

Spin Chain for Quantum Strings

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Abstract

We review and compare the integrable structures in $\mathcal{N} = 4$ gauge theory and string theory on $AdS_5 \times S^5$. Recently, Bethe ansätze for gauge theory/weak coupling and string theory/strong coupling were proposed to describe scaling dimensions in the $\mathfrak{su}(2)$ subsector. Here we investigate the Bethe equations for quantum string theory, naively extrapolated to weak coupling. Excitingly, we find a spin chain Hamiltonian similar, but not equal, to the gauge theory dilatation operator.

1 Introduction

When computing scaling dimensions of local operators in $U(N)$ $\mathcal{N} = 4$ gauge theory one can, for convenience, restrict to a number of subsectors. The smallest, nontrivial one is the $\mathfrak{su}(2)$ subsector with only two scalar fields \mathcal{Z} and ϕ . A local operator in field theory can now be interpreted as a state of a spin- $\frac{1}{2}$ $\mathfrak{su}(2)$ spin chain, e.g.

$$\text{Tr } \mathcal{Z} \mathcal{Z} \phi \mathcal{Z} \mathcal{Z} \mathcal{Z} \phi \phi \mathcal{Z} = |\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\rangle \quad (1)$$

In this picture, the planar dilatation operator \mathfrak{D} , which measures gauge theory scaling dimensions, maps to the spin chain Hamiltonian \mathcal{H}

$$\mathfrak{D}(g) = \mathcal{L} + g^2 \mathcal{H}(g) + \mathcal{O}(1/N), \quad g^2 = \frac{g_{\text{YM}}^2 N}{8\pi^2}, \quad (2)$$

where \mathcal{L} counts the number of spin chain sites. At leading order, $g = 0$, Minahan and Zarembo have shown that \mathcal{H} (alias the one-loop dilatation operator) is the Hamiltonian of the Heisenberg spin chain [1]. This Hamiltonian is integrable, i.e. it is part of a tower of local charges \mathcal{Q}_r , $\mathcal{H} = \mathcal{Q}_2$, which commute with the $\mathfrak{su}(2)$ generators \mathfrak{J} and with each other (at $g = 0$)

$$[\mathfrak{J}, \mathcal{Q}_r(g)] = 0, \quad [\mathcal{Q}_r(g), \mathcal{Q}_s(g)] = 0. \quad (3)$$

In [2] it was subsequently shown that integrability extends to next-to-leading order in g (two-loops) and conjectured that (3) might hold exactly in perturbation theory or even beyond. Based on three basic assumptions [2, 3]

- (i) integrability,
- (ii) proper scaling in the thermodynamic limit and
- (iii) constraints from Feynman diagrams,

it was possible to construct a *unique* spin chain Hamiltonian up to at least fourth order in g^2 (five-loops) [4].¹ This proposal has been confirmed at three-loops by several independent methods: An algebraic construction in a larger subsector [6], a direct computation in QCD [7] which can apparently be lifted to $\mathcal{N} = 4$ SYM [8] and a two-loop computation which can be lifted to three-loops by means of multiplet splitting [9].²

The physical interest in gauge theory scaling dimensions lies in the AdS/CFT correspondence which relates them to energies of string configurations on $AdS_5 \times S^5$. The availability of three-loop gauge theory results lead to very precise comparisons within the (near) BMN and spinning strings proposals [11]. While agreement was demonstrated up to two-loops, see e.g. [12–14] (c.f. [15] for a review), it soon emerged that there are discrepancies starting at three-loops [16, 17]. In [4] these were argued to be due to an order-of-limits problem and, when so-called wrapping interactions are taken into account properly, agreement might be restored. Therefore, the AdS/CFT correspondence is not in danger, but a direct, perturbative comparison of the sort proposed in [11] is invalidated.³

¹For a review on the dilatation operator, its construction and integrability, see [5].

²Also a direct computation in a matrix quantum mechanics closely related to $\mathcal{N} = 4$ yields the same result [10].

³Nevertheless, in the strict (i.e. planar, leading $1/J$) BMN limit, perturbative gauge and string results do seem to agree.

Here, various Bethe ansätze, for classical as well as for quantum models, either enable the comparison (spinning strings) or, at least, simplify it drastically (near-BMN). After reviewing the most up-to-date Bethe ansätze, for gauge theory as well as for string theory, we shall consider their weak-coupling regime. While for gauge theory the Bethe ansatz is known to be equivalent to a spin chain Hamiltonian, we show that the same is true even for string theory! This is remarkable because the string Bethe equations are sufficiently different from common Bethe equations. It is possible that the novel spin chain underlying the string Bethe ansatz is a suitable description of string theory at weak coupling.

2 Bethe Ansätze

The Bethe ansatz is a means of finding eigenvalues $E = Q_2, Q_r$ of the Hamiltonian and commuting charges $\mathcal{H} = \mathcal{Q}_2, \mathcal{Q}_r$ on an integrable system by solving a set of algebraic equations. Within the Bethe ansatz, a state is represented by a set of Bethe roots $\{u_k\}$ which specify the rapidities of the magnon spin-waves making up the state. The charge eigenvalues Q_r are the sums of the contributions $q_r(u_k)$ from the individual spin waves. In both Bethe ansätze, for gauge theory [4] and string theory [18], the charges are given by the same expressions⁴

$$Q_r = \sum_{k=1}^K q_r(u_k), \quad q_r(u_k) = \frac{i}{r-1} \left(\frac{1}{(x_k^+)^{r-1}} - \frac{1}{(x_k^-)^{r-1}} \right). \quad (4)$$

Here we have defined x_k^\pm as additional representations of the Bethe roots u_k via⁵

$$x_k^\pm = x(u \pm \frac{i}{2}), \quad x(u) = \frac{1}{2}u + \frac{1}{2}u\sqrt{1 - 2g^2/u^2}, \quad u(x) = x + \frac{g^2}{2x}. \quad (5)$$

In other words, by specifying one of u_k, x_k^+, x_k^- , the others are defined by (5).

The Bethe equation [4] for gauge theory is a modification of the one for the Inozemtsev spin chain [19, 17], it reads⁶

$$\left(\frac{x_k^+}{x_k^-} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^K \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}} = \prod_{\substack{j=1 \\ j \neq k}}^K \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (6)$$

These equations have to be solved subject to the constraints that there are neither roots at infinity nor coinciding roots. Furthermore the cyclicity or level matching constraint $\mathcal{Q}_1 = 2\pi m$ with $q_1(u_k) = -i \log x_k^+/x_k^-$ has to be obeyed for physical states. Then the Q_r give the eigenvalues of a highest weight state in a representation $[L - 2K]$ of $\mathfrak{su}(2)$. The

⁴It turns out that already (i) and (ii) together imply this form of charge eigenvalues Q_r .

⁵The map between x and u is a double covering. We shall use the branch $x_k^\pm \approx u_k \pm \frac{i}{2}$ for $g \approx 0$.

⁶The two products are equivalent upon (5).

above equations reproduce the spectrum of the gauge theory spin chain model, which is known up to $\mathcal{O}(g^8)$ (five-loop), up to wrapping order $\mathcal{O}(g^{2L-2})$.

Bethe equations for quantum strings were proposed in [18]^{7 8}

$$\begin{aligned} \left(\frac{x_k^+}{x_k^-}\right)^L &= \prod_{\substack{j=1 \\ j \neq k}}^K \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - \frac{g^2}{2x_k^- x_j^+}}{1 - \frac{g^2}{2x_k^+ x_j^-}} \left(\frac{1 - \frac{g^2}{2x_k^- x_j^+}}{1 - \frac{g^2}{2x_k^+ x_j^-}} \frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}} \right)^{2i(u_k - u_j)} \\ &= \prod_{\substack{j=1 \\ j \neq k}}^K \frac{u_k - u_j + i}{u_k - u_j - i} \exp \left(2i \sum_{r=2}^{\infty} \left(\frac{1}{2}g^2\right)^r (q_r(u_k) q_{r+1}(u_j) - q_{r+1}(u_k) q_r(u_j)) \right). \end{aligned} \quad (7)$$

These equations were designed to match with the equations for the classical string sigma model [20] in the thermodynamic limit. They also give correct predictions for dimensions of strings in near plane-waves [16, 21, 22]. More excitingly, they reproduce a generic \sqrt{g} -scaling for dimensions at strong coupling. As far as we know, these equations yield correct string results for large g and L [18], nevertheless they are also reasonable equations in the weak coupling regime for g and L small. In fact, it is easy to see that (7) agrees with (6) at $\mathcal{O}(g^2)$ (two-loops). In the remainder of this text we will investigate the perturbative regime of both Bethe ansätze. For simplicity of notation, we shall distinguish between the gauge (6) and string equations (7), although there is no indication that the string equations reproduce string theory results for small g .

3 Spin Chains

The string equations (7) are substantially different from common Bethe equations, but at least their abstract form remains the same: They consist of single particle propagation (l.h.s.) and two-particle scattering (r.h.s.) terms which are functionally more complicated than usual. Furthermore, the equations agree with gauge theory (and the corresponding spin chain) up to $\mathcal{O}(g^2)$. All this suggests that there might also be a spin chain formulation for string theory. Now it was shown that conditions (i-iii) yield a unique answer up to and including at least $\mathcal{O}(g^8)$ and it does not agree with string theory. Therefore we need to relax one of the conditions. Clearly we cannot modify conditions (i) or (ii) because the string equations manifestly have these properties. However there is no reason to rely on condition (iii) for string theory. This condition splits up into two statements (iiia) and (iiib), see [2, 3, 5] for a detailed discussion of (i-iii). Condition (iiia) limits the range of the interaction at $\mathcal{O}(g^{2\ell-2})$ to $\ell + 1$ neighbouring spin sites, whereas (iiib) limits the number of adjacent permutations to ℓ .

⁷The two products are equivalent upon summation over r . Note that the second factor in the first product is inverted as compared to (6).

⁸It would be important to find a transfer matrix from which the Bethe equations follow as in the case of the gauge equations [4].

Let us now drop condition (*iiib*) and compute the most general Hamiltonian satisfying (*i,ii,iiia*). In the notation using adjacent permutations, see [2, 5],

$$\{p_1, p_2, \dots\} = \sum_{p=1}^L \mathcal{P}_{p+p_1, p+p_1+1} \mathcal{P}_{p+p_2, p+p_2+1} \dots \quad (8)$$

we find the three-loop Hamiltonian (see Tab. 2 for the five-loop contribution, Tab. 3 for an alternative representation and Tab. 4 for the four-loop third charge)

$$\begin{aligned} \mathcal{H}(g) &= \sum_{\ell=1}^{\infty} g^{2\ell-2} \mathcal{H}_{2\ell-2}, \\ \mathcal{H}_0 &= \{\} - \{1\}, \\ \mathcal{H}_2 &= -2\{\} + 3\{1\} - \frac{1}{2}(\{1, 2\} + \{2, 1\}), \\ \mathcal{H}_4 &= +(\frac{15}{2} - \frac{1}{2}c_4)\{\} + (-13 + \frac{3}{2}c_4)\{1\} + (\frac{1}{2} - \frac{3}{4}c_4)\{1, 3\} + (3 - \frac{1}{2}c_4)(\{1, 2\} + \{2, 1\}) \\ &\quad + \frac{1}{2}c_4(\{1, 3, 2\} + \{2, 1, 3\}) - \frac{1}{2}(\{1, 2, 3\} + \{3, 2, 1\}) - \frac{1}{4}c_4\{2, 1, 3, 2\}. \end{aligned} \quad (9)$$

Note that the interaction $\{2, 1, 3, 2\}$ in \mathcal{H}_4 is composed from four adjacent permutations whereas (*iiib*) would allow for only three. This forces $c_4 = 0$ for gauge theory.

We can now try to adjust the free parameter c_4 to match the spectrum of $\mathcal{H}(g)$ with the string Bethe equations. Remarkably, this appears to be possible and we find $c_4 = 1$! E.g. for the state with $L = 4, K = 2$ the Hamiltonian yields

$$D = 4 + \frac{3g_{\text{YM}}^2 N}{4\pi^2} - \frac{3g_{\text{YM}}^4 N^2}{16\pi^4} + \frac{(42 - 9c_4)g_{\text{YM}}^6 N^3}{512\pi^6}, \quad (10)$$

while the string Bethe equations predict $33 = 42 - 9$ for the last coefficient. To give further support to this observation, we have extended the analysis to four and five loops.⁹ There we find one and two free parameters, c_6 and $c_{8a,b}$, which can again be adjusted to agree with the predictions from (7), i.e. $c_6 = c_{8a,b} = 0$. For the comparisons we have used an extensive list of two-excitation states, see Tab. 1 for their energy formula. To check our results we have used one paired three-excitation state at $L = 7$ and found agreement

$$D^\pm = 7 + \frac{5g_{\text{YM}}^2 N}{8\pi^2} - \frac{15g_{\text{YM}}^4 N^2}{128\pi^4} + \frac{95g_{\text{YM}}^6 N^3}{2048\pi^6} - \frac{155g_{\text{YM}}^8 N^4}{6144\pi^8} + \frac{16335g_{\text{YM}}^{10} N^5}{1048576\pi^{10}}. \quad (11)$$

Finally, we have repeated the numerical comparison of [13] between spinning strings and higher-loop spin chains for the above model with $c_4 = 1$. For the three-loop dimensions δ_3'' at $J = 4, 8, 12, 16$ (to be compared to Table 1 in [13]) we find the values 0.068651, 0.068938, 0.106585, 0.126870, respectively. The dimension extrapolated to $J = \infty$ as described in [13] is 0.184... which is in 2% agreement with the energy 0.181347 of spinning stings and thus with the Bethe ansatz (7).

⁹Investigating the form of the five-loop Hamiltonian and charges, we notice that the number of adjacent permutations for \mathcal{Q}_r at $\mathcal{O}(g^{2\ell-2})$ is limited to $r + 2\ell - 4$ (plus an additional one at $\ell = 1$) as opposed to $r + \ell - 2$ in (*iiib*).

$$D_n^J = J + 2 + \sum_{\ell=1}^{\infty} \left(\frac{g_{\text{YM}}^2 N}{\pi^2} \sin^2 \frac{\pi n}{J+1} \right)^\ell \left(\frac{(-1)^{\ell-1} (2\ell-2)!}{4^{\ell-1} \ell! (\ell-1)!} + \sum_{k,l=1}^{\ell-1} c_{\ell,k,l} \frac{\cos^{2l} \frac{\pi n}{J+1}}{(J+1)^k} \right),$$

$$c_{2,1,1} = -1,$$

$$c_{3,k,l} = \begin{pmatrix} +\frac{3}{4} - \frac{1}{2}c_4 & +\frac{1}{2} \\ -\frac{3}{4} & +\frac{5}{2} \end{pmatrix},$$

$$c_{4,k,l} = \begin{pmatrix} -\frac{5}{8} + \frac{7}{12}c_4 - c_6 & -\frac{5}{12} + \frac{1}{6}c_4 + 4c_6 & -\frac{1}{3} \\ +\frac{3}{4} - \frac{3}{4}c_4 & -\frac{7}{4} + 3c_4 & -\frac{7}{2} \\ -\frac{1}{2} & +\frac{59}{12} & -\frac{49}{6} \end{pmatrix},$$

$$c_{5,k,l} = \begin{pmatrix} +\frac{35}{64} - \frac{5}{8}c_4 + \frac{1}{32}c_4^2 + \frac{11}{4}c_6 + \frac{1}{3}c_{8a} - \frac{1}{2}c_{8b} & +\frac{35}{96} - \frac{11}{48}c_4 - 7c_6 - 6c_{8b} & +\frac{7}{24} - \frac{1}{12}c_4 + 8c_{8b} & +\frac{1}{4} \\ -\frac{45}{64} + \frac{17}{16}c_4 - \frac{3}{16}c_4^2 - \frac{3}{2}c_6 & +\frac{185}{96} - \frac{47}{12}c_4 + \frac{7}{8}c_4^2 + 17c_6 & +\frac{131}{48} - \frac{10}{3}c_4 - 32c_6 & +\frac{33}{8} \\ +\frac{5}{8} - \frac{3}{4}c_4 & -\frac{125}{24} + \frac{17}{2}c_4 & -\frac{13}{24} - 16c_4 & +\frac{81}{4} \\ -\frac{5}{16} & +\frac{305}{48} & -\frac{1319}{48} & +\frac{243}{8} \end{pmatrix}.$$

Table 1: Two excitation formula up to five loops with $J = L - K = L - 2$. For gauge theory set $c_4 = c_6 = c_{8a} = c_{8b} = 0$. For the string Bethe ansatz set $c_4 = 1$, $c_6 = c_{8a} = c_{8b} = 0$.

4 Discussion

Here we have presented a perturbative spin chain Hamiltonian that agrees with the Bethe ansatz for quantum string theory extrapolated to weak coupling. It is very similar to the gauge theory Hamiltonian, but has a slightly extended form of interactions. In terms of physics the two Hamiltonians are quite different: The gauge Hamiltonian does not perturbatively reproduce the near plane-wave and spinning strings results. It was argued that agreement may be restored when order-of-limits or, more explicitly, wrapping effects are taken into account [4]. In contradistinction, the string Hamiltonian naively agrees with near plane-waves and spinning strings in perturbation theory! This is somewhat disappointing, as one might have hoped that the ‘strange’ string Bethe equations would not have yielded a sensible spin chain at weak coupling. Their form would thus have had to be altered to give a spin chain at weak coupling (presumably precisely the gauge theory spin chain). At this point such an interpolating Bethe ansatz does not appear necessary for consistency reasons any longer. Instead, it is a logical possibility that string theory is described by (7) at all values of g and in particular by (9) with $c_4 = 1$ at weak coupling.¹⁰ This would be disastrous for the AdS/CFT correspondence which requires $c_4 = 0$ to achieve agreement with gauge theory. A test of this option might be achieved by computing higher $1/J$ corrections in near plane-wave string theory¹¹ and comparing them to (7). However, it is also possible that $c_4 = 1$ effectively sums up all putative wrapping effects and thus describes string theory only in the thermodynamic limit. In this case, the equivalence between (7) and (9) is still a remarkable mathematical result which clearly deserves further investigation.

¹⁰It was shown that the spectrum of string states is compatible with the spectrum of gauge theory local operators when a suitable dimension formula is chosen [23].

¹¹See [24] for first steps in this direction.

Unfortunately, we do not yet know a good condition (*iib'*) to replace (*iib*) which would fix the parameters values c_4, \dots uniquely.¹² Certainly, we can adjust them to (7), but this is not very satisfactory when we try to generalise our result from the $\mathfrak{su}(2)$ subsector to bigger ones. For instance, the Hamiltonian (9) is also a restriction of the supersymmetric $\mathfrak{su}(2|3)$ spin chain investigated in [6] when we set¹³

$$\sigma_1 = -\frac{3}{2}c_4, \quad \sigma_2 = \frac{1}{8}c_4, \quad \sigma_3 = \frac{1}{8}c_4, \quad \sigma_4 = 0. \quad (12)$$

Here it would be interesting to compare to the predictions for fermionic states in near plane-wave string theory [22]. In the absence of a higher-loop Bethe ansatz, the virial and coherent methods [25] would be very helpful.

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¹²It would be interesting to find a Bethe ansatz for arbitrary parameter values, if it exists at all.

¹³Note that $\sigma_2 = \frac{1}{8}c_4$ in [6] is equivalent to $\sigma_2 = \frac{3}{4}c_4$ in [5].

$$\begin{aligned}
\mathcal{H}_0 &= +\{\} - \{1\} \\
\mathcal{H}_2 &= -2\{\} + 3\{1\} - \frac{1}{2}(\{1, 2\} + \{2, 1\}) \\
\mathcal{H}_4 &= +\left(+\frac{15}{2} - \frac{1}{2}c_4\right)\{\} \\
&\quad + (-13 + \frac{3}{2}c_4)\{1\} \\
&\quad + \left(\frac{1}{2} - \frac{3}{4}c_4\right)\{1, 3\} \\
&\quad + \left(+3 - \frac{1}{2}c_4\right)(\{1, 2\} + \{2, 1\}) \\
&\quad + \left(+\frac{1}{2}c_4\right)(\{1, 3, 2\} + \{2, 1, 3\}) \\
&\quad + \left(-\frac{1}{2}\right)(\{1, 2, 3\} + \{3, 2, 1\}) \\
&\quad + \left(-\frac{1}{4}c_4\right)\{2, 1, 3, 2\} \\
\mathcal{H}_6 &= +(-35 + \frac{13}{4}c_4 - 6c_6)\{\} \\
&\quad + (+67 + 4\alpha_6)\{1\} \\
&\quad + \left(-\frac{21}{4} - 2\alpha_6 + \frac{11}{12}c_4 + 4c_6\right)\{1, 3\} \\
&\quad + \left(-\frac{1}{4} + \frac{1}{8}c_4 - 8c_6\right)\{1, 4\} \\
&\quad + \left(-\frac{151}{8} - 4\alpha_6 - \frac{79}{12}c_4 + 16c_6\right)(\{1, 2\} + \{2, 1\}) \\
&\quad + (+2\alpha_6 + c_4 - 12c_6)(\{1, 3, 2\} + \{2, 1, 3\}) \\
&\quad + \left(+\frac{1}{4} - \frac{5}{12}c_4 + 2c_6\right)(\{1, 2, 4\} + \{1, 3, 4\} + \{1, 4, 3\} + \{2, 1, 4\}) \\
&\quad + (+6 + 2\alpha_6 + \frac{61}{12}c_4 - 10c_6)(\{1, 2, 3\} + \{3, 2, 1\}) \\
&\quad + \left(-\frac{3}{4} - 2\alpha_6 - \frac{19}{6}c_4 + 14c_6\right)\{2, 1, 3, 2\} \\
&\quad + \left(+\frac{9}{8} + 2\alpha_6 + \frac{47}{8}c_4 - 9c_6\right)(\{1, 3, 2, 4\} + \{2, 1, 4, 3\}) \\
&\quad + \left(-\frac{1}{2} - \alpha_6 - \frac{49}{8}c_4 + 6c_6\right)(\{1, 2, 4, 3\} + \{1, 4, 3, 2\} + \{2, 1, 3, 4\} + \{3, 2, 1, 4\}) \\
&\quad + \left(-\frac{5}{8}\right)(\{1, 2, 3, 4\} + \{4, 3, 2, 1\}) \\
&\quad + \left(-\frac{21}{4}c_4 - 2c_6\right)(\{1, 3, 2, 4, 3\} + \{2, 1, 3, 2, 4\} + \{2, 1, 4, 3, 2\} + \{3, 2, 1, 4, 3\}) \\
&\quad + \left(+\frac{1}{24}c_4 + c_6\right)(\{2, 1, 3, 2, 4, 3\} + \{3, 2, 1, 4, 3, 2\}) \\
\mathcal{H}_8 &= +\left(+\frac{1479}{8} + 4\alpha_{8d} - \frac{39}{2}c_4 + \frac{3}{16}c_4^2 + 86c_6 + 4c_{8a} - 10c_{8b}\right)\{\} \\
&\quad + \left(-\frac{1043}{4} - 12\alpha_6 + 4\alpha_{8a} - 16\alpha_{8d} + \frac{535}{16}c_4 - \frac{3}{8}c_4^2 - 293c_6 - 12c_{8a} + 40c_{8b}\right)\{1\} \\
&\quad + (-19 + 8\alpha_6 - 2\alpha_{8a} - 4\alpha_{8b} + 8\alpha_{8d} - \frac{1667}{96}c_4 + \frac{51}{32}c_4^2 + \frac{177}{2}c_6 + \frac{10}{3}c_{8a} - 38c_{8b})\{1, 3\} \\
&\quad + (+5 + 2\alpha_6 + 4\alpha_{8b} + 4\alpha_{8c} + 8\alpha_{8d} - \frac{5}{3}c_4 - \frac{19}{16}c_4^2 + 103c_6 - \frac{8}{3}c_{8a} + 16c_{8b})\{1, 4\} \\
&\quad + \left(+\frac{1}{8} + \frac{1}{16}c_4 + \frac{1}{8}c_4^2 + 10c_6 + 10c_{8b}\right)\{1, 5\} \\
&\quad + (+11\alpha_6 - 4\alpha_{8a} + 2\alpha_{8c} + 6\alpha_{8d} + \frac{47}{32}c_4 + \frac{3}{8}c_4^2 + 81c_6 + \frac{10}{3}c_{8a} - 28c_{8b})(\{1, 2\} + \{2, 1\}) \\
&\quad + \left(-\frac{1}{4} + \frac{5}{12}c_4 - \frac{9}{16}c_4^2 - 6c_6 + 8c_{8b}\right)\{1, 3, 5\} \\
&\quad + \left(+\frac{251}{4} - 5\alpha_6 + 2\alpha_{8a} - 2\alpha_{8c} + \frac{277}{16}c_4 - \frac{7}{4}c_4^2 - \frac{37}{2}c_6 - 6c_{8a} + 43c_{8b}\right)(\{1, 3, 2\} + \{2, 1, 3\}) \\
&\quad + (-3 - \alpha_6 - 2\alpha_{8c} - 4\alpha_{8d} + \frac{143}{48}c_4 + \frac{1}{32}c_4^2 - \frac{49}{2}c_6 + \frac{8}{3}c_{8a} - 2c_{8b})(\{1, 2, 4\} + \{1, 3, 4\} + \{1, 4, 3\} + \{2, 1, 4\}) \\
&\quad + \left(-\frac{1}{8} + \frac{5}{48}c_4 - \frac{1}{16}c_4^2 - \frac{11}{2}c_6 - 2c_{8b}\right)(\{1, 2, 5\} + \{1, 4, 5\} + \{1, 5, 4\} + \{2, 1, 5\}) \\
&\quad + \left(+\frac{41}{4} - 6\alpha_6 + 2\alpha_{8a} - 4\alpha_{8c} - 2\alpha_{8d} - \frac{935}{96}c_4 + \frac{11}{32}c_4^2 - \frac{15}{2}c_6 + \frac{8}{3}c_{8a} - 3c_{8b}\right)(\{1, 2, 3\} + \{3, 2, 1\}) \\
&\quad + \left(-\frac{107}{2} + 4\alpha_6 - 2\alpha_{8a} - 8\alpha_{8d} - \frac{171}{32}c_4 + \frac{35}{32}c_4^2 - \frac{25}{2}c_6 + \frac{26}{3}c_{8a} - 28c_{8b}\right)\{2, 1, 3, 2\} \\
&\quad + \left(+\frac{1}{4} + \alpha_{8b} - \frac{19}{32}c_4 + \frac{3}{8}c_4^2 + 2c_6 - 7c_{8b}\right)(\{1, 3, 2, 5\} + \{1, 3, 5, 4\} + \{1, 4, 3, 5\} + \{2, 1, 3, 5\}) \\
&\quad + \left(+\frac{183}{4} - 6\alpha_6 + 2\alpha_{8a} - 2\alpha_{8b} + 2\alpha_{8d} - \frac{321}{16}c_4 + \frac{7}{8}c_4^2 - \frac{19}{2}c_6 - 2c_{8a} + c_{8b}\right)(\{1, 3, 2, 4\} + \{2, 1, 4, 3\}) \\
&\quad + \left(-\frac{3}{4} - 2\alpha_{8b} + \frac{53}{32}c_4 - \frac{1}{16}c_4^2 - 18c_6 - 6c_{8a} + 24c_{8b}\right)(\{1, 2, 5, 4\} + \{2, 1, 4, 5\}) \\
&\quad + (+1 + 2\alpha_{8b} - \frac{17}{8}c_4 + \frac{1}{8}c_4^2 + 22c_6 + 6c_{8a} - 23c_{8b})(\{1, 2, 4, 5\} + \{2, 1, 5, 4\}) \\
&\quad + \left(-\frac{5}{2} + \frac{5}{2}\alpha_6 - \alpha_{8a} + \alpha_{8b} + 3\alpha_{8c} - \alpha_{8d}\right)(\{1, 2, 4, 3\} + \{1, 4, 3, 2\} + \{2, 1, 3, 4\} + \{3, 2, 1, 4\}) \\
&\quad + (-\alpha_{8b})(\{1, 2, 3, 5\} + \{1, 3, 4, 5\} + \{1, 5, 4, 3\} + \{3, 2, 1, 5\}) \\
&\quad + \left(+\frac{35}{4} + \alpha_6 + 2\alpha_{8c} + 2\alpha_{8d} + \frac{7}{8}c_4 - \frac{1}{32}c_4^2 - \frac{3}{2}c_6 - 2c_{8a} + 3c_{8b}\right)(\{1, 2, 3, 4\} + \{4, 3, 2, 1\}) \\
&\quad + \left(+\frac{1}{8}c_4 - \frac{3}{16}c_4^2 + c_6 + 4c_{8b}\right)(\{1, 4, 3, 5, 4\} + \{2, 1, 3, 2, 5\}) \\
&\quad + \left(-\frac{7}{8} - \alpha_6 + 2\alpha_{8c} - \frac{19}{4}c_4 - \frac{1}{4}c_4^2 + 12c_6\right)(\{1, 4, 3, 2, 5\} + \{2, 1, 3, 5, 4\}) \\
&\quad + \left(+\frac{1}{2} + \alpha_6 + \frac{49}{24}c_4 - \frac{3}{16}c_4^2 + \frac{13}{2}c_6 + 4c_{8a} - 14c_{8b}\right)(\{1, 3, 2, 5, 4\} + \{2, 1, 4, 3, 5\}) \\
&\quad + \left(+\frac{5}{8} + \frac{1}{2}\alpha_6 - \alpha_{8c} + 3\alpha_{8d} + \frac{333}{64}c_4 - \frac{11}{32}c_4^2 + 14c_6 - \frac{5}{3}c_{8a} - 2c_{8b}\right)(\{1, 3, 2, 4, 3\} + \{2, 1, 3, 2, 4\} + \{2, 1, 4, 3, 2\} + \{3, 2, 1, 4, 3\}) \\
&\quad + \left(+\frac{1}{4} - 2\alpha_{8c} + \frac{51}{32}c_4 + \frac{7}{32}c_4^2 + 7c_6 + 4c_{8a} - 17c_{8b}\right)(\{1, 2, 5, 4, 3\} + \{3, 2, 1, 4, 5\}) \\
&\quad + \left(+\frac{1}{4} + \frac{\alpha_6}{2} + \alpha_{8c} + \frac{277}{192}c_4 - \frac{9}{64}c_4^2 - \frac{39}{4}c_6 - 4c_{8a} + \frac{29}{2}c_{8b}\right)(\{1, 2, 4, 3, 5\} + \{1, 3, 2, 4, 5\} + \{2, 1, 5, 4, 3\} + \{3, 2, 1, 5, 4\}) \\
&\quad + \left(-\frac{1}{2}\alpha_6 - \alpha_{8c}\right)(\{1, 2, 3, 5, 4\} + \{1, 5, 4, 3, 2\} + \{2, 1, 3, 4, 5\} + \{4, 3, 2, 1, 5\}) \\
&\quad + \left(-\frac{7}{8}\right)(\{1, 2, 3, 4, 5\} + \{5, 4, 3, 2, 1\}) \\
&\quad + \left(+\alpha_{8d} + \frac{47}{192}c_4 + \frac{1}{8}c_4^2 - 2c_6 - c_{8a} + 3c_{8b}\right)(\{1, 4, 3, 2, 5, 4\} + \{2, 1, 3, 2, 5, 4\} + \{2, 1, 4, 3, 2, 5\} + \{2, 1, 4, 3, 5, 4\}) \\
&\quad + (-2\alpha_{8d} - c_4 + \frac{5}{32}c_4^2 - 9c_6 + \frac{4}{3}c_{8a} + 2c_{8b})(\{2, 1, 3, 2, 4, 3\} + \{3, 2, 1, 4, 3, 2\}) \\
&\quad + (-2\alpha_{8d} - \frac{3}{8}c_4 - 4c_6 + 8c_{8b})(\{1, 3, 2, 5, 4, 3\} + \{3, 2, 1, 4, 3, 5\}) \\
&\quad + (+2\alpha_{8d})(\{1, 3, 2, 4, 3, 5\} + \{3, 2, 1, 5, 4, 3\}) \\
&\quad + (-\alpha_{8d} - \frac{31}{64}c_4 + c_{8a})(\{1, 2, 4, 3, 5, 4\} + \{2, 1, 3, 2, 4, 5\} + \{2, 1, 5, 4, 3, 2\} + \{4, 3, 2, 1, 5, 4\}) \\
&\quad + \left(+\frac{1}{24}c_4 - \frac{1}{16}c_4^2 - 4c_{8b}\right)\{2, 1, 4, 3, 2, 5, 4\} \\
&\quad + \left(+\frac{5}{192}c_4 - \frac{1}{64}c_4^2 + \frac{1}{4}c_6 - \frac{3}{2}c_{8b}\right)(\{1, 4, 3, 2, 5, 4, 3\} + \{2, 1, 3, 2, 5, 4, 3\} + \{3, 2, 1, 4, 3, 2, 5\} + \{3, 2, 1, 4, 3, 5, 4\}) \\
&\quad + \left(+\frac{11}{192}c_4 + \frac{1}{64}c_4^2 + \frac{3}{4}c_6 - \frac{5}{2}c_{8b}\right)(\{1, 3, 2, 4, 3, 5, 4\} + \{2, 1, 3, 2, 4, 3, 5\} + \{3, 2, 1, 5, 4, 3, 2\} + \{4, 3, 2, 1, 5, 4, 3\}) \\
&\quad + \left(-\frac{1}{96}c_4 + c_{8b}\right)(\{2, 1, 4, 3, 2, 5, 4, 3\} + \{3, 2, 1, 4, 3, 2, 5, 4\}) \\
&\quad + \left(-\frac{1}{96}c_4 + c_{8b}\right)(\{2, 1, 3, 2, 4, 3, 5, 4\} + \{4, 3, 2, 1, 5, 4, 3, 2\})
\end{aligned}$$

Table 2: Interpolating five-loop Hamiltonian. For gauge theory set $c_4 = c_6 = c_{8a} = c_{8b} = 0$. For the string Bethe ansatz set $c_4 = 1$, $c_6 = c_{8a} = c_{8b} = 0$. The parameters α do not influence the spectrum.

$$\begin{aligned}
\mathcal{H}_{0,12} &= 1 - \mathcal{P}_{12} = \frac{1}{2} - \frac{1}{2}\bar{\sigma}_1 \cdot \bar{\sigma}_2 \\
\mathcal{H}_{2,123} &= -2\mathcal{H}_{0,12} + \frac{1}{2}\mathcal{H}_{0,13} \\
\mathcal{H}_{4,1234} &= +\frac{15}{2}\mathcal{H}_{0,12} - 3\mathcal{H}_{0,13} + \frac{1}{2}\mathcal{H}_{0,14} \\
&\quad + \left(-\frac{1}{2} - \frac{1}{2}c_4\right)\mathcal{H}_{0,14}\mathcal{H}_{0,23} \\
&\quad + \left(+\frac{1}{2} + \frac{1}{4}c_4\right)\mathcal{H}_{0,13}\mathcal{H}_{0,24} \\
&\quad + \left(-\frac{1}{4}c_4\right)\mathcal{H}_{0,12}\mathcal{H}_{0,34} \\
\mathcal{H}_{6,12345} &= -35\mathcal{H}_{0,12} + \frac{35}{2}\mathcal{H}_{0,13} - 5\mathcal{H}_{0,14} + \frac{5}{8}\mathcal{H}_{0,15} \\
&\quad + \left(+\frac{17}{4} - 2\alpha_6 - \frac{7}{4}c_4 + 12c_6\right)\mathcal{H}_{0,14}\mathcal{H}_{0,23} \\
&\quad + \left(-\frac{7}{8} - \alpha_6 - \frac{7}{24}c_4 + 4c_6\right)\mathcal{H}_{0,15}\mathcal{H}_{0,23} \\
&\quad + \left(-5 - \frac{7}{4}c_4\right)\mathcal{H}_{0,13}\mathcal{H}_{0,24} \\
&\quad + \left(+\frac{3}{4} + 2\alpha_6 + \frac{16}{3}c_4 - 10c_6\right)\mathcal{H}_{0,15}\mathcal{H}_{0,24} \\
&\quad + \left(+\frac{7}{8} + \alpha_6 + \frac{11}{4}c_4 - 6c_6\right)\mathcal{H}_{0,13}\mathcal{H}_{0,25} \\
&\quad + \left(-\frac{3}{4} - 2\alpha_6 - \frac{127}{24}c_4 + 11c_6\right)\mathcal{H}_{0,14}\mathcal{H}_{0,25} \\
&\quad + \left(+\frac{3}{4} + 2\alpha_6 + \frac{29}{4}c_4 - 12c_6\right)\mathcal{H}_{0,12}\mathcal{H}_{0,34} \\
&\quad + \left(-\frac{7}{8} - \alpha_6 - \frac{7}{24}c_4 + 4c_6\right)\mathcal{H}_{0,15}\mathcal{H}_{0,34} \\
&\quad + \left(-\frac{1}{24}c_4 + 2c_6\right)\mathcal{H}_{0,12}\mathcal{H}_{0,35} \\
&\quad + \left(+\frac{7}{8} + \alpha_6 + \frac{11}{4}c_4 - 6c_6\right)\mathcal{H}_{0,14}\mathcal{H}_{0,35} \\
&\quad + \left(-\frac{1}{8}c_4 - 3c_6\right)\mathcal{H}_{0,12}\mathcal{H}_{0,45} \\
&\quad + \left(-\frac{1}{24}c_4 + 2c_6\right)\mathcal{H}_{0,13}\mathcal{H}_{0,45} \\
\mathcal{H}_{8,123456} &= +\frac{735}{8}\mathcal{H}_{0,12} - 105\mathcal{H}_{0,13} + \frac{315}{8}\mathcal{H}_{0,14} - \frac{35}{4}\mathcal{H}_{0,15} + \frac{7}{8}\mathcal{H}_{0,16} \\
&\quad + \left(-\frac{725}{8} + 5\alpha_6 - 2\alpha_{8a} - 2\alpha_{8c} - 2\alpha_{8d} - \frac{355}{48}c_4 + \frac{39}{32}c_4^2 + \frac{35}{2}c_6 + 12c_{8a} - 48c_{8b}\right)\mathcal{H}_{0,14}\mathcal{H}_{0,23} \\
&\quad + \left(+\frac{307}{8} - 2\alpha_{8d} + \frac{503}{48}c_4 - \frac{19}{32}c_4^2 - \frac{37}{2}c_6 - 4c_{8a} + 24c_{8b}\right)\mathcal{H}_{0,13}\mathcal{H}_{0,24} \\
&\quad + \left(+\frac{429}{8} - 4\alpha_6 + 2\alpha_{8a} + 2\alpha_{8b} + 2\alpha_{8c} + 2\alpha_{8d} - \frac{671}{48}c_4 + \frac{11}{32}c_4^2 + 14c_6 + \frac{8}{3}c_{8a} - 14c_{8b}\right)\mathcal{H}_{0,15}\mathcal{H}_{0,24} \\
&\quad + \left(-\frac{429}{8} + 4\alpha_6 - 2\alpha_{8a} - 2\alpha_{8b} - 2\alpha_{8c} - 4\alpha_{8d} + \frac{1249}{96}c_4 - \frac{7}{32}c_4^2 - \frac{45}{2}c_6 - \frac{4}{3}c_{8a} + 15c_{8b}\right)\mathcal{H}_{0,14}\mathcal{H}_{0,25} \\
&\quad + \left(-\frac{1}{8} + 2\alpha_{8b} + 4\alpha_{8c} + 2\alpha_{8d} - \frac{107}{24}c_4 - \frac{3}{16}c_4^2 + \frac{23}{2}c_6 - 2c_{8a} + 3c_{8b}\right)\mathcal{H}_{0,16}\mathcal{H}_{0,25} \\
&\quad + \left(+\frac{1}{8} - 2\alpha_{8b} - 4\alpha_{8c} - 2\alpha_{8d} + \frac{427}{96}c_4 + \frac{3}{16}c_4^2 - \frac{23}{2}c_6 + 2c_{8a} - 2c_{8b}\right)\mathcal{H}_{0,15}\mathcal{H}_{0,26} \\
&\quad + \left(+\frac{409}{8} - 5\alpha_6 + 2\alpha_{8a} - 4\alpha_{8b} - 2\alpha_{8c} - \frac{1099}{48}c_4 + \frac{23}{32}c_4^2 + \frac{49}{2}c_6 + 4c_{8a} - 20c_{8b}\right)\mathcal{H}_{0,12}\mathcal{H}_{0,34} \\
&\quad + \left(-\frac{3}{8} - \alpha_6 + 2\alpha_{8b} + 2\alpha_{8c} + 2\alpha_{8d} - \frac{71}{12}c_4 - \frac{1}{32}c_4^2 + 21c_6 + 2c_{8a} - 11c_{8b}\right)\mathcal{H}_{0,16}\mathcal{H}_{0,34} \\
&\quad + \left(-\frac{3}{4} - 4\alpha_{8b} - 4\alpha_{8c} - 4\alpha_{8d} + \frac{49}{8}c_4 + \frac{1}{16}c_4^2 - 34c_6 - 4c_{8a} + 26c_{8b}\right)\mathcal{H}_{0,16}\mathcal{H}_{0,25}\mathcal{H}_{0,34} \\
&\quad + \left(+\frac{3}{4} + 4\alpha_{8b} + 4\alpha_{8c} + 4\alpha_{8d} - \frac{49}{8}c_4 - \frac{1}{16}c_4^2 + 34c_6 + 4c_{8a} - 26c_{8b}\right)\mathcal{H}_{0,15}\mathcal{H}_{0,26}\mathcal{H}_{0,34} \\
&\quad + \left(-\frac{309}{16} + \frac{5}{2}\alpha_6 - \alpha_{8a} - \alpha_{8b} - 2\alpha_{8c} - \alpha_{8d} + \frac{51}{4}c_4 - \frac{29}{64}c_4^2 - 14c_6 - \frac{14}{3}c_{8a} + 21c_{8b}\right)(\mathcal{H}_{0,15}\mathcal{H}_{0,23} + \mathcal{H}_{0,15}\mathcal{H}_{0,34}) \\
&\quad + \left(+\frac{3}{4} + 4\alpha_{8b} + 4\alpha_{8c} - \frac{105}{16}c_4 - \frac{1}{8}c_4^2 + 29c_6 + 4c_{8a} - 16c_{8b}\right)\mathcal{H}_{0,16}\mathcal{H}_{0,24}\mathcal{H}_{0,35} \\
&\quad + \left(+\frac{319}{16} - 2\alpha_6 + \alpha_{8a} + \alpha_{8b} + \alpha_{8c} + 3\alpha_{8d} - \frac{1463}{192}c_4 + \frac{5}{32}c_4^2 + \frac{89}{4}c_6 + 2c_{8a} - \frac{33}{2}c_{8b}\right)(\mathcal{H}_{0,13}\mathcal{H}_{0,25} + \mathcal{H}_{0,14}\mathcal{H}_{0,35}) \\
&\quad + \left(-\frac{5}{16} + \frac{\alpha_6}{2} - 2\alpha_{8b} - 2\alpha_{8c} + \frac{295}{64}c_4 + \frac{1}{32}c_4^2 - \frac{69}{4}c_6 - 2c_{8a} + \frac{15}{2}c_{8b}\right)(\mathcal{H}_{0,16}\mathcal{H}_{0,24} + \mathcal{H}_{0,16}\mathcal{H}_{0,35}) \\
&\quad + \left(+\frac{3}{8} + \alpha_6 - 2\alpha_{8b} - 2\alpha_{8c} + \frac{71}{12}c_4 + \frac{1}{32}c_4^2 - 21c_6 - 2c_{8a} + 11c_{8b}\right)\mathcal{H}_{0,14}\mathcal{H}_{0,36} \\
&\quad + \left(+\frac{3}{4} + 4\alpha_{8b} + 4\alpha_{8c} - \frac{13}{2}c_4 - \frac{1}{16}c_4^2 + 30c_6 + 4c_{8a} - 18c_{8b}\right)\mathcal{H}_{0,14}\mathcal{H}_{0,25}\mathcal{H}_{0,36} \\
&\quad + \left(+\frac{5}{16} - \frac{1}{2}\alpha_6 + 2\alpha_{8b} + 2\alpha_{8c} - \frac{437}{96}c_4 - \frac{1}{64}c_4^2 + 18c_6 + 2c_{8a} - 10c_{8b}\right)(\mathcal{H}_{0,14}\mathcal{H}_{0,26} + \mathcal{H}_{0,15}\mathcal{H}_{0,36}) \\
&\quad + \left(-\frac{3}{4} - 4\alpha_{8b} - 4\alpha_{8c} + \frac{209}{32}c_4 + \frac{3}{32}c_4^2 - \frac{59}{2}c_6 - 4c_{8a} + 17c_{8b}\right)(\mathcal{H}_{0,14}\mathcal{H}_{0,26}\mathcal{H}_{0,35} + \mathcal{H}_{0,15}\mathcal{H}_{0,24}\mathcal{H}_{0,36}) \\
&\quad + \left(+\frac{3}{8} + 2\alpha_{8b} + 2\alpha_{8c} + 2\alpha_{8d} - \frac{95}{48}c_4 - \frac{3}{4}c_4^2 + 44c_6 + 6c_{8b}\right)\mathcal{H}_{0,12}\mathcal{H}_{0,45} \\
&\quad + \left(+\frac{3}{4} + 4\alpha_{8d} + \frac{115}{48}c_4 - 18c_6 - 8c_{8a} + 22c_{8b}\right)\mathcal{H}_{0,16}\mathcal{H}_{0,23}\mathcal{H}_{0,45} \\
&\quad + \left(-\frac{1}{16} - \frac{1}{2}\alpha_6 + \alpha_{8b} + 2\alpha_{8c} - \frac{59}{24}c_4 + \frac{7}{64}c_4^2 - 22c_6 - \frac{10}{3}c_{8a} + 11c_{8b}\right)(\mathcal{H}_{0,12}\mathcal{H}_{0,35} + \mathcal{H}_{0,13}\mathcal{H}_{0,45}) \\
&\quad + \left(-\frac{7}{16} - \alpha_{8c} - 2\alpha_{8d} - \frac{29}{192}c_4 + \frac{5}{32}c_4^2 + \frac{23}{2}c_6 + 4c_{8a} - \frac{21}{2}c_{8b}\right)(\mathcal{H}_{0,16}\mathcal{H}_{0,23} + \mathcal{H}_{0,16}\mathcal{H}_{0,45}) \\
&\quad + \left(-\frac{3}{8} - \frac{7}{16}c_4 + \frac{3}{32}c_4^2 + 9c_6 + 2c_{8a} - 9c_{8b}\right)\mathcal{H}_{0,13}\mathcal{H}_{0,46} \\
&\quad + \left(+\frac{3}{4} + \frac{11}{12}c_4 - \frac{3}{16}c_4^2 - 14c_6 - 4c_{8a} + 18c_{8b}\right)\mathcal{H}_{0,13}\mathcal{H}_{0,25}\mathcal{H}_{0,46} \\
&\quad + \left(+\frac{7}{16} + \alpha_{8c} + \alpha_{8d} - \frac{1}{3}c_4 - \frac{5}{32}c_4^2 - \frac{23}{4}c_6 - 3c_{8a} + \frac{21}{2}c_{8b}\right)(\mathcal{H}_{0,13}\mathcal{H}_{0,26} + \mathcal{H}_{0,15}\mathcal{H}_{0,46}) \\
&\quad + \left(-\frac{3}{4} - 2\alpha_{8d} - \frac{161}{96}c_4 + \frac{1}{8}c_4^2 + 16c_6 + 6c_{8a} - 18c_{8b}\right)(\mathcal{H}_{0,13}\mathcal{H}_{0,26}\mathcal{H}_{0,45} + \mathcal{H}_{0,15}\mathcal{H}_{0,23}\mathcal{H}_{0,46}) \\
&\quad + \left(-\frac{1}{32}c_4 - \frac{1}{32}c_4^2 + 3c_{8b}\right)\mathcal{H}_{0,12}\mathcal{H}_{0,56} \\
&\quad + \left(-\frac{1}{24}c_4 + \frac{1}{16}c_4^2 + 4c_{8b}\right)\mathcal{H}_{0,12}\mathcal{H}_{0,34}\mathcal{H}_{0,56} \\
&\quad + \left(+\frac{3}{16} + \alpha_{8b} + \alpha_{8c} - \frac{35}{24}c_4 - \frac{1}{64}c_4^2 + 2c_6 - 2c_{8b}\right)(\mathcal{H}_{0,12}\mathcal{H}_{0,46} + \mathcal{H}_{0,13}\mathcal{H}_{0,56}) \\
&\quad + \left(-\frac{3}{16} - \alpha_{8b} - \alpha_{8c} + \frac{67}{48}c_4 + \frac{3}{64}c_4^2 - \frac{5}{2}c_6\right)(\mathcal{H}_{0,12}\mathcal{H}_{0,36} + \mathcal{H}_{0,14}\mathcal{H}_{0,56}) \\
&\quad + \left(+\frac{3}{8} + 2\alpha_{8b} + 2\alpha_{8c} - \frac{67}{24}c_4 + \frac{1}{32}c_4^2 + 7c_6 - 4c_{8b}\right)(\mathcal{H}_{0,12}\mathcal{H}_{0,36}\mathcal{H}_{0,45} + \mathcal{H}_{0,14}\mathcal{H}_{0,23}\mathcal{H}_{0,56}) \\
&\quad + \left(-\frac{3}{8} - 2\alpha_{8b} - 2\alpha_{8c} + \frac{17}{6}c_4 + \frac{1}{32}c_4^2 - 7c_6\right)(\mathcal{H}_{0,12}\mathcal{H}_{0,35}\mathcal{H}_{0,46} + \mathcal{H}_{0,13}\mathcal{H}_{0,24}\mathcal{H}_{0,56})
\end{aligned}$$

Table 3: Alternative representation of the five-loop Hamiltonian using non-overlapping permutations. For gauge theory set $c_4 = c_6 = c_{8a} = c_{8b} = 0$. For the string Bethe ansatz set $c_4 = 1$, $c_6 = c_{8a} = c_{8b} = 0$. The parameters α do not influence the spectrum.

$$\begin{aligned}
\mathcal{Q}_{3,0} &= +\left(+\frac{i}{2}\right)(\{1, 2\} - \{2, 1\}) \\
\mathcal{Q}_{3,2} &= +(-2i)(\{1, 2\} - \{2, 1\}) \\
&\quad + \left(+\frac{i}{2}\right)(\{1, 2, 3\} - \{3, 2, 1\}) \\
\mathcal{Q}_{3,4} &= +\left(+\frac{73i}{8} - \frac{i}{2}c_4\right)(\{1, 2\} - \{2, 1\}) \\
&\quad + \left(-\frac{i}{4} + \frac{3i}{8}c_4\right)(\{1, 2, 4\} + \{1, 3, 4\} - \{1, 4, 3\} - \{2, 1, 4\}) \\
&\quad + \left(-\frac{7i}{2} + \frac{i}{4}c_4\right)(\{1, 2, 3\} - \{3, 2, 1\}) \\
&\quad + \left(-\frac{i}{8} - \frac{i}{2}c_4\right)(\{1, 3, 2, 4\} - \{2, 1, 4, 3\}) \\
&\quad + \left(-\frac{i}{8} - \frac{i}{4}c_4\right)(\{1, 2, 4, 3\} - \{1, 4, 3, 2\} + \{2, 1, 3, 4\} - \{3, 2, 1, 4\}) \\
&\quad + \left(+\frac{5i}{8}\right)(\{1, 2, 3, 4\} - \{4, 3, 2, 1\}) \\
&\quad + \left(+\frac{i}{8}c_4\right)(\{1, 3, 2, 4, 3\} + \{2, 1, 3, 2, 4\} - \{2, 1, 4, 3, 2\} - \{3, 2, 1, 4, 3\}) \\
\mathcal{Q}_{3,6} &= +\left(-\frac{373i}{8} - i\alpha_6 + \frac{31i}{24}c_4 - 8ic_6\right)(\{1, 2\} - \{2, 1\}) \\
&\quad + \left(+\frac{5i}{2} - \frac{41i}{12}c_4 + 5ic_6\right)(\{1, 2, 4\} + \{1, 3, 4\} - \{1, 4, 3\} - \{2, 1, 4\}) \\
&\quad + \left(+\frac{i}{8} - \frac{i}{12}c_4 + 4ic_6\right)(\{1, 2, 5\} + \{1, 4, 5\} - \{1, 5, 4\} - \{2, 1, 5\}) \\
&\quad + \left(+\frac{45i}{2} + 2i\alpha_6 + 3ic_4 - 6ic_6\right)(\{1, 2, 3\} - \{3, 2, 1\}) \\
&\quad + \left(+\frac{i}{48}c_4 - ic_6\right)(\{1, 3, 2, 5\} - \{1, 3, 5, 4\} + \{1, 4, 3, 5\} - \{2, 1, 3, 5\}) \\
&\quad + \left(+2i + \frac{14i}{3}c_4 - 2ic_6\right)(\{1, 3, 2, 4\} - \{2, 1, 4, 3\}) \\
&\quad + \left(-\frac{i}{4} + \frac{5i}{12}c_4 - 2ic_6\right)(\{1, 2, 4, 5\} - \{2, 1, 5, 4\}) \\
&\quad + \left(+\frac{23i}{16} - \frac{i}{2}\alpha_6 + \frac{15i}{16}c_4 + \frac{3i}{2}c_6\right)(\{1, 2, 4, 3\} - \{1, 4, 3, 2\} + \{2, 1, 3, 4\} - \{3, 2, 1, 4\}) \\
&\quad + \left(-\frac{i}{4} + \frac{19i}{48}c_4 - ic_6\right)(\{1, 2, 3, 5\} + \{1, 3, 4, 5\} - \{1, 5, 4, 3\} - \{3, 2, 1, 5\}) \\
&\quad + \left(-\frac{57i}{8} - i\alpha_6 - \frac{29i}{12}c_4 + 5ic_6\right)(\{1, 2, 3, 4\} - \{4, 3, 2, 1\}) \\
&\quad + \left(+\frac{i}{2} + i\alpha_6 + \frac{21i}{8}c_4 - 6ic_6\right)(\{1, 4, 3, 2, 5\} - \{2, 1, 3, 5, 4\}) \\
&\quad + \left(+\frac{i}{16} + \frac{i}{2}\alpha_6 + \frac{i}{16}c_4 - 3ic_6\right)(\{1, 3, 2, 4, 3\} + \{2, 1, 3, 2, 4\} - \{2, 1, 4, 3, 2\} - \{3, 2, 1, 4, 3\}) \\
&\quad + \left(-\frac{7i}{16} - \frac{i}{2}\alpha_6 - \frac{15i}{8}c_4 + \frac{3i}{2}c_6\right)(\{1, 2, 4, 3, 5\} + \{1, 3, 2, 4, 5\} - \{2, 1, 5, 4, 3\} - \{3, 2, 1, 5, 4\}) \\
&\quad + \left(+\frac{i}{16} + \frac{i}{2}\alpha_6 + \frac{17i}{16}c_4 - 3ic_6\right)(\{1, 2, 3, 5, 4\} - \{1, 5, 4, 3, 2\} + \{2, 1, 3, 4, 5\} - \{4, 3, 2, 1, 5\}) \\
&\quad + \left(+\frac{7i}{8}\right)(\{1, 2, 3, 4, 5\} - \{5, 4, 3, 2, 1\}) \\
&\quad + \left(+\frac{i}{24}c_4 + ic_6\right)(\{1, 4, 3, 2, 5, 4\} - \{2, 1, 3, 2, 5, 4\} + \{2, 1, 4, 3, 2, 5\} - \{2, 1, 4, 3, 5, 4\}) \\
&\quad + \left(+\frac{i}{8}c_4\right)(\{2, 1, 3, 2, 4, 3\} - \{3, 2, 1, 4, 3, 2\}) \\
&\quad + \left(+\frac{5i}{24}c_4 + 2ic_6\right)(\{1, 3, 2, 4, 3, 5\} - \{3, 2, 1, 5, 4, 3\}) \\
&\quad + \left(+\frac{i}{6}c_4 + ic_6\right)(\{1, 2, 4, 3, 5, 4\} + \{2, 1, 3, 2, 4, 5\} - \{2, 1, 5, 4, 3, 2\} - \{4, 3, 2, 1, 5, 4\}) \\
&\quad + \left(-\frac{i}{48}c_4 - \frac{i}{2}c_6\right)(\{1, 4, 3, 2, 5, 4, 3\} - \{2, 1, 3, 2, 5, 4, 3\} + \{3, 2, 1, 4, 3, 2, 5\} - \{3, 2, 1, 4, 3, 5, 4\}) \\
&\quad + \left(-\frac{i}{48}c_4 - \frac{i}{2}c_6\right)(\{1, 3, 2, 4, 3, 5, 4\} + \{2, 1, 3, 2, 4, 3, 5\} - \{3, 2, 1, 5, 4, 3, 2\} - \{4, 3, 2, 1, 5, 4, 3\})
\end{aligned}$$

Table 4: Interpolating four-loop third charge. For gauge theory set $c_4 = c_6 = 0$. For the string Bethe ansatz set $c_4 = 1, c_6 = 0$. The parameter α_6 does not influence the spectrum.