\gamma\mathcal{H} + \beta M - S(r_+), \tag{26}$$

where M is given in (22) and S is the usual entropy in three dimensions, given in (18). $\mathcal{H} = 0$ is one of the equations of motion. By construction, this action has an extremum when evaluated on solutions with β fixed. The on-shell value of I only depends on β. The value of M is such that I has an extremum.

Since the bulk contribution is proportional to a constraint, the on-shell value of the action is

$$I[\beta] = \beta M - S(M), \tag{27}$$

where r_+ is written as a function of the total energy M using the solution.[7] M is not fixed but has to be chosen such that I has an extremum, that is, the first law is satisfied,

$$\beta\delta M = \delta S. \tag{28}$$

B. The Smarr relation

We are now ready to prove our main result. We go back to the expression for the scaling charge C given in (13). Comparing (13) and (22) we conclude that the scaling charge is proportional to the total mass. Evaluating C at infinity we get the exact relation

$$C = 8\beta MG. \tag{29}$$

On the other hand, since C is $r-$ independent we can also evaluate it at the horizon $h(r_+) = 0$ to get

$$C = 2\pi r_+. \tag{30}$$

[7]There is a nontrivial assumption here, namely, that r_+ depends only on M, and not on the scalar field parameter a. We prove in the next section that black holes exists only for special values of $a = a(M)$, and hence a is not an independent parameter.

Here we have used the condition of absence of conical singularities (17). Comparing the values of C at infinity and at the horizon we find the equation

$$\beta M = \frac{1}{2}\frac{2\pi r_+}{4G} \tag{31}$$

representing the nonrotating version of (1). This relation is satisfied for any black hole solution with or without scalar field. Of course, this is also true for the BTZ vacuum black hole, as can be readily checked. The rotating case, leading to (1), will be indicated in Sec. VI.

C. Positivity of energy

We prove now that a hairy black hole can exist only if the total mass M is positive. To see this we first note that the field $\gamma(r)$ does not change sign in the whole range $r_+ \leq r \leq \infty$. In fact, directly from the equations of motion (8) we can write the formal solution

$$\gamma(r) = \gamma_0 e^{\int^r ds 8s\phi'(s)^2} \tag{32}$$

where γ_0 is an arbitrary integration constant. This expression for γ is manifestly positive, if γ_0 is positive. Now, the scaling charge evaluated at the horizon and at infinity gives the equation (we relax here the condition (17) and consider either Minkowskian or Euclidean signature)

$$16G\gamma_\infty M = \gamma_+ h'_+ r_+, \tag{33}$$

where the subscript + indicates the corresponding function evaluated at r_+. The function $h(r)$ must be positive outside the horizon, and vanishes at r_+. This means that $h'_+ > 0$. Since $\gamma(r)$ does not change sign and r_+ is positive, we conclude that this equation requires

$$M > 0. \tag{34}$$

D. Temperature and specific heat

Combining (31) and (18) and the first law we derive the general relation

$$8MG = \kappa_0^2 r_+^2, \tag{35}$$

where κ_0 is an arbitrary (dimensionless) integration constant, with no variation. This constant cannot be computed from this analysis and depends on the details of the potential, as well as all other fixed parameters. For example, for the BTZ black hole $\kappa_0 = 1$ while for the potential (4) considered in [1] one finds that κ_0 becomes a complicated function of ν.

All thermodynamical properties can now be extracted, for example, the temperature as a function of the mass gives

$$\frac{1}{\beta} = T = \kappa_0\frac{\sqrt{2MG}}{\pi}, \tag{36}$$

as announced. We can also check that the specific heat,

$$c = \frac{\partial M}{\partial T} = \frac{2\pi r_+}{4G}, \tag{37}$$

becomes equal to the entropy (this was also noted by [1] in their particular example).

V. A CLOSER LOOK AT THE HAIRY BLACK HOLE

A hairy black hole is, by definition, a solution to the Einstein + matter system displaying a regular horizon. In particular, the value of the matter field $\phi(r)$ at the horizon must be finite. We have argued in the previous section that, if the black hole exists, then the Smarr (1) relation is satisfied. However, we have said very little about the conditions for the existence of a black hole.

The condition of regularity at the horizon imposes constraints on the solutions which can be analyzed using scale invariance. In this section we will prove that for a given value of η [see (25)], the values of the total mass M and the parameter a have to be fine tuned in order to have a regular black hole. This means that, apart from η which acts as an external parameter with no variation, the only free parameter in the black hole spectrum is the total energy M.

Consider the set of equations of motion (8). We would like to find a solution displaying a regular event horizon $h(r_+) = 0$. At the point $r = r_+$, the matter field, and its derivatives up to some sufficiently high order, must be finite. In particular,

$$\phi(r_+) = \phi_0. \tag{38}$$

Define the new field

$$\chi(r) = \phi(r) - \phi_0 \tag{39}$$

which vanishes at the horizon, $\chi(r_+) = 0$. Near the horizon the new field χ is small and hence, without specifying $V(\phi)$, we write the near horizon series,

$$V(\chi) = v_0 + v_1\chi + v_2\chi^2 + \cdots \tag{40}$$

where the constants v_i depend on the potential V and ϕ_0.

Under these conditions, the fields h, γ, ϕ have the following series expansions near the horizon

$$h = h_1(r - r_+) + h_2(r - r_+)^2 + \cdots \tag{41}$$

$$\gamma = \gamma_0 + \gamma_1(r - r_+) + \gamma_2(r - r_+)^2 + \cdots \tag{42}$$

$$\chi = \chi_1(r - r_+) + \chi_2(r - r_+)^2 + \cdots \tag{43}$$

Recall that in the Euclidean formalism the values of h' and γ at $r = r_+$ are linked by (17), that is $h_1\gamma_0 = 4\pi$. Our conclusions, however, do not depend on the signature.

We have assumed that no fractional powers or logs are present because they would induce divergences in the derivatives of the fields.

We now plug this series expansion into the equations of motion and solve for the coefficients order by order. This is a straightforward exercise that we do not display here. The important comment is that all coefficients are fixed in terms of ϕ_0 and r_+ (recall that ϕ_0 enters in the coefficient v_0 in the series (40) for the potential). There are thus 2 arbitrary constants at the horizon:

$$\text{Horizon data}: \{\phi_0, r_+\}, \tag{44}$$

as opposed to the series analysis at infinity with

$$\text{Asymptotic data}: \{\eta, a, M\}. \tag{45}$$

What happens here is that the series expansion (42) is not the most general one. There exists other solutions with logs or fractional powers (probably depending on the potential), which are not contained in the regular ansatz.[8]

We conclude that if one integrates from infinity to the horizon, the values of a, η and M must be fine tuned in order to reach a regular event horizon. Conversely, if one integrates from the horizon, prescribing the values of r_+ and ϕ_0, one gets at infinity a surface in the η, a, M space. Actually, we can say something else. We shall prove now that η only depends on the value of ϕ_0, and not on r_+,

$$\eta = \eta(\phi_0). \tag{46}$$

To see this, suppose we are given a solution to the equations of motion, $h(r), \gamma(r), \phi(r)$ displaying a regular event horizon. Using scale invariance we can provide immediately another exact solution to the equations by the simple transformation

$$\tilde{h}(r) = \sigma^2 h(r/\sigma) \tag{47}$$

$$\tilde{\gamma}(r) = \sigma^{-2}\gamma(r/\sigma) \tag{48}$$

$$\tilde{\phi}(r) = \phi(r/\sigma) \tag{49}$$

The new solution is a different one! If the horizon in the first solution was at $r = r_+$, then the location of the horizon in the second solution is at

$$\tilde{r}_+ = \sigma r_+. \tag{50}$$

In fact, $\tilde{h}(\tilde{r}_+) = 0$. This means that acting with scale transformations, we can cover all possible values of r_+. On the other hand, the value of ϕ_0 remains unchanged since

$$\tilde{\phi}(\tilde{r}_+) = \phi(r_+) = \phi_0. \tag{51}$$

Acting with scale transformations we thus cover all solutions with a given value of ϕ_0. Now, scale transformations act on the asymptotic parameters leaving η invariant. We thus conclude that the asymptotic parameter η is in one-to-one correspondence with the value of ϕ at the horizon

$$\phi_0 \Leftrightarrow \eta. \tag{52}$$

[8] This has also been remarked in [31].

For a given value of η, the value of ϕ_0 is determined. In the example of [1], $\eta = -2/3$ and $\phi_0 = \tanh^{-1}(1/\sqrt{3})$.

Recall that η is fixed in the action principle, and acts as an external parameter. For fixed η (and hence ϕ_0), the remaining degrees of freedom are M and a, at infinity, and r_+ at the horizon. This means that if one integrates from the horizon, varying the values of r_+, one obtains at infinity a curve in the M, a plane. As we have shown this curve will cover only the $M > 0$ half plane. Of course, for different values of η, the curve changes.

VI. ADDING ANGULAR MOMENTUM

We will now extend the discussion of the thermodynamics to black holes with angular momentum. This requires a change of the ansatz for the metric (6) to

$$ds_E^2 = \gamma(r)^2 h(r) dt^2 + \frac{dr^2}{h(r)} + r^2(d\varphi + n(r)dt)^2, \quad \phi = \phi(r). \tag{53}$$

The reduced action is

$$I[h, \gamma, n, \phi] = \frac{1}{8G}\int dr\Big\{\gamma\Big(-\frac{2p^2}{r^3} + h' + 8rh\phi'^2 + 16rV\Big) + 2np'\Big\} + B. \tag{54}$$

Here $p = \pi^r_\varphi = -\frac{r^3}{2\gamma}n'$. The bulk term of the action vanishes on shell. The equation of motion for n gives $p =$ const. By shifting the angular coordinate we arrange for $n(r_+) = 0$.

The action is invariant under (9)–(12), augmented by

$$\tilde{p}(\tilde{r}) = \sigma^2 p(r), \tag{55}$$

$$\tilde{n}(\tilde{r}) = \sigma^{-2} n(r). \tag{56}$$

This leads to the following radially conserved Noether charge

$$C = \gamma\Big(-h + \frac{1}{2}h'r + 8r^2h\phi'^2\Big) - 2np. \tag{57}$$

One checks that indeed $C' = 0$ by virtue of the equations of motion.

The boundary terms B must again be chosen such that δB cancels the boundary terms which appear when one extremizes the action. One finds

$$\begin{aligned}\delta B &= -\frac{1}{8G}\{(\beta(\delta h + 16rh\phi'\delta\phi) + 2n\delta p\}\Big|_{r=\infty} + \frac{1}{8G}(\gamma\delta h + 2n\delta p)\Big|_{r=r_+} \\ &\equiv \beta(\delta M + \Omega\delta J) - \delta S\end{aligned} \tag{58}$$

Here we have used that $h(r_+) = 0$ and the definitions $\gamma(\infty) \equiv \beta, n(\infty) \equiv \beta\Omega$. It follows from the equations of motion that β and Ω are finite. The first two terms are the contribution from $r = \infty$, the last term is the contribution from the horizon. Replacing once more the functional variations by those which follow from the scaling properties of the fields, combined with the fact that M and J have weight two, one finds

$$C = 8G\beta(M + \Omega J). \tag{59}$$

From the contribution at $r = r_+$ we find again Eq. (18), i.e. $S = \frac{2\pi r_+}{4G}$.

In order to find a relation between M, J and S, we use the fact that C is a constant. While its expression at $r = \infty$ was used to relate it to M and J, we now use its expression at the horizon to relate it to S. This leads to

$$\beta(M + \Omega J) = \frac{1}{2}S. \tag{60}$$

as promised. One easily verifies that this relation is satisfied for the rotating BTZ black hole.

$D = 4$

In four dimensions the equations of motion have a similar structure, although there are important differences. For reasons which will become clear very soon, we make the general ansatz for the metric

$$ds^2 = -\gamma^2 h dt^2 + \frac{dr^2}{h} + r^2 d\Omega_k, \tag{61}$$

where the "sphere" $d\Omega_k$ is either a 2-sphere, a 2-torus or a higher genus surface,

$$d\Omega_k = \begin{cases} d\theta^2 + \sin^2\theta d\phi^2, & k = 1 \\ dx^2 + dy^2, & k = 0 \\ d\theta^2 + \sinh^2\theta d\phi^2, & k = -1. \end{cases} \tag{62}$$

Black holes with unusual topologies were first discussed in [19,20].

The ansatz (61) leads to the reduced action

$$I[h, \gamma, \phi] = -\frac{(t_2 - t_1)}{8\tilde{G}}\int dr\gamma(rh' + h - k + 8r^2h\phi'^2 + 16r^2V(\phi)) + B. \tag{63}$$

We have introduced the notation $\tilde{G} = \frac{4\pi G}{V_k}$ with $V_k = \int d\Omega_k$. The horizon area is then $V_k r_+^2$. Varying γ, h, ϕ one obtains the equations of motion

$$rh' + h - k + 8r^2h\phi'^2 + 16r^2V(\phi) = 0, \tag{64}$$

$$-\gamma' + 8r\gamma\phi'^2 = 0, \tag{65}$$

$$-(r^2\gamma h\phi')' + r^2\gamma V_{,\phi} = 0. \tag{66}$$

These equations are similar to those in three dimensions, (8), except for the constant k appearing in (64). This constant, which is a fixed number associated to the sphere's curvature, spoils scale invariance.[9]

However, for the torus topology, $k = 0$, the equations are scale invariant and we can immediately generalize the discussion from $d = 3$ to $d = 4$,[10] in particular due to scale invariance there is a radially conserved charge. Because of the invariance of the action under the replacements $(r, h(r), \gamma(r), \phi(r)) \rightarrow (\tilde{r}, \tilde{h}(\tilde{r}), \tilde{\gamma}(\tilde{r}), \tilde{\phi}(\tilde{r}))$ with [c.f. (9)–(12)]

$$\tilde{r} = \sigma r \tag{67}$$

$$\tilde{h}(\tilde{r}) = \sigma^2 h(r) \tag{68}$$

$$\tilde{\gamma}(\tilde{r}) = \sigma^{-3}\gamma(r) \tag{69}$$

$$\tilde{\phi}(\tilde{r}) = \phi(r) \tag{70}$$

one finds

$$C = \gamma(r^2 h' - 2h + 8r^3 h\phi'^2) \tag{71}$$

with $C' = 0$ by virtue of the equations of motion.

Equation (18) for the entropy is now

$$S = \frac{4\pi r_+^2}{4\tilde{G}} = \frac{V_0 r_+^2}{4G}. \tag{72}$$

and

$$\delta M = -\frac{V_0}{8\pi G}(r\delta h + 8r^2 h\phi'\delta\phi)\Big|_{r=\infty} \tag{73}$$

Using (70) and (68), and the fact that M now has scaling weight three, one finds from (73) and from comparing with (71) the relation

$$C = \frac{24\pi G\beta}{V_0} M. \tag{74}$$

On the other hand, evaluation of C at the horizon gives

$$C = 4\pi r_+^2. \tag{75}$$

Comparison of (74) and (75) leads to the relation

$$\beta M = \frac{V_0 r_+^2}{6G}. \tag{76}$$

In place of (35) one now finds

$$8\pi G M = V_0 \kappa_0^3 r_+^3 \tag{77}$$

and the specific heat can be computed to be twice the entropy and for the temperature one finds

$$T = \frac{3}{2}\pi^{-2/3}\kappa_0^2\left(\frac{GM}{V_0}\right)^{1/3}. \tag{78}$$

We stress once more that these results are valid for arbitrary potentials as long as they lead to a solution for the scalar field which vanishes asymptotically. The specific form of the potential only enters through the integration constant κ_0.

The proof of positivity of M proceeds in exactly the same way as in $d = 3$. It depends crucially on the existence of the scaling charge, i.e. on considering the case $k = 0$. In fact, negative mass hairy black holes for $k = 1$ have recently been constructed in [32].

It is now straightforward to check that for a constant potential $V = -3$, i.e. in the presence of a cosmological constant but no scalar field, one finds the above results with $\kappa_0 = 1$ as one easily verifies given the explicit solution

$$h = r^2 - \frac{8\pi G m}{V_0 r} \tag{79}$$

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[9] Note that if one replaces $\gamma = \lambda'$ and varies λ, the piece $\lambda' k$ is a boundary term and the action becomes scale invariant. The space of solutions has an extra integration constant, and for particular values of that constant, the original equations are recovered. The main obstruction to follow up this idea is the relativistic version of the modified equations of motion.

[10] The generalization to arbitrary d is straightforward if we take for $d\Omega_{d-2}$ the volume element of a flat torus.

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