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Elements of Loop Quantum Cosmology

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Abstract

The expansion of our universe, when followed backward in time, implies that it emerged from a phase of huge density, the big bang. These stages are so extreme that classical general relativity combined with matter theories is not able to describe them properly, and one has to refer to quantum gravity. A complete quantization of gravity has not yet been developed, but there are many results about key properties to be expected. When applied to cosmology, a consistent picture of the early universe arises which is free of the classical pathologies and has implications for the generation of structure which are potentially observable in the near future.

1 Introduction

General relativity provides us with an extremely successful description of the structure of our universe on large scales, with many confirmations by macroscopic experiments and so far no conflict with observations. The resulting picture, when applied to early stages of cosmology, suggests that the universe had a beginning a finite time ago, at a point where space, matter, and also time itself were created. Thus, it does not even make sense to ask what was there before since "before" does not exist at all. At very early stages, space was small such that there were huge energy densities to be diluted in the later expansion of the universe that is still experienced today. In order to explain also the structure that we see in the form of galaxies in the correct statistical distribution, the universe not only needs to expand but do so in an accelerated manner, a so-called inflationary period, in its early stages. With this additional input, usually by introducing inflation with exponential acceleration [1, 2, 3] lasting long enough to expand the scale factor $a(t)$, the radius of the universe at a given time t, by a ratio $a_{\rm final}/a_{\rm initial} > e^{60}$. The resulting seeds for structure after the inflationary phase can be observed in the anisotropy spectrum of the cosmic microwave background (most recently of the WMAP satellite [4]), which agrees well with theoretical predictions over a large range of scales.

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Nonetheless, there are problems remaining with the overall picture. The beginning was extremely violent with conditions such as diverging energy densities and tidal forces under which no theory can prevail. This is also true for general relativity itself which led to this conclusion in the first place: according to the singularity theorems any solution to general relativity, under reasonable conditions on the form of matter, must have a singularity in the past or future [5]. There, space degenerates, e.g. to a single point in cosmology, and energy densities and tidal forces diverge. From the observed expansion of our current universe one can conclude that according to general relativity there must have been such a singularity in the past (which does not rule out further possible singularities in the future). This is exactly what is usually referred to as the "beginning" of the universe, but from the discussion it is clear that the singularity does not so much present a boundary to the universe as a boundary to the classical theory: The theory predicts conditions under which it has to break down and is thus incomplete. Here it is important that the singularity in fact lies only a finite time in the past rather than an infinite distance away, which could be acceptable. A definitive conclusion about a possible beginning can therefore be reached only if a more complete theory is found which is able to describe these very early stages meaningfully.

Physically, one can understand the inevitable presence of singularities in general relativity by the characteristic property of classical gravitation being always attractive. In the backward evolution in which the universe contracts, there is, once matter has collapsed to a certain size, simply no repulsive force strong enough to prevent the total collapse into a singularity. A similar behavior happens when not all the matter in the universe but only that in a given region collapses to a small size, leading to the formation of black holes which also are singular.

This is the main problem which has to be resolved before one can call our picture of the universe complete. Moreover, there are other problematic issues in what we described so far. Inflation has to be introduced into the picture, which currently is done by assuming a special field, the inflaton, in addition to the matter we know. In contrast to other matter, its properties must be very exotic so as to ensure accelerated expansion which with Einstein's equations is possible only if there is negative pressure. This is achieved by choosing a special potential and initial conditions for the inflaton, but there is no fundamental explanation of the nature of the inflaton and its properties. Finally, there are some details in the anisotropy spectrum which are hard to bring in agreement with theoretical models. In particular, there seems to be less structure on large scales than expected, referred to as a loss of power.

2 Classical Cosmology

In classical cosmology one usually assumes space to be homogeneous and isotropic, which is an excellent approximation on large scales today. The metric of space is then solely determined by the scale factor $a(t)$ which gives the size of the universe at any given time t. The function $a(t)$ describes the expansion or contraction of space in a way dictated by the Friedmann equation [6]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho(a) \tag{1}$$

which is the reduction of Einstein's equations under the assumption of isotropy. In this equation, G is the gravitational constant and $\rho(a)$ the energy density of whatever matter we have in the universe. Once the matter content is chosen and $\rho(a)$ is known, one can solve the Friedmann equation in order to obtain $a(t)$.

As an example we consider the case of radiation which can be described phenomenologically by the energy density $\rho(a) \propto a^{-4}$. This is only a phenomenological description since it ignores the fundamental formulation of electrodynamics of the Maxwell field. Instead of using the Maxwell Hamiltonian in order to define the energy density, which would complicate the situation by introducing the electromagnetic fields with new field equations coupled to the Friedmann equation, one uses the fact that on large scales the energy density of radiation is diluted by the expansion and in addition red-shifted. This leads to a behavior proportional a^{-3} from dilution times a^{-1} from redshift. In this example we then solve the Friedmann equation $\dot{a} \propto a^{-1}$ by $a(t) \propto \sqrt{t - t_0}$ with a constant of integration t_0. This demonstrates the occurrence of singularities: For any solution there is a time $t = t_0$ where the size of space vanishes and the energy density $\rho(a(t_0))$ diverges. At this point not only the matter system becomes unphysical, but also the gravitational evolution breaks down: When the right hand side of (1) diverges at some time t_0, we cannot follow the evolution further by setting up an initial value problem there and integrating the equation. We can thus only learn that there is a singularity in the classical theory, but do not obtain any information as to what is happening there and beyond. These are the two related but not identical features of a singularity: energy densities diverge and the evolution breaks down.

One could think that the problem comes from too strong idealizations such as symmetry assumptions or the phenomenological description of matter. That this is not the case follows from the singularity theorems which do not depend on these assumptions. One can also illustrate the singularity problem with a field theoretic rather than phenomenological description of matter. For simplicity we now assume that matter is provided by a scalar ϕ whose energy density then follows from the Hamiltonian

$$\rho(a) = a^{-3} H(a) = a^{-3}\left(\tfrac{1}{2} a^{-3} p_\phi^2 + a^3 V(\phi)\right) \tag{2}$$

with the scalar momentum p_ϕ and potential $V(\phi)$. At small scale factors a, there still is a diverging factor a^{-3} in the kinetic term which we recognized as being responsible for the singularity before. Since this term dominates over the non-diverging potential term, we still cannot escape the singularity by using this more fundamental description of matter. This is true unless we manage to arrange the evolution of the scalar in such a way that $p_\phi \to 0$ when $a \to 0$ in just the right way for the kinetic term not to diverge. This is difficult to arrange in general, but is exactly what is attempted in slow-roll inflation (though with a different motivation, and not necessarily all the way up to the classical singularity).

For the evolution of p_ϕ we need the scalar equation of motion, which can be derived from the Hamiltonian H in (2) via $\dot{\phi} = \{\phi, H\}$ and $\dot{p}_\phi = \{p_\phi, H\}$. This results in the isotropic Klein–Gordon equation in a time-dependent background determined by $a(t)$,

$$\ddot{\phi} + 3\dot{a}a^{-1}\dot{\phi} + V'(\phi) = 0\,. \tag{3}$$

In an expanding space with positive $\dot{a}$ the second term implies friction such that, if we assume the potential $V'(\phi)$ to be flat enough, ϕ will change only slowly (slow-roll). Thus, $\dot{\phi}$ and $p_\phi = a^3\dot{\phi}$ are small and at least for some time we can ignore the kinetic term in the energy density. Moreover, since ϕ changes only slowly we can regard the potential $V(\phi)$ as a constant Λ which again allows us to solve the Friedmann equation with $\rho(a) = \Lambda$. The solution $a \propto \exp(\sqrt{8\pi G\Lambda/3}\,t)$ is inflationary since $\ddot{a} > 0$ and non-singular: a becomes zero only in the limit $t \to -\infty$.

Thus, we now have a mechanism to drive a phase of accelerated expansion important for observations of structure. However, this expansion must be long enough, which means that the phase of slowly rolling ϕ must be long. This can be achieved only if the potential is very flat and ϕ starts sufficiently far away from its potential minimum. Flatness means that the ratio of $V(\phi_{\text{initial}})$ and ϕ_{initial} must be of the order 10^{-10}, while ϕ_{initial} must be huge, of the order of the Planck mass [7]. These assumptions are necessary for agreement with observations, but are in need of more fundamental explanations.

Moreover, inflation alone does not solve the singularity problem [8]. The non-singular solution we just obtained was derived under the approximation that the kinetic term can be ignored when $\dot{\phi}$ is small. This is true in a certain range of a, depending on how small $\dot{\phi}$ really is, but never very close to $a = 0$. Eventually, even with slow-roll conditions, the diverging a^{-3} will dominate and lead to a singularity.

3 Quantum Gravity

For decades, quantum gravity has been expected to complete the picture which is related to well-known properties of quantum mechanics in the presence of a non-zero $\hbar$.

3.1 Indications

First, in analogy to the singularity problem in gravity, where everything falls into a singularity in finite time, there is the instability problem of a classical hydrogen atom, where the electron would fall into the nucleus after a brief time. From quantum mechanics we know how the instability problem is solved: There is a finite ground state energy $E_0 = -\frac{1}{2}me^4/\hbar^2$, implying that the electron cannot radiate away all its energy and not fall further once it reaches the ground state. From the expression for E_0 one can see that quantum theory with its non-zero $\hbar$ is essential for this to happen: When $\hbar \to 0$ in a classical limit, $E_0 \to -\infty$ which brings us back to the classical instability. One expects a similar role to be played by the Planck length $\ell_{\rm P} = \sqrt{8\pi G\hbar/c^3} \approx 10^{-35}$m which is tiny but non-zero in quantum theory. If, just for dimensional reasons, densities are bounded by $\ell_{\rm P}^{-3}$, this would be finite in quantum gravity but diverge in the classical limit.

Secondly, a classical treatment of black body radiation suggests the Rayleigh–Jeans law according to which the spectral density behaves as $\rho(\lambda) \propto \lambda^{-4}$ as a function of the wave length. This is unacceptable since the divergence at small wave lengths leads to an infinite total energy. Here, quantum mechanics solves the problem by cutting off the divergence with Planck's formula which has a maximum at a wave length $\lambda_{\rm max} \sim h/kT$ and approaches zero at smaller scales. Again, in the classical limit $\lambda_{\rm max}$ becomes zero and the expression diverges.

In cosmology the situation is similar for matter in the whole universe rather than a cavity. Energy densities as a function of the scale factor behave as, e.g., a^{-3} if matter is just diluted or a^{-4} if there is an additional redshift factor. In all cases, the energy density diverges at small scales, comparable to the Rayleigh–Jeans law. Inflation already provides an indication that the behavior must be different at small scales. Indeed, inflation can only be achieved with negative pressure, while all matter whose energy falls off as a^{-k} with non-negative k has positive pressure. This can easily be seen from the thermodynamical definition of pressure as the negative change of energy with volume. Negative pressure then requires the energy to increase with the scale factor at least at small scales where inflation is required (e.g., an energy Λa^3 for exponential inflation). This could be reconciled with standard forms of matter if there is an analog to Planck's formula, which interpolates between decreasing behavior at large scales and a behavior increasing from zero at small scales, with a maximum in between.

3.2 Early Quantum Cosmology

Since the isotropic reduction of general relativity leads to a system with finitely many degrees of freedom, one can in a first attempt try quantum mechanics to quantize it. Starting with the Friedmann equation (1) and replacing $\dot{a}$ by its momentum $p_a = 3a\dot{a}/8\pi G$ gives a Hamiltonian which is quadratic in the momentum and can be quantized easily to an operator acting on a wave function depending on the gravitational variable a and possibly matter fields ϕ. The usual Schrödinger representation yields the Wheeler–DeWitt equation [9, 10]

$$\frac{3}{2}\left(-\frac{1}{9}\ell_{\rm P}^4 a^{-1}\frac{\partial}{\partial a}a^{-1}\frac{\partial}{\partial a}\right)a\psi(a,\phi) = 8\pi G\hat{H}_\phi(a)\psi(a,\phi) \tag{4}$$

with the matter Hamiltonian $\hat{H}_\phi(a)$. This system is different from usual quantum mechanics in that there are factor ordering ambiguities in the kinetic term, and that there is no derivative with respect to coordinate time t. The latter fact is a consequence of general covariance: the Hamiltonian is a constraint equation restricting allowed states $\psi(a,\phi)$, rather than a Hamiltonian generating evolution in coordinate time. Nevertheless, one can interpret equation (4) as an evolution equation in the scale factor a, which is then called internal time. The left hand side thus becomes a second order time derivative, and it means that the evolution of matter is measured relationally with respect to the expansion or contraction of the universe, rather than absolutely in coordinate time.

Straightforward quantization thus gives us a quantum evolution equation, and we can now check what this implies for the singularity. If we look at the equation for $a = 0$, we notice first that the matter Hamiltonian still leads to diverging energy densities. If we quantize (2), we replace p_ϕ by a derivative, but the singular dependence on a does not change; a^{-3} would simply become a multiplication operator acting on the wave function. Moreover, $a = 0$ remains a singular point of the quantum evolution equation in internal time. There is nothing from the theory which tells us what physically happens at the singularity or beyond (baring intuitive pictures which have been developed from this perspective [11, 12]).

So one has to ask what went wrong with our expectations that quantizing gravity should help. The answer is that quantum theory itself did not necessarily fail, but only our simple implementation. Indeed, what we used was just quantum mechanics, while quantum gravity has many consistency conditions to be fulfilled which makes constructing it so complicated. At the time when this formalism was first applied there was in fact no corresponding full quantum theory of gravity which could have guided developments. In such a simple case as isotropic cosmology, most of these consistency conditions trivialize and one can easily overlook important issues. There are many choices in quantizing an unknown system, and tacitly making one choice can easily lead in a wrong direction.

Fortunately, the situation has changed with the development of strong candidates for quantum gravity. This then allows us to reconsider the singularity and other problems from the point of view of the full theory, making sure that also in a simpler cosmological context only those steps are undertaken that have an analog in the full theory.

3.3 Loop quantum gravity

Singularities are physically extreme and require special properties of any theory aimed at tackling them. First, there are always strong fields (classically diverging) which requires a non-perturbative treatment. Moreover, classically we expect space to degenerate at the singularity, for instance a single point in an isotropic model. This means that we cannot take the presence of a classical geometry to measure distances for granted, which is technically expressed as background independence. A non-perturbative and background independent quantization of gravity is available in the form of loop quantum gravity [13, 14, 15], which by now is understood well enough in order to be applicable in physically interesting situations.

Here, we only mention salient features of the theory which will turn out to be important for cosmology. The first one is the kind of basic variables used, which are the Ashtekar connection [16, 17] describing the curvature of space and a densitized triad describing the metric by a collection of three orthonormal vectors in each point. These variables are important since they allow a background independent representation of the theory, where the connection A_a^i is integrated to holonomies

$$h_e(A) = \mathcal{P}\exp\int_e A_a^i \tau_i \dot{e}^a \mathrm{d}t \tag{5}$$

along curves e in space and the densitized triad E^a_i to fluxes

$$F_S(E) = \int_S E^a_i \tau^i n_a \mathrm{d}^2 y \tag{6}$$

along surfaces S. (In these expressions, $\dot{e}^a$ denotes the tangent vector to a curve and n_a the conormal to a surface, both of which are defined without reference to a background metric. Moreover, $\tau_j = -\frac{1}{2} i\sigma_j$ in terms of Pauli matrices). While usual quantum field theory techniques rest on the presence of a background metric, for instance in order to decompose a field in its Fourier modes and define a vacuum state and particles, this is no longer available in quantum gravity where the metric itself must be turned into an operator. On the other hand, some integration is necessary since the fields themselves are distributional in quantum field theory and do not allow a well-defined representation. This "smearing" with respect to a background metric has to be replaced by some other integration sufficient for resulting in honest operators [18, 19]. This is achieved by the integrations in (5) and (6), which similarly lead to a well-defined quantum representation. Usual Fock spaces in perturbative quantum field theory are thereby replaced by the loop representation, where an orthonormal basis is given by spin network states [20].

This shows that choosing basic variables for a theory to quantize has implications for the resulting representation. Connections and densitized triads can naturally be smeared along curves and surfaces without using a background metric and then represented on a Hilbert space. Requiring diffeomorphism invariance, which means that a background independent theory must not change under deformations of space (which can be interpreted as changes of coordinates), even selects a unique representation [21, 22, 23, 24, 25]. These are basic properties of loop quantum gravity, recognized as important requirements for a background independent quantization. Already here we can see differences to the Wheeler–DeWitt quantization, where the metric is used as a basic variable and then quantized as in quantum mechanics. This is possible in the model but not in a full theory, and in fact we will see later that a loop quantization will give a representation inequivalent to the Wheeler–DeWitt quantization.

The basic properties of the representation have further consequences. Holonomies and fluxes act as well-defined operators, and fluxes have discrete spectra. Since spatial geometry is determined by the densitized triad, spatial geometry is discrete, too, with discrete spectra for, e.g., the area and volume operator [26, 27, 28]. The geometry of space-time is more complicated to understand since this is a dynamical situation which requires solving the Hamiltonian constraint. This is the analog of the Wheeler–DeWitt equation in the full theory and is the quantization of Einstein's dynamical equations. There are candidates for such operators [29], well-defined even in the presence of matter [30] which in usual quantum field theory would contribute divergent matter Hamiltonians. Not surprisingly, the full situation is hard to analyze, which is already the case classically, without assuming simplifications from symmetries. We will thus return to symmetric, in particular isotropic models, but with the new perspective provided by the full theory of loop quantum gravity.

4 Quantum cosmology

Symmetries can be introduced in loop quantum gravity at the level of states and basic operators [31, 32, 33], such that it is not necessary to reduce the classical theory first and then quantize as in the Wheeler–DeWitt quantization. Instead, one can view the procedure as quantizing first and then introducing symmetries which ensures that consistency conditions of quantum gravity are observed in the first step before one considers treatable situations. In particular, the quantum representation derives from symmetric states and basic operators, while the Hamiltonian constraint can be obtained with constructions analogous to those in the full theory. This allows us to reconsider the singularity problem, now with methods from full quantum gravity. In fact, symmetric models present a class of systems which can often be treated explicitly while still being representative for general phenomena. For instance, the prime examples of singular situations in gravity, and some of the most widely studied physical applications, are already obtained in isotropic or spherically symmetric systems, which allow access to cosmology and black holes.

4.1 Representation

Before discussing the quantum level we reformulate isotropic cosmology in connection and triad variables instead of a. The role of the scale factor is now played by the densitized triad component p with $|p| = a^2$ whose canonical momentum is the isotropic connection component $c = -\frac{1}{2}\dot{a}$ with $\{c, p\} = 8\pi G/3$. The main difference to metric variables is the fact that p, unlike a, can take both signs with $\mathrm{sgn}p$ being the orientation of space. This is a consequence of having to use triad variables which not only know about the size of space but also its orientation (depending on whether the set of orthonormal vectors is left or right handed).

States in the full theory are usually written in the connection representation as functions of holonomies. Following the reduction procedure for an isotropic symmetry group leads to orthonormal states which are functions of the isotropic connection component c and given by [34]

$$\langle c|\mu\rangle = e^{i\mu c/2} \qquad \mu \in \mathbb{R}\,. \tag{7}$$

On these states the basic variables p and c are represented by

$$\hat{p}|\mu\rangle = \tfrac{1}{6}\ell_{\mathrm{P}}^2\mu|\mu\rangle \tag{8}$$

$$\widehat{e^{i\mu' c/2}}|\mu\rangle = |\mu+\mu'\rangle \tag{9}$$

with the properties:

1. $[\widehat{e^{i\mu' c/2}}, \hat{p}] = -\frac{1}{6}\ell_{\mathrm{P}}^2\mu'\widehat{e^{-i\mu' c/2}} = i\hbar(\{e^{i\mu' c/2}, p\})^\wedge$,
2. $\hat{p}$ has a discrete spectrum and
3. only exponentials $e^{i\mu' c/2}$ of c are represented, not c directly.

These statements deserve further explanation: First, the classical Poisson relations between the basic variables are indeed represented correctly, turning the Poisson brackets into commutators divided by $i\hbar$. On this representation, the set of eigenvalues of $\hat{p}$ is the full real line since μ can take arbitrary real values. Nevertheless, the spectrum of $\hat{p}$ is discrete in the technical sense that eigenstates of $\hat{p}$ are normalizable. This is indeed the case in this non-separable Hilbert space where (7) defines an orthonormal basis. The last property follows since the exponentials are not continuous in the label μ', for otherwise one could simply take the derivative with respect to μ' at $\mu' = 0$ and obtain an operator for c. The discontinuity can be seen, e.g., from

$$\langle\mu|\widehat{e^{i\mu' c/2}}|\mu\rangle = \delta_{0,\mu'}$$

which is not continuous.

These properties are quite unfamiliar from quantum mechanics, and indeed the representation is inequivalent to the Schrödinger representation (the discontinuity of the c-exponential evading the Stone–von Neumann theorem which usually implies uniqueness of the representation). In fact, the loop representation is inequivalent to the Wheeler-DeWitt quantization which just assumed a Schrödinger like quantization. In view of the fact that the phase space of our system is spanned by c and p with $\{c, p\} \propto 1$ just as in classical mechanics, the question arises how such a difference in the quantum formulation arises.

As a mathematical problem the basic step of quantization occurs as follows: given the classical Poisson algebra of observables Q and P with $\{Q, P\} = 1$, how can we define a representation of the observables on a Hilbert space such that the Poisson relations become commutator relations and complex conjugation, meaning that Q and P are real, becomes adjointness? The problem is mathematically much better defined if one uses the bounded expressions e^{isQ} and $e^{it\hbar^{-1}P}$ instead of the unbounded Q and P, which still allows us to distinguish any two points in the whole phase space. The basic objects e^{isQ} and $e^{it\hbar^{-1}P}$ upon quantization will then not commute but fulfill the commutation relation (Weyl algebra)

$$e^{isQ}e^{it\hbar^{-1}P} = e^{ist}e^{it\hbar^{-1}P}e^{isQ} \tag{10}$$

as unitary operators on a Hilbert space.

In the Schrödinger representation this is done by using a Hilbert space $L^2(\mathbb{R}, \mathrm{d}q)$ of square integrable functions $\psi(q)$ with $\int_{\mathbb{R}} \mathrm{d}q|\psi(q)|^2$ finite. The representation of basic operators is

$$\begin{aligned} e^{isQ}\psi(q) &= e^{isq}\psi(q) \\ e^{it\hbar^{-1}P}\psi(q) &= \psi(q+t) \end{aligned}$$

which indeed are unitary and fulfill the required commutation relation. Moreover, the operator families as functions of s and t are continuous and we can take the derivatives in $s = 0$ and $t = 0$, respectively:

$$-i\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} e^{isQ} = q$$

$$-i\hbar \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{it\hbar^{-1}P} \quad = \quad \hat{p} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}q}\,.$$

This is the familiar representation of quantum mechanics which, according to the Stone–von Neumann theorem is unique under the condition that e^{isQ} and $e^{it\hbar^{-1}P}$ are indeed continuous in both s and t.

The latter condition is commonly taken for granted in quantum mechanics, but in general there is no underlying physical or mathematical reason. It is easy to define representations violating continuity in s or t, for instance if we use a Hilbert space $\ell^2(\mathbb{R})$ where states are again maps ψ_q from the real line to complex numbers but with norm $\sum_q |\psi_q|^2$ which implies that normalizable ψ_q can be non-zero for at most countably many q. We obtain a representation with basic operators

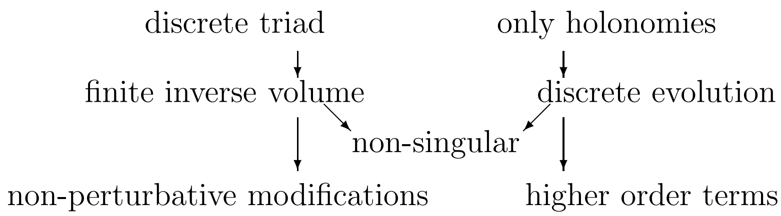
$$\begin{aligned} e^{isQ}\psi_q &= e^{isq}\psi_q \\ e^{it\hbar^{-1}P}\psi_q &= \psi_{q+t} \end{aligned}$$

which is of the same form as before. However, due to the different Hilbert space the second operator $e^{it\hbar^{-1}P}$ is no longer continuous in t which can be checked as in the case of $e^{i\mu c/2}$. In fact, the representation for Q and P is isomorphic to that of p and c used before, where a general state $|\psi\rangle = \sum_\mu \psi_\mu|\mu\rangle$ has coefficients ψ_μ in $\ell^2(\mathbb{R})$.

This explains mathematically why different, inequivalent representations are possible, but what are the physical reasons for using different representations in quantum mechanics and quantum cosmology? In quantum mechanics it turns out that the choice of representation is not that important and is mostly being done for reasons of familiarity with the standard choice. Physical differences between inequivalent representations can only occur at very high energies [35] which are not probed by available experiments and do not affect characteristic quantum effects related to the ground state or excited states. Thus, quantum mechanics as we know it can well be formulated in an inequivalent representation, and also in quantum field theory this can be done and even be useful [36].

In quantum cosmology we have a different situation where it is the high energies which are essential. We do not have direct observations of this regime, but from conceptual considerations such as the singularity issue we have learned which problems we have to face. The classical singularity leads to the highest energies one can imagine, and it is here where the question of which representation to choose becomes essential. As shown by the failure of the Wheeler–DeWitt quantization in trying to remove the singularity, the Schrödinger representation is inappropriate for quantum cosmology. The representation underlying loop quantum cosmology, on the other hand, implies very different properties which become important at high energies and can shed new light on the singularity problem.

Moreover, by design of the symmetric models as derived from the full theory, we have the same basic properties of a loop representation in cosmological models and the full situation where they were recognized as being important for a background independent quantization: discrete fluxes $\hat{F}_S(E)$ or $\hat{p}$ and a representation only of holonomies $h_e(A)$ or $e^{i\mu c/2}$ but not of connection components A_a^i or c. These basic properties have far-reaching consequences as discussed in what follows [37]:



By this reliable quantization of representative and physically interesting models with a known relation to full quantum gravity we are finally able to resolve long-standing issues such as the singularity problem.

4.2 Quantum evolution

We will first look at the quantum evolution equation which we obtain as the quantized Friedmann equation. This is modeled on the Hamiltonian constraint of the full theory such that we can also draw some conclusions for the viability of the full constraint.

4.2.1 Difference equation

The constraint equation will be imposed on states of the form $|\psi\rangle = \sum_\mu \psi_\mu |\mu\rangle$ with summation over countably many values of μ. Since the states $|\mu\rangle$ are eigenstates of the triad operator, the coefficients ψ_μ which can also depend on matter fields such as a scalar ϕ represent the state in the triad representation, analogous to $\psi(a,\phi)$ before. For the constraint operator we again need operators for the conjugate of p, related to $\dot{a}$ in the Friedmann equation. Since this is now the exponential of c, which on basis states acts by shifting the label, it translates to a finite shift in the labels of coefficients $\psi_\mu(\phi)$. Plugging together all ingredients for a quantization of (1) along the lines of the constraint in the full theory leads to the difference equation [38, 34]

$$\begin{aligned}&(V_{\mu+5}-V_{\mu+3})\psi_{\mu+4}(\phi)-2(V_{\mu+1}-V_{\mu-1})\psi_\mu(\phi)\\&+(V_{\mu-3}-V_{\mu-5})\psi_{\mu-4}(\phi)=-\tfrac{4}{3}\pi G\ell_{\mathrm{P}}^2\hat{H}_{\mathrm{matter}}(\mu)\psi_\mu(\phi)\end{aligned}\qquad(11)$$

with volume eigenvalues $V_\mu = (\ell_{\mathrm{P}}^2|\mu|/6)^{3/2}$ obtained from the volume operator $\hat{V} = |\hat{p}|^{3/2}$, and the matter Hamiltonian $\hat{H}_{\mathrm{matter}}(\mu)$.

We again have a constraint equation which does not generate evolution in coordinate time but can be seen as evolution in internal time. Instead of the continuous variable a we now have the label μ which only jumps in discrete steps. As for the singularity issue, there is a further difference to the Wheeler–DeWitt equation since now the classical singularity is located at $p = 0$ which is in the interior rather than at the boundary of the configuration space. Nevertheless, the classical evolution in the variable p breaks down at $p = 0$ and there is still a singularity. In quantum theory, however, the situation is very different: while the Wheeler–DeWitt equation does not solve the singularity problem, the difference equation (11) uniquely evolves a wave function from some initial values at positive μ, say, to negative μ. Thus, the evolution does not break down at the classical singularity and can rather be continued beyond it. Quantum gravity is thus a theory which is more complete than classical general relativity and is free of limitations set by classical singularities.

An intuitive picture of what replaces the classical singularity can be obtained from considering evolution in μ as before. For negative μ, the volume V_μ decreases with increasing μ while V_μ increases for positive μ. This leads to the picture of a collapsing universe before it reaches the classical big bang singularity and re-expands. While at large scales the classical description is good [39], when the universe is small close to the classical singularity it starts to break down and has to be replaced by discrete quantum geometry. The resulting quantum evolution does not break down, in contrast to the classical space-time picture which dissolves. Using the fact that the sign of μ, which defines the orientation of space, changes during the transition through the classical singularity one can conclude that the universe turns its inside out during the process. This can have consequences for realistic matter Hamiltonians which violate parity symmetry.

4.2.2 Meaning of the wave function

An important issue in quantum gravity which is still outstanding even in isotropic models is the interpretation of the wave function and its relation to the problem of time. In the usual interpretation of quantum mechanics the wave function determines probabilities for measurements made by an observer outside the quantum system. Quantum gravity and cosmology, however, are thought of as theories for the quantum behavior of a whole universe such that, by definition, there cannot be an observer outside the quantum system. Accordingly, the question of how to interpret the wave function in quantum cosmology is more complicated. One can avoid the separation into a classical and quantum part of the problem in quantum mechanics by the theory of decoherence which can explain how a world perceived as classical emerges from the fundamental quantum description [40]. The degree of "classicality" is related to the number of degrees of freedom which do not contribute significantly to the evolution but interact with the system nonetheless. Averaging over those degrees of freedom, provided there are enough of them, then leads to a classical picture. This demonstrates why macroscopic bodies under usual circumstances are perceived as classical while in the microscopic world, where a small number of degrees of freedom is sufficient to capture crucial properties of a system, quantum mechanics prevails. This idea has been adapted to cosmology, where a large universe comes with many degrees of freedom such as small inhomogeneities which are not of much relevance for the overall evolution. This is different, however, in a small universe where quantum behavior becomes dominant.

Thus, one can avoid the presence of an observer outside the quantum system. The quantum system is described by its wave function, and in some circumstances one can approximate the evolution by a quantum part being looked at by classical observers within the same system. Properties are then encoded in a relational way: the wave function of the whole system contains information about everything including possible observers. Now, the question has shifted from a conceptual one — how to describe the system if no outside observers can be available — to a technical one. One needs to understand how information can be extracted from the wave function and used to develop schemes for intuitive pictures or potentially observable effects. This is particularly pressing in the very early universe where everything including what we usually know as space and time are quantum and no familiar background to lean on is left.

One lesson is that evolution should be thought of as relational by determining probabilities for one degree of freedom under the condition that another degree of freedom has a certain value. If the reference degree of freedom (such as the direction of the hand of a clock) plays a distinguished role for determining the evolution of others, it is called internal time: it is not an absolute time outside the quantum system as in quantum mechanics, and not a coordinate time as in general relativity which could be changed by coordinate transformations. Rather, it is one of the physical degrees of freedom whose evolution is determined by the dynamical laws and which shows how other degrees of freedom change by interacting with them. From this picture it is clear that no external observer is necessary to read off the clock or other measurement devices, such that it is ideally suited to cosmology. What is also clear is that now internal time depends on what we choose it to be, and different questions require different choices. For a lab experiment the hand of a clock would be a good internal time and, when the clock is sufficiently isolated from the physical fields used in the experiment and other outside influence, will not be different from an absolute time except that it is mathematically more complicated to describe. The same clock, on the other hand, will not be good for describing the universe when we imagine to approach a classical singularity. It will simply not withstand the extreme physical conditions, dissolve, and its parts will behave in a complicated irregular manner ill-suited for the description of evolution. Instead, one has to use more global objects which depend on what is going on in the whole universe.

Close to a classical singularity, where one expects monotonic expansion or contraction, the spatial volume of the universe is just the right quantity as internal time. A wave function then determines relationally how matter fields or other gravitational degrees of freedom change with respect to the expansion or contraction of the universe. In our case, this is encoded in the wave function $\psi_\mu(\phi)$ depending on internal time μ (which through the volume defines the size of the universe but also spatial orientation) and matter fields ϕ. By showing that it is subject to a difference equation in μ which does not stop at the classical singularity $\mu = 0$ we have seen that relational probabilities are defined for all internal times without breaking down anywhere. This shows the absence of singularities and allows developing intuitive pictures, but does not make detailed predictions before relational probabilities are indeed computed and shown how to be observable at least in principle.

Here, we encounter the main issue in the role of the wave function: we have a relational scheme to understand what the wave function should mean but the probability measure to be used, called the physical inner product, is not known so far. We already used a Hilbert space which we needed to define the basic operators and the quantized Hamiltonian constraint, where wave functions ψ_μ, which by definition are non-zero for at most countably many values $\mu \in \mathbb{R}$, have the inner product $\langle\psi|\psi'\rangle = \sum_\mu \bar{\psi}_\mu \psi'_\mu$. This is called the kinematical inner product which is used for setting up the quantum theory. But unlike in quantum mechanics where the kinematical inner product is also used as physical inner product for the probability interpretation of the wave function, in quantum gravity the physical inner product must be expected to be different. This occurs because the quantum evolution equation (11) in internal time is a constraint equation rather than an evolution equation in an external absolute time parameter. Solutions to this constraint in general are not normalizable in the kinematical inner product such that a new physical inner product on the solution space has to be found. There are detailed schemes for a derivation, but despite some progress [41, 42] they are difficult to apply even in isotropic cosmological models and research is still ongoing. An alternative route to extract physical statements will be discussed in Sec. 4.4 together with the main results.

A related issue, which is also of relevance for the classical limit of the theory is that of oscillations on small scales of the wave function. Being subject to a difference equation means that ψ_μ is not necessarily smooth but can change rapidly when μ changes by a small amount even when the volume is large. In such a regime one expects classical behavior, but small scale oscillations imply that the wave function is sensitive to the Planck scale. There are also other issues related to the fact that now a difference rather than differential equation provides the fundamental law [43]. Before the physical inner product is known one cannot say if these oscillations would imply any effect observable today, but one can still study the mathematical problem of if and when solutions with suppressed oscillations exist. This is easy to answer in the affirmative for isotropic models subject to (11) where in some cases one even obtains a unique wave function [44, 45]. However, already in other homogeneous but anisotropic models the issue is much more complicated to analyze [46, 47].

In a more general situation than homogeneous cosmology there is an additional complication even if the physical inner product would be known. In general, it is very difficult to find an internal time to capture the evolution of a complicated quantum system, which is called the problem of time in general relativity. In cosmology, the volume is a good internal time to understand the singularity, but it would not be good for the whole history if the universe experiences a recollapse where the volume would not be monotonic. This is even more complicated in inhomogeneous situations such as the collapse of matter into a black hole. Since we used internal time μ to show how quantum geometry evolves through the classical singularity, it seems that the singularity problem in general cannot be solved before the problem of time is understood. Fortunately, while the availability of an internal time simplifies the analysis, requirements on a good choice can be relaxed for the singularity problem. An internal time provides us with an interpretation of the constraint equation as an evolution equation, but the singularity problem can be phrased independently of this as the problem to extend the wave function on the space of metrics or triads. This implies weaker requirements and also situations can be analyzed where no internal time is known. The task then is to find conditions which characterize a classical singularity, analogous to $p = 0$ in isotropic cosmology, and find an evolution parameter which at least in individual parts of an inhomogeneous singularity allows to see how the system can move through it. Inhomogeneous cases are now under study but only partially understood so far, such that in the next section we return to isotropic cosmology.

4.3 Densities

In the previous discussion we have not yet mentioned the matter Hamiltonian on the right hand side, which diverges classically and in the Wheeler–DeWitt quantization when we reach the singularity. If this were the case here, the discrete quantum evolution would break down, too. However, as we will see now the matter Hamiltonian does not diverge, which is again a consequence of the loop representation.

4.3.1 Quantization

For the matter Hamiltonian we need to quantize the matter field and in quantum gravity also coefficients such as a^{-3} in the kinetic term which now become operators, too. In the Wheeler–DeWitt quantization where a is a multiplication operator, a^{-3} is unbounded and diverges at the classical singularity. In loop quantum cosmology we have the basic operator $\hat{p}$ which one can use to construct a quantization of a^{-3}. However, a straightforward quantization fails since, as one of the basic properties, $\hat{p}$ has a discrete spectrum containing zero. In this case, there is no densely defined inverse operator which one could use. This seems to indicate that the situation is even worse: an operator for the kinetic term would not only be unbounded but not even be well-defined. The situation is much better, however, when one tries other quantizations which are more indirect. For non-basic operators such as a^{-3} there are usually many ways to quantize, all starting from the same classical expression. What we can do here, suggested by constructions in the full theory [30], is to rewrite a^{-3} in a classically equivalent way as

$$a^{-3} = (\pi^{-1}G^{-1}\mathrm{tr}\tau_3 e^{c\tau_3}\{e^{-c\tau_3}, \sqrt{V}\})^6$$

where we only need a readily available positive power of $\hat{p}$. Moreover, exponentials of c are basic operators, where we just used su(2) notation $e^{c\tau_3} = \cos\frac{1}{2}c + 2\tau_3\sin\frac{1}{2}c$ in order to bring the expression closer to what one would have in the full theory, and the Poisson bracket will become a commutator in quantum theory.

This procedure, after taking the trace, leads to a densely defined operator for a^{-3} despite the nonexistence of an inverse of $\hat{p}$ [48]:

$$\widehat{a^{-3}} = \left(8i\ell_{\mathrm{P}}^{-2}(\sin\tfrac{1}{2}c\sqrt{\hat{V}}\cos\tfrac{1}{2}c - \cos\tfrac{1}{2}c\sqrt{\hat{V}}\sin\tfrac{1}{2}c)\right)^6 . \qquad (12)$$

That this operator is indeed finite can be seen from its action on states $|\mu\rangle$ which follows from that of the basic operators:

$$\widehat{a^{-3}}|\mu\rangle = \left(4\ell_{\mathrm{P}}^{-2}(\sqrt{V_{\mu+1}} - \sqrt{V_{\mu-1}})\right)^6 |\mu\rangle \qquad (13)$$

immediately showing the eigenvalues which are all finite. In particular, at $\mu = 0$ where we would have the classical singularity the density operator does not diverge but is zero.

This finiteness of densities finally confirms the non-singular evolution since the matter Hamiltonian

$$\hat{H}_{\mathrm{matter}} = \tfrac{1}{2}\widehat{a^{-3}}\hat{p}_\phi^2 + \hat{V}V(\phi) \qquad (14)$$

in the example of a scalar is well-defined even on the classically singular state $|0\rangle$. The same argument applies for other matter Hamiltonians since only the general structure of kinetic and potential terms is used.

4.3.2 Confirmation of indications

The finiteness of the operator is a consequence of the loop representation which forced us to take a detour in quantizing inverse powers of the scale factor. A more physical understanding can be obtained by exploiting the fact that there are quantization ambiguities in this non-basic operator. This comes from the rewriting procedure which is possible in many classically equivalent ways, which all lead to different operators. Important properties such as the finiteness and the approach to the classical limit at large volume are robust under the ambiguities, but finer details can change. The most important choices one can make are selecting the representation j of SU(2) holonomies before taking the trace [49, 50] and the power l of $|p|$ in the Poisson bracket [51]. These values are restricted by the requirement that j is a half-integer ($j = 1/2$ in the above choice) and $0 < l < 1$ to obtain a well-defined inverse power of a ($l = 3/4$ above). The resulting eigenvalues can be computed explicitly and be approximated by the formula [50, 51]

$$(a^{-3})_{\rm eff} = a^{-3} p_l(a^2/a_{\rm max}^2)^{3/(2-2l)} \tag{15}$$

where $a_{\rm max} = \sqrt{j/3}\,\ell_{\rm P}$ depends on the first ambiguity parameter and the function

$$\begin{aligned} p_l(q) \;=\; & \frac{3}{2l} q^{1-l} \left((l+2)^{-1} \left((q+1)^{l+2} - |q-1|^{l+2} \right) \right. \\ & \left. - (l+1)^{-1} q \left((q+1)^{l+1} - \mathrm{sgn}(q-1)|q-1|^{l+1} \right) \right) . \end{aligned} \tag{16}$$

on the second.

The function $p_l(q)$, shown in Fig. 1, approaches one for $q \gg 1$, has a maximum close to $q = 1$ and drops off as q^{2-l} for $q \ll 1$. This shows that $(a^{-3})_{\rm eff}$ approaches the classical behavior a^{-3} at large scales $a \gg a_{\rm max}$, has a maximum around $a_{\rm max}$ and falls off like $(a^{-3})_{\rm eff} \sim a^{3/(1-l)}$ for $a \ll a_{\rm max}$. The peak value can be approximated, e.g. for $j = 1/2$, by $(a^{-3})_{\rm eff}(a_{\rm max}) \sim 3l^{-1}2^{-l}(1-3^{-l})^{3/(2-2l)}\ell_{\rm P}^{-3}$ which indeed shows that densities are bounded by inverse powers of the Planck length such that they are finite in quantum gravity but diverge in the classical limit. This confirms our qualitative expectations from the hydrogen atom, while details of the coefficients depend on the quantization.

Similarly, densities are seen to have a peak at $a_{\rm max}$ whose position is given by the Planck length (and an ambiguity parameter). Above the peak we have the classical behavior of an inverse power, while below the peak the density increases from zero. As suggested by the behavior of radiation in a cavity whose spectral energy density

$$\rho_T(\lambda) = 8\pi h \lambda^{-5} (e^{h/kT\lambda} - 1)^{-1} = h\lambda^{-5} f(\lambda/\lambda_{\rm max})$$

can, analogously to (15), be expressed as the diverging behavior λ^{-5} multiplied with a cut-off function $f(y) = 8\pi/((5/(5-x))^{1/y} - 1)$ with $x = 5 + W(-5e^{-5})$ (in terms of the

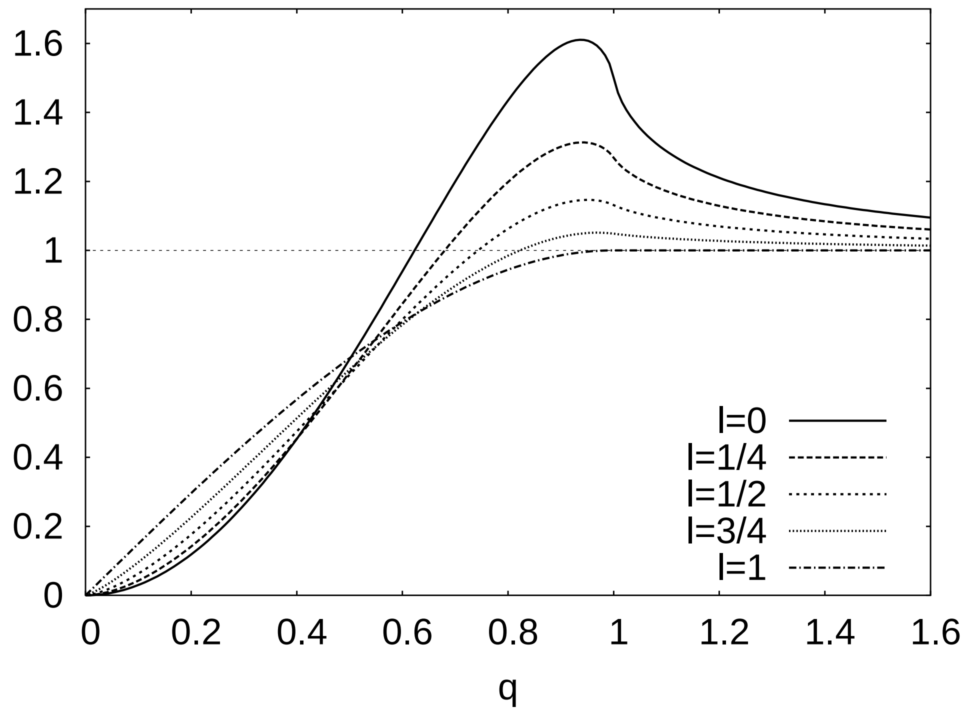


Figure 1: The function $p_l(q)$ in (16) for some values of l, including the limiting cases $l = 0$ and $l = 1$.

Lambert function $W(x)$, the inverse function of xe^x) and $\lambda_{\max} = h/xkT$, we obtain an interpolation between increasing behavior at small scales and decreasing behavior at large scales in such a way that the classical divergence is cut off.

We thus have an interpolation between increasing behavior necessary for negative pressure and inflation and the classical decreasing behavior (Fig. 2). Any matter density turns to increasing behavior at sufficiently small scales without the need to introduce an inflaton field with tailor-made properties. In the following section we will see the implications for cosmological evolution by studying effective classical equations incorporating this characteristic loop effect of modified densities at small scales.

4.4 Phenomenology

The quantum difference equation (11) is rather complicated to study in particular in the presence of matter fields and, as discussed in Sec. 4.2.2, difficult to interpret in a fully quantum regime. It is thus helpful to work with effective equations, comparable conceptually to effective actions in field theories, which are easier to handle and more familiar to interpret but still show important quantum effects. This can be done systematically [34, 52, 53], starting with the Hamiltonian constraint operator, resulting in different types of correction terms whose significance in given regimes can be estimated or studied nu-

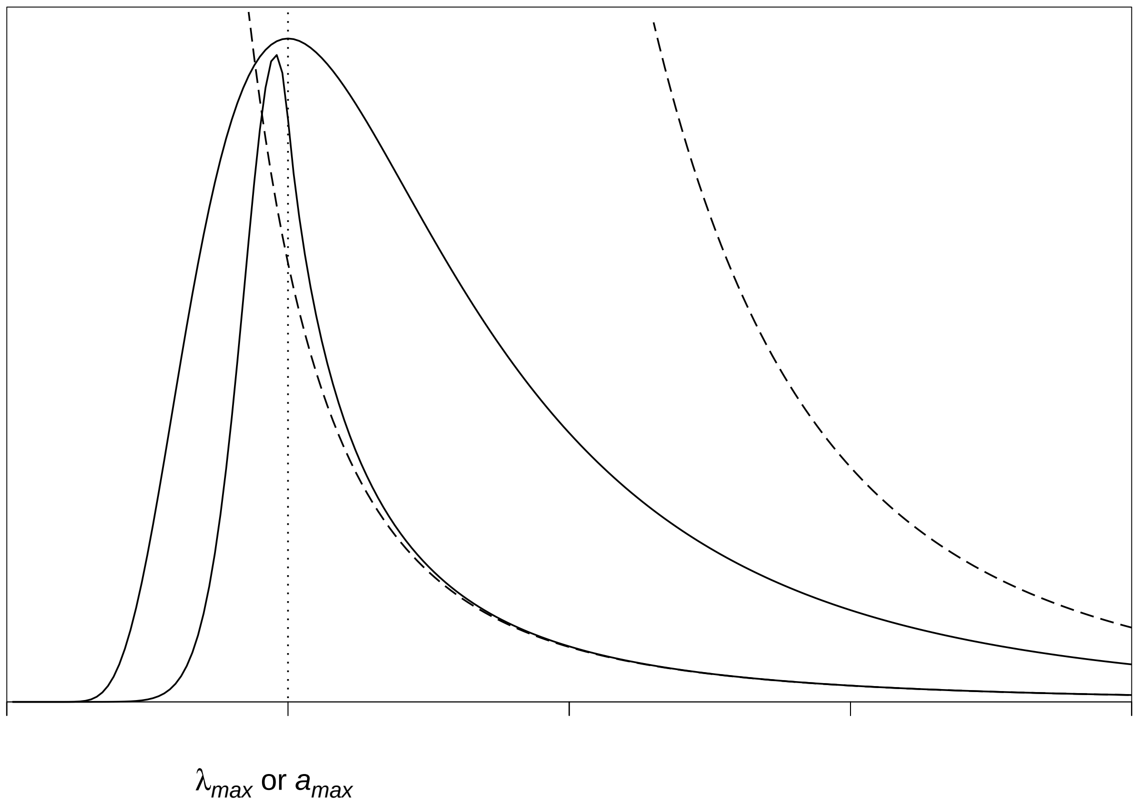


Figure 2: Comparison between the spectral energy density of black body radiation (wide curve) and an effective geometrical density with their large scale approximations (dashed).

merically [54]. There are perturbative corrections to the Friedmann equation of higher order form in $\dot{a}$, or of higher derivative, in the gravitational part on the left hand side, but also modifications in the matter Hamiltonian since the density in its kinetic term behaves differently at small scales. The latter corrections are mainly non-perturbative since the full expression for $(a^{-3})_{\rm eff}$ depends on the inverse Planck length, and their range can be extended if the parameter j is rather large. For these reasons, those corrections are most important and we focus on them from now on.

The effective Friedmann equation then takes the form

$$a\dot{a}^2 = \tfrac{8\pi}{3} G \left(\tfrac{1}{2}(a^{-3})_{\rm eff}\, p_\phi^2 + a^3 V(\phi)\right) \tag{17}$$

with $(a^{-3})_{\rm eff}$ as in (15) with a choice of ambiguity parameters. Since the matter Hamiltonian does not just act as a source for the gravitational field on the right hand side of the Friedmann equation, but also generates Hamiltonian equations of motion, the modification entails further changes in the matter equations of motion. The Klein–Gordon equation (3) then takes the effective form

$$\ddot{\phi} = \dot{\phi}\, \dot{a} \frac{\mathrm{d}\log(a^{-3})_{\rm eff}}{\mathrm{d}a} - a^3 (a^{-3})_{\rm eff} V'(\phi) \tag{18}$$

and finally there is the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(a^{-3}d(a)_{\rm eff}^{-1}\dot{\phi}^2\left(1-\tfrac{1}{4}a\frac{{\rm d}\log(a^3d(a)_{\rm eff})}{{\rm d}a}\right)-V(\phi)\right) \tag{19}$$

which follows from the above equation and the continuity equation of matter.

4.4.1 Bounces

The resulting equations can be studied numerically or with qualitative analytic techniques. We first note that the right hand side of (17) behaves differently at small scales since it increases with a at fixed ϕ and p_ϕ. Viewing this equation as analogous to a constant energy equation in classical mechanics with kinetic term $\dot{a}^2$ and potential term $\mathcal{V}(a) := -\frac{8\pi}{3}Ga^{-1}\left(\frac{1}{2}(a^{-3})_{\rm eff}\, p_\phi^2 + a^3V(\phi)\right)$ illustrates the classically attractive nature of gravity: The dominant part of this potential behaves like $-a^{-4}$ which is increasing. Treating the scale factor analogously to the position of a classical particle shows that a will be driven toward smaller values, implying attraction of matter and energy in the universe. This changes when we approach smaller scales and take into account the quantum modification. Below the peak of the effective density the classical potential $\mathcal{V}(a)$ will now decrease, $-\mathcal{V}(a)$ behaving like a positive power of a. This implies that the scale factor will be repelled away from $a = 0$ such that there is now a small-scale repulsive component to the gravitational force if we allow for quantum effects. The collapse of matter can then be prevented if repulsion is taken into account, which indeed can be observed in some models where the effective classical equations alone are sufficient to demonstrate singularity-free evolution.

This happens by the occurrence of bounces where a turns around from contracting to expanding behavior. Thus, $\dot{a} = 0$ and $\ddot{a} > 0$. The first condition is not always realizable, as follows from the Friedmann equation (1). In particular, when the scalar potential is non-negative there is no bounce, which is not changed by the effective density. There are then two possibilities for bounces in isotropic models, the first one if space has positive curvature rather than being flat as assumed here [55, 56], the second one with a scalar potential which can become negative [57, 58]. Both cases allow $\dot{a} = 0$ even in the classical case, but this always corresponds to a maximum rather than minimum. This can easily be seen for the case of negative potential from the Raychaudhuri equation (19) which in the classical case implies negative $\ddot{a}$. With the modification, however, the additional term in the equation provides a positive contribution which can become large enough for $\ddot{a}$ to become positive at a value of $\dot{a} = 0$ such that there is a bounce.

This provides intuitive explanations for the absence of singularities from quantum gravity, but not a general one. The generic presence of bounces depends on details of the model such as its matter content or which correction terms are being used [59, 60], and even with the effective modifications there are always models which classically remain singular. Thus, the only general argument for absence of singularities remains the quantum one based on the difference equation (11), where the conclusion is model independent.

4.4.2 Inflation

A repulsive contribution to the gravitational force can not only explain the absence of singularities, but also enhances the expansion of the universe on scales close to the classical singularity. Thus, as seen also in Fig. 3 the universe accelerates just from quantum effects, providing a mechanism for inflation without choosing special matter.

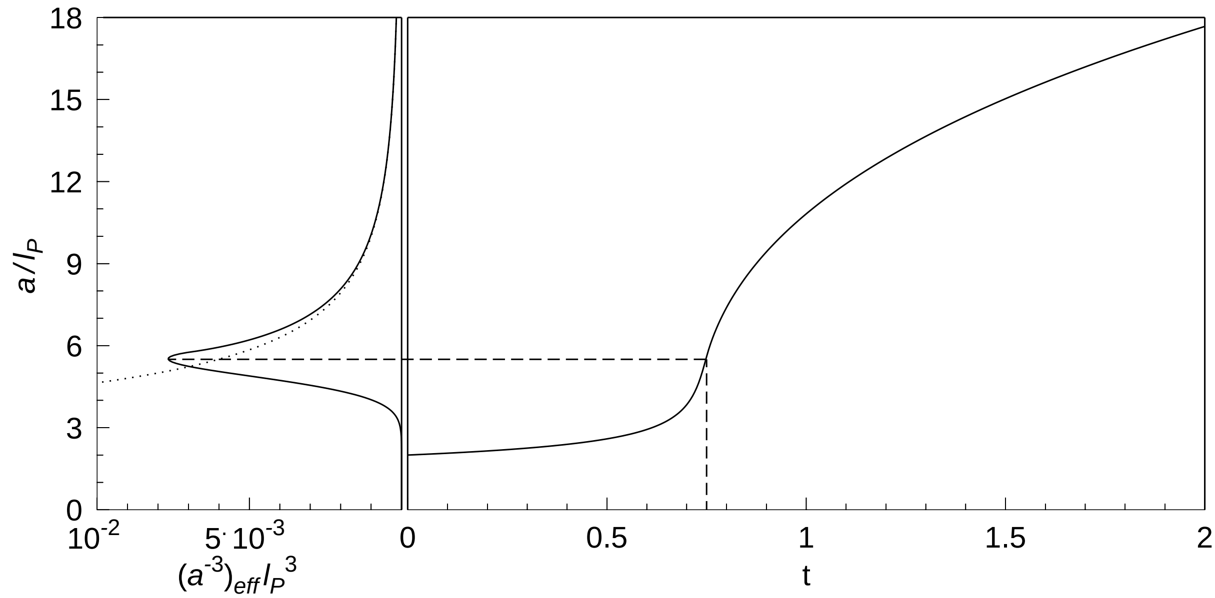


Figure 3: Numerical solution to the effective Friedmann equation (17) with a vanishing scalar potential. While the modification in the density on the left is active the expansion is accelerated, which stops automatically once the universe expands to a size above the peak in the effective density.

Via the generation of structure, inflationary phases of the universe can have an imprint on the observable cosmic microwave background. Observations imply that the predicted power spectrum of anisotropies must be nearly independent of the scale on which the anisotropies are probed, which implies that the inflationary phase responsible for structure formation must be close to exponential acceleration. This is true for slow-roll inflation, but also for the inflationary phase obtained from the effective density once a non-zero scalar potential is taken into account [61]. For more detailed comparisons between theory and observations one needs to consider how inhomogeneous fields evolve, which already requires us to relax the strong symmetry assumption of homogeneity. The necessary methods are not well-developed at the current stage (see [33, 62, 63] for the basic formulation), but preliminary calculations of implications on the power spectrum have been undertaken nonetheless. Ref. [64] indicates that loop inflation can be distinguished from simple inflaton models because the power depends differently on scales.

It turns out that this loop phase alone can provide a sufficient amount of inflation only for unnatural choices of parameters (such as extremely large j), and those cases are even ruled out by observations already. At this point, the modified matter dynamics of (18) and its $\dot{\phi}$-term becomes important. Classically, it is a friction term which is used for slow-roll inflation. But in the modified regime at small scales the sign of the term changes once $(a^{-3})_{\rm eff}$ is increasing. Thus, at those small scales friction turns into antifriction and matter is driven up its potential if it has a non-zero initial momentum (even a tiny one, e.g., from quantum fluctuations). After the loop phase matter fields slow down and roll back toward their minima, driving additional inflation. The potentials need not be very special since structure formation in the first phase and providing a large universe happen by different mechanisms. When matter fields reach their minima they start to oscillate and usual re-heating to obtain hot matter can commence.

Loop quantum cosmology thus presents a viable alternative to usual inflaton models which is not in conflict with current observations but can be verified or ruled out with future experiments such as the Planck satellite. Its attractive feature is that it does not require the introduction of an inflaton field with its special properties, but provides a fundamental explanation of acceleration from quantum gravity. This scenario is thus encouraging, but so far has not been developed to the same extent as inflaton models.

Even if we assume the presence of an inflaton field are its properties less special than in the purely classical treatment. We still need to assume a potential which is sufficiently flat, but there is now an explanation of initial values far away from the minimum. For this we again use the effective Klein–Gordon equation and the fact that ϕ is driven up its potential. One can then check that for usual inflaton potentials the value of typical initial conditions, as a function of chosen ambiguity parameters and initial fluctuations of the scalar, is just what one needs for sufficient inflation in a wide range [65, 66]. After the modifications in the density subside, the inflaton keeps moving up the potential from its initial push, but is now slowed down by the friction term. Eventually, it will stop and turn around, entering a slow roll phase in its approach to the potential minimum. Thus, the whole history of the expansion is described by a consistent model as illustrated in Fig. 4, not just the slow roll phase after the inflaton has already obtained its large initial values.

One may think that such a second phase of slow-roll inflation washes away potential quantum gravity effects from the early expansion. That this is not necessarily the case has been shown in [65], based on the fact that around the turning point of the inflaton the slow-roll conditions are violated. In this scenario, structure we see today on the largest scales was created at the earliest stages of the second inflationary phase since it was enlarged by the full inflationary phase. If the second inflationary regime did not last too long, these scales are just observable today where in fact the observed loss of power can be taken as an indication of early violations of slow-roll expansion. Thus, loop quantum cosmology can provide an explanation, among others, for the suppression of power on large scales.

There are diverse scenarios since different phases of inflation can be combined, and eventually the right one has to be determined from observations. One can also combine bounces and inflationary regimes in order to obtain cyclic universes which eventually reach a long phase of accelerated expansion [67]. In particular, this allows the conctruction of models which start close to a simple, static initial state and, after a series of cycles, automatically reach values of the scalar to start inflation. In this way, a semiclassical non-

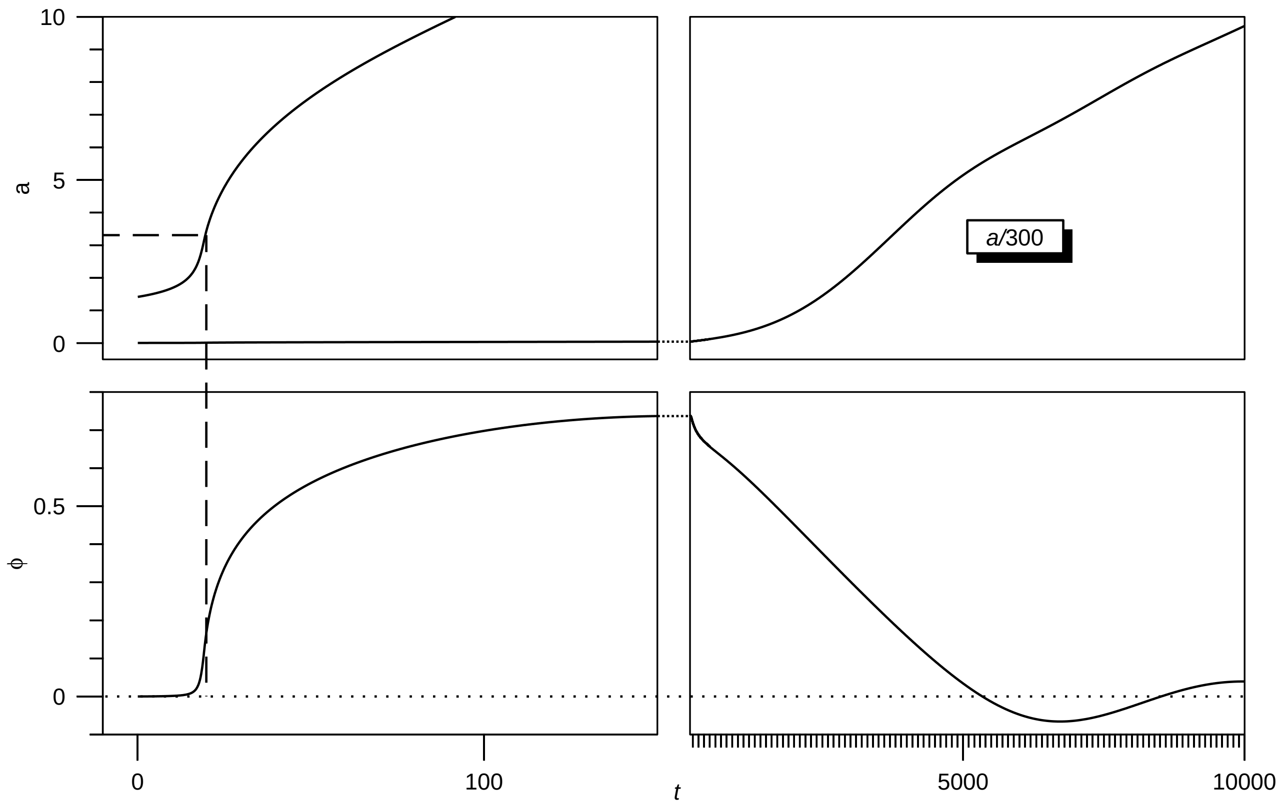


Figure 4: History of the scale factor (top) and inflaton (bottom) with the left hand side in slow motion. (Tics on the right horizontal axis mark increments in t by 100.) The upper right data are rescaled so as to fit on the same plot. Units for a and ϕ are Planck units, and parameters are not necessarily realistic but chosen for plotting purposes. Dashed lines mark the time and scale factor where classical behavior of $(a^{-3})_{\rm eff}$ starts.

singular inflationary model [68, 69, 70] is formulated which evades the singularity theorem of [8].

Current observations are already beginning to rule out certain, very large values of the ambiguity parameter j such that from future data one can expect much tighter limits. In all these scenarios the non-perturbative modification of the density is important, which is a characteristic feature of loop quantum cosmology. At larger scales above the peak there are also perturbative corrections which imply small changes in the cosmological expansion and the evolution of field modes. This has recently been investigated [71] with the conclusion that potential effects on the power spectrum would be too small to be noticed by the next generation satellites. The best candidates for observable effects from quantum gravity thus remain the non-perturbative modifications in effective densities.

5 Conclusions and Outlook

What we have described is a consistent picture of the universe which is not only observationally viable but also mathematically well-defined and non-singular. There are instances where quantum gravity is essential, and others where it is helpful in achieving important effects. The background independent quantization employed here is very efficient: There are a few basic properties, such as the discreteness of spatial geometry and the representation only of exponentials of curvature, which are behind a variety of applications. Throughout all the developments, those properties have been known to be essential for mathematical consistency before they were recognized as being responsible for physical phenomena.

For instance, for the singularity issue the basic properties are all needed in the way they turn out to be realized. First, the theory had to be based on densitized triad variables which now not only provides us with the sign of orientation, and thus two sides of the classical singularity, but also in more complicated models positions the classical singularity in phase space such that it becomes accessible by quantum evolution. Then, the discreteness of spatial geometry encoded in triad operators and the representation of exponentials of curvature play together in the right way to remove divergences in densities and extend the quantum evolution through the classical singularity. These features allow general results about the absence of singularities without any new or artificial ingredients, and lead to a natural solution of a long-standing problem which has eluded previous attempts for decades. Symmetry assumptions are still important in order to be able to perform the calculations, but they can now be weakened considerably and are not responsible for physical implications. The essential step is to base the symmetry reduction on a candidate for full quantum gravity which is background independent so as to allow studying quantum geometry purely.

Absence of singularities in this context is a rather general statement about the possibility to extend a quantum wave function through a regime which classically would appear as a singularity. More explicit questions, such as what kind of new region one is evolving to and whether it again becomes classical or retains traces of the evolution through a quantum regime, depend on details of the relevant constraint operators. This includes, for instance, quantization ambiguities and the question whether a symmetric operator has to be used. The latter aspect is also important for technical concepts such as a physical inner product.

Here we discussed only isotropic models which are classically described solely by the scale factor determining the size of space. But a more realistic situation has to take into account also the shape of space, and changes of the distribution of geometry and matter between different points of space. The methods we used have been extended to homogeneous models, allowing for anisotropic spaces, and recently to some inhomogeneous ones, defined by spherical symmetry and some forms of cylindrical symmetry. In all cases, essential aspects of the general mechanism for removing classical singularities which has first been seen only in the simple isotropic models are known to be realized.[1] Moreover, in the more complicated systems it is acting much more non-trivially, again with the right ingredients provided by the background independent quantization. Nevertheless, since the inhomogeneous constraints are much more complicated to analyze, absence of singularities for them has not yet been proven completely. The inhomogeneous systems now also allow access to black hole and gravitational wave models such that their quantum geometry can be studied, too.

Effective equations are a useful tool to study quantum effects in a more familiar setting given by classical equations of motion. They show diverse effects whose usefulness in cosmological phenomenology is often surprising. Also here, the effects were known to occur from the quantization and the transfer into effective classical equations, before they turned out to be helpful. In addition to inflationary scenarios and bounces which one can see in isotropic cosmologies, modified densities have more implications in less symmetric models. The anisotropic Bianchi IX model, for instance, is classically chaotic which is assumed to play a role in the complicated approach to a classical singularity [75]. With the effective modifications the dynamics changes and simplifies, removing the classical chaos [76]. This has implications for the effective approach to a classical singularity and can provide a more consistent picture of general singularities [77]. Effective classical equations can also be used to study the collapse of matter to a black hole, with modifications in the development of classical singularities and horizons [78]. This can now also be studied with inhomogeneous quantum models which allow new applications for black holes and cosmological phenomenology where the evolution of inhomogeneities is of interest in the context of structure formation.

With these models there will be new effects not just in cosmology but also for black holes and other systems which further check the overall consistency of the theory. Moreover, a better understanding of inhomogeneities evolving in a cosmological background will give us a much better computational handle on signatures in the cosmic microwave or even gravitational wave background, which may soon be testable with a new generation of observations. One may wonder how it can be possible to observe quantum gravity effects, given that the Planck scale is so many orders of magnitude away from scales accessible by

[1] This does not refer to the boundedness of densities or curvature components for *all geometries*, which is known not to be present in anisotropic models [72, 73] or even on some degenerate configurations in the full theory [74]. What is relevant is the behavior on configurations seen along the dynamical evolution.

today's experiments. The difference in scales, however, does not preclude the observation of indirect effects even though direct measurements on the discreteness scale are impossible, as illustrated by a well-known example: Brownian motion allows to draw conclusions about the atomic structure of matter and its size by observations on much larger scales [79]. Similarly, cosmological observations can carry information on quantum gravity effects which otherwise would manifest themselves only at the Planck scale.

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