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A black hole mass threshold from non-singular quantum gravitational collapse

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Quantum gravity is expected to remove the classical singularity that arises as the end-state of gravitational collapse. To investigate this, we work with a simple toy model of a collapsing homogeneous scalar field. We show that non-perturbative semi-classical effects of Loop Quantum Gravity cause a bounce and remove the classical black hole singularity. Furthermore, we find a critical threshold scale, below which no horizon forms – quantum gravity may exclude very small astrophysical black holes.

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Singularity formation during gravitational collapse signals the breakdown of classical general relativity. In a more complete theory of quantum gravity the singularity should be removed. However, a satisfactory quantum gravity theory has yet to be developed. In addition, the dynamics of general collapse is very complicated. Thus we can only expect to make partial progress in tackling the problem. We need a candidate quantum gravity theory and a collapse model that is simple enough to be tractable.

A non-perturbative approach to quantizing gravity is loop quantum gravity or quantum geometry [1], whose successes include prediction of the Bekenstein-Hawking entropy formula [2]. When applied to the early universe, loop quantum effects can remove the big bang singularity [3]. A natural question is: do these effects also remove the black hole singularity as the end-state of collapse? Answering this question in full generality is not currently feasible, since techniques to handle inhomogeneous dynamical systems are still under development – and quantum gravity corrections are further expected to introduce some non-static effects. We thus consider a simple toy model of a collapsing homogeneous massless scalar field. Classically, this model always produces a black hole, but we show that loop quantum effects change this situation dramatically.

Since we do not yet know the semi-classical non-perturbative effects in the inhomogeneous case, we are unable to extend our analysis fully from the interior to the exterior. However, constraints are imposed through the matching conditions, so that quantum effects can be carried into the exterior partially and indirectly. The collapsing homogeneous scalar field cannot be matched to a Schwarzschild exterior because the pressure does not vanish at the boundary. But in any case, we expect that quantum effects will include non-stationary corrections (which should be negligible for large black holes). So it is reasonable to match the interior to a non-stationary spherically symmetric exterior. The generalized Vaidya metric provides a reasonable starting point. It is sufficiently general to allow for a broad range of behavior, including non-stationary radiative effects.

We first review the classical collapse, and show the inevitable existence of a black hole singularity covered by a horizon, for any initial mass. The interior metric is

$$ds^2 = -dt^2 + a(t)^2(1+r^2/4)^{-2}\left[dr^2 + r^2 d\Omega^2\right] , \quad (1)$$

and the massless scalar field $\phi(t)$ has pressure and energy density $p = \rho = \frac{1}{2}\dot{\phi}^2$. The Friedmann equation is

$$\dot{a}^2/a^2 = 4\pi\ell_{\rm p}^2\,\dot{\phi}^2/3 - 1/a^2\,. \quad (2)$$

The Klein-Gordon equation, $a\ddot{\phi} + 3\dot{a}\,\dot{\phi} = 0$, has solution

$$\dot{\phi} = L/a^3\,, \quad (3)$$

where L is a length scale associated with the maximal size of the collapse region, since the Friedmann equation (2) implies

$$a \le a_{\rm m} \equiv (4\pi/3)^{1/4}\sqrt{\ell_{\rm p}L}\,. \quad (4)$$

At the singularity $a \to 0$, we have $\dot{\phi}, \rho \to \infty$. The solution of the Friedmann equation is

$$t - t_0 = a_{\rm m}\int_{a/a_{\rm m}}^{a_0/a_{\rm m}} \frac{b^2 db}{\sqrt{1-b^4}}\,, \quad (5)$$

where $a_0 (\le a_{\rm m})$ gives the initial size of the collapse region at the initial time t_0. The singularity $a = 0$ is covered by a horizon (see below), and is reached in a finite proper time for any a_0:

$$\frac{1}{a_{\rm m}}(t_{\rm s} - t_0) < \int_0^1 \frac{db}{\sqrt{1-b^4}} = \frac{1}{\sqrt{2}}F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right), \quad (6)$$

where F is an elliptic integral of the first kind.

We now consider the non-perturbative modifications to the dynamics introduced by loop quantum gravity. The quantization scheme introduces a fundamental length scale

$$\ell_* = 0.28\sqrt{j}\,\ell_{\rm p}\,, \tag{7}$$

where $j(>1)$ is a half-integer that is freely specifiable. For $a < \ell_*$, the dynamics is increasingly different from general relativity. For $a \lesssim \ell_{\rm p}$, the continuum approximation to the spacetime geometry begins to break down, and the fully quantum gravity regime is reached. In the intermediate regime $\ell_{\rm p} \lesssim a \lesssim \ell_*$, loop quantum effects may be treated semi-classically, i.e., the spacetime metric behaves classically, while the dynamics acquires non-perturbative modifications to general relativity [4]. The non-perturbative semi-classical regime exists provided $\ell_* \gg \ell_{\rm p}$, i.e., for $j \gg 1$.

The key feature of the loop quantization scheme is the prediction that the geometrical density, $1/a^3$, does not diverge as $a \to 0$, but remains finite. The expectation values of the density operator are approximated by

$$d_j(a) = D(a)\, a^{-3}\,, \tag{8}$$

where the loop quantum correction factor is [5]

$$D(a) = (8/77)^6\, q^{3/2}\Big\{7\Big[(q+1)^{11/4} - |q-1|^{11/4}\Big] - 11q\Big[(q+1)^{7/4} - {\rm sgn}\,(q-1)|q-1|^{7/4}\Big]\Big\}^6, \tag{9}$$

with $q \equiv a^2/\ell_*^2$. In the classical limit we recover the expected behavior of the density, while the quantum regime shows a radical departure from classical behavior:

$$a \gg \ell_* \Rightarrow D \approx 1\,,\; a \ll \ell_* \Rightarrow D \approx (12/7)^6\,(a/\ell_*)^{15}. \tag{10}$$

Then d_j remains finite as $a \to 0$, unlike in conventional quantum cosmology, thus evading the problem of the big-bang singularity in a closed model [6]. Intuitively, one can think of the modified behavior as meaning that gravity, which is classically always attractive, becomes repulsive at small scales when quantized. This effect can produce a bounce where classically there would be a singularity, and can also provide a new mechanism for high-energy inflationary acceleration [7]. In the semi-classical regime (where the spectrum can be treated as continuous), d_j has a smooth transition from classical to quantum behavior, varying from a^{-3} to a^{12}.

In loop quantum gravity the Hamiltonian of a scalar field in a closed universe is

$$\mathcal{H} = a^3 V(\phi) + d_j\, P_\phi^2/2\,, \quad P_\phi = d_j^{-1}\dot\phi\,, \tag{11}$$

where P_ϕ is the momentum canonically conjugate to ϕ. This leads to a modified Friedmann equation [7, 8],

$$\frac{\dot a^2}{a^2} = \frac{8\pi\ell_{\rm p}^2}{3}\left[V(\phi) + \frac{1}{2D}\dot\phi^2\right] - \frac{1}{a^2}\,, \tag{12}$$

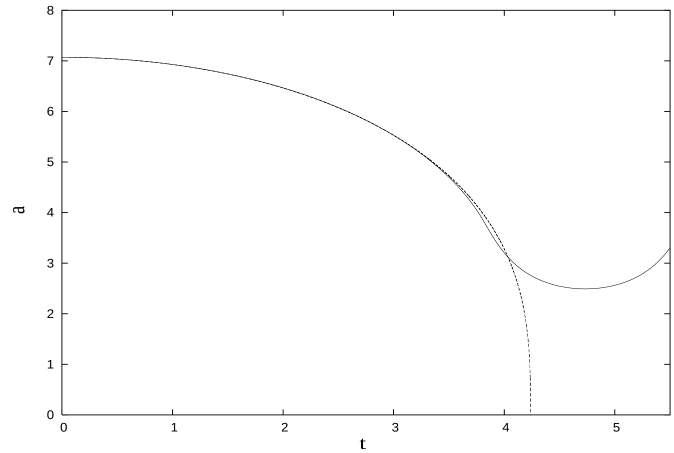

FIG. 1: The scale factor $a(t)$ of the collapsing interior, for classical (dashed) and semi-classical quantum dynamics (solid).

and a modified Klein-Gordon equation [9]

$$\ddot\phi + 3\,\frac{\dot a}{a}\,(1-\alpha)\,\dot\phi + DV(\phi) = 0\,, \quad \alpha \equiv \frac{a\dot D}{3\dot a D}\,. \tag{13}$$

For $a \ll \ell_*$, we have $\alpha \to 5$, whereas classically $D = 1$ and hence $\alpha = 0$. Thus in the semi-classical regime, $0 < \alpha \le 5$.

For a massless scalar field, $V = 0$, the solution of Eq. (13), generalizing Eq. (3), is

$$\dot\phi = L d_j(a)\,, \tag{14}$$

so that $P_\phi = L =$const. Then the Friedmann equation becomes

$$\dot a^2 + 1 = D(a)(a_{\rm m}/a)^4\,. \tag{15}$$

The energy density and pressure are modified as $\rho = \dot\phi^2/2D$, $p = \dot\phi^2\,(1-\alpha)/2D$, so that

$$w \equiv p/\rho = 1 - \alpha\,. \tag{16}$$

(The modified ρ and p satisfy the usual conservation equation if ϕ satisfies the modified Klein-Gordon equation.) Since α varies from 0 to 5 as a decreases, the $\dot\phi$ term in Eq. (13), which classically behaves as *anti-frictional* during collapse, starts to behave as *frictional* when $\alpha > 1$. Thus, contrary to classical behavior, where $\dot\phi$ increases as a decreases, in the semi-classical regime the scalar field starts slowing down with collapse. In fact at $\alpha = 2$ the magnitude of the frictional term becomes exactly equal to the classical anti-frictional term. Thereafter at smaller values of the scale factor the term becomes increasingly frictional and the collapse further slows down, and may turn around.

The point where $\alpha = 2$ is also the point beyond which the null energy condition is violated: $w < -1$, by Eq. (16). Violations of the null energy condition by quantum gravity effects are to be expected, and in loop quantum gravity this occurs for $\alpha > 2$, when the scalar field effectively behaves as a "phantom" field.

In order to see qualitatively how the non-perturbative frictional quantum effects remove the classical singularity, we assume that, over a small interval of scale factor, we can take $\alpha \approx$constant, so that $D \approx D_* \, (a/\ell_*)^{3\alpha}$, where D_* is a dimensionless constant. By Eq. (14),

$$\dot{\phi} \approx L D_* \ell_*^{-3\alpha} \, a^{3(\alpha-1)} \,, \tag{17}$$

which shows how the kinetic energy decreases with decreasing a when $\alpha > 1$, contrary to the classical case. The modified Friedmann equation (15) gives

$$\dot{a}^2 \approx \left(a_{\rm m}^4 \ell_*^{-3\alpha} D_*\right) a^{3\alpha-4} - 1 \,. \tag{18}$$

In general relativity, where $\alpha = 0$ and $D_* = 1$, this shows that for $a < a_{\rm m}$, there is no turning point in a, i.e., $\dot{a} \neq 0$. With loop quantum effects, for $\alpha > \frac{4}{3}$, the equation $\dot{a}(t_{\rm c}) = 0$ has a solution, $a_{\rm c} \approx \left(\ell_*^{3\alpha}/D_* a_{\rm m}^4\right)^{1/(3\alpha-4)} \ll a_{\rm m}$. Thus the collapse leads to a bounce and singularity avoidance. The numerical integration of the modified Friedmann and Klein-Gordon equations confirms the qualitative analysis, and the results are illustrated in Fig. 1. As is clear from the figure, the classical curve (dashed line) hits the singularity in finite time, whereas the quantum-corrected curve bounces and avoids the singularity. The key question is whether a horizon forms in the quantum-corrected collapse.

The formation or avoidance of the singularity $a = 0$ is independent of the matching to the exterior. But in order to understand horizon formation in the semi-classical quantum case, we need to impose the matching conditions. Since the pressure is nonzero at the boundary, given in comoving coordinates by $r = R =$ constant, the interior cannot be directly matched to a static Schwarzschild exterior. However we can match to an intermediate non-stationary region – for example, a generalized Vaidya region [10],

$$ds^2 = -\left[1 - 2M(v,\chi)/\chi\right] dv^2 + 2 dv d\chi + \chi^2 d\Omega^2 \,. \tag{19}$$

The usual Vaidya mass $M/\ell_{\rm p}^2$ is generalized so that $\partial M/\partial\chi$ may be nonzero. The total mass as measured by an observer at asymptotic infinity is $m = m_M + m_\phi$, where m_M is the effective total mass in the generalized Vaidya region, and $m_\phi = \int \rho dV$ is the interior mass. By Eqs. (1), (4) and (14),

$$\frac{m_\phi}{m_{\rm p}} = \frac{3a}{2\ell_{\rm p}} \left(\frac{a_{\rm m}}{a}\right)^4 D(a) \left[\tan^{-1}\frac{R}{2} - \frac{R(1-R^2/4)}{2(1+R^2/4)^2}\right]. \tag{20}$$

Since we do not specify the matter content in the exterior, and we do not know the quantum-corrected field equations there, we cannot determine $M(v,\chi)$ and thus m_M.

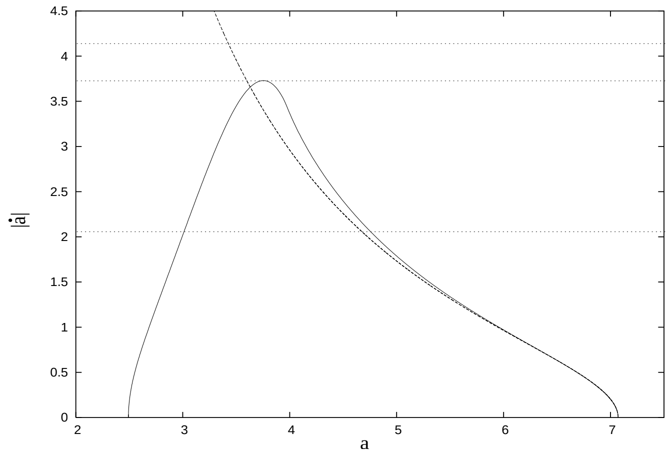

FIG. 2: The speed of collapse, $|\dot{a}|$, against the scale factor a, for the evolution shown in Fig. 1, up to the bounce. The dashed curve is for classical dynamics and semi-classical quantum dynamics gives the solid curve. The horizontal dotted lines correspond to different values of R in Eq. (25): for the upper line there is no horizon in the quantum-corrected case, the middle line corresponds to the threshold for a horizon, the lower line to the case of an inner and outer horizon.

However, as we discuss below, we can still draw qualitative conclusions about the behavior of horizons close to the matter shells.

Matching the first and second fundamental forms at the boundary, we obtain

$$\chi(v) \; = \; Ra(t)/(1+R^2/4) \,, \tag{21}$$
$$dv/dt \; = \; (1+R^2/4)/(1-R^2/4-R\dot{a}) \,, \tag{22}$$
$$2M \; = \; aR^3(\dot{a}^2+1)/(1+R^2/4)^3 \,, \tag{23}$$
$$-M_{,v} \; = \; \chi_{,vv} + (1-2M/\chi-3\chi_{,v})(M/\chi-M_{,\chi}) . \tag{24}$$

The exterior region can contain trapped surfaces when the condition $2M(v,\chi) = \chi$ is satisfied. We evaluate this condition at the matching surface, using Eqs. (21) and (23), to obtain,

$$|\dot{a}| = R^{-1}(1-R^2/4) \,. \tag{25}$$

When this value is reached, a dynamical horizon [11] intersects the matching surface. This always occurs classically since during the collapse $|\dot{a}|$ varies from zero to infinity. With the modified dynamics, however, $|\dot{a}|$ is *bounded* throughout the evolution, so that *it depends on initial conditions whether or not a horizon forms.* This is illustrated in Fig. 2. Moreover, after the bounce, $\dot{a}$ grows again, so that the condition can be satisfied a second time. This results in a picture where the bounce, replacing the classical singularity, may be shrouded by an evaporating dynamical horizon outside, as shown in Fig. 3. There will be a second point where the horizon condition is satisfied since $|\dot{a}|$ decreases between the peak of $d_j(a)$ and the bounce.

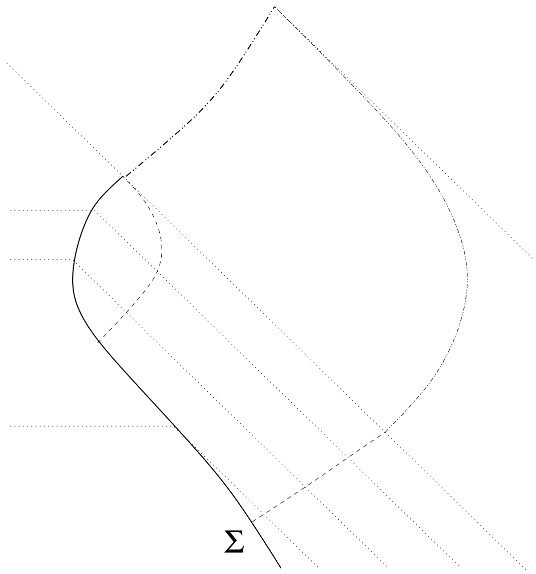

FIG. 3: Eddington-Finkelstein diagram of the collapse, with boundary Σ. Dotted lines show constant v (outside) and constant t (inside). Quantum modifications imply a bounce of the collapsing field, which for large enough mass is covered by an inner and outer evaporating horizon (dashed). A single matching suffices only until the inner horizon disappears. The dot-dash curves correspond to the subsequent evolution which is not determined in our model.

When it intersects the matching surface, the horizon is always null, as follows from Eq. (24). Its later behavior depends on the details of the outer region, which can not be determined here. Nevertheless, one can expect that both horizons will become timelike and evaporate. Horizon evaporation in this model does not only come from Hawking radiation, which may be included effectively in the outside matter content, but also from violations of energy conditions around the bounce, which may lead to effective outgoing negative energy.

The model is not able to specify the future of the system after it re-emerges out of the horizon. Equation (22) shows that dv/dt diverges if and only if $\dot{a} > 0$ and the matching surface becomes trapped. Thus, we can describe the collapse with a single matching until a horizon disappears, at which point the interior t ceases to be a good coordinate. One has to continue with a second matched region to analyze the future of the system, but this is beyond the scope of our model.

The qualitative picture that emerges from our toy model is thus the following:

• We do obtain black holes, i.e., dynamical horizons, for large masses, but they contain a bounce of the infalling matter rather than a singularity. For large mass, violations of energy conditions are initially small and the evaporation takes a long time, so that there are only small deviations from classical results.

• For small enough mass however, black holes do not form; horizons do not develop during collapse and the bounce is uncovered. The critical threshold scale for horizon formation is given by the turning point in the $|\dot{a}|$ curve. By Eqs. (9) and (15), the critical scale is

$$a_{\rm crit} = 0.987\ell_* = 0.276\sqrt{j}\,\ell_{\rm p}\,. \tag{26}$$

The corresponding threshold mass for the black hole is $m_{\rm crit} = m_M + m_\phi(a_{\rm crit})$, but we are unable to compute this mass because the exterior dynamics remains undetermined.

Our estimates may be strongly influenced by the simplifications we are forced to impose on the problem. However, the qualitative features that we have found should be robust. In particular, they mean that there could be *lower bounds on the masses of black holes that form by gravitational collapse*. In particular, this could rule out primordial black holes below the threshold mass, and thus modify estimates of the Hawking radiation effects from very small black holes.

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