Splitting strings and chains

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Abstract: We review a study of the semiclassical decay of macroscopic spinning strings in $\mathrm{AdS}_5 \times S^5$ as well as its dual gauge theory description. The conservation of the infinite tower of commuting charges in the semiclassical string σ-model description of the process suggests that the decay channel of maximal probability should preserve integrability in the gauge theory.

1 Introduction

Probing the AdS/CFT correspondence in the regime where the string coupling constant g_s is non-vanishing is obviously a relevant task. It identifies string splitting and joining interactions with non-planar diagrams in the dual gauge theory. In recent studies of the correspondence it has proved very fruitful to consider a limit of large quantum numbers in both theories, enabling detailed comparisons. The celebrated Berenstein, Maldacena and Nastase limit [1] considers the sector in which one angular momentum J_1 on the five sphere becomes large. Here quantitative control on the interacting string sector is available [2].

In [3] we addressed the question whether this control over the non-planar gauge theory/interacting string sector could be extended to the situation where two angular momenta on the five sphere J_1 and J_2 become large. In the free string situation this limit corresponds to large, macroscopic spinning strings in $\mathrm{AdS}_5 \times S^5$. The energies of these strings and the anomalous dimensions of the dual gauge theory operators agree in leading loop orders for the planar gauge theory, but not much has been done in th non-planar sector.

The central question addressed in our work [3] is what can be said about $g_s \neq 0$ effects for large spinning strings. Although the quantum computation on $\mathrm{AdS}_5 \times S^5$ cannot be done at present, it is possible to analyze the decay semi-classically. In *flat* space-time, the semi-classical decay of macroscopic strings was analyzed in detail by Iengo and Russo [4]. In the semi-classical approach, one starts with a classical, rotating closed string solution. At a given time $\tau = 0$, the string can spontaneously split if two points σ and σ' on the string coincide in target space, and if their velocities agree. The string described by these boundary conditions, $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma')$ and $\dot{X}^\mu(\tau,\sigma) = \dot{X}^\mu(\tau,\sigma')$, then forms a "figure eight". The splitting is realized by declaring that from $\tau = 0$ onward, each of the two string pieces ("left and right" from the overlapping point), *separately* satisfy periodic boundary conditions. The initial conditions on the positions and velocities of

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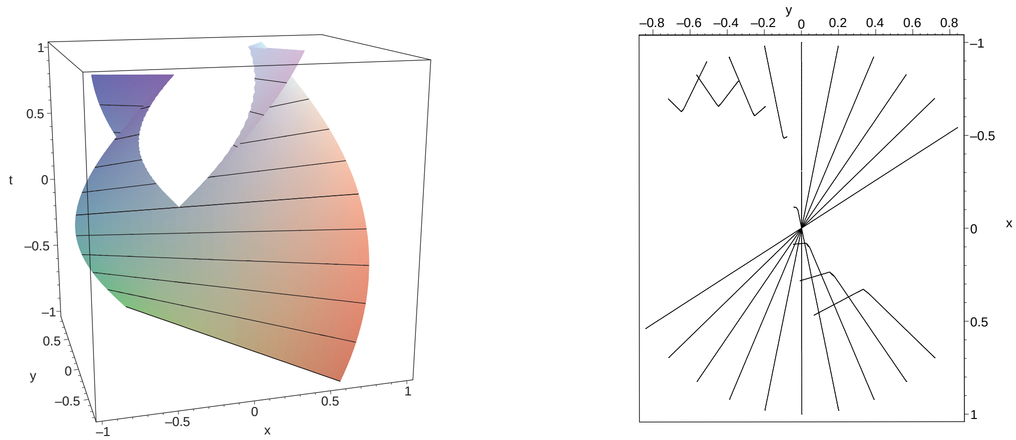


Figure 1: Semi-classical decay of a folded, rotating string in flat space-time, following [4]. The plot on the right shows snapshots at various values of τ. The outgoing pieces exhibit kinks, which propagate outward along the strings. New momenta $P^I_x = -P^{II}_x$ are generated in the decay process.

the outgoing pieces are simply taken to be those of the incoming string at the moment of splitting. The effect of the splitting propagates with the speed of light along the outgoing pieces, leading to kink-like shapes (see figure 1).

The relations between the energies and angular momenta of the outgoing strings are determined completely by conservation laws, i.e. one does not need to derive the explicit string shapes in order to obtain these relations. From the relations between the charges one can then produce a curve in, for instance, the plane spanned by the masses M_I and M_{II} of the outgoing string pieces. In flat space-time, this curve can be compared with a *full quantum* string computation of the decay rate. It has been shown that the quantum decay rate, as a function of the outgoing masses, reaches its maximum very close to the curve obtained from the classical analysis (see figure 2).

In the present paper we will review the analysis of the decay of semi-classical strings on $\text{AdS}_5 \times S^5$ presented in [3]. The goal is to produce predictions which can in principle be verified on the gauge theory side. We will focus on the folded string which is rotating on the S^5 factor of the background [5].

2 String splitting in AdS/CFT

Let us consider the spontaneous splitting of the solution of Frolov and Tseytlin [5]. The two-spin string solution is given by the equations

$$t = \kappa\tau\,, \quad \rho = 0\,, \quad \gamma = \frac{\pi}{2}\,, \quad \varphi_3 = 0\,, \quad \varphi_1 = w_1\tau\,, \quad \varphi_2 = w_2\tau\,, \quad \psi = \psi(\sigma)\,, \qquad (1)$$

where κ, w_1 and w_2 are constants. The equation which determines the profile of $\psi(\sigma)$ is

$$\psi'^2 = w_{21}^2(\sin^2\psi_0 - \sin^2\psi)\,, \quad w_{21}^2 \equiv w_2^2 - w_1^2 \geq 0\,. \qquad (2)$$

Here the constant ψ_0 corresponds to the target-space length of the folded string. The charges carried by the string are given by

$$\mathcal{E} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\dot{X}^0\,, \quad \mathcal{J}_{ij} = \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}(X_i\dot{X}_j - X_j\dot{X}_i)\,, \quad (i,j = 1\cdots 4)\,. \qquad (3)$$

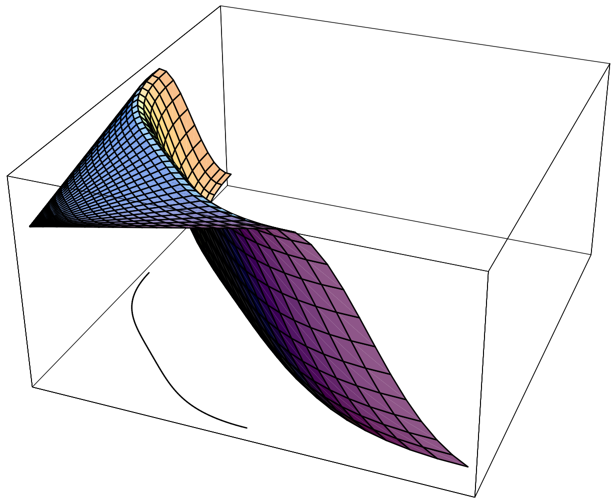


Figure 2: Sketch of the relation between the semi-classical and the full quantum calculations. The surface depicts the quantum decay amplitude over the (horizontal) plane spanned by the mass-square of the two outgoing strings, $(M_I)^2$ and $(M_{II})^2$. The amplitude reaches its maximum over the curve allowed by semi-classical decay.

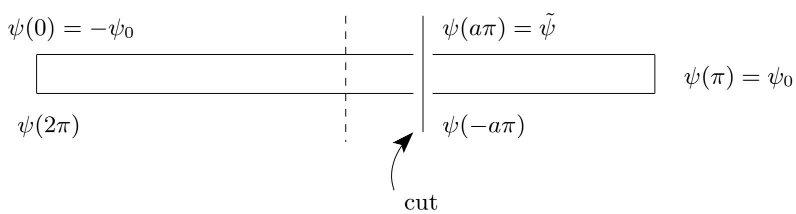
Before the decay, these charges evaluate to

$$\mathcal{E} = \sqrt{w_2^2 \sin^2\psi_0 + w_1^2 \cos^2\psi_0}\,, \quad \mathcal{J}_{12} = \frac{2\omega_1}{\pi\,\omega_{21}} E(q)\,, \quad \mathcal{J}_{34} = \frac{2\omega_2}{\pi\,\omega_{21}} (K(q) - E(q))\,, \tag{4}$$

where we defined $q \equiv \sin^2\psi_0$. The parameters ω_1 and ω_2 have no analogue on the gauge theory side, but they can be eliminated completely, producing a relation between the physical quantities,

$$\mathcal{E} = \mathcal{J}\,\mathcal{E}_0(\alpha) + \frac{\mathcal{E}_1(\alpha)}{\mathcal{J}} + \frac{\mathcal{E}_2(\alpha)}{\mathcal{J}^3} + \ldots\,, \tag{5}$$

where the coefficients $\mathcal{E}_i$ are explicitly computable functions, which depend on the filling fraction $\alpha = \mathcal{J}_{34}/\mathcal{J}$. Let us now consider the splitting process. We choose a parameterization on the world-sheet of the string which is depicted in the figure below,



The charges of the strings *after* the decay can be obtained by using the solution of the string *before* the decay, but integrating the charge densities over the lengths of each piece of string separately (i.e. $\sigma \in [-\pi a, \pi a]$ for the first piece and $\sigma \in [-\pi, -\pi a] \cup [\pi a, \pi]$ for the second one). This is consistent, as the initial conditions generated from the unsplit solution for the outgoing two string pieces are consistent, i.e. obey the Virasoro constraint.

The energies get distributed simply according to

$$\mathcal{E}^I = \kappa\, a\,, \qquad \mathcal{E}^{II} = \kappa\,(1-a)\,, \tag{6}$$

while the angular momenta J_{12} and J_{34} get distributed between the outgoing string pieces I and II according to (similar expressions hold for piece II; see [3])

$$\mathcal{J}_{12}^I = \frac{\omega_1}{\pi\omega_{21}} \left(E(q) + E(x;q)\right) , \qquad \mathcal{J}_{34}^I = \frac{\omega_2}{\pi\omega_{21}} \left(K(q) - E(q) + F(x;q) - E(x;q)\right) \quad (7)$$

where $x := \arcsin(\frac{\sin\tilde{\psi}}{\sin\psi_0})$. Some of the remaining angular momenta which vanish before the split now become non-zero for the outgoing strings,

$$\mathcal{J}_{14}^I = -\mathcal{J}_{14}^{II} = -\frac{w_2}{\pi w_{21}}\sqrt{\sin^2\psi_0 - \sin^2\tilde{\psi}}\,, \quad \mathcal{J}_{23}^I = -\mathcal{J}_{23}^{II} = \frac{w_1}{\pi w_{21}}\sqrt{\sin^2\psi_0 - \sin^2\tilde{\psi}}\,. \quad (8)$$

The sum of each of these momenta is zero in accordance with the conservation laws.

3 Invariant physical data

The goal now is to eliminate the parameters x and q related to the splitting point and initial string length, and express all conserved charges in terms of a minimal set of independent ones. The split introduces only one extra free parameter, namely the point x at which the string splits, while the number of measurable charges doubles: α^I,α^{II}, $\mathcal{J}^I$ and $\mathcal{J}^{II}$. Hence after the split, the number of dependent quantities, as well as the number of functional relations between them (which should be compared to the gauge theory) is larger.

The first functional relation we want to establish is the relation between the two angular momenta carried by the first part of the string,

$$\beta_{12} := \frac{\mathcal{J}_{12}^I}{\mathcal{J}_{12}}\,, \qquad \beta_{34} := \frac{\mathcal{J}_{34}^I}{\mathcal{J}_{34}}\,, \qquad \text{with} \qquad \mathcal{J} := \underbrace{\mathcal{J}_{12}^I + \mathcal{J}_{12}^{II}}_{=:\mathcal{J}_{12}} + \underbrace{\mathcal{J}_{34}^I + \mathcal{J}_{34}^{II}}_{=:\mathcal{J}_{34}}\,. \qquad (9)$$

Combining equations (7) with equations (4) one deduces that

$$\beta_{12} = \frac{1}{2}\left(1 + \frac{E(x;q)}{E(q)}\right), \quad \beta_{34} = \frac{1}{2}\left(1 + \frac{F(x;q) - E(x,q)}{K(q) - E(q)}\right). \qquad (10)$$

The parameter q appearing in these equations is determined by the unsplit string. However, the splitting point x should now be eliminated by a combination of global charges of the outgoing strings. We decided to choose β_{12} as the new free physical parameter of the splitting process. Using an expansion of x in $1/\mathcal{J}^2$,

$$x = x_0 + \frac{x_1}{\mathcal{J}^2} + \frac{x_2}{\mathcal{J}^4} + \dots\,, \qquad (11)$$

one can find the coefficients $\beta_{12}(x_0, q_0)$ and $x_1(x_0, q_0, q_1)$. Substituting the expansion for q and x in the second equation of (10), one is left with the functional relation $\beta_{34} = \beta_{34}(\beta_{12}, \alpha, \mathcal{J})$, given as a series in $1/\mathcal{J}$. See figure 3a.

One might wonder whether from the gauge-theory perspective it makes sense for the splitting parameter x and the outgoing angular momentum fraction β_{34} to be dependent on $\mathcal{J}$. After all, the splitting Hamiltonian commutes with the R-charge operators $\mathcal{J}_{12}$ and $\mathcal{J}_{34}$. Hence, going up higher in perturbation theory should not induce coupling-constant dependent modifications to the R-charges of the outgoing strings. However, the semi-classical string calculation captures only a part (namely the maximum) of the

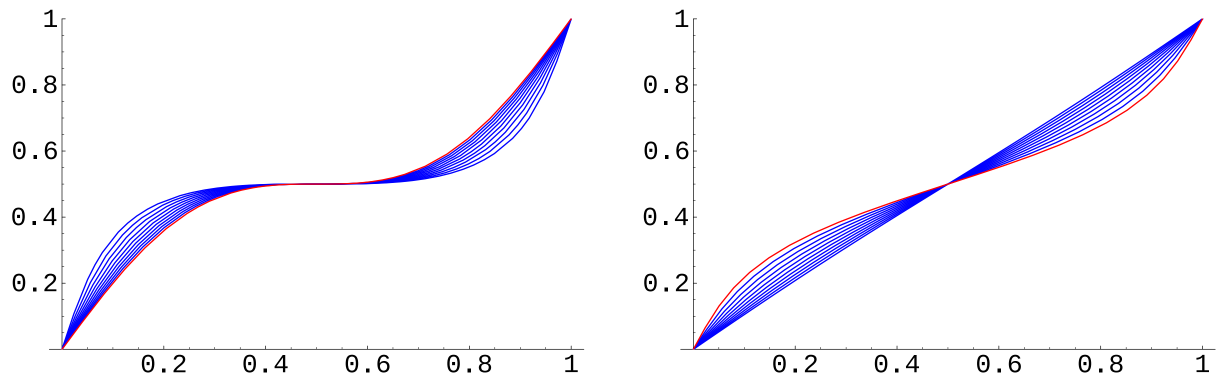


Figure 3: Figure on the left: plot of the relation between β_{12} (horizontal) and β_{34}^0 (vertical) as defined in (9). The various curves correspond to various values for the filling fraction $\alpha \in [0.05, \ldots 0.5]$. Note the symmetry with respect to the point $(0.5, 0.5)$ as a consequence of the geometry of the folded string $(\psi(\sigma) = \psi(\pi - \sigma))$. The point $(1, 1)$ corresponds to the unsplit string. Figure on the right: the energy $\mathcal{E}_0^I$ of the first outgoing string as a function of β_{12}. The straight line corresponds to $\alpha = 0.5$.

full quantum surface of the decay process. The position of the maximum varies as we go higher up in perturbation theory. At each order in perturbation theory, the most probable outgoing string with fixed $\mathcal{J}_{12}^I$ is carrying a different $\mathcal{J}_{34}^I$. This effectively means that the maximal probability varies with $\mathcal{J}$.

The second functional relation we want to obtain is a relation between the energy of the first outgoing piece $\mathcal{E}^I$ and the parameters $(\mathcal{J}, \alpha, \beta_{12})$. Following similar steps as in the previous case, it is easy to derive an expression for $\mathcal{E}^{I/II}$,

$$\mathcal{E}^{I/II} = \mathcal{J}\, \mathcal{E}_0^{I/II} + \mathcal{E}_1^{I/II}\, \frac{1}{\mathcal{J}} + \ldots \, . \tag{12}$$

The first coefficient in the expansion is given by

$$\mathcal{E}_0^I = \frac{\mathcal{J}_{12}^I}{\mathcal{J}} + \frac{\mathcal{J}_{34}^{I,0}}{\mathcal{J}} = (1-\alpha)\,\beta_{12} + \alpha\,\beta_{34}^0 \,, \tag{13}$$

and is in agreement with the (trivial) gauge theory prediction: the two decay products (single trace operators) have engineering dimensions J_{12}^I and $J_{34}^{I,0}$. In figure 3b we plot the energy of the first string piece as a function of β_{12}, for various filling fractions. The coefficient at order $1/\mathcal{J}$ of (12) can be obtained as well, and yields a prediction of the anomalous dimension at one loop of the first decay product (single trace operator) in the dual gauge theory. Further relations can be found in [3].

Thus far we have only discussed the behavior of the string energy and angular momenta under the decay process. However, the classical string sigma model is known to possess an infinite number of local, conserved and commuting charges Q_n due to its integrability [6, 7, 8]. These were written down explicitly in the work of [9] for the folded string solution in terms of a generating functional. On the other hand, one does not expect the string sigma model to remain integrable once string interactions are included (i.e. when $g_s \neq 0$). This may be seen explicitly from the dual gauge theory side: non-planar graphs break the integrability of the planar theory. Nevertheless it is obvious that, for the semi-classical decay process we are studying here, the higher charges Q_n *are* conserved. This conservation follows from the same logic that was used for the calculation of the energy and angular momenta. If the initial charges are given via a charge density as $Q_n =$

$\int \mathrm{d}\sigma\, q_n(\sigma,\tau)$, then the charges of the outgoing strings after the split are simply

$$Q_n^I = \int_0^{2\pi a} \mathrm{d}\sigma\, q_n(\sigma,\tau)\,, \quad Q_n^{II} = Q_n - Q^I\,. \tag{14}$$

Here one uses the charge densities $q_n(\sigma,\tau)$ *before* the split. Generating functional commuting charges of the outgoing strings have been explicitly computed in [3].

How is this result to be reconciled with the breakdown of integrability at $g_s \neq 0$? Again we need to remember that the quantum string decay leads to a full surface of possible decay channels, which generically will not preserve the charges beyond Q_2. A subset of channels will, however, preserve all Q_n. It is precisely this subsector which should capture the semiclassical string decay analyzed in the previous subsections and is expected to dominate the decay amplitude.

4 Splitting processes in the dual gauge theory

Let us now turn to the discussion of the splitting process in the dual gauge theory. In the large-N limit, the dilatation operator of $\mathcal{N}=4$ super-Yang-Mills factorizes as the product of a universal space-time dependent factor times a combinatorial factor acting on the fields inside composite operators. The string splitting vertex is encoded in the non-planar piece of this dilatation operator. In the relevant SU(2) sector of two chiral complex scalar adjoint fields Z and W the (space-time independent part of the) dilatation operator is known to be [3, 10]

$$D_2 = -\frac{g_{\mathrm{YM}}^2}{8\pi^2}\,\mathrm{Tr}[Z,W][\check{Z},\check{W}]\,, \tag{15}$$

where $\check{Z}_{ab} := \delta/\delta Z_{ba}$ is the matrix derivative. The action of this operator can be expressed in the language of spin chains, by considering the action of D_2 on two fields in an arbitrary single trace operator $\mathrm{Tr}(WAZB)$. One finds

$$D_2 \circ \mathrm{Tr}(WAZB) = \\ \frac{g_{\mathrm{YM}}^2}{8\pi^2}\,\mathrm{Tr}\,A\Big(\mathrm{Tr}(WZB)-\mathrm{Tr}(ZWB)\Big) + \frac{g_{\mathrm{YM}}^2}{8\pi^2}\,\mathrm{Tr}\,B\Big(\mathrm{Tr}(ZWA)-\mathrm{Tr}(WZA)\Big)\,. \tag{16}$$

The planar (nearest neighbor) contribution is obtained when A is the identity operator, leading to the Heisenberg $\mathrm{XXX}_{1/2}$ model [11], and the remaining part forms the splitting Hamiltonian,

$$D_2^{\mathrm{planar}} = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\sum_{i=1}^{L}(\delta_{i,i+1}-P_{i,i+1})\,, \qquad D_2^{\mathrm{splitting}} = \frac{g_{\mathrm{YM}}^2}{8\pi^2}\sum_{i,j}(\delta_{i,j}-P_{i,j})\,\mathcal{S}_{ij}\,, \tag{17}$$

with $P_{i,j}$ the permutation operator permuting the fields (spins) at sites i and j. The splitting operator $\mathcal{S}_{ij}$ acts in a somewhat complicated way on the sites i and j [3]. That is, we have a Heisenberg exchange interaction multiplied by a chain splitting operation.

While the dilatation operator is thus under control, the initial gauge theory operator dual to the single folded string solution with angular momenta $\mathcal{J}_{12}$ and $\mathcal{J}_{34}$ is less understood. The dual gauge operator may be written as

$$\mathrm{Tr}(Z^{\mathcal{J}_{12}}W^{\mathcal{J}_{34}}) + \ldots \tag{18}$$

where the dots stand for suitable permutations of the Z and W's – which are of essential importance for the evaluation of decay amplitudes! The spin chain picture has proved to be very efficient for the task of diagonalizing $D_2^{\rm planar}$ for long operators ($\mathcal{J} \to \infty$) with the technology of the Bethe ansatz. This technology allows one to find energy eigenvalues,

$$D_2^{\rm planar}\,|\psi\rangle = \frac{g_{\rm YM}^2 N}{2\pi^2} \sum_{i=1}^{\mathcal{J}_{34}} \sin^2\left(\frac{p_i}{2}\right)\,|\psi\rangle\,. \tag{19}$$

where p_i are the quasi-momenta. For this problem, one does not have to write down the eigenstate. For the splitting process, however, one would need this state explicitly. Denote by $|\{m_1, m_2, \ldots, m_{J_{34}}\}\rangle_L$ the single trace operator of length L with W's appearing at positions m_i. The eigenstate is then [12],

$$|\psi\rangle = \tag{20}$$
$$\sum_{\substack{1\le m_1<m_2<\ldots\\ \ldots<m_{\mathcal{J}_{34}}\le L}} \sum_{\mathcal{P}\in {\rm Perm}_{\mathcal{J}_{34}}} \exp\Big[i\sum_{i=1}^{\mathcal{J}_{34}} p_{\mathcal{P}(i)}\cdot m_i + \frac{i}{2}\sum_{i<j}^{\mathcal{J}_{34}} \varphi_{\mathcal{P}(i),\mathcal{P}(j)}\Big]\,\Big|\{m_1, m_2, \ldots, m_{\mathcal{J}_{34}}\}\Big\rangle_L$$

φ_{ij} are the scattering phases respectively, and the second sum is over all $\mathcal{J}_{34}!$ permutations of the labels $\{1, 2, 3, \ldots, \mathcal{J}_{34}\}$. In order to make contact to our semiclassical string considerations we need to take the thermodynamic limit $L, \mathcal{J}_{34} \to \infty$ with $\mathcal{J}_{34}/L = \alpha$ fixed. Due to the unknown structure of the continuum limit of the permutation group the Bethe wave function (not to mention the action of the splitting Hamiltonian) becomes a monstrous object in this limit. This is in stark contrast to the Bethe equations, which actually simplify in the same limit. This is the core of the problem which hampers a direct analytic computation of the splitting in the gauge theory. In principle one could attempt to address this problem numerically. Here however, one faces technical limitations, as the minimal length of the spin chain for which distinguishable structures limiting to the continuum folded string configuration start to emerge is 26 (with half filling fraction) [13]. The corresponding wave function $|\psi\rangle$ contains roughly $4 \cdot 10^5$ terms, many of which have coefficients of the same order.

There are two key properties which one *can* verify in the dual gauge theory, or examine in some detail in certain toy calculations which exemplify the general logic of the quantum decay [3]. The first one concerns the SU(2) structure of the decay products. This symmetry is realized through the operators

$$J_z \equiv J_{12} - J_{34} = {\rm Tr}(W\check{W} - Z\check{Z})\,, \qquad J_+ = {\rm Tr}(W\check{Z})\,, \qquad J_- = {\rm Tr}(Z\check{W})\,. \tag{21}$$

The total spin and the J_z charge of a given initial state is conserved in the decay process. However, a highest-weight state will generically not decay into the product of two highest-weight states: from the semi-classical calculation we see that the decay products are not rigid, but turn on an infinite number of modes. The dual statement is that the "decay products" in the gauge theory are no longer highest-weight states. The second property concerns the higher local charges of the Heisenberg $\text{XXX}_{1/2}$ chain [14]. These higher charges are generically not preserved in this decay process. However, in the thermodynamic limit we expect the decay to be dominated by the channels which *do* preserve all higher charges.

5 Outlook

We have reviewed the computation of the semi-classical decay of strings in $\mathrm{AdS}_5 \times S^5$ and the formalism for the dual gauge theory computation [3]. The complexity of the Bethe wave function is the main obstacle against making a direct comparison. One possible simplification can perhaps be obtained by using the coherent state wave function. However, a potential problem in this approach seems to arise from the inability to write down wave functions for the outgoing strings. An additional guideline for a better analytic understanding is the existence of the higher local charges. The decay channels in which these charges are conserved are expected to correspond to semi-classical decay, and form only a small subsector of all possible channels.

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