

IIB supergravity and E_{10}

Axel Kleinschmidt and Hermann Nicolai

Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)
Mühlenberg 1, D-14476 Golm, Germany
axel.kleinschmidt,hermann.nicolai@aei.mpg.de

Abstract

We analyse the geodesic $E_{10}/K(E_{10})$ σ -model in a level decomposition w.r.t. the $A_8 \times A_1$ subalgebra of E_{10} , adapted to the bosonic sector of type IIB supergravity, whose $SL(2, \mathbb{R})$ symmetry is identified with the A_1 factor. The bosonic supergravity equations of motion, when restricted to zeroth and first order spatial gradients, are shown to match with the σ -model equations of motion up to level $\ell = 4$. Remarkably, the self-duality of the five-form field strength is implied by E_{10} and the matching.

1 Introduction

The simple and essentially unique geodesic Lagrangian describing a null world line in the infinite-dimensional coset manifold $E_{10}/K(E_{10})$ has been shown to reproduce the dynamics of the bosonic sector of eleven-dimensional supergravity in the vicinity of a space-like singularity [1, 2]. This result was subsequently extended to *massive* IIA supergravity in [3], where also parts of the fermionic sector were treated for the first time. A main ingredient in the derivation of these results was the level decomposition of E_{10} w.r.t. the A_9 and D_9 subalgebras of E_{10} , respectively. Here, we extend these results to type IIB supergravity, and demonstrate that this model as well can be incorporated into the $E_{10}/K(E_{10})$ σ -model within the framework proposed in [1], by making a level decomposition w.r.t. the $A_8 \times A_1$ subalgebra of E_{10} . As we will explain, this decomposition is precisely adapted to type IIB supergravity, in that it gives rise to the field representation content of IIB supergravity, where the A_1 factor becomes identified with the continuous $SL(2, \mathbb{R})$ symmetry of the IIB theory. Furthermore, the bosonic supergravity equations of motion, when restricted to zeroth and first order spatial gradients, match with the σ -model equations of motion up to and including level $\ell = 4$. Perhaps our most important new result here is that the self-duality of the five-form field strength is implied by the dynamical matching between the $E_{10}/K(E_{10})$ σ -model and the supergravity equations of motion, and does not require local supersymmetry or some other extraneous argument for its explanation.

Related results had been obtained previously in the framework of another, and conceptually different proposal, according to which it is the ‘very extended’ Kac–Moody algebra E_{11} that underlies $D = 11$ supergravity or a suitable extension thereof [4]. This proposal can likewise be extended to massive IIA, and to IIB supergravity [5, 6], and consistency with a level decomposition of the adjoint representation of E_{11} was shown in [7]. By contrast, the present construction shows that the hyperbolic Kac–Moody algebra E_{10} is already ‘big enough’ by itself to accommodate all the maximally supersymmetric theories in $D = 10$ and $D = 11$. Unlike E_{10} , E_{11} does not allow for an action unless one introduces an unphysical extraneous coordinate [8]. In the absence of an action principle, the self-duality restriction on the the five-form field strength, as well as the mutual duality between the three- and seven-form field strengths must be imposed as an extra requirement.

Combining the known results, we can summarize the correspondence between the maximally supersymmetric theories and the maximal rank regular subalgebras of E_{10} as follows

$$\begin{array}{lll}
A_9 \subset E_{10} & \iff & D = 11 \text{ supergravity} \\
D_9 \subset E_{10} & \iff & \text{massive IIA supergravity} \\
A_8 \times A_1 \subset E_{10} & \iff & \text{IIB supergravity}
\end{array}$$

The decompositions of E_{10} w.r.t. its other rank 9 regular subalgebras $A_{D-2} \times E_{11-D}$ (for $D = 3, \dots, 9$) will similarly reproduce the representation content of maximal supergravities in D space-time dimensions as the lowest level representations. The first factor here is identified with the $SL(D-1)$ acting on the spatial vielbein, while E_{11-D} is the Cremmer–Julia hidden symmetry [9]. The only missing, but perhaps the most interesting, piece in this analysis is the decomposition w.r.t. the *affine* subalgebra E_9 obtained in the reduction to two dimensions.

In [3], we have shown that, with the exception of the space-filling D9-brane, E_{10} can accommodate all D-branes, such that NSNS and RR fields are associated with even and odd levels, respectively, in the D_9 decomposition of E_{10} .¹ These results are confirmed by the present investigation, but with the important difference that NSNS and RR fields occur in the same $SL(2, \mathbb{R})$ multiplets, and hence transform into one another under the action of $SL(2, \mathbb{R})$. The precise assignment of these fields to parts of the E_{10} structure is one of the main results of the present work.

This article is structured as follows. First, we deduce the generators of E_{10} and their relations as appropriate for the $A_8 \times A_1$ decomposition up to $\ell = 4$. From this we deduce the σ -model dynamics which are then shown to be equivalent to the reduced IIB supergravity equations if the fields are identified in the right way.

2 \mathfrak{e}_{10} relations in $\mathfrak{gl}(9) \oplus \mathfrak{so}(2, 1)$ form

Our analysis is based on a decomposition of the hyperbolic Kac–Moody algebra \mathfrak{e}_{10} under its $\mathfrak{gl}(9) \oplus \mathfrak{so}(2, 1)$ subalgebra, as indicated in figure 1. In the table below we list the field content on the first five levels. All representations occur with outer multiplicity one. For the decomposition technique we refer readers to [1, 10, 11, 7].

¹The possible relevance of the D9-brane for this algebraic analysis was emphasized to us by S. Chaudhuri.

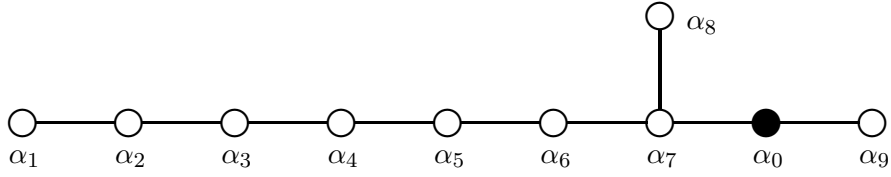


Figure 1: Dynkin diagram of \mathfrak{e}_{10} with nodes marked for a decomposition under $\mathfrak{gl}(9) \oplus \mathfrak{so}(2, 1)$.

ℓ	$A_8 \oplus A_1$ representation	Generator	Interpretation
1	$[0, 0, 0, 0, 0, 0, 1, 0] \otimes \mathbf{2}$	$E^{a_1 a_2}{}_{\alpha}$	F1/D1
2	$[0, 0, 0, 0, 1, 0, 0, 0] \otimes \mathbf{1}$	$E^{a_1 \dots a_4}$	D3
3	$[0, 0, 1, 0, 0, 0, 0, 0] \otimes \mathbf{2}$	$E^{a_1 \dots a_6}{}_{\alpha}$	NS5/D5
4	$[0, 1, 0, 0, 0, 0, 0, 1] \otimes \mathbf{1}$	$E^{a_1 \dots a_7 a_8}$	KK Monopole
5	$[1, 0, 0, 0, 0, 0, 0, 0] \otimes \mathbf{3}$	$E^{a_1 \dots a_8}{}_i$	NS7/D7/*
	$[1, 0, 0, 0, 0, 0, 1, 0] \otimes \mathbf{2}$?
	$[0, 0, 0, 0, 0, 0, 0, 1] \otimes \mathbf{2}$?

The $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$ representation is labelled by its dimension rather than by its Dynkin labels for ease of notation. For $\ell = 0$ the content is simply the adjoint representation of the subalgebra $\mathfrak{gl}(9) \oplus \mathfrak{so}(2, 1)$ (which includes the Cartan subalgebra generator h_0 of the omitted black node). We have also listed the generators for the first four levels, whose complete commutation relations will be worked out below.

In the last column of the table we indicate the interpretation of these representations in terms of branes of the IIB theory, as suggested by the representation structure and the study of finite-dimensional U-duality groups [12]. Level $\ell = 0$ contains the $SL(2, \mathbb{R})/SO(2)$ coset degrees of freedom which in supergravity are parametrized by the axion (the source of the D(-1) instanton) and the dilaton, and a third $SO(2)$ gauge degree of freedom which can be gauged to zero. At level $\ell = 1$ we find the fundamental (F1) string and Dirichlet (D1) string, which transform as a doublet under $SL(2, \mathbb{R})$ [13]. Evidently the $SL(2, \mathbb{R})$ symmetry thus mixes NSNS and RR type fields. This feature persists for the other Dp-branes of IIB string theory which couple to the $(p+1)$ -form RR potentials, with the exception of the rank-4 potential on $\ell = 2$ which is an $SL(2, \mathbb{R})$ singlet, consistent with the S-self-duality of the D3-brane. For $\ell = 3$ we get a doublet of five-branes, while at $\ell = 4$ we find a triplet of seven-branes, the best known of which is the D7-brane (other members of the multiplet have also been investigated in the literature [14, 15]). The asterisk indicates a subtlety for the seven branes arising from issues of gauge fixing: the corresponding fields are here interpreted as dual to the spatial gradients of the three $\ell = 0$ scalars, hence the gauge fixing implies a constraint on the charges associated with the three generators. We have also included the level $\ell = 5$ representations because this is the lowest level where a difference between E_{10} and E_{11} occurs, and because of their relevance to nine-branes. There the situation is much less clear, and will be commented upon in the concluding section.

We now study the commutation relations of the corresponding generators up to $|\ell| = 4$.²

²The commutators for the positive level ($0 \leq \ell \leq 3$) generators and two of the three anti-symmetric eight-forms on $\ell = 4$ were already given in [5], with the generators in representations of $GL(10)$, but without manifest $SL(2, \mathbb{R})$ covariance. The algebra G_{IIB} introduced there is a truncation of the $\ell \geq 0$

Level $\ell = 0$: The generators are K^a_b and J_i with relations

$$[K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b, \quad (1)$$

$$[J_i, J_j] = \epsilon_{ij}^k J_k, \quad (2)$$

$$[J_i, K^a_b] = 0. \quad (3)$$

The indices take values $a = 1, \dots, 9$; $i = 1, 2, 3$ and the $\mathfrak{so}(2, 1)$ metric is $\eta = \text{diag}(-, +, -)$ such that explicitly

$$[J_1, J_2] = -J_3, \quad [J_2, J_3] = -J_1, \quad [J_3, J_1] = J_2. \quad (4)$$

The identification with the Chevalley basis of \mathfrak{e}_{10} on the subalgebra nodes is

$$e_a = K^a_{a+1}, \quad f_a = K^{a+1}_a, \quad h_a = K^a_a - K^{a+1}_{a+1}, \quad (5)$$

for $a = 1, \dots, 8$. The trace $K \equiv \sum_{a=1}^9 K^a_a$ is

$$K = -8h_1 - 16h_2 - 24h_3 - 32h_4 - 40h_5 - 48h_6 - 56h_7 - 28h_8 - 18h_9 - 36h_0. \quad (6)$$

For the ‘decoupled’ $\mathfrak{sl}(2, \mathbb{R})$ node 9, the identifications are

$$e_9 = J_+, \quad f_9 = J_-, \quad h_9 = 2J_3, \quad (7)$$

where $J_{\pm} = J_1 \pm J_2$. The maximal compact subalgebra $\mathfrak{ke}_{10} \subset \mathfrak{e}_{10}$ consists of all ‘antisymmetric’ elements of \mathfrak{e}_{10} , where the generalized transposition is defined as

$$x^T := -\theta(x) \quad , \quad x \in \mathfrak{e}_{10}, \quad (8)$$

and θ is the Chevalley involution. With this definition, the antisymmetric elements at level $\ell = 0$ are $K^a_b - K^b_a$ and J_2 ; they generate the compact level $\ell = 0$ subalgebra $\mathfrak{so}(9) \oplus \mathfrak{so}(2) \subset \mathfrak{ke}_{10}$. The symmetric elements at $\ell = 0$ are $K^a_b + K^b_a$, and J_1 and J_3 .

A two-dimensional representation of $\mathfrak{so}(2, 1)$ is furnished by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

We introduce indices $\alpha, \beta = 1, 2$ for this representation by writing $(\sigma_i)^{\alpha}_{\beta}$. The tensor $\epsilon^{\alpha\beta}$ equals σ_2 as a set of numbers and satisfies the condition $\epsilon^{\alpha\gamma}\epsilon_{\beta\delta}(\sigma_i)^{\delta}_{\gamma} = (\sigma_i)^{\alpha}_{\beta}$. The inverse $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$ satisfies $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. We also define $\sigma^i \equiv \eta^{ij}\sigma_j$.

The normalizations of the generators within \mathfrak{e}_{10} are

$$\langle K^a_b | K^c_d \rangle = \delta_d^a \delta_b^c - \delta_b^a \delta_d^c, \quad \langle J_i | J_j \rangle = -\frac{1}{2} \eta_{ij}. \quad (10)$$

consistent with the standard normalization of the Chevalley generators. From this it can be shown that all compact (i.e. anti-symmetric) generators have negative norm as expected.

Level $\ell = 1$: The only representation is an $\mathfrak{so}(2, 1)$ doublet of $\mathfrak{gl}(9)$ two-forms, denoted by E^{ab}_{α} and transforming under $\mathfrak{gl}(9) \oplus \mathfrak{so}(2, 1)$ as

$$[K^a_b, E^{cd}_{\alpha}] = -2\delta_b^{[c} E^{d]a}_{\alpha}, \quad [J_i, E^{cd}_{\alpha}] = \frac{1}{2} (\sigma_i)^{\beta}_{\alpha} E^{cd}_{\beta}. \quad (11)$$

Borel subalgebra of E_{11} , but does not correspond to a consistent truncation of E_{11} if the negative level generators are also included.

The transposed field is

$$F_{ab}{}^\alpha := (E^{ab}{}_\alpha)^T \quad (12)$$

and satisfies

$$[K^a{}_b, F_{cd}{}^\alpha] = 2\delta_{[c}^a F_{d]b}{}^\alpha, \quad [J_i, F_{cd}{}^\alpha] = -\frac{1}{2}(\sigma_i)^\alpha{}_\beta F_{cd}{}^\beta. \quad (13)$$

The identification of the remaining Chevalley generators yields

$$e_0 = E^{89}{}_2, \quad f_0 = F_{89}{}^2, \quad h_0 = -\frac{1}{4}K + K^8{}_8 + K^9{}_9 - J_3, \quad (14)$$

where h_0 is already identified in $\mathfrak{gl}(9) \oplus \mathfrak{so}(2, 1)$ through (5)–(7).

Demanding the normalization

$$\langle E^{ab}{}_\alpha | F_{cd}{}^\beta \rangle = 2\delta_{cd}^{ab} \delta_\alpha^\beta \quad (15)$$

leads to the commutator

$$[E^{ab}{}_\alpha, F_{cd}{}^\beta] = -\frac{1}{2}\delta_\alpha^\beta \delta_{cd}^{ab} K + 4\delta_\alpha^\beta \delta_{[c}^{[a} K^{b]}_{d]} - 2\delta_{cd}^{ab} (\sigma^i)^\beta{}_\alpha J_i. \quad (16)$$

Level $\ell = 2$: The only representation is an $\mathfrak{so}(2, 1)$ singlet, transforming as an anti-symmetric rank four tensor under $\mathfrak{gl}(9)$. We denote it by $E^{a_1 \dots a_4}$, and its transpose by $F_{a_1 \dots a_4} := (E^{a_1 \dots a_4})^T$. They are obtained by commuting two $|\ell| = 1$ elements:

$$[E^{ab}{}_\alpha, E^{cd}{}_\beta] = \epsilon_{\alpha\beta} E^{abcd}, \quad [F_{ab}{}^\alpha, F_{cd}{}^\beta] = \epsilon^{\alpha\beta} F_{abcd}. \quad (17)$$

The normalization of commutators is consistent with

$$\langle E^{a_1 \dots a_4} | F_{b_1 \dots b_4} \rangle = 4! \delta_{b_1 \dots b_4}^{a_1 \dots a_4}. \quad (18)$$

The remaining commutation relations up to $|\ell| \leq 2$ are

$$[E^{a_1 \dots a_4}, F_{b_1 b_2}{}^\beta] = 12\epsilon^{\beta\alpha} \delta_{b_1 b_2}^{[a_1 a_2} E^{a_3 a_4]}{}_\alpha, \quad (19)$$

$$[F_{b_1 \dots b_4}, E^{a_1 a_2}{}_\alpha] = 12\epsilon_{\alpha\beta} \delta_{[b_1 b_2}^{a_1 a_2} F_{b_3 b_4]}{}^\beta, \quad (20)$$

$$[E^{a_1 \dots a_4}, F_{b_1 \dots b_4}] = -12\delta_{b_1 \dots b_4}^{a_1 \dots a_4} K + 96\delta_{[b_1 \dots b_3}^{[a_1 \dots a_3} K^{a_4]}_{b_4]}. \quad (21)$$

Note that there is no term proportional to J_i in the last commutator which is in agreement with the IIB supergravity structure as we will see.

Level $\ell = 3$: The only representation is an $\mathfrak{so}(2, 1)$ doublet of six-forms under $\mathfrak{gl}(9)$. The generators $E^{a_1 \dots a_6}{}_\alpha$ and their transpose $F_{a_1 \dots a_6}{}^\alpha$ are obtained via

$$[E^{a_1 a_2}{}_\alpha, E^{a_3 \dots a_6}] = E^{a_1 \dots a_6}{}_\alpha, \quad [F_{a_1 a_2}{}^\alpha, F_{a_3 \dots a_6}] = -F_{a_1 \dots a_6}{}^\alpha, \quad (22)$$

consistent with the normalization

$$\langle E^{a_1 \dots a_6}{}_\alpha | F_{b_1 \dots b_6}{}^\beta \rangle = 6! \delta_\alpha^\beta \delta_{b_1 \dots b_6}^{a_1 \dots a_6}. \quad (23)$$

The remaining relations are

$$[E^{a_1 a_2}{}_{\alpha}, F_{b_1 \dots b_6}{}^{\beta}] = -30 \delta_{\alpha}^{\beta} \delta_{[b_1 b_2]}^{a_1 a_2} F_{b_3 \dots b_6}, \quad (24)$$

$$[F_{b_1 b_2}{}^{\beta}, E^{a_1 \dots a_6}{}_{\alpha}] = 30 \delta_{\alpha}^{\beta} \delta_{b_1 b_2}^{[a_1 a_2} E^{a_3 \dots a_6]}, \quad (25)$$

$$[E^{a_1 \dots a_4}, F_{b_1 \dots b_6}{}^{\beta}] = 360 \delta_{[b_1 \dots b_4]}^{a_1 \dots a_4} F_{b_5 b_6}{}^{\beta}, \quad (26)$$

$$[F_{b_1 \dots b_4}, E^{a_1 \dots a_6}{}_{\alpha}] = -360 \delta_{b_1 \dots b_4}^{[a_1 \dots a_4} E^{a_5 a_6]}{}_{\alpha}, \quad (27)$$

$$[E^{a_1 \dots a_6}{}_{\alpha}, F_{b_1 \dots b_6}{}^{\beta}] = -540 \delta_{\alpha}^{\beta} \delta_{b_1 \dots b_6}^{a_1 \dots a_6} K + 4320 \delta_{\alpha}^{\beta} \delta_{[b_1 \dots b_5]}^{[a_1 \dots a_5} K^{a_6]}{}_{b_6}] - 720 \delta_{b_1 \dots b_6}^{a_1 \dots a_6} (\sigma^i)^{\beta}{}_{\alpha} J_i. \quad (28)$$

Level $\ell = 4$: The fields are a tensor of mixed A_8 Young symmetry, usually called the ‘dual graviton’, transforming as a singlet under $\mathfrak{so}(2, 1)$, and a fully anti-symmetric A_8 eight-form, transforming as a triplet under $\mathfrak{so}(2, 1)$. They are obtained by

$$[E^{a_1 a_2}{}_{\alpha}, E^{a_3 \dots a_8}{}_{\beta}] = \frac{1}{6} \epsilon_{\alpha\beta} E^{a_1 a_2 [a_3 \dots a_7 | a_8]} + E^{a_1 \dots a_8}{}_i (\epsilon \sigma^i)_{\alpha\beta}. \quad (29)$$

Therefore

$$E^{a_1 \dots a_8}{}_i = -\frac{1}{2} (\sigma_i \epsilon)^{\alpha\beta} [E^{[a_1 a_2}{}_{\alpha}, E^{a_3 \dots a_8]}{}_{\beta}], \quad (30)$$

$$E^{a_1 \dots a_7 | a_8} = -63 \epsilon^{\alpha\beta} [E^{[a_1 a_2}{}_{\alpha}, E^{a_3 \dots a_7 | a_8]}{}_{\beta}]. \quad (31)$$

This is consistent with the normalizations

$$\langle E^{a_1 \dots a_8}{}_i | F_{b_1 \dots b_8}{}^j \rangle = \frac{1}{2} \cdot 8! \delta_i^j \delta_{b_1 \dots b_8}^{a_1 \dots a_8}, \quad (32)$$

$$\langle E^{a_1 \dots a_7 | a_8} | F_{b_1 \dots b_7 | b_8} \rangle = \frac{7 \cdot 7!}{8} (\delta_{b_1 \dots b_7}^{a_1 \dots a_7} \delta_{b_8}^{a_8} + \delta_{b_8}^{[a_1} \delta_{[b_1 \dots b_6}^{a_2 \dots a_7]} \delta_{b_7]}^{a_8]}). \quad (33)$$

The remaining commutation relations for the second representation $E^{a_1 \dots a_8}{}_i$ are

$$[E^{a_1 \dots a_8}{}_i, F_{b_1 b_2}{}^{\beta}] = 28 (\sigma_i \epsilon)^{\beta\gamma} \delta_{b_1 b_2}^{[a_1 a_2} E^{a_3 \dots a_8]}{}_{\gamma}, \quad (34)$$

$$[E^{a_1 \dots a_8}{}_i, F_{b_1 \dots b_4}] = 0, \quad (35)$$

$$[E^{a_1 \dots a_8}{}_i, F_{b_1 \dots b_6}{}^{\beta}] = -\frac{1}{4} \cdot 8! (\sigma_i \epsilon)^{\beta\gamma} \delta_{b_1 \dots b_6}^{[a_1 \dots a_6} E^{a_2 a_8]}{}_{\gamma}, \quad (36)$$

$$[E^{a_1 \dots a_8}{}_i, F_{b_1 \dots b_8}{}^j] = -\frac{1}{2} \cdot 8! \delta_i^j \delta_{b_1 \dots b_8}^{a_1 \dots a_8} K + 4 \cdot 8! \delta_i^j \delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} K^{a_8]}{}_{b_8]} - \frac{1}{2} \cdot 8! \epsilon_i{}^{jk} J_k, \quad (37)$$

and the corresponding relations for the transposed fields. We note that the anti-symmetrized commutator $[E^{[a_1 \dots a_4}, E^{a_5 \dots a_8]}]$ vanishes, consistent with the E_7^{++} subsector.

The commutation relations involving $E^{a_1 \dots a_7 | a_8}$ are

$$[E^{a_1 \dots a_7 | a_8}, F_{b_1 b_2}{}^{\beta}] = +378 \epsilon^{\beta\alpha} (\delta_{b_1 b_2}^{[a_1 a_2} E^{a_3 \dots a_7 | a_8]}{}_{\alpha} + \delta_{b_1 b_2}^{a_8 [a_1} E^{a_2 \dots a_7]}{}_{\alpha]}), \quad (38)$$

$$[E^{a_1 \dots a_7 | a_8}, F_{b_1 \dots b_4}] = 1890 (\delta_{b_1 \dots b_4}^{[a_1 \dots a_4} E^{a_5 \dots a_7 | a_8]} + \delta_{b_1 \dots b_4}^{a_8 [a_1 \dots a_3} E^{a_4 \dots a_7]}), \quad (39)$$

$$[E^{a_1 \dots a_7 | a_8}, F_{b_1 \dots b_6}{}^{\beta}] = 45360 \epsilon^{\beta\alpha} (\delta_{b_1 \dots b_6}^{[a_1 \dots a_6} E^{a_7 | a_8]}{}_{\alpha} + \delta_{b_1 \dots b_6}^{a_8 [a_1 \dots a_5} E^{a_6 a_7]}{}_{\alpha]}), \quad (40)$$

$$[X_{a_1 \dots a_7 | a_8} E^{a_1 \dots a_7 | a_8}, F_{b_1 \dots b_7 | b_8}] = -7! (X_{b_1 \dots b_7 | b_8} K - X_{b_1 \dots b_7 | c} K^c{}_{b_8} - 7 K^c{}_{[b_1} X_{b_2 \dots b_7 | c] b_8}), \quad (41)$$

where we have introduced an auxiliary tensor $X_{a_1 \dots a_7 | a_8}$ to simplify the expression in the last line on the right hand side. Besides the transposed relations of the above, we also find

$$[E^{a_1 \dots a_8}{}_i, F_{b_1 \dots b_7 | b_8}] = 0. \quad (42)$$

3 σ -model equations of motion

In this section, we work out the σ -model equations of motion, using the formulation in terms of $K(E_{10})$ orthonormal frames developed in [2] and [3] for the A_9 and D_9 level decompositions of E_{10} , respectively. Accordingly, we parametrize the coset space in terms of a ‘matrix’ $\mathcal{V} \equiv \mathcal{V}(A(t)) \in E_{10}$. Here $A(t)$ are ‘local coordinates’ on the infinite dimensional coset manifold $E_{10}/K(E_{10})$. Making use of the local $K(E_{10})$ invariance, a convenient choice of gauge is the Borel type triangular gauge, with the fields $A = A^{(\ell)}$ for $\ell \geq 0$ to parametrize the $E_{10}/K(E_{10})$ coset space. The scalar fields $A(t)$ couple via the Lie algebra-valued ‘velocity’

$$\begin{aligned} \partial_t \mathcal{V} \mathcal{V}^{-1} &= P_{ab}^{(0)} S^{ab} + Q_{ab}^{(0)} J^{ab} + Q^{(0)} J_2 + P^{(0)\hat{i}} J_{\hat{i}} + \frac{1}{2} P_{a_1 a_2}^{(1)\alpha} E^{a_1 a_2}{}_{\alpha} \\ &+ \frac{1}{4!} P_{a_1 \dots a_4}^{(2)} E^{a_1 \dots a_4} + \frac{1}{6!} P_{a_1 \dots a_6}^{(3)\alpha} E^{a_1 \dots a_6}{}_{\alpha} + \frac{1}{7!} P_{a_1 \dots a_7 | a_8}^{(4)} E^{a_1 \dots a_7 | a_8} \\ &+ \frac{1}{8!} P_{a_1 \dots a_8}^{(4)}{}^i E^{a_1 \dots a_8}{}_i + \dots \in \mathfrak{e}_{10} \end{aligned} \quad (43)$$

where hatted indices \hat{i} label the $SL(2, \mathbb{R})/SO(2)$ coset generators and hence take only the values $\hat{i} = 1, 3$:

$$P^{(0)\hat{i}} J_{\hat{i}} \equiv P^{(0)1} J_1 + P^{(0)3} J_3. \quad (44)$$

Splitting the ‘velocity’ as $\partial_t \mathcal{V} \mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P}$, where $\mathcal{Q} \in \mathfrak{k}_{\mathfrak{e}_{10}}$ is the $K(E_{10})$ gauge connection and $\mathcal{P} \in \mathfrak{e} \ominus \mathfrak{k}_{\mathfrak{e}_{10}}$ in the coset, we write

$$\mathcal{Q} = Q_{ab}^{(0)} J^{ab} + Q^{(0)} J_2 + \sum_{\ell > 0} P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} - F^{(\ell)}), \quad (45)$$

$$\mathcal{P} = P_{ab}^{(0)} S^{ab} + P^{(0)\hat{i}} J_{\hat{i}} + \sum_{\ell > 0} P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} + F^{(\ell)}), \quad (46)$$

with $F^{(\ell)} := (E^{(\ell)})^T$, $J^{ab} = \frac{1}{2}(K^a{}_b - K^b{}_a)$ and $S^{ab} = \frac{1}{2}(K^a{}_b + K^b{}_a)$ and the higher level contributions are indicated schematically.

Following [2] we define the ‘covariant’ derivative for $\ell > 0$

$$\begin{aligned} \mathcal{D}^{(0)} P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} + F^{(\ell)}) &= \partial_t P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} + F^{(\ell)}) - [Q_{ab}^{(0)} J^{ab}, P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} + F^{(\ell)})] \\ &- [P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} - F^{(\ell)}), P_{ab}^{(0)} S^{ab}] \\ &- [Q^{(0)} J_2, P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} + F^{(\ell)})] - [P^{(\ell)} \star \frac{1}{2} (E^{(\ell)} - F^{(\ell)}), P^{(0)\hat{i}} J_{\hat{i}}]. \end{aligned} \quad (47)$$

This expression is covariant with respect to both $\mathfrak{so}(9)$ and $\mathfrak{so}(2)$. The analogous covariant derivatives for $\ell = 0$ are

$$(\mathcal{D}^{(0)} P_{ab}^{(0)}) S^{ab} = \partial_t P_{ab}^{(0)} S^{ab} - [Q_{ab}^{(0)} J^{ab}, P_{cd}^{(0)} S^{cd}], \quad (48)$$

$$(\mathcal{D}^{(0)} P^{(0)\hat{i}}) J_{\hat{i}} = \partial_t P^{(0)\hat{i}} J_{\hat{i}} - [Q^{(0)} J_2, P^{(0)\hat{i}} J_{\hat{i}}]. \quad (49)$$

This reflects the fact that there are no terms coupling the orthogonal summands $\mathfrak{so}(9)$ and $\mathfrak{so}(2)$.

The geodesic σ -model Lagrangian reads

$$\mathcal{L}(t) = \frac{1}{4}n(t)^{-1}\langle\mathcal{P}(t)|\mathcal{P}(t)\rangle \quad (50)$$

where t is an affine parameter ('time'), and $\langle\cdot|\cdot\rangle$ is the standard bilinear form on the Kac Moody algebra. This Lagrangian is unique because for infinite dimensional Kac Moody algebras the only invariant form is quadratic. The above Lagrangian gives rise to the equation of motion

$$n\partial_t(n^{-1}\mathcal{P}) = [\mathcal{Q}, \mathcal{P}]. \quad (51)$$

The Lagrange multiplier ('lapse') n is needed for invariance under reparametrizations of the time coordinate t and ensures that the motion on the coset manifold takes place on a null geodesic. In the truncation to $|\ell| \leq 4$ we set

$$P^{(5)} = P^{(6)} = \dots = 0 \quad (52)$$

With the commutation relations derived above we find

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P_{ab}^{(0)}) &= -\frac{1}{16}\delta_{ab}P_{cd}^{(1)\alpha}P_{cd}^{(1)\alpha} + \frac{1}{2}P_{ac}^{(1)\alpha}P_{bc}^{(1)\alpha} - \frac{1}{96}\delta_{ab}P_{c_1\dots c_4}^{(2)}P_{c_1\dots c_4}^{(2)} \\ &+ \frac{1}{12}P_{ac_1\dots c_3}^{(2)}P_{bc_1\dots c_3}^{(2)} - \frac{1}{16\cdot 5!}\delta_{ab}P_{c_1\dots c_6}^{(3)\alpha}P_{c_1\dots c_6}^{(3)\alpha} \\ &+ \frac{1}{2\cdot 5!}P_{ac_1\dots c_5}^{(3)\alpha}P_{bc_1\dots c_5}^{(3)\alpha} - \frac{1}{2\cdot 7!}\delta_{ab}P_{c_1\dots c_7|c_8}^{(4)}P_{c_1\dots c_7|c_8}^{(4)} \\ &+ \frac{1}{2\cdot 7!}P_{c_1\dots c_7|a}^{(4)}P_{c_1\dots c_7|b}^{(4)} + \frac{1}{2\cdot 6!}P_{ac_1\dots c_6|d}^{(4)}P_{bc_1\dots c_6|d}^{(4)} \\ &- \frac{1}{4\cdot 8!}\delta_{ab}P_{c_1\dots c_8}^{(4)\ i}P_{c_1\dots c_8}^{(4)\ i} + \frac{1}{4\cdot 7!}P_{ac_1\dots c_7}^{(4)\ i}P_{bc_1\dots c_7}^{(4)\ i}, \quad (53) \end{aligned}$$

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P^{(0)\hat{i}}) &= \left(-\frac{1}{4}P_{ab}^{(1)\alpha}P_{ab}^{(1)\beta} - \frac{1}{2\cdot 6!}P_{a_1\dots a_6}^{(3)\alpha}P_{a_1\dots a_6}^{(3)\beta}\right)(\sigma^{\hat{i}})^{\beta\alpha} \\ &+ \frac{1}{2\cdot 8!}\epsilon^{\hat{i}2\hat{j}}P_{c_1\dots c_8}^{(4)\ 2}P_{c_1\dots c_8}^{(4)\ \hat{j}} \quad (54) \end{aligned}$$

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P_{ab}^{(1)\alpha}) &= \frac{1}{4}\epsilon_{\alpha\beta}P_{abcd}^{(2)}P_{cd}^{(1)\beta} + \frac{1}{2\cdot 4!}P_{abc_1\dots c_4}^{(3)\alpha}P_{c_1\dots c_4}^{(2)} \\ &+ \frac{1}{160}\epsilon^{\alpha\beta}\left(P_{c_1\dots c_6}^{(3)\beta}P_{c_1\dots c_6[a|b]}^{(4)} + P_{c_1\dots c_6}^{(3)\beta}P_{a[bc_1\dots c_5|c_6]}^{(4)}\right) \\ &- \frac{1}{4\cdot 6!}(\sigma_i\epsilon)^{\alpha\beta}P_{c_1\dots c_6ab}^{(4)\ i}P_{c_1\dots c_6}^{(3)\ \beta}, \quad (55) \end{aligned}$$

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P_{a_1\dots a_4}^{(2)}) &= -\frac{1}{4}P_{a_1\dots a_4b_1b_2}^{(3)\alpha}P_{b_1b_2}^{(1)\beta}\epsilon_{\alpha\beta} \\ &+ \frac{1}{128}\left(P_{c_1\dots c_4}^{(2)}P_{c_1\dots c_4[a_1\dots a_3|a_4]}^{(4)} + P_{c_1\dots c_4}^{(2)}P_{a_1\dots a_4c_1\dots c_3|c_4}^{(4)}\right), \quad (56) \end{aligned}$$

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P_{a_1\dots a_6}^{(3)\alpha}) &= -\frac{3}{160}\epsilon^{\alpha\beta}\left(P_{c_1c_2}^{(2)\beta}P_{c_1c_2[a_1\dots a_5|a_6]}^{(4)} + P_{c_1c_2}^{(2)\beta}P_{c_1a_1\dots a_6|c_2}^{(4)}\right) \\ &+ \frac{1}{8}(\epsilon\sigma_i)^{\alpha\beta}P_{a_1\dots a_6c_1c_2}^{(4)\ i}P_{c_1c_2}^{(1)\beta}, \quad (57) \end{aligned}$$

$$n\mathcal{D}^{(0)}(n^{-1}P_{a_1\dots a_7|a_8}^{(4)}) = 0, \quad (58)$$

$$n\mathcal{D}^{(0)}(n^{-1}P_{a_1\dots a_8}^{(4)\ i}) = 0. \quad (59)$$

Indices on the same level are contracted with the Euclidean flat metrics of $\mathfrak{so}(9)$ and $\mathfrak{so}(2)$; in particular, the indices α, β, \dots are no longer contracted in an $SL(2)$ invariant way.

The consistency of the truncation (52) is ensured by the same arguments as in [2]. Note that although this requires only a finite number of non-vanishing $P^{(\ell)}$, the ‘unendlichsein’ \mathcal{V} parametrized by the $E_{10}/K(E_{10})$ coset coordinates $A(t)$ needs to evolve correctly to ensure the vanishing of $P^{(\ell)}$ for $|\ell| > 4$. This involves the full structure of E_{10} .

4 Comparison with IIB supergravity

We now compare the equations (53)–(59) with the type IIB supergravity equations of motion [16, 17]. We will use conventions similar to those of [18, 19] in order to make the $SL(2, \mathbb{R})$ invariance transparent.

As is well known, IIB supergravity requires the following bosonic fields. For the zehnbein we choose a pseudo-Gaussian gauge with lapse N and vanishing shift

$$E_M{}^A = \begin{pmatrix} N & 0 \\ 0 & e_m{}^a \end{pmatrix}. \quad (60)$$

In addition, there are an $SL(2, \mathbb{R})$ doublet of two-form potentials $A_{2, \tilde{\alpha}}$ ($\tilde{\alpha} = 1, 2$) and an $SL(2, \mathbb{R})$ singlet four-form B_4 with self-dual field strength given by

$$H_5 = dB_4 + \frac{1}{4} \epsilon^{\tilde{\alpha}\tilde{\beta}} A_{2, \tilde{\alpha}} dA_{2, \tilde{\beta}} = {}^*H_5 \quad (61)$$

We will drop the rank indices on H_5 and dA_2 in the remainder. Last but not least, there are two scalar fields ϕ and χ (dilaton and axion) which, in a convenient triangular gauge, parametrize the coset $SL(2, \mathbb{R})/SO(2)$ according to [19]

$$\mathcal{E} = \begin{pmatrix} e^{\phi/2} & \chi e^{\phi/2} \\ 0 & e^{-\phi/2} \end{pmatrix}. \quad (62)$$

The matrix $\mathcal{E}_\alpha{}^{\tilde{\alpha}}$ can be thought of as an internal zweibein, which serves to convert global $SL(2, \mathbb{R})$ indices $\tilde{\alpha}, \tilde{\beta}, \dots$ into local $SO(2)$ indices α, β, \dots , in complete analogy with the spatial neunbein $e_m{}^a$ which converts global $GL(9)$ indices to local (Lorentz) $SO(9)$ indices, and vice versa. Thus, the model possesses a local $\mathfrak{so}(9) \oplus \mathfrak{so}(2)$ symmetry, which will be directly identified with the $\mathfrak{so}(9) \oplus \mathfrak{so}(2)$ subalgebra of \mathfrak{e}_{10} in the $A_8 \times A_1$ decomposition of E_{10} developed in the foregoing section. The scalar field ϕ and χ appear in the IIB supergravity Lagrangian via

$$\partial_M \mathcal{E} \mathcal{E}^{-1} = R_M J_2 + S_M^{\tilde{i}} J_{\tilde{i}} = \partial_M \phi J_2 + \partial_M \chi e^{\phi} J_+ \quad (63)$$

The derivation of the IIB equations of motion is greatly facilitated by the $SL(2, \mathbb{R})$ symmetry which fixes many couplings uniquely. Converting the global indices to local indices we write for the doublet of two-forms

$$(\mathcal{E}dA)_{ABC\alpha} = \mathcal{E}_\alpha{}^{\tilde{\alpha}} E_A{}^M E_B{}^N E_C{}^P \partial_{[M} A_{NP], \tilde{\alpha}}, \quad (64)$$

such that

$$(\mathcal{E}dA)_{ABC1} = e^{\phi/2} (dA)_{ABC, \tilde{1}} + \chi e^{\phi/2} (dA)_{ABC, \tilde{2}}, \quad (65)$$

$$(\mathcal{E}dA)_{ABC2} = e^{-\phi/2} (dA)_{ABC, \tilde{2}}, \quad (66)$$

where $\tilde{1}$ and $\tilde{2}$ are global $SL(2, \mathbb{R})$ indices.

For the comparison with the above σ -model equations of motion it is most convenient to write the bosonic field equations of IIB supergravity in terms of *flat* $SO(1, 9) \times SO(2)$ indices, viz.

$$R_{AB} = -\frac{1}{4}S_A^{\tilde{i}}S_{iB} + \frac{1}{48}H_A{}^{C_1\dots C_4}H_{BC_1\dots C_4} - \frac{1}{480}\eta_{AB}H^{C_1\dots C_5}H_{C_1\dots C_5} \\ + \frac{1}{4}(\mathcal{E}dA)_A{}^{CD\alpha}(\mathcal{E}dA)_{BCD}{}^\alpha - \frac{1}{48}\eta_{AB}(\mathcal{E}dA)^{BCD\alpha}(\mathcal{E}dA)_{BCD}{}^\alpha, \quad (67)$$

$$D^A S_A^{\tilde{i}} = \frac{1}{12}(\mathcal{E}dA)_{BCD}{}^\alpha(\mathcal{E}dA)^{BCD\beta}(\sigma^{\tilde{i}})^\alpha{}_\beta, \quad (68)$$

for the Einstein equation and the coset $SL(2, \mathbb{R})/SO(2)$. D^A is the $SO(1, 9) \times SO(2)$ covariant derivative; splitting the divergence into time and space components, it reads

$$D^A S_A^{\tilde{i}} = \partial^0 S_0^{\tilde{i}} + \partial^a S_a^{\tilde{i}} + \omega^0{}_a{}^0 S_a^{\tilde{i}} + \omega^a{}_a{}^0 S_0^{\tilde{i}} + \omega^a{}_a{}^b S_b^{\tilde{i}} \\ + \frac{1}{2}\epsilon^{i2j}(R^0 S_0^{\tilde{j}} + R^a S_a^{\tilde{j}}), \quad (69)$$

where $\omega_A{}_{BC}$ is the usual spin connection. For the form potentials we obtain

$$D^A H_{AC_1\dots C_4} = -\frac{1}{144}\epsilon_{\alpha\beta}\epsilon_{C_1\dots C_4}{}^{D_1\dots D_6}(\mathcal{E}dA)_{D_1 D_2 D_3}{}^\alpha(\mathcal{E}dA)_{D_4 D_5 D_6}{}^\beta, \quad (70)$$

$$D^A(\mathcal{E}dA)_{ABC}{}^\alpha = -\frac{1}{4}\epsilon^{i2j}S^{A\tilde{i}}(\sigma_j)^\alpha{}_\beta(\mathcal{E}dA)_{ABC}{}^\beta + \frac{1}{12}H_{BC}{}^{DEF}\epsilon^{\alpha\beta}(\mathcal{E}dA)_{DEF}{}^\beta. \quad (71)$$

Here, it is not necessary to restrict $H_{C_1\dots C_5}$ to be self-dual. This condition can be imposed additionally by hand since it is consistent with the Bianchi identity for B_4 from (61):

$$D_{[A_1} H_{A_2\dots A_6]} = -\frac{1}{4}\epsilon_{\alpha\beta}(\mathcal{E}dA)_{[A_1 A_2 A_3}{}^\alpha(\mathcal{E}dA)_{A_4 A_5 A_6]}{}^\beta. \quad (72)$$

The Bianchi identities for the two-forms are

$$D_{[A_1}(dA)_{A_2 A_3 A_4]}{}^{\tilde{\alpha}} = 0. \quad (73)$$

We can now directly verify our main claim, that the bosonic equations of motion of IIB supergravity when reduced to one (time) dimension and the E_{10} σ -model equations when truncated to $|\ell| \leq 4$ coincide if one makes the following identifications between the t -dependent σ -model quantities up to $\ell = 4$, and the IIB supergravity quantities evaluated at a fixed, but arbitrarily chosen spatial point $\mathbf{x} = \mathbf{x}_0$:

$$n(t) = N e^{-1}(t, \mathbf{x}_0), \quad (74)$$

$$P_{ab}^{(0)}(t) = e_{(a}{}^m \partial_t e_{mb)}(t, \mathbf{x}_0), \quad (75)$$

$$Q_{ab}^{(0)}(t) = e_{[a}{}^m \partial_t e_{mb]}(t, \mathbf{x}_0), \quad (76)$$

$$P^{(0)1}(t) = S_t^1(t, \mathbf{x}_0) = e^\phi \partial_t \chi(t, \mathbf{x}_0), \quad (77)$$

$$P^{(0)3}(t) = S_t^3(t, \mathbf{x}_0) = \partial_t \phi(t, \mathbf{x}_0), \quad (78)$$

$$Q^{(0)}(t) = R_t(t, \mathbf{x}_0) = e^\phi \partial_t \chi(t, \mathbf{x}_0), \quad (79)$$

$$P_{a_1 a_2}^{(1)\alpha}(t) = \mathcal{E}_\alpha{}^{\tilde{\alpha}} e_a{}^m e_b{}^n \partial_t A_{mn, \tilde{\alpha}}(t, \mathbf{x}_0), \quad (80)$$

$$P_{a_1 \dots a_4}^{(2)}(t) = e_{a_1}^{m_1} \dots e_{a_4}^{m_4} H_{tm_1 \dots m_4}(t, \mathbf{x}_0), \quad (81)$$

$$P_{a_1 \dots a_6}^{(3)\ \alpha}(t) = \frac{1}{3!} n e \epsilon_{a_1 \dots a_6 c_1 \dots c_3} e_{c_1}^{m_1} \dots e_{c_3}^{m_3} \mathcal{E}_\alpha^{\tilde{\alpha}} \partial_{m_1} A_{m_2 \dots m_3, \tilde{\alpha}}(t, \mathbf{x}_0) \quad (82)$$

$$P_{a_1 \dots a_7 | a_8}^{(4)}(t) = \frac{1}{2} n e \epsilon_{a_1 \dots a_7 b c} \tilde{\Omega}_{bc|a_8}(t, \mathbf{x}_0), \quad (83)$$

$$P_{a_1 \dots a_8}^{(4)\ i}(t) = n e \epsilon_{a_1 \dots a_8 b} \mathcal{S}_b^i(t, \mathbf{x}_0), \quad (84)$$

$$P_{a_1 \dots a_8}^{(4)\ 2}(t) = n e \epsilon_{a_1 \dots a_8 b} R_b^i(t, \mathbf{x}_0). \quad (85)$$

Here, $e = \det(e_m^a)$ and $\tilde{\Omega}_{ab|c}$ is the traceless part of the anholonomy, the trace part has been gauged to zero [2]. Spatial derivatives of the lapse N are neglected in this approximation and hence, for example, out of the three terms in the covariant derivative (69) involving the spin connection only the second term survives. For the $SL(2, \mathbb{R})/SO(2)$ coset we have adopted the parametrization above. The $SL(2, \mathbb{R})$ symmetry plays an important part in this identification: for instance, the formula for $P_{a_1 a_2}^{(1)\ \alpha}$ must contain the neunbein and the zweibein \mathcal{E} in precisely the indicated form in order to be compatible with the symmetries.

With these identifications, eq. (53) coincides with the Einstein equation (67), if we recall from [2] that $n\mathcal{D}^{(0)}(n^{-1}P_{ab}^{(0)})$ can be directly identified with the part of the spatial Ricci tensor containing only time derivatives, $-N^2 R_{ab}^{(0)}$, cf. Eqn. (4.66) of [2]. The contribution from $\ell = 4$ on the r.h.s. of the $\ell = 0$ equation of motion gives the leading terms of R_{ab} in the first spatial derivatives of the vielbein, but there appears a mismatch involving the contribution from $P_{a_1 \dots a_8}^{(4)\ i}$ to the Einstein equation, analogous to the one involving the subleading terms from the curvature as in [2]. The coset eq. (54) is mapped to (68) in the reduction. The equations of motion (71) and the Bianchi identities (73) are mapped to the σ -model equations (55) and (57), respectively. Equations (58) and (59) are related to the factorization of the vielbeine of the two cosets in the vicinity of a space-like singularity [2].

Remarkably, the self-duality of the five-form field strength H_5 is built into E_{10} . The right hand side of (71) is expanded to

$$H_{bcdef} \epsilon^{\alpha\beta} (\mathcal{E}dA)_{def}{}^\beta - 3N^2 H_{bcdet} \epsilon^{\alpha\beta} (\mathcal{E}dA)_{det}{}^\beta. \quad (86)$$

The second term is precisely of the form of the $[(\ell = 2), (\ell = 1)]$ term generated by E_{10} in (55) but the first term appears to be in conflict with E_{10} . This puzzle is resolved by restricting H_5 to be self-dual such that

$$H_{bcdef} \epsilon^{\alpha\beta} (\mathcal{E}dA)_{def}{}^\beta = \frac{1}{4!} N^{-1} \epsilon_{bcdefc_1 \dots c_4} H_{tc_1 \dots c_4} \epsilon^{\alpha\beta} (\mathcal{E}dA)_{def}{}^\beta, \quad (87)$$

and then is recognized as the $[(\ell = 3), (\ell = 2)]$ contribution to (55). Likewise, matching eq. (56) with (70) requires the self-duality of the five-form field strength and then can be read either as the Bianchi identity or the equation of motion. Therefore, E_{10} is related to IIB dynamically only for self-dual field strength of the five-form.

5 Conclusions

As we have shown the geodesic action of the $E_{10}/K(E_{10})$ σ -model based on the standard bilinear form for E_{10} , with manifest $SL(2, \mathbb{R})$ symmetry together with the representation content up to $\ell \leq 4$ implies the self-duality constraint of the five-form field strength of IIB supergravity. In the standard derivation of the bosonic IIB equations of motion this

self-duality cannot be deduced from the bosonic symmetries, but becomes necessary only when one adds fermions to make the theory locally supersymmetric. In addition all the other fields of type IIB supergravity theory can be accommodated naturally, and their dynamics is equivalent to those in the σ -model, at least up to the level considered here.

In contradistinction to the non-hyperbolic algebra E_{11} , the hyperbolic Kac Moody algebra E_{10} does *not* allow for a source of the D9-brane of IIB string theory in any natural way [3, 7]. If one insists that the D9-brane is essential for string dualities, one would therefore conclude that E_{10} is ‘too small’ [20]. The D9-brane couples to a ten-form potential, which in ten space-time dimensions has vanishing field strength and hence no dynamical degrees of freedom (a generalization of the IIB theory with a *doublet* of ten-form potentials has been given in [21]). In eleven dimensions, on the other hand, the equation of motion for such a rank-ten potential implies constancy of the associated field strength. This is a well-known mechanism for generating masses and a cosmological constant in (super-)gravity [22]. Yet, a cosmological term in eleven dimensions is inconsistent with 32 supersymmetries, and therefore such a modification of $D = 11$ supergravity does not appear to exist [23]. Disregarding this fact for the moment, an appropriate field transforming in a singlet representation of $SL(10)$ can be identified in an A_9 decomposition of E_{11} [7, 4]. In terms of a D_{10} decomposition of E_{11} analogous to [3], the corresponding generator is part of an $SO(10, 10)$ multiplet which includes all RR potentials. To recover these representations from the perspective taken here, we have to decompose E_{11} under $A_9 \times A_1$, with $SL(10) \subset SO(10, 10)$. Indeed, one finds new $SL(10)$ singlet representations at level $\ell = 5$, which are absent in E_{10} , and contain the D9-brane generator. However, the relevant representation is an $SL(2, \mathbb{R})$ *quadruplet*. Therefore, E_{11} would predict the existence of *four* nine-brane objects, transforming under $SL(2, \mathbb{R})$, whereas current superstring wisdom seems to be compatible only with a *doublet* of such objects [24, 14, 21].

Finally, we mention that it is believed that the continuous $SL(2, \mathbb{R})$ symmetry of IIB supergravity is broken to $SL(2, \mathbb{Z})$ by quantum effects in the string theory. Similar effects have recently been discussed for the full $E_{10}(\mathbb{R}) \rightarrow E_{10}(\mathbb{Z})$ in [25]. The inclusion of fermions along the lines of [3] seems straight-forward, with the local $SO(9)$ appearing here being identified with the diagonal subgroup of $SO(9, 9)$.

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