Relativistic Elastostatics I: Bodies in Rigid Rotation

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Abstract

We consider elastic bodies in rigid rotation, both nonrelativistically and in special relativity. Assuming a body to be in its natural state in the absence of rotation, we prove the existence of solutions to the elastic field equations for small angular velocity.

Keywords: elastic bodies, rotation

1 Introduction

The field of relativistic elasticity is still in its infancy. This work is part of a program where we set up the field equations and prove existence theorems for some of the most basic problems, both dynamical and time-independent. In the present paper we study the equilibrium of an elastic body in rigid uniform rotation. Interestingly the nonrelativistic case of our result seems to be unknown, so we have to treat this also.

We will be interested in equilibrium configurations of ideal elastic solids which are subject to the centrifugal force but otherwise free. The natural boundary condition, then, is that the so-called "normal traction", i.e. the components of the stress tensor normal to the surface of the body, be zero. The location of this boundary of the region-in-space occupied by the elastic body can not be given freely, but is part of the sought-for solution. It is thus preferable to work in the material ("Lagrangian") representation where the maps describing configurations go from the material space - whose boundary is fixed - into physical space, with Neumann-type boundary conditions. We note in passing that it is the material representation which is used almost universally in standard nonrelativistic elasticity. Since the elasticity literature is hard to digest for workers with a background in relativity, we have made an effort to make this paper reasonably self-contained, by formulating the necessary concepts and computations in the framework of a Lagrangian field theory.

Nonrelativistic elasticity in the time-independent case takes the form "$\mathcal{E}+\mathcal{F}=0$", where $\mathcal{E}$ is a 2nd order partial differential operator acting on the configuration. This operator, usually called elasticity operator, is a quasilinear, elliptic operator on Euclidean space, with coefficients depending on the elastic material. The force $\mathcal{F}$, usually called "load", depends on the problem at hand of course - in our case it is the centrifugal force. In special relativity this picture essentially survives with the complication that $\mathcal{E}$ now becomes a PDO living on the space of trajectories of a "helical" Killing vector with the natural (curved) metric which this 3-space inherits from Minkowski space. For this reason - and for use in future work on elasticity in GR - we base our work here on a formulation of $\mathcal{E}$ on the background of an arbitrary curved 3-space.

In Sect.2 we describe our setup for static elasticity, which is the curved-space generalization of the standard nonrelativistic theory for hyperelastic materials in the time independent case. The static elasticity operator is viewed as the Euler-Lagrange expression for a certain action principle. This is done merely for convenience, since the action principle facilitates the calculations necessary for moving back and forth between the spatial and the material picture. We also write down, in the material picture, the linearized elasticity operator at a "natural state", i.e. at a solution of the field equations with zero stress (whence zero body force). When the physical space has Killing vectors, this operator, with the obvious choice of function spaces, is neither injective nor surjective. Rather its range has to satisfy certain integral constraints (often called "equilibration conditions" in the literature) on the force involving the Killing vectors and the natural configuration. There is a related fact concerning the full elasticity operator: any configuration and any force have to satisfy equilibration conditions, if the spatial metric has Killing vectors. This lack of surjectivity of the elasticity operator generalizes a well-known fact in standard mechanics: forces acting on an otherwise free body at rest have to be such that the total force and the total torque be zero.

In Sect.3 we derive the equations governing relativistic rotating elastic bodies. We do this by means of "dimensional reduction" of the general time dependent theory laid out in our previous work [2], where this reduction is carried out w.r. to the helical Killing vector corresponding to rigid rotation with angular frequency ω. The resulting action functional (rather: "energy functional") is in fact more

general than the framework of Sect.2: There automatically appears a force term in the form of a multiplicative function, namely the norm of the Killing vector which, in the case at hand, is essentially the centrifugal potential for frequency ω. As a complication, for purely relativistic reasons, the coefficients of the elasticity operator have a dependence on ω, which they inherit from the curved spatial metric, i.e. the natural metric arising by quotienting the Minkowski metric by the action of the helical Killing vector. To avoid confusion we wish to stress that, under these circumstances, it is the full equation $\mathcal{E} + \mathcal{F} = 0$, which is obtained as the Euler-Lagrange condition for the energy functional.
Taking the formal limit $c \to \infty$ of the field equations, one obtains the nonrelativistic equations. These of course have the form "flat elasticity operator + force =0", where the second term, i.e. the centrifugal force, is linear in ω^2. In Sect.3 we solve these equations for small ω and configurations close to the natural one. An immediate application of the implicit function theorem is of course forbidden by the lack of surjectivity of the linearized elasticity operator in flat space. This is a well-known problem in elasticity, often resulting in bifurcation phenomena which have led to a lot of difficult work (see e.g. Sect. 7 of [9]). Our problem, luckily, turns out to be simpler. We first note that the equilibration conditions, when $\omega \neq 0$, require the configuration to be such that the center of mass in physical space lie on the rotation axis and that this rotation axis coincide with one of the principal axes of inertia: We ab initio impose these conditions on our allowed configurations (including of course the natural configuration, i.e. the undeformed body). The resulting space of configurations turns out to be a smooth manifold near the natural state (which it would not be if there were bifurcations). After a suitable projection of the equations to this manifold, reminiscent of bifurcation theory, the problem can then be solved using the standard implicit function theorem, provided the constitutive law (expressed in terms of the so-called "stored-energy function") satifies the condition of "uniform pointwise stability", which is valid for standard elastic materials.
In Sect.4 we solve the relativistic problem. It has the form "quotient space elasticity operator + relativistic centrifugal force = 0". For $\omega = 0$ the quotient space metric is Euclidean and the relativistic centrifugal force goes to zero like ω^2 for small ω. If one now uses equilibration w. r. to "$1/\omega^2\times$ relativistic centrifugal force", a new situation seems to arise: for $\omega = 0$ we get the same conditions as before. But for $\omega \neq 0$ we get none: the only Killing vectors of the spatial metric are now ∂_ϕ and ∂_3, and for those the equilibration conditions turn out to be automatically satisfied. Thus, in order to be able to use an implicit function argument at $\omega = 0$, we resort to brute force: we split off the non-flat part of the elasticity operator, which vanishes like ω^2, and view it as a contribution to the force. Now much the same goes through as in the nonrelativistic case. However, in order to be guaranteed that the set of equilibrated configurations is again a manifold, we have to assume, in addition to the constitutive condition for the nonrelativistic case, that some characteristic velocity of the system, in essence

some upper bound on the sound velocity, be sufficiently small compared to c. Being guaranteed, by the theorems of Sect.4 and 5, that solutions exist for small ω, we finally, in Sect.6, calculate these explicitly to linear order in ω^2, for a material which is isotropic in its natural state and for an undeformed body which is an ellipsoid with rotational symmetry about the rotation axis. In the nonrelativistic limit our results agree with the ones found by [4], see [8].

2 The static elasticity operator

The basic setup underlying time-independent situations in both nonrelativistic and relativistic elasticity is as follows: We consider maps between two 3-dimensional Riemannian manifolds given by $f : (N, h_{ij}) \mapsto (\Omega, V_{ABC})$, with the manifold N describing physical space with smooth metric h_{ij} and Ω a domain in $\mathbb{R}^3$ (open, connected, bounded) with smooth boundary $\partial\Omega$ (not necessarily connected) and V_{ABC} a smooth volume form on $\bar{\Omega}$. The domain Ω, called "body" or "material space", is to be t hought of as the collection of particles making up the elastic body prior to the action of any external forces, stresses, etc.[1] Thus f is the "back-to-labels-map", its inverse $\Phi : \Omega \to N$ is called a configuration. In nonrelativistic elasticity (N, h_{ij}) is Euclidean space. (In the case studied here of a relativistic body rotating at angular frequency ω, the metric h_{ij} will be the one coming from the Minkowski metric on $\mathbb{R}^4$ with the action of the helical Killing vector $\partial_t + \omega\partial_\phi$ quotiented out.) The maps f are required to be one-one and orientation-preserving, i.e. the function n, defined for each f by

$$(\partial_i f^A)(x)(\partial_j f^B)(x)(\partial_k f^C)(x)V_{ABC}(f(x)) = n(x)\varepsilon_{ijk}(x), \tag{1}$$

is positive. The physical interpretation of f is that of the density of particle number. We are using coordinates X^A on Ω and coordinates x^i on N. The three form ε_{ijk} is the metric volume element in N associated with h_{ij} and the three form $V_{ABC}(X) = V(X)\epsilon_{ABC}$ with $V > 0$ and $\epsilon_{123} = 1$ is the volume element on Ω. (We think of V as having physical dimension $[mass/volume]$.) Put differently the definition (1) says that $nh^{\frac{1}{2}} = V det(\partial f)$, where $h = det(h_{ij})$. An elastic body will be specified by a constitutional law, as follows. There is given a scalar function of maps f called stored-energy function (of physical dimension $[velocity]^2$) $w = w(f, \partial f, x)$, smooth in all its arguments. Covariance of w under spatial diffeomorphisms requires (see e.g. [2]) that w be of the form [2]

[1] It would be unnatural for our purposes to add further structure to the body at this stage - such as a flat metric, as is common in the literature. Once a choice of reference map ("state") has been made, there is of course a metric defined on Ω by the push-forward of h_{ij} under this map.

[2] In standard nonrelativistic elasticity diffeomorphism invariance is replaced by requiring the validity of the so-called "principle of material frame indifference" (see e.g. [9]).

$$w = w(H^{AB}, f^C), \tag{2}$$

smooth in its arguments, where

$$H^{AB} = (\partial_i f^A)(\partial_j f^B)h^{ij}. \tag{3}$$

By virtue of our assumptions H^{AB} is positive definite. It thus has an inverse H_{AB}. The Cauchy stress tensor $\sigma_i{}^j$ associated with w is defined as follows. Think of the function $L = nw$ as a Lagrangian density, i.e. consider the action $S[f]$

$$S = \int_M n\, w\, h^{\frac{1}{2}}\, d^3x. \tag{4}$$

Then σ_{ij} is the Cauchy stress tensor

$$-\sigma_{ij} = 2\frac{\partial(nw)}{\partial h^{ij}} - nwh_{ij} = 2n\frac{\partial w}{\partial h^{ij}} \tag{5}$$

The mixed tensor $\sigma_i{}^j$ is the same as the "canonical stress tensor" $\sigma_i{}^j = nw\delta_i{}^j - \frac{\partial(nw)}{\partial(\partial_j f^A)}\,(\partial_i f^A)$ corresponding to the energy functional (4), which can also be written as

$$\sigma_i{}^j = -n\frac{\partial w}{\partial(\partial_j f^A)}\,(\partial_i f^A) \tag{6}$$

We next define the elasticity operator $\mathcal{E}$ by

$$n\mathcal{E}_i[f] = D_j\, \sigma_i{}^j, \tag{7}$$

where D_i is the covariant derivative w.r. to h_{ij}. Clearly $\mathcal{E}_i$ is a second-order quasilinear partial differential operator. Its relation with our variational principle, as can be checked in a straightforward fashion, is given by

$$n\mathcal{E}_i = (\partial_i f^A)\, \mathcal{E}_A, \tag{8}$$

where $\mathcal{E}_A$ is the Euler-Lagrange expression corresponding to the energy functional S, i.e.

$$-\mathcal{E}_A = h^{-\frac{1}{2}}\partial_j\left(h^{\frac{1}{2}}\frac{\partial(nw)}{\partial(\partial_j f^A)}\right) - n\frac{\partial w}{\partial f^A} \tag{9}$$

Having written down the elasticity operator in the "spatial" (i.e. maps go from space into body) representation, we have to spell out the boundary conditions. For a free body these are the conditions of "vanishing surface traction", namely:

$$\sigma_i{}^j n_j|_{f^{-1}(\partial\Omega)} = 0, \tag{10}$$

where n_i is an outward co-normal of the surface $f^{-1}(\partial\Omega)$. The awkward feature of the boundary condition Eq.(10) of involving both the map f and its inverse

$\Phi = f^{-1}$ strongly suggests that one go over to the "material" representation which uses Φ as the basic dependent variable. Note that such a change of representation should not be confused with a change of coordinates on either N or Ω.
The field equations in the material representation are easiest to derive by again starting from the energy Eq.(4) which now reads

$$S[\Phi] = \int_\Omega wV d^3X, \tag{11}$$

where the argument H^{AB} in w should be interpreted as a function of $(\Phi, \partial\Phi)$. As such it can be viewed as the inverse of the pull-back metric

$$H_{AB}(X) = (\partial_A \Phi^i)(X)(\partial_B \Phi^j)(X) h_{ij}(\Phi(X)) \tag{12}$$

of h_{ij} under Φ. In the elasticity literature H_{AB} is called the "(right) Cauchy-Green strain tensor". Note that the material Lagrangian contains the dependent variable Φ only through h_{ij} and has arbitrary dependence on the independent variable X, whereas the spatial Lagrangian contains the independent variable x only through h_{ij} and may depend on f in an arbitrary fashion. The material Euler-Lagrange expression, by virtue of Eq.(8) and the relation

$$\delta\Phi^i(f(x)) = -(\partial_A \Phi^i)(f(x))\delta f^A(x), \tag{13}$$

is nothing but $\mathcal{E}_i$, expressed in terms of Φ rather than f. Explicitly one finds that

$$\mathcal{E}_i[\Phi] = V^{-1}\partial_A \left(V \frac{\partial w}{\partial(\partial_A \Phi^i)} \right) - \frac{\partial w}{\partial \Phi^i} \tag{14}$$

A calculation shows that Eq.(14) can be written more concisely as

$$\mathcal{E}_i = \nabla_A \sigma_i{}^A, \tag{15}$$

where $\sigma_i{}^A$ is the "first Piola stress tensor" given by

$$\sigma_i{}^A = \frac{\partial w}{\partial(\partial_A \Phi^i)} \tag{16}$$

and ∇_A is given by

$$(\nabla_A \sigma_i{}^A)(X) = V^{-1}(X)\partial_A(V(X)\sigma_i{}^A(X)) - \Gamma^k_{ij}(\Phi(X))\sigma_k{}^A(X)(\partial_A \Phi^j)(X) \tag{17}$$

with Γ^i_{jk} being the Christoffel symbols of the metric h_{ij}. Note that no metric on Ω is required, just the volume form V_{ABC} The connection between the Piola and the Cauchy stress tensor is given by

$$n(\Phi(X))\sigma_i{}^A(X) = \Psi^A{}_j(X)\sigma_i{}^j(\Phi(X)), \tag{18}$$

where

$$\Psi^A{}_i(X)(\partial_B \Phi^i)(X) = \delta^A{}_B, \tag{19}$$

or $\Psi^A{}_i(X) = (\partial_i f^A)(\Phi(X))$. Thus the above derivation has in particular recovered a version of the so-called "Piola identity", namely

$$(n^{-1} D_j \sigma_i{}^j)(\Phi(X)) = (\nabla_A \sigma_i{}^A)(X) \tag{20}$$

The boundary conditions in the material picture take the form

$$\sigma_i \doteqdot \sigma_i{}^A n_A|_{\partial\Omega} = 0, \tag{21}$$

where n_A is an outward co-normal of $\partial\Omega$. From Eq.(16) it follows that

$$\sigma_i{}^A = \frac{\partial w}{\partial(\partial_A \Phi^i)} = -2H^{AC} \Psi^B{}_i \frac{\partial w}{\partial H^{BC}}. \tag{22}$$

From this we infer that the second-order terms of $\mathcal{E}_i$ are given by

$$\begin{aligned} \partial_A \sigma_i{}^A = [&\Psi^C{}_i H^{AB} H_{CD} \sigma_j{}^D + 2\Psi^B{}_{(i} \sigma_{j)}{}^A + \\ &+ 2H^{AE} H^{BF} \Psi^C{}_i \psi^D{}_j L_{CEDF}](\partial_A \partial_B \Phi^j) + l.o., \end{aligned} \tag{23}$$

where

$$L_{ABCD} = \left(\frac{\partial^2 w(H^{EF}, X)}{\partial H^{AB} \partial H^{CD}} \right). \tag{24}$$

The problems considered in elastostatics are usually written as

$$\mathcal{E}_i[\Phi] + \mathcal{F}_i[\Phi] = 0 \tag{25}$$

The quantity $\mathcal{F}_i$ is called "load". It will be convenient to use a more general terminology with respect to loads and elasticity operators. Namely, we call the (nonlinear) elasticity operator E the assignment, to any allowable map Φ, of the pair $(\mathcal{E}_i, \sigma_i)$, where σ_i is the function on $\partial\Omega$ given by $\sigma_i = \sigma_i{}^A n_A|_{\partial\Omega}$. Similarly a load F is a pair of functions $(\mathcal{F}_i, \tau_i)$ on $\Omega \times \partial\Omega$, both of which may depend on Φ (otherwise the load is called "dead"), and may do so in a nonlocal fashion.

There is a condition on loads in order for Eq.(25) to have solutions, which will play an important role and which arises as follows: Let $\xi^i(x)$ be a Killing vector of the metric h_{ij} on N. Then, in order for a map Φ to be solution of the extended version of Eq.(25), i.e. $E + F = 0$, the load $F = (\mathcal{F}_i, \tau_i)$ has to satisfy

$$\int_\Omega (\xi^i \circ \Phi)\, \mathcal{F}_i \, dV + \int_{\partial\Omega} (\xi^i \circ \Phi)\, \tau_i \, dO = 0. \tag{26}$$

The integrals in Eq.(26)[3] are with respect to the volume form V_{ABC} on Ω. The proof of Eq.(26) is a straightforward verification based on the Stokes theorem

[3] More precisely, the quantities σ_i and τ_i should be interpreted as two-forms on $\partial\Omega$ with σ_i corresponding to the pull-back to $\partial\Omega$ of the two-form $\sigma_i{}^A V_{ABC}$. Similarly the first term in Eq.(26) should be interpreted as the integral over Ω of the three-form arising by replacing $\mathcal{F}_i$ with $\mathcal{F}_i V_{ABC}$.

and the following identity

$$[\partial_A(\xi^i \circ \Phi) + \Gamma^i_{jk}(\xi^j \circ \Phi)(\partial_A \Phi^k)]\sigma_i{}^A = 0, \tag{27}$$

which in turn follows from the Killing equation for ξ together with Eq.(18) or Eq.(22) (i.e. Eq.(27) is the material version of $(D^i \xi^j)\sigma_{ij} = 0$). Loads satisfying Eq.(26) for all Killing vectors on $\Phi(\bar{\Omega})$ are said to be "equilibrated at Φ". Similarly, for given load L, a map Φ is called equilibrated w.r. to L, if (26) is valid for all Killing vectors.
We now introduce a reference configuration. This will simply be some given invertible, orientation-preserving map $\bar{\Phi}$ from $\bar{\Omega}$ to N, obtained by restriction to $\bar{\Omega}$ of a smooth function defined in a neighbourhood of Ω with smooth inverse on its image. We in addition require $\bar{\Phi}^{-1}$ to have $n = \overset{\circ}{\rho} = const$ or, in other words, that

$$V(X) = \overset{\circ}{\rho}\, \bar{H}^{\frac{1}{2}}(X) \tag{28}$$

with $\bar{H} = det(\bar{H}_{AB})$, $\bar{H}_{AB}$ being the pull-back under $\bar{\Phi}$ of h_{ij}. The constant $\overset{\circ}{\rho}$, usually called the mass density in the reference configuration, in our convention will drop out of all our equations. The Eq.(15) can now be written as

$$\mathcal{E}_i = \bar{D}_A \sigma_i{}^A \tag{29}$$

with $\bar{D}_A$ the two-point covariant derivative (see [9]) referring to the metric $\bar{H}_{AB}$ on $\bar{\Omega}$ and h_{ij} on N [4]. Given a reference configuration one can define, for any configuration Φ, the matrix $\bar{\mathcal{H}}$ given by $\bar{\mathcal{H}}^A{}_B = \bar{H}_{BC} H^{AC}$. Much of elasticity theory concerns isotropic materials, which are described by stored-energy functions depending on H^{AB} only via the principal invariants of $\bar{\mathcal{H}}$. We will not need that assumption however.
In this work we will always require that the reference configuration be unstressed, i.e. $\bar{\sigma}_i{}^A = 0$. By the identity (22) unstressedness is equivalent to

$$\left(\frac{\partial w(H^{CD}, X)}{\partial H^{AB}}\right)|_{\Phi=\bar{\Phi}(X)} = 0. \tag{30}$$

The subscript $\Phi = \bar{\Phi}(X)$ in Eq.(30) is understood in the sense that Φ and $\partial\Phi$ should be replaced by the values respectively of $\bar{\Phi}$ and $\partial\bar{\Phi}$ at X. Clearly there holds

$$\bar{\mathcal{E}}_i = \mathcal{E}_i[\bar{\Phi}] = 0. \tag{31}$$

We will need the linearization of the Piola tensor in the reference configuration. This is given by

$$\delta\sigma_i{}^A = -2\bar{H}^{AB}\bar{\Psi}^C{}_i \bar{L}_{BCDE}\, \delta H^{DE}, \tag{32}$$

[4] In contrast to [9] these metrics are not chosen independently, but are isometric under the reference configuration.

where $\bar{L}_{ABCD} = L_{ABCB}|_{\Phi=\bar{\Phi}(X)}$ and δH^{AB} is the first-order linear PDO acting on $\delta\Phi^i(X)$ given by

$$\delta H^{AB} = \mathcal{L}_{\delta\Phi}\bar{H}^{AB}, \tag{33}$$

and the Lie derivative in Eq.(33) is taken with respect to the vector field $\delta\Phi^A(X) = \bar{\Psi}^A{}_i(X)\delta\Phi^i(X)$ on Ω. More explicitly Eq.(33) can be written as

$$\delta\bar{H}^{AB} = -2\,\bar{D}^{(A}\delta\Phi^{B)} = -2\bar{\Psi}^{(A}{}_i\,\bar{D}^{B)}\delta\Phi^i, \tag{34}$$

since $\bar{D}_A$ annihilates $\bar{\Psi}^B{}_i$ (see footnote [4)]). We remark that in the so-called "pre-stressed case" of linearization at a reference configuration which solves Eq.(31) without being stress-free there arises an expression more complicated than (32) for $\delta\sigma_i{}^A$ and hence for the linearized elasticity operator.
One easily sees that δH^{AB} vanishes if and only if $\delta\Phi^i$ is of the form $\delta\Phi^i(X) = \xi^i(\bar{\Phi}(X))$ for ξ^i a Killing vector on $(\bar{\Phi}(\bar{\Omega}) \subset N, h_{ij})$. The linearization of the elasticity operator E introduced after Eq.(25) is the second-order operator

$$\delta E : \Phi \mapsto (\bar{D}_A\,\delta\sigma_i{}^A, (\delta\sigma_i{}^A n_A)|_{\partial\Omega}) \tag{35}$$

with $\delta\sigma_i{}^A$ given be Eq.'s (32,34). Clearly, functions $(\mathcal{F}_i, \tau_i)$ on $\Omega \times \partial\Omega$, in order to lie in the range of δE, have to be equilibrated at $\bar{\Phi}$. Now to the kernel. It is immediate that the kernel of δE contains all elements $\delta\Phi$ which are "Killing" in the above sense or, equivalently, of the form $\delta\Phi^i(X) = (\partial_A\bar{\Phi}^i)(X)\eta^A(X)$, where η^A a Killing vector on $(\bar{\Omega}, \bar{G}_{AB})$. This is an expression, on the linearized level, of the following fact: given an unstressed state $\bar{\Phi}$, then, for any isometry F of N, the map $F \circ \bar{\Phi}$ is also unstressed and hence a solution.
We now suppose the stored-energy function w to be such that δE is "uniformly pointwise stable". This means that, for $X \in \bar{\Omega}$,

$$\bar{L}_{ABCD}(X)M^{AB}M^{CD} > 0 \quad \forall\, M^{AB} = M^{(AB)} \neq 0. \tag{36}$$

For example in the case of an isotropic homogenous material there are constants λ and μ, such that

$$4\rho_0\,\bar{L}_{ABCD} = \lambda\bar{H}_{AB}\bar{H}_{CD} + 2\mu\bar{H}_{C(A}\bar{H}_{B)D}, \tag{37}$$

and in that case uniform pointwise stability is equivalent to the conditions

$$\mu > 0,\ 3\lambda + 2\mu > 0. \tag{38}$$

By a standard integration-by-parts argument, it follows from uniform pointwise stability that any element $\delta\Phi$ in the kernel of the map δE is Killing. Thus the map δE has a kernel of the dimension of that of the isometry group of $(\bar{\Phi}(\Omega), h_{ij})$ and a cokernel of that same dimension. The latter fact can e.g. be inferred from the formal self-adjointness of δE, as follows: raising the index i in $\delta\bar{\sigma}_i{}^A$ with $\bar{h}^{ij} = h^{ij} \circ \bar{\Phi}$, E can be viewed as map from functions $\delta\Phi^i(X)$

into themselves, which is formally self-adjoint w.r. to the inner product given by $\langle \delta\mu, \delta\Phi\rangle = \int_\Omega \bar{h}_{ij}(\delta\mu^i)(\delta\Phi^j)d^3V + \int_{\delta\Omega} \bar{h}_{ij}(\delta\mu^i)(\delta\Phi^j)dO$, using the symmetry $\bar{L}_{ABCB} = \bar{L}_{CDAB}$ [5].
It follows from these observations, that, with an appropriate choice of function spaces, the operator δL, viewed as a map from some complement of Killing elements at $\bar{\Phi}$ to loads equilibrated at $\bar{\Phi}$, is an isomorphism [6].

3 Rigidly rotating bodies

In order to derive the equations governing a rigidly rotating elastic body we start from the time-dependent theory as laid out in [2]. Back-to-label maps now go from a relativistic spacetime $(M, g_{\mu\nu})$ to (Ω, V_{ABC}). The condition of invertibility in the time-independent case is replaced by the condition that there is a unique-up-to-sign timelike vector field u^μ with $u^\mu u^\nu g_{\mu\nu} = -1$, so that

$$u^\mu(\partial_\mu f^A) = 0 \tag{39}$$

The quantity n, in the time-dependent setting, is defined by

$$(\partial_\mu f^A)(\partial_\nu f^B)(\partial_\lambda f^C)V_{ABC} = n\varepsilon_{\mu\nu\lambda\rho}u^\rho. \tag{40}$$

The spacetime action for the elastic field is given by

$$S = \int n\epsilon(-g)^{\frac{1}{2}}d^4x, \tag{41}$$

where ϵ is a function of $H^{AB} = (\partial_\mu f^A)(\partial_\nu f^B)g^{\mu\nu}$. We now suppose the maps f are time-independent in the sense that u^μ is proportional to an everywhere timelike Killing vector field ξ^μ. Furthermore suppose $M = \mathbb{R}^1 \times N$ with N the quotient of M by the isometry group generated by ξ^μ. The natural metric on N is given by $h_{ij} = g_{ij} - g_{0i}g_{oj}/g_{00}$ in local coordinates (t, x^i) on $\mathbb{R}^1 \times N$. Since $g^{ij} = h^{ij}$, with h^{ij} the inverse of h_{ij}, we have that the f's, viewed as maps $f : (N, h_{ij}) \mapsto (\Omega, V_{ABC})$, H^{AB} and n bear the same relationship to each other as the quantities of the same name in the previous chapter. Since, furthermore, $(-g)^{\frac{1}{2}} = (-g_{00})^{\frac{1}{2}}h^{\frac{1}{2}}$ we find that

$$S = \int n\epsilon(-g_{00})^{\frac{1}{2}}h^{\frac{1}{2}}d^3x\, dt, \tag{42}$$

[5]This formal selfadjointness runs under the name of the "Betti reciprocity theorem" in standard elasticity.

[6]The necessary Fredholm theory is fairly standard in a Hilbert space (i.e. $W^{2,2}$-)setting (see Ref. [12]). However in this paper we need $W^{2,p}$ with $p > 3$, and this can be found e.g. in [11] in the case when (N, h_{ij}) is flat. Luckily this is all we require for the present purposes.

and the reduced action reads

$$S = \int n\,\epsilon(H^{AB}, f^C)(-g_{00})^{\frac{1}{2}} h^{\frac{1}{2}} d^3x. \tag{43}$$

We write ϵ as

$$\epsilon = c^2 + w \tag{44}$$

and g_{00} as

$$-g_{00} = c^2 e^{\frac{2U}{c^2}}. \tag{45}$$

The resulting field equations turn out to be equivalent to

$$-e^{-\frac{U}{c^2}} D_j(e^{\frac{U}{c^2}} \sigma_i{}^j) + n(1 + \frac{w}{c^2}) D_i U = 0 \tag{46}$$

in the spatial picture and

$$-e^{-\frac{U}{c^2}} \nabla_A(e^{\frac{U}{c^2}} \sigma_i{}^A) + (1 + \frac{w}{c^2}) D_i U = 0 \tag{47}$$

in the material picture. We now specialize to bodies in SRT which rigidly rotate at angular speed ω. Thus we take $(M, g_{\mu\nu})$ to be $(\mathbb{R}^4, \eta_{\mu\nu})$ with

$$\eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + \delta_{ij}\, dx^i dx^j \tag{48}$$

and $\xi^\mu \partial_\mu$ to be

$$\xi^\mu \partial_\mu = \partial_t + \omega\, \partial_\phi. \tag{49}$$

There results

$$e^{\frac{2U}{c^2}} = 1 - \frac{\omega^2 r^2}{c^2} \tag{50}$$

and

$$h_{ij}(x^k; \omega) dx^i dx^j = \delta_{ij}\, dx^i dx^j + \frac{\omega^2}{c^2 - \omega^2 r^2}(x^1 dx^2 - x^2 dx^1)^2, \tag{51}$$

where $r^2 = (x^1)^2 + (x^2)^2$. We restrict ourselves to the region where $r < \frac{c}{\omega}$, i.e. inside the timelike cylinder on which ξ^μ gets null. Note that $\frac{1}{c^2}$ enters Eq.(46), resp. Eq.(47), explicitly, but $\frac{\omega^2}{c^2}$ enters also implicitly through U and via the h_{ij}-dependence both of the covariant derivative and of that of H^{AB} appearing in w, n and $\sigma_i{}^j$, resp. $\sigma_i{}^A$.

4 The nonrelativistic Theorem

Taking the formal limit of Eq.(47) as $c \to \infty$ we find

$$V^{-1} \partial_A(V \sigma_i{}^A) + \omega^2(\Phi^1, \Phi^2, 0) = 0. \tag{52}$$

Here ${\sigma_i}^A$ is given in terms of w by Eq.(16), with $H^{AB} = {\Psi^A}_i {\Psi^B}_j \, \delta^{ij}$. As usual the boundary condition is that

$$\sigma_i = ({\sigma_i}^A n_A)|_{\partial\Omega} = 0 \tag{53}$$

Let $\bar{\Phi}$ be an unstressed reference configuration. It follows that $\bar{\Phi}$ is a solution of Eq.(52) and of the boundary condition Eq.(53). We can, and will, choose coordinates X^A on Ω so that $\bar{\Phi}$ is the identity map, i.e. $X^A = \bar{\Phi}^A(x^i) = {\delta^A}_i x^i$. From Eq.(28) we have that $V = const$ in these coordinates. Thus, apart from an overall minus-sign, our field equation is of the form (25). We are seeking solutions Φ of Eq.'s (52,53) for small values of ω which coincide with $\bar{\Phi}$ for $\omega = 0$. We have the conditions (26) for each element of the Lie algebra of the Euclidean group. These six conditions, in the presence of (53), amount to the statement that, for any configuration Φ solving the field equations and the boundary condition, the total centrifugal force and the total centrifugal torque be zero. The former of these conditions, namely

$$\omega^2 \int_\Omega \Phi^1 d^3X = \omega^2 \int_\Omega \Phi^2 d^3X = 0 \tag{54}$$

states that the center of mass has to be on the axis of rotation. The vanishing of the total torque reads

$$\omega^2 \int_\Omega \Phi^3 \Phi^1 d^3X = \omega^2 \int_\Omega \Phi^3 \Phi^2 d^3X = 0, \tag{55}$$

and this means that the axis of rotation is an eigendirection of

$$\Theta^{ij} = \int_\Omega (\delta^{ij}\delta_{kl}\Phi^k\Phi^l - \Phi^i\Phi^j) d^3X, \tag{56}$$

apart from a factor $\int_\Omega \bar{\rho}\, d^3X$ the tensor of inertia. Hence rigid rotation is only possible through a principal axis of inertia of the rotating object.
The above conditions depend on the configuration Φ we want to determine. If one linearizes at the stress free configuration a kernel and range of the linearized elasticity operator appears, as explained in Sect.2, and one can not use the implicit function theorem directly. In the known existence theorems this problem was dealt with as follows:
Stoppelli [10], who gave the first existence theorem for dead loads, in a first step projects the equation on the range of the linearized operator. In a second step he shows that one can use the invariance the linearized operator under motions to construct, from the "projected solution", a solution of the original equation. This is similar to the method of Liapunov-Schmidt reduction in bifurcation theory. By a similar technique, Valent [11] solves various problems including ones with life loads.

We propose here yet another method which rests on the fact that our life load is a differentiable function of the configuration. Therefore we can construct a priori a manifold of configurations which are equilibrated for the centrifugal force. Using only these configurations the implicit function theorem directly gives the solution for small ω, provided our stress free initial configuration $\bar{\Phi}$ is equilibrated for the centrifugal force and has three different moments of inertia.

We now fix the spaces we will work in to be the standard ones in static elasticity problems. We assume that the boundary of our body Ω is at least C^1, i.e. there are no corners. We take the components of the configurations in $W^{2,p}(\Omega, \mathbb{R}^3)$ for $p > 3$. Then Φ is in $C^1(\bar{\Omega})$ and the boundary traction $\sigma_i{}^A n_A$ is in $W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$.
We can choose a neighbourhood $\mathcal{C}$ of $\bar{\Phi}$ in $W^{2,p}(\Omega, \mathbb{R}^3)$ small enough so that each $\Phi \in \mathcal{C}$ has a C^1-inverse (see p.224 of [5]). The elasticity operator E of Sect.2, given by

$$E : \Phi \in \mathcal{C} \mapsto (\partial_A \sigma_i{}^A, \sigma_i) \in W^{0,p}(\Omega, \mathbb{R}^3) \times W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3) \tag{57}$$

is differentiable. (For the first factor in (57), see Appendix A, for the second factor, see Remark (6.5) on p.78 of [11].) We call

$$L = W^{0,p}(\Omega, \mathbb{R}^3) \times W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3) \tag{58}$$

the "load space" and its elements $(\mathcal{F}_i, \tau_i)$. The load map $F : \mathcal{C} \longrightarrow L$ is given by $\mathcal{F}_i = \omega^2 \mathcal{Z}_i$ together with $\tau_i = 0$. The form of the centrifugal force $\mathcal{Z}_i = -(\Phi^1, \Phi^2, 0)$ and the principle of material frame indifference imply (see footnote [5]) that, given any solution (Φ^1, Φ^2, Φ^3) of $E + F = 0$, $(\Phi^1, \Phi^2, \Phi^3 + c)$ and $(\Phi'^1, \Phi'^2, \Phi^3)$ where (Φ'^1, Φ'^2) are a rotation of (Φ^1, Φ^2) in the (x^1, x^2)-plane, are also solutions. We fix this freedom by imposing the conditions

$$\mathcal{C}^0 := \{\Phi | \Phi \in \mathcal{C}, \Phi^3(0) = 0,\ (\partial_2 \Phi^1)(0) = (\partial_1 \Phi^2)(0)\}, \tag{59}$$

where we have also assumed that Ω has been chosen such that $0 \in \Omega$. Clearly $\mathcal{C}^0$ is a C^1-(in fact: analytic) submanifold of $\mathcal{C}$. Finally we want to restrict ourselves to configurations, which are equilibrated w.r.t. the centrifugal force, i.e. we define

$$\mathcal{C}^0_{\mathcal{Z}} := \{\Phi | \Phi \in \mathcal{C}^0, \int_\Omega \Phi^1 d^3X = \int_\Omega \Phi^2 d^3X = \int_\Omega \Phi^1 \Phi^3 d^3X = \int_\Omega \Phi^2 \Phi^3 d^3X = 0\}. \tag{60}$$

Suppose furthermore that $\bar{\Phi} \in \mathcal{C}^0_{\mathcal{Z}}$. Explicitly this means that

$$\int_\Omega X^1 d^3X = \int_\Omega X^2 d^3X = \int_\Omega X^1 X^3 d^3X = \int_\Omega X^2 X^3 d^3X = 0 \tag{61}$$

Note that the last two equations in (61) mean that the rotation axis concides with some fixed but arbitrarily chosen principal axes in the given reference configuration. Then we want to show that $\mathcal{C}^0_{\mathcal{Z}}$ is a submanifold of finite codimension

in $\mathcal{C}^0$.
The map H, which sends a configuration to the values of the four integrals in Eq.(60) is a differentiable map from $\mathcal{C}^0$ to $\mathbb{R}^4$. Hence $\mathcal{C}^0_{\mathcal{Z}} \subset \mathcal{C}^0$ has codimension 4 near $\bar{\Phi}$ if the four differential 1–forms defined by the four conditions above are linearly independent at $\bar{\Phi}$. These 1–forms are

$$A(\delta\Phi) = \int_\Omega \delta\Phi^1 d^3X \ , \ B(\delta\Phi) = \int_\Omega \delta\Phi^2 d^3X \tag{62}$$

and

$$C(\delta\Phi) = \int_\Omega (X^1\delta\Phi^3 + X^3\delta\Phi^1) d^3X \ , \ D(\delta\Phi) = \int_\Omega (X^2\delta\Phi^3 + X^3\delta\Phi^2) d^3X \tag{63}$$

Suppose a linear combination exists, i.e.

$$a\,A(\delta\Phi) + b\,B(\delta\Phi) + c\,C(\delta\Phi) + d\,D(\delta\Phi) = 0 \tag{64}$$

for constants $(a, b, c, d) \in \mathbb{R}^4$, which vanishes for all $\delta\Phi^i \in W^{2,p}(\Omega, \mathbb{R}^3)$ in the tangent space of $\mathcal{C}^0_{\mathcal{Z}}$ at $\bar{\Phi}$, i.e. satisfying $\delta\Phi^3(0) = 0, (\partial_1\delta\Phi^2)(0) = (\partial_2\delta\Phi^1)(0)$. Choosing $\delta\Phi^1 = \delta\Phi^2 = 0$ we obtain

$$\int_\Omega \left(cX^1 + dX^2\right) \delta\Phi^3 \, d^3X = 0. \tag{65}$$

Choosing $\delta\Phi^3 = X^1$ and successively $\delta\Phi^3 = X^2$ and using the Schwarz inequality, we infer $c = d = 0$. Similarly we obtain $a = b = 0$ from considering $\delta\Phi^2 = (X^2)^2$ and $\delta\Phi^1 = (X^1)^2$. Maximal codimension at $\bar{\Phi}$ implies maximal codimension nearby. Therefore the level sets of H are closed submanifolds with the property that the tangent space splits into an infinite dimensional closed subspace tangent to $H = const$ and some finite dimensional subspace (see e.g. Theorem 3.5.4 of [1]). We note that the above proof works for any $\bar{\Phi}$ which is equilibrated. No condition like "no axis of equilibrium" (see [11]) appears at this point. We collect the above rsults in the

Lemma 1: The configurations in $\mathcal{C}^0_{\mathcal{Z}}$ form a differentiable closed submanifold of codimension 4 in $\mathcal{C}^0$ for Φ sufficiently close to the identity.

We can say more about possible finite dimensional complements of $\mathcal{C}^0_{\mathcal{Z}}$. Let us assume the equilibrium configuration $(\Omega, \bar{\Phi})$ to be such that all three moments inertia are different, hence the principal axes of inertia are uniquely determined. This implies that any small rotation around the x^1 or x^2 – axis and translation in x^1, x^2 – directions will destroy equilibration! Therefore the corresponding 4 Killing vectors span a complement of $\mathcal{C}^0_{\mathcal{Z}}$ at $\bar{\Phi}$.

We now turn to the nonlinear map

$$\mathcal{G} : \mathcal{C}^0_{\mathcal{Z}} \times R \to L \tag{66}$$

defined by the l.h. side of Eq.(52), i.e.

$$\mathcal{G}[\Phi;\omega] = \big(\mathcal{E}_i[\Phi] + \omega^2 \mathcal{Z}_i[\Phi],\ \sigma_i[\Phi]\big) \tag{67}$$

We have that $\mathcal{G}[\bar\Phi, 0] = 0$. We want to solve the equation $\mathcal{G}[\Phi,\omega] = 0$ for small ω by the implicit function theorem. The map $\mathcal{G}$ is differentiable (see Appendix A). The derivative of $\mathcal{G}$ with respect to Φ at $\bar\Phi$ and $\omega = 0$ is the linearized elasticity operator together with the linearized normal traction on the boundary, i.e.

$$\frac{D\mathcal{G}}{D\Phi}[\Phi;\omega]|_{(\bar\Phi;0)} = \left(\frac{D\mathcal{E}_i}{D\Phi}[\Phi]|_{\bar\Phi}\ ,\frac{D\sigma_i}{D\Phi}[\Phi]|_{\bar\Phi}\right) = \delta E : T_{\bar\Phi}(\mathcal{C}^0_{\mathcal{Z}}) \to L. \tag{68}$$

We now recall the discussion of Sect.2. If we consider the elasticity operator δE as a map $\mathcal{C} \to L$ we have Ker(δE)= Killing vectors and Range(δE) $= L_{\bar\Phi}$, the loads equilibrated at $\bar\Phi$. Hence on the tangent space of $\mathcal{C}^0_{\mathcal{Z}}$ the kernel is trivial as we showed above. To deal with the range we proceed as follows: Choose a (6–dimensional) complement S_6 to $L_{\bar\Phi}$ which defines a unique linear projection

$$P : L = L_{\bar\Phi} \oplus S_6 \to L_{\bar\Phi} \tag{69}$$

As is done in bifurcation theory we can solve the "projected equation" i.e.

$$P \circ (\mathcal{E}_i[\Phi] + \omega^2\mathcal{Z}_i[\Phi]) = 0, \quad P \circ \sigma_i[\Phi] = 0 \tag{70}$$

by the implicit function theorem for small $\omega \neq 0$ because the derivative of $P \circ E$, namely $P \circ \delta E$, is an isomorphism at $\bar\Phi$. Denote this configuration by Φ_ω. Finally we show that Φ_ω already solves our problem: fix any configuration $\Phi \in \mathcal{C}^0_{\mathcal{Z}}$ and consider the codimension-6-linear-subspace $L_\Phi \subset L$ of all loads equilibrated at Φ. These are all $(\mathcal{F}_i, \tau_i)$ satisfying

$$\int_\Omega \mathcal{F}_i\, d^3X + \int_{\partial\Omega} \tau_i\, dO = 0\ ,\int_\Omega (\Phi\wedge\mathcal{F})_i\, d^3X + \int_{\partial\Omega} (\Phi\wedge\tau)_i\, dO = 0 \tag{71}$$

The projection P, when restricted to L_Φ, defines an isomorphism $P_\phi : L_\Phi \to L_{\bar\Phi}$ (see [7]) because, for configurations Φ near $\bar\Phi$, the complement S_6 of $L_{\bar\Phi}$ is still a complement of L_Φ. In particular L_Φ and $L_{\bar\Phi}$ just intersect at the origin. For all $\Phi \in \mathcal{C}^0_{\mathcal{Z}}$ we have

$$(\mathcal{E}_i[\Phi], \sigma_i[\Phi]) \in L_\Phi \tag{72}$$

Hence this holds for Φ_ω. Since $\Phi_\omega \in \mathcal{C}^0_{\mathcal{Z}}$ there holds

$$(\mathcal{Z}_i[\Phi_\omega], 0) \in L_{\Phi_\omega} \tag{73}$$

by construction of $\mathcal{C}^0_{\mathcal{Z}}$. Hence we obtain

$$(\mathcal{E}_i[\Phi_\omega] + \omega^2\mathcal{Z}_i[\Phi_\omega], \sigma_i[\Phi_\omega]) \in L_{\Phi_\omega} \tag{74}$$

Our candidate solution Φ_ω satisfies $P \circ (\mathcal{E}_i[\Phi_\omega] + \omega^2 \mathcal{Z}_i[\Phi_\omega]) = 0$ and $P \circ \sigma_i[\Phi_\omega] = 0$. Consequently

$$\mathcal{E}_i[\Phi_\omega] + \omega^2 \mathcal{Z}_i[\Phi_\omega] = 0, \;\; \sigma_i[\Phi_\omega] = 0. \tag{75}$$

Hence Φ_ω is a solution, and we have proved the

Theorem 1: Let the domain Ω and the map $\bar{\Phi}$ be such that all three moments of inertia are different and that one of the principal axes concide with the rotation axis. Let the stored-energy function w be uniformly pointwise stable, i.e. satisfy the inequality (36). Then there is, in a neighbourhood of $(\bar{\Phi}, \omega = 0) \in \mathcal{C}^0_{\mathcal{Z}} \times \mathbb{R}$, a unique element Φ_ω in $\mathcal{C}^0_{\mathcal{Z}}$, solving the equation (52) together with the boundary condition (53).

We remark that the condition on the natural configuration, in Theorem 1, of having three different moments of inertia I_1, I_2, I_3, is not necessary for the theorem to go through. In Sect.7 we consider a natural configuration which, by virtue of its axisymmetry with respect to the rotation axis, has $I_1 = I_2 \neq I_3$, with the eigenvectors for I_1 and I_2 orthogonal to the rotation axis. Then Theorem 1 remains true, with the proviso that the rotation axis is chosen to coincide with the I_3- axis, since the only additional freedom of performing continuous motions is that under ∂_ϕ and ∂_3, and that has already been frozen out in $\mathcal{C}^0_{\mathcal{Z}}$.

Consider finally the case of a spherical top, i.e. where all eigenvalues of the tensor of inertia coincide. An example is given by taking Ω to have the geometry of a cube (w.r. to the the metric $\bar{H}_{AB}$). Our above proof works also in this case: We now have a 2–dimensional intersection of the kernel of the linearized elasticity operator with the tangent space of $\mathcal{C}^0_{\mathcal{Z}}$ at $\bar{\Phi}$. We can solve by the implicit function theorem if we fix an element of the kernel. The cube is equilibrated for the centrifugal force for any rotation axis which goes through its center of mass. Hence there is a 2-paramter family of possibilities for $\bar{\Phi}$. For the cube these $\bar{\Phi}$'s will determine physically distinct non-linear solutions. On the other hand, for a sphere, the trivial spherical top, different choices of $\bar{\Phi}$ lead to nonlinear solutions which are just rotations of each other. In general, the interplay between the symmetry of the tensor of inertia and the symmetry of the domain Ω determines which of the solutions we obtain by selecting an element of the kernel are different

Remark: If the stored-energy function is analytic, we obtain maps between the function spaces which are also analytic. The analytic implicit function theorem implies that the family Φ_ω is analytic in ω^2, i.e. we have a converging Signorini expansion. Furthermore elliptic regularity in that case implies analyticity of the solutions in Ω.

5 Non relativistic, self gravitating, rotating

Combining the result of the last section with our paper on a static self gravitating body [3], it is straightforward to obtain an existence theorem for a slowly rotating, weakly gravitating body. Namely, if we add the gravitational force to the centrifugal force, the equation to be solved takes the form

$$\mathcal{E}_i + \omega^2 \mathcal{Z}_i - G\rho_0 \int_\Omega \frac{\Phi_i(X) - \Phi_i(X')}{|\Phi(X) - \Phi(X')|^3} d^3X' = 0 \tag{76}$$

The important point, then, about self-gravity is that the last term in Eq.(76) is automatically equilibrated for all configurations. One can then basically proceed as in the last section [7].

6 Relativistic case

A specific model is again characterized by a choice of stored energy $w = w(H^{AB}, X)$. The main complication is that H^{AB} now refers to a curved spatial metric which depends on ω. In moving back and forth between the relativistic and the nonrelativistic theory, it is natural to require that the stored-energy functions, for a given material, be the same. It follows that

$$\left(\frac{\partial w(H^{CD}, X)}{\partial H^{AB}}\right)|_{(\Phi=\bar{\Phi}(X);\omega=0)} = 0 \tag{77}$$

and that

$$\bar{L}_{ABCD} = \left(\frac{\partial^2 w(H^{EF}, X)}{\partial H^{AB}\partial H^{CD}}\right)|_{(\Phi=\bar{\Phi}(X);\omega=0)} \tag{78}$$

be pointwise stable in the sense of Eq.(36). As opposed to the nonrelativistic case the equations are not invariant under adding a constant to w. We thus assume that

$$w(H^{AB}, X)|_{(\Phi=\bar{\Phi}(X);\omega=0)} = 0 \tag{79}$$

As in the previous section we assume coordinates X^A to be chosen such that $\bar{\Phi}$ is the identity map. In the relativistic case this requires that $(X^1)^2 + (X^2)^2 < \frac{c^2}{\omega^2}$ for all $(X^1, X^2, X^3) \in \bar{\Omega}$.

We will treat the relativistic case by splitting off the nonrelativistic elasticity operator in Eq.(47) and putting all remaining terms into the load. This we do as follows: We can write

$$e^{\frac{U}{c^2}} \sigma_i{}^A = \overset{\circ}{\sigma}_i{}^A + \omega^2 \widetilde{\sigma}_i{}^A, \tag{80}$$

where $\widetilde{\sigma}_i{}^A = \widetilde{\sigma}_i{}^A(\Phi, \partial\Phi; \omega)$ and $\overset{\circ}{\sigma}_i{}^A = \overset{\circ}{\sigma}_i{}^A|_{\omega=0}$, and analogously

$$(e^{\frac{U}{c^2}} \sigma_i{}^A n_A)|_{\partial\Omega} = \overset{\circ}{\sigma}_i + \omega^2 \widetilde{\sigma}_i \tag{81}$$

[7] We take this opportunity to point that, on p.111 of [3], we required the stored-energy function to satify the so-called Legendre-Hadamard condition, whereas we should have required the stronger condition (36) employed here, in order for the theorem stated there to be true.

for the surface traction. The boundary conditions now take the form

$$\overset{\circ}{\sigma}_i = -\omega^2 \widetilde{\sigma}_i \tag{82}$$

By virtue of Eq.(77) we have that

$$\overset{\circ}{\sigma}_i{}^A[\bar{\Phi}] = 0. \tag{83}$$

The field equations (47) can be written as

$$\partial_A \overset{\circ}{\sigma}_i{}^A + \omega^2(\mathcal{Y}_i + \mathcal{Z}_i) = 0, \tag{84}$$

where

$$\mathcal{Y}_i = \partial_A \widetilde{\sigma}_i{}^A - \frac{1}{\omega^2}\left(1 - \frac{\omega^2 r^2}{c^2}\right)^{\frac{1}{2}} (\partial_A \Phi^j)\Gamma^k_{ji}\, \sigma_k{}^A \tag{85}$$

and

$$\mathcal{Z}_i = \left(1 - \frac{\omega^2 r^2}{c^2}\right)^{-\frac{1}{2}} \left(1 + \frac{w}{c^2}\right) \partial_i \left(\frac{r^2}{2}\right) \tag{86}$$

where it is understood that the functions r and Γ^k_{ij} are evaluated at the points $\Phi(X) \in N$. Note that the second term in Eq.(85) is regular also at $\omega = 0$, due to Eq.(51). Furthermore, using (83) and (79), there holds

$$\mathcal{Z}_i[\Phi;\omega]|_{(\bar{\Phi},0)} = (X^1, X^2, 0). \tag{87}$$

We want to solve Eq.(84) by viewing the first term as the (unperturbed) elasticity operator $\overset{\circ}{\mathcal{E}}_i$. (Note that the flat-space operator $\overset{\circ}{\mathcal{E}}_i$, though "unperturbed", is still nonlinear.) The remaining terms in Eq.(84) form the load, i.e.

$$\mathcal{F}_i = \omega^2(\mathcal{Z}_i + \mathcal{Y}_i) \tag{88}$$

together with

$$\tau_i = \omega^2 \widetilde{\sigma}_i. \tag{89}$$

Thus the equilibration conditions Eq.(26), dividing by ω^2 and turning the surface integral to a volume integral, take the form

$$\begin{aligned} 0 = &- \int_\Omega \partial_A(\xi^i \circ \Phi)\widetilde{\sigma}_i{}^A d^3X + \\ &- \int_\Omega (\xi^i \circ \Phi) \left(1 - \frac{\omega^2 r^2}{c^2}\right)^{\frac{1}{2}} \frac{1}{\omega^2}\Gamma^k_{ji}\, \sigma_k{}^A(\partial_A \Phi^j) + \\ &+ \int_\Omega (\xi^i \circ \Phi)\, \mathcal{Z}_i[\Phi;\omega]\, d^3X. \end{aligned} \tag{90}$$

The vectors ξ^i in Eq.(90) run through the Euclidean Killing vectors on N. We now claim that, as in the nonrelativistic case, 2 of these 6 conditions are identities, namely, if ξ^i is either ∂_3 or ∂_ϕ. To see this, recall that the conditions (90) are equivalent to the relations

$$\int_\Omega \xi^i \, [\frac{1}{\omega^2} D_A \sigma_i{}^A + \mathcal{Z}_i] \, d^3X = 0 \tag{91}$$

By virtue of the axial symmetry of r^2, the second term on the right in (91) gives zero when ξ is ∂_3 or ∂_ϕ. The first term, using the boundary conditions and that ∂_3 and ∂_ϕ are Killing vectors also of the ω-dependent curved metric h_{ij}, gives also zero for $\omega \neq 0$, whence for $\omega = 0$, by continuity. This proves the above claim. We next have to look at the different terms in the integrand of Eq.(90) at $\omega = 0$. We have that

$$\mathcal{Z}_i|_{\omega=0} \; = (1 + \frac{\overset{\circ}{w}}{c^2}) \, \partial_i \left(\frac{r^2}{2} \right). \tag{92}$$

Furthermore we find that

$$\frac{1}{\omega^2} \Gamma^k_{ij} \sigma_k{}^A|_{\omega=0} = \frac{1}{c^2} \lambda^k_{ij} \; \overset{\circ}{\sigma}_k{}^A, \tag{93}$$

where the quantities λ^k_{ij} are linear functions of $\Phi(X)$ with constant coefficients and, using Eq.'s (50,51,80) and Eq.(22),

$$\widetilde{\sigma}_i{}^A|_{\omega=0} = -\frac{r^2}{2c^2} \; \overset{\circ}{\sigma}_i{}^A - \frac{2}{c^2} \; \overset{\circ}{K}{}^{AC} \; \overset{\circ}{\sigma}_i{}^B \; \overset{\circ}{H}_{BC} - \frac{2}{c^2} \; \overset{\circ}{H}{}^{AC} \Psi^B{}_i \; \overset{\circ}{K}{}^{DE} \; \overset{\circ}{L}_{BCDE}, \tag{94}$$

where $\overset{\circ}{K}{}^{AB} = \frac{d}{d\omega^2} H^{AB}|_{\omega=0} = \Psi^A{}_i \Psi^B{}_j \kappa^{ij}$ and κ^{ij} are quadratic functions of $\Phi(X)$ with constant coefficients, $\overset{\circ}{H}_{AB} = H_{AB}|_{\omega=0}$ and $\overset{\circ}{L}_{ABCD} = L_{ABCD}|_{\omega=0}$. Note that all quantities with superscript $\circ$ depend on $\partial\Phi(X)$, but not on $\Phi(X)$. We next evaluate the equilibration conditions in the reference configuration. The first term in Eq.(90) gives no contribution at $\Phi = \bar{\Phi}$, since the first two terms in (94) vanish and the third term contributes zero, due to the symmetries of L_{ABCD} and the Killing equation for ξ. The second term in (90) is also zero in the reference configuration. The third term, finally, is identical with its nonrelativistic value, by Eq.(79). It follows that the requirements in the previous section on the reference configuration can remain unchanged. It remains to compute the derivative at $\Phi = \bar{\Phi}$ of the function $H[\phi]$ given by the four functions resulting by inserting into the r.h. side of (90) the Killing vectors $\xi = \partial_1, \xi = \partial_2, \xi = x^3\partial_1 - x^1\partial_3, \xi = x^3\partial_2 - x^2\partial_3$ on N. The derivative at $\Phi = \bar{\Phi}$ of the last term in (90), using (92), the stressfreeness of $\bar{\Phi}$ and (79), is the same as that in the nonrelativistic case. The explicit form of the remaining terms does not matter except that they are linear, with coefficients some given functions of X, in the quantities $\frac{1}{c^2}\bar{L}_{ABCD}(X)$. The quantities $\frac{1}{c^2}\bar{L}_{ABCDEF}(X)$, where

$$\bar{L}_{ABCDEF} = \left(\frac{\partial^3 w}{\partial H^{AB} \, \partial H^{CD} \, \partial H^{EF}} \right) |_{[\Phi=\bar{\Phi}(X);\; \omega=0]}. \tag{95}$$

appear in the derivative of Eq.(94), but do not contribute to DH at $\Phi = \bar{\Phi}$, again using the symmetry and the flat-space Killing equation for ξ. Now recall the discussion of Sect.4. Evaluating the nonrelativistic $DH|_{\bar{\Phi}}$ on some test functions $\delta\Phi$, we arrived at certain positive expressions. These can bounded from below by a (dimensionful) quantity, say β, which solely depends on the geometry of the domain Ω. Similarly, the contribution to $DH|_{\bar{\Phi}}$ of the relativistic terms just discussed can be bounded from above by a geometrical quantity, say γ times $\frac{1}{c^2}|\bar{L}|$, where $|\bar{L}|$ is some upper bound for the components of $\bar{L}_{ABCD}$. It follows that there is a dimensionless number α, which depends only on the geometry of Ω, so that $DH|_{[\Phi=\bar{\Phi};\, \omega=0]}$ is nonzero provided that

$$\frac{|\bar{L}|}{c^2} < \alpha = \frac{\beta}{\gamma} \tag{96}$$

We now follow the pattern of the discussion in Sect.4 as much as possible. We assume the domain Ω to lie strictly inside the cylinder $(X^1)^2 + (X^2)^2 = \frac{c^2}{\epsilon^2}$. We choose a neighbourhood ${}^{\epsilon}\mathcal{D}$ of the identity in $\mathcal{C}$, small enough so that $\Phi(\Omega) \subset N_\omega$ for all $0 \leq \omega < \epsilon$. Here N_ω is the subset in $\mathbb{R}^3$ with $r < \frac{c}{\omega}$. We then restrict ${}^{\epsilon}\mathcal{D}$ to ${}^{\epsilon}\mathcal{D}^0$, by imposing the conditions Eq.(59). The elasticity operator E is still the nonrelativistic one, namely $\Phi \mapsto (\mathcal{E}_i = \overset{\circ}{\mathcal{E}}_i = -\partial_A \overset{\circ}{\sigma}_i{}^A, \overset{\circ}{\sigma}_i)$. The load map F, consisting previously of $\mathcal{F}_i = \omega^2(\Phi^1, \Phi^2, 0)$ together with $\tau_i = 0$, is replaced by $\mathcal{F}_i = \omega^2 \mathcal{Z}'_i = \omega^2(\mathcal{Y}_i + \mathcal{Z}_i)$, with $\mathcal{Z}, \mathcal{Y}$ according to Eq.'s (85,86), together with $\tau_i = \omega^2 \widetilde{\sigma}_i$. The important difference is that $\mathcal{Z}'_i$ depends on ω and on $(\partial\Phi, \partial\partial\Phi)$ with $\partial\partial\Phi$ appearing only linearly as required by the second result in Appendix A. Let us denote by ${}^{\epsilon}\mathcal{D}^0_{\mathcal{Z}'} \times (-\epsilon, \epsilon)$ the set of configurations in ${}^{\epsilon}\mathcal{D}^0$ and values $\omega \in (-\epsilon, \epsilon)$ satisfying the 4 relativistic equilibration conditions (i.e. the zero-level set of the function H described above, which involves $\mathcal{Z}'_i$). Our above discussion, together with the inverse function theorem, shows the following

Lemma 2: Suppose the inequality (96) is valid. Then the set ${}^{\epsilon}\mathcal{D}^0_{\mathcal{Z}'} \times (-\epsilon, \epsilon)$, for Φ sufficiently close to the identity and ω sufficiently close to zero, is a C^1-submanifold of codimension 4 in the Banach space ${}^{\epsilon}\mathcal{D}^0 \times \mathbb{R}$.

We now consider the equation $P \circ (E + F) = 0$ on ${}^{\epsilon}\mathcal{D}^0_{\mathcal{Z}'} \times (-\epsilon, \epsilon)$ with the projection map P defined exactly as before. The remainder of the argument is completely analogous to Sect.4, and we obtain the

Theorem 2: Let the stored-energy function w satisfy Eq.(79) and Eq.(77), and let the constants $\bar{L}_{ABCD}$ defined by (78) satisfy the inequalities (36) and (96). Suppose, finally, $(\Omega, \bar{\Phi})$ to be such that the three principal axes are different and that one of them coincides with the rotation axis. Then there is, in a neighbourhood of $(\bar{\Phi}, 0) \in {}^{\epsilon}\mathcal{D}^0_{\mathcal{Z}'} \times (-\epsilon, \epsilon)$, a unique element (Φ_ω, ω) which solves the equations (84) together with the boundary condition (82).

7 Linearized solutions

In this section we want to present an explicit solution which is linearized in ω^2. We take as the background solution a stress free ellipsoid of the form

$$X^2 + Y^2 + \epsilon^2 Z^2 = R^2 \tag{97}$$

The material is assumed to be isotropic in its natural state. Thus we have that $\bar{L}_{ABCD}$ defined by Eq.(78) satisfies (see Eq. (37))

$$4\rho_0\, \bar{L}_{ABCD} = \lambda\delta_{AB}\delta_{CD} + 2\mu\delta_{C(A}\delta_{B)D}, \tag{98}$$

and the inequalities (38), namely

$$\mu > 0,\ 3\lambda + 2\mu > 0. \tag{99}$$

Consider the family of solution of (82) determined by the implicit function theorem and parametrized by ω^2. The linearization in ω^2 satisfies the equation

$$\partial_A\ \delta\overset{0}{\sigma}_i{}^A + \bar{\mathfrak{Y}}_i + \bar{\mathfrak{Z}}_i = 0, \tag{100}$$

where, again, the bar means evaluation at $\Phi = \bar{\Phi}$ and $\omega = 0$.
We write

$$\delta\Phi^i = \frac{d}{d\omega^2}\Phi^i|_{(\omega=0} \tag{101}$$

and $\delta\overset{0}{\sigma}_i{}^A$ is determined from (32,33,34). We obtain, using (98),(77)and (34), that

$$\delta\overset{0}{\sigma}_{iA} = \frac{1}{\rho_0}[\mu(\partial_A\delta\Phi_i + \partial_i\delta\Phi_A + \lambda\delta_{iA}\partial_k\delta\Phi^k]. \tag{102}$$

Note that, by the convention to view $\bar{\Phi}$ as the identity map, $\partial_A\bar{\Phi}^i = \delta_i{}^A$ and $\bar{H}_{AB} = \delta_{AB}$. Eq.(102) leads to the standard nonrelativistic operator of linearized elasticity, i.e.

$$\partial_A\ \delta\overset{0}{\sigma}_i{}^A = \frac{1}{\rho_0}[\mu\Delta\delta\Phi_i + (\mu+\lambda)\partial_i\partial_k\delta\Phi^k]. \tag{103}$$

For the remaining terms in the equation we obtain from (85,86)

$$\bar{\mathfrak{Y}}_i = \frac{1}{\rho_0 c^2}(\lambda - \mu)(X^1, X^2, 0) \tag{104}$$

$$\bar{\mathfrak{Z}}_i = (X^1, X^2, 0) \tag{105}$$

The boundary conditions at $(X^1)^2 + (X^2)^2 + \epsilon^2(X^3)^2 = R^2$ are

$$(\delta\overset{0}{\sigma}_i{}^A + \widetilde{\sigma}_i{}^A)n_A|_{\partial\Omega} = 0 \tag{106}$$

where we can take $n_A dX^A = X^1 dX^1 + X^2 dX^2 + \epsilon^2 X^3 dX^3$ and $\widetilde{\sigma}_i$ has to be evaluated at $\Phi = \bar{\Phi}$ and $\omega = 0$ using Eq.(94).
We can replace λ by the Poisson number σ of elasticity defined by

$$\lambda = \frac{2\mu\sigma}{1-2\sigma} \tag{107}$$

which satisfies

$$-1 < \sigma < \frac{1}{2} \tag{108}$$

Then μ drops out from the boundary conditions and in the linearized equations μ appears only as $\frac{\rho_0}{\mu}$. As the "force is equilibrated", the linearization of the boundary value problem has a unique solution. For the non relativistic case a solution of the above equations can be found in [8] going back to [4]. We make to following ansatz for $\delta\Phi^i$:

$$\delta\Phi^1 = X^1[a_1 + a_2((X^1)^2 + (X^2)^2) + a_3(X^3)^2] \tag{109}$$

$$\delta\Phi^2 = X^2[a_1 + a_2((X^1)^2 + (X^2)^2) + a_3(X^3)^2] \tag{110}$$

$$\delta\Phi^3 = X^3[a_4 + a_5((X^1)^2 + (X^2)^2) + a_6(X^3)^2] \tag{111}$$

The rational behind this ansatz is that we have a linear PDE-problem with constant coefficients, a polynomial (in fact:linear) right-hand side and boundary conditions on an algebraic surface. Thus the solution should also be polynomial. In fact, inserting the ansatz Eq.(109,110,111) into the equations and the boundary conditions we obtain by a lengthy calculation a linear inhomogeneous system $Qa = C$ for $a = (a_1, ..a_6)$ (note that 2 of these equations come from the field equation and 4 conditions come from the boundary conditions). Using Maple we find

$$Q = \begin{bmatrix} 2\,\lambda\,\varepsilon^2\,,\,4\,\lambda\,R^2\,\varepsilon^2\,,\,2\,\mu\,R^2\,,\,(2\,\mu+\lambda)\,\varepsilon^2\,,\,(4\,\mu+\lambda)\,R^2\,\varepsilon^2\,,\,0 \\ 0\,,\,-4\,\lambda\,\varepsilon^4\,,\,(-2\,\mu+2\,\lambda)\,\varepsilon^2\,,\,0\,,\,(-2\,\mu-\lambda)\,\varepsilon^4-2\,\mu\,\varepsilon^2\,,\,(6\,\mu+3\,\lambda)\,\varepsilon^2 \\ 0\,,\,0\,,\,4\,\lambda+4\,\mu\,,\,0\,,\,4\,\mu\,,\,12\,\mu+6\,\lambda \\ 0\,,\,16\,\mu+8\,\lambda\,,\,2\,\mu\,,\,0\,,\,2\,\lambda+2\,\mu\,,\,0 \\ 2\,\lambda+2\,\mu\,,\,4\,\lambda\,R^2+6\,\mu\,R^2\,,\,0\,,\,\lambda\,,\,\lambda\,R^2\,,\,0 \\ 0\,,\,(-6\,\mu-4\,\lambda)\,\varepsilon^2\,,\,4\,\mu+2\,\lambda\,,\,0\,,\,(-\lambda+2\,\mu)\,\varepsilon^2\,,\,3\,\lambda \end{bmatrix}$$

with determinant

$$det(Q) = -48\,\varepsilon^4\,\mu^3\,(2\,\mu+3\,\lambda)\,(2\,\mu+\lambda)\,(6\,\varepsilon^4\,\mu+64\,\mu+11\,\lambda\,\varepsilon^4+64\,\lambda+20\,\lambda\,\varepsilon^2)$$

and

$$C := [-\frac{\lambda R^2}{2c^2}, \frac{\lambda\epsilon^2}{2c^2}, 0, -\rho_0 + \frac{(\mu-\lambda)}{c^2}, \frac{-\lambda R^2}{2c^2}, \frac{\lambda\epsilon^2}{c^2}]$$

Here are some observations concerning this linear system:

1.) For positive μ, λ obeying (99), we can solve the linear system.
2.) The velocity of light, c, appears only on the right-hand side of the linear system and we can write the solution as the non–relativistic solution plus a relativistic correction term proportional to $\frac{1}{c^2}$. For $c \to \infty$ the vector C greatly simplifies.
3.) The case $\lambda = 0$, i.e. $\sigma = \frac{\lambda}{2\lambda+2\mu}$, is also much simpler. In particular, the only relativistic correction is a change of the "effective density", $-\rho_0 \to -\rho_0 + c^{-2}(\mu - \lambda)$.
The solution can be given in closed form. We begin with the simplest case, i.e. $\epsilon = 1$: the deformation of a sphere. We find that

$$a_1 = \frac{2}{5}\,\frac{(-3+2\,\sigma+3\,\sigma^2)\,R^2}{(\sigma-1)\,(5\,\sigma+7)\,(\sigma+1)}\,\frac{\rho_0}{\mu} \tag{112}$$

$$a_2 = -\frac{1}{10}\,\frac{(-4+3\,\sigma+5\,\sigma^2)}{(5\,\sigma+7)\,(\sigma-1)}\,\frac{\rho_0}{\mu} \tag{113}$$

$$a_3 = -\frac{1}{10}\,\frac{(-9+8\,\sigma+5\,\sigma^2)}{(5\,\sigma+7)\,(\sigma-1)}\,\frac{\rho_0}{\mu} \tag{114}$$

$$a_4 = -\frac{1}{10}\,\frac{(-3-18\,\sigma+3\,\sigma^2+10\,\sigma^3)\,R^2}{(\sigma-1)\,(5\,\sigma+7)\,(\sigma+1)}\,\frac{\rho_0}{\mu} \tag{115}$$

$$a_5 = \frac{1}{5}\,\frac{(\sigma-3)}{(5\,\sigma+7)\,(\sigma-1)}\,\frac{\rho_0}{\mu} \tag{116}$$

$$a_6 = -\frac{1}{10}\,\frac{(1+3\,\sigma)}{(5\,\sigma+7)\,(\sigma-1)}\,\frac{\rho_0}{\mu} \tag{117}$$

The change of a point X is given by $\omega^2 \delta\Phi^i(X)$. On the equator and north pole one finds:

$$\delta\Phi^1(0,R,0) = R(a_1 + a_2 R^2) = -\frac{1}{10}\,\frac{(5\,\sigma^2+\sigma-8)\,R^2}{(\sigma+1)\,(5\,\sigma+7)}\,\frac{\rho_0}{\mu} \tag{118}$$

$$\delta\Phi^3(0,0,R) = R(a_4 + a_6 R^2) = -\frac{1}{5}\,\frac{(5\,\sigma^2+8\,\sigma+1)\,R^2}{(\sigma+1)\,(5\,\sigma+7)}\,\frac{\rho_0}{\mu}. \tag{119}$$

Note that the north pole can move outward when $-1 < \sigma < -\frac{4-\sqrt{15}}{5}$, but for physical materials σ will be positive. Next we give the relativistic corrections which we denote by b_i . They are independent of μ:

$$b_1 = -\frac{3}{10}\,\frac{R^2\,(5\,\sigma^2+3\,\sigma-4)}{(5\,\sigma+7)\,(\sigma-1)\,c^2}$$

$$b_2 = \frac{1}{10}\,\frac{5\,\sigma^2+11\,\sigma-4}{(5\,\sigma+7)\,(\sigma-1)\,c^2}$$

$$b_3 = \frac{3}{10}\,\frac{10\,\sigma^2 - 3\,\sigma - 3}{(5\,\sigma + 7)\,(\sigma - 1)\,c^2}$$

$$b_4 = -\frac{3}{10}\,\frac{R^2\,(3\,\sigma + 1)}{(5\,\sigma + 7)\,(\sigma - 1)\,c^2}$$

$$b_5 = -\frac{3}{10}\,\frac{5\,\sigma^2 - 7\,\sigma - 2}{(5\,\sigma + 7)\,(\sigma - 1)\,c^2}$$

$$b_6 = \frac{1}{10}\,\frac{1 + \sigma + 10\,\sigma^2}{(5\,\sigma + 7)\,(\sigma - 1)\,c^2}$$

The change of a point on the equator and the north pole are:

$$\delta\Phi^1 = -\frac{2}{5}\,\frac{(15\,\sigma^3 + 6\,\sigma^2 - 17\,\sigma + 2)\,R^3}{(-1 + 2\,\sigma)\,(\sigma + 1)\,(5\,\sigma + 7)c^2}$$

$$\delta\Phi^3 = -\frac{1}{5}\,\frac{(30\,\sigma^3 + 47\,\sigma^2 - 4\,\sigma - 1)\,R^3}{(-1 + 2\,\sigma)\,(\sigma + 1)\,(5\,\sigma + 7)c^2}$$

To show the effect of the ellipticity, we give just a_1 :

$$a_1 := \frac{\rho_0 R^2}{2}\,\frac{(6 + 3\,\varepsilon^2 + 3\,\varepsilon^4 + 30\,\sigma^2 + \varepsilon^4\,\sigma^2 + \varepsilon^4\,\sigma^3 - 27\,\sigma^2\,\varepsilon^2 + 23\,\sigma^3\,\varepsilon^2 - 5\,\varepsilon^4\,\sigma - 28\,\sigma + \sigma\,\varepsilon^2)}{\mu\,(\sigma - 1)\,(\sigma + 1)\,(10\,\varepsilon^4\,\sigma^2 + 40\,\sigma^2\,\varepsilon^2 - 8\,\sigma\,\varepsilon^2 + 48\,\sigma + 5\,\varepsilon^4\,\sigma - 16 - 11\,\varepsilon^4 - 8\,\varepsilon^2)}$$

The limits $\epsilon = \infty$ exist. They approximate a very flat ellipsoid. The formulas are comparable to the nonrelativistic case.

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A Some Functional Analysis

We collect some differentiability statements which can be easily extracted from Ref. [11]. First consider maps $\hat{f}$ of the form

$$\hat{f} : \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R}^3 \tag{1}$$

Let $\Phi \in W^{2,p}(\Omega, \mathbb{R}^3)$, $p > 3$. Define the "Nemitsky" operator $f : W^{2,p}(\Omega, \mathbb{R}^3) \to W^{1,p}(\Omega, \mathbb{R}^3)$ given by

$$f_i(\Phi)(X) := \hat{f}_i(\Phi(X), \partial\Phi(X)) \tag{2}$$

Then f is C^1 if $\hat{f}$ is C^1 and f is C^ω if $\hat{f}$ is C^ω.

Secondly, consider maps g of the form

$$\hat{g} : \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R}^6 \times \mathbb{R}^9. \tag{3}$$

Then the quasilinear operator g

$$g_i(\Phi)(X) := \hat{g}^{kl}_{ij}(\Phi(X), \partial\Phi(X))\partial_k\partial_l\Phi^j, \tag{4}$$

viewed as a map $g : W^{2,p}(\Omega, \mathbb{R}^3) \to W^{0,p}(\Omega, \mathbb{R}^3)$, is C^1 if $\hat{f}$ is C^1 and C^ω if $\hat{f}$ is C^ω.
An elementary statement used in the body of the paper is the following: If Φ is in L^p with $p \geq 1$, then the map $\mu : L^p \longrightarrow \mathbb{R}$ sending Φ into $\int_\Omega \Phi \, d^3X$ is continuous, thus analytic by linearity.

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