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Theories with Memory

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Abstract

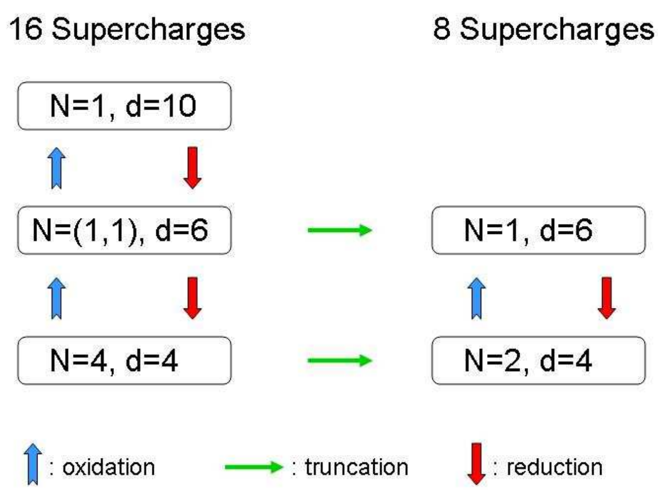
Dimensionally reduced supersymmetric theories retain a great deal of information regarding their higher dimensional origins. In superspace, this "memory" allows us to restore the action governing a reduced theory to that describing its higher-dimensional progenitor. We illustrate this by restoring four-dimensional $\mathcal{N} = 4$ Yang-Mills to its six-dimensional parent, $\mathcal{N} = (1, 1)$ Yang-Mills. Supersymmetric truncation is introduced into this framework and used to obtain the $\mathcal{N} = 1$ action in six dimensions. We work in light-cone superspace, dealing exclusively with physical degrees of freedom.

1 Introduction

Dimensionally reduced supersymmetric theories retain a great deal of information regarding their higher dimensional origins. ($\mathcal{N}=4, d=4$) SuperYang-Mills is a good example of a theory with such "memory". The six scalars in its spectrum serve as signatures of a lost (compactified) $SO(6)$ while its $SU(4)$ spinors assemble nicely into a single eight-spinor: $\mathbf{4}_{1/2}+\bar{\mathbf{4}}_{-1/2} \;=\; \mathbf{8}_s$ of $SO(8)$. Its spectrum naturally favors reformulation in ten dimensions with a single supersymmetry.

In superspace, this allows us to "oxidize" (restore) four-dimensional $\mathcal{N}=4$ Yang-Mills into its fully ten-dimensional parent, $\mathcal{N}=1$ Yang-Mills. This is achieved by simply generalizing the $d=4$ transverse space derivatives [1].

The six scalars in the $\mathcal{N}=4$ spectrum could equally well be thought of as 2 + 4 scalars with the first two signaling a six-dimensional progenitor. Indeed, we will show that the $\mathcal{N}=4$ action can be restored to that describing six-dimensional $\mathcal{N}=(1,1)$ Yang-Mills. Supersymmetric truncation is introduced into this framework and used (as illustrated below) to obtain the ($\mathcal{N}=1, d=6$) action.



In six dimensions, massless particles are classified according to the little group $SO(4)$. Our focus (in this paper) will be mainly on the $\mathcal{N}=(1,1)$ Yang-Mills theory in six dimensions. This theory has 16 supercharges and may be obtained by reduction from ten-dimensional $\mathcal{N}=1$ Yang-Mills. The relevant little group decomposition is

$$SO(8) \supset SO(4) \times SO(4) .$$

The first $SO(4) \sim SU(2) \times SU(2)$ is the little group in six dimensions while the second represents the R-symmetry in the theory. The bosonic fields in the

$\mathcal{N}=(1,1)$ theory include a gauge field and four scalars. Under the little group, its spectrum transforms as

$$(\,\mathbf{2},\,\mathbf{2}\,)\,+\,4\cdot(\,\mathbf{1},\,\mathbf{1}\,)\,+\,2\cdot(\,\mathbf{2},\,\mathbf{1}\,)\,+\,2\cdot(\,\mathbf{1},\,\mathbf{2}\,)\ .$$

There is a second theory in six dimensions with 16 supercharges. This is the superconformally-invariant $\mathcal{N}=(2,0)$ theory characterized by an $SO(5)$ R symmetry. The spectrum of this theory transforms as

$$(\,\mathbf{3}\,,\,1\,)\,+\,5\cdot(\,\mathbf{1},\,\mathbf{1}\,)\,+\,4\cdot(\,\mathbf{2},\,\mathbf{1}\,)\ ,$$

with a bosonic spectrum comprised of a self-dual antisymmetric tensor and five scalars. One of the aims of the present paper is to construct the six-dimensional SuperPoincaré algebra with a view to tackling this theory in the future. In reference [2], we developed an algorithm to construct the entire $(\mathcal{N}=4,d=4)$ action starting from a single Ansatz for its dynamical supersymmetry generator. This was possible thanks to the vast $PSU(2,2|4)$ symmetry in that theory. The hope is that a similar approach will offer insights into the structure of other symmetry-laden theories like the superconformal ones in three [3, 4, 5] and six [6, 7, 8] dimensions.

1.1 Brief Summary of the Proposal

In Section **2**, we review the formulation of $(\mathcal{N}=4,d=4)$ SuperYang-Mills in light-cone superspace. It will turn out that the entire action can be written in terms of a single superfield as

$$\int d^4x\int d^4\theta\, d^4\bar{\theta}\,\mathcal{L}\ ,$$

where

$$\begin{aligned}\mathcal{L}\ &=\ -\bar{\phi}\,\frac{\Box_4}{\partial^{+2}}\,\phi\ +\frac{4g}{3}\,f^{abc}\Big(\frac{1}{\partial^+}\,\bar{\phi}^a\,\phi^b\,\bar{\partial}\,\phi^c\,+\,\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\partial\,\overline{\phi}^c\Big)\\&\quad -g^2f^{abc}\,f^{ade}\Big(\frac{1}{\partial^+}(\phi^b\,\partial^+\phi^c)\frac{1}{\partial^+}\,(\bar{\phi}^d\,\partial^+\,\bar{\phi}^e)+\frac{1}{2}\,\phi^b\bar{\phi}^c\,\phi^d\,\bar{\phi}^e\Big)\ .\end{aligned}$$

The basic idea is to modify the above action such that it describes the six-dimensional $\mathcal{N}=(1,1)$ theory. This modification (sections **3** and **4**) is done in three steps;

- Introduce two new coordinates (and their derivatives)
- Allow ϕ and $\overline{\phi}$ to depend on these new coordinates

• Generalize the $d = 4$ transverse derivatives ∂ and $\bar{\partial}$ to incorporate the new derivatives.

The new "generalized" derivatives (∇ and $\overline{\nabla}$) are defined in section **4.1**. Thus our proposal for the action governing the $d = 6$ $\mathcal{N} = (1, 1)$ theory is

$$\int d^6x \int d^4\theta \, d^4\bar{\theta} \, \mathcal{L} \; ,$$

where

$$\begin{aligned} \mathcal{L} \;\; = \;\; & -\bar{\phi} \, \frac{\Box_6}{\partial^{+2}} \, \phi \; + \frac{4g}{3} \, f^{abc} \left(\frac{1}{\partial^+} \, \bar{\phi}^a \, \phi^b \, \overline{\nabla} \, \phi^c + \frac{1}{\partial^+} \, \phi^a \, \bar{\phi}^b \, \nabla \, \bar{\phi}^c \right) \\ & -g^2 f^{abc} \, f^{ade} \left(\frac{1}{\partial^+} (\phi^b \, \partial^+ \phi^c) \frac{1}{\partial^+} \, (\bar{\phi}^d \, \partial^+ \, \bar{\phi}^e) + \frac{1}{2} \, \phi^b \bar{\phi}^c \, \phi^d \, \bar{\phi}^e \right) \; . \end{aligned}$$

We will prove that this action is invariant under the six-dimensional Super-Poincaré algebra. This proof is presented in sections **4.1** and **4.2**.

Finally, in section **5** this six-dimensional $\mathcal{N} = (1, 1)$ theory (with 16 supercharges) is truncated to obtain the $(\mathcal{N} = 1, d = 6)$ theory with 8 supercharges. Supersymmetric truncation [9] is based on the fact that

$$\int d^6x \, d^{\mathcal{N}}\theta \, d^{\mathcal{N}}\bar{\theta} \, \mathcal{L} \; \propto \; \int d^6x \, d^{\mathcal{N}-1}\theta \, d^{\mathcal{N}-1}\bar{\theta} \, \bar{d}_{\mathcal{N}} \, d^{\mathcal{N}} \, \mathcal{L} \, |_{\theta^{\mathcal{N}} = \bar{\theta}_{\mathcal{N}} = 0} \; .$$

Six-dimensional $\mathcal{N} = 1$ Yang-Mills has been studied previously in harmonic superspace by Howe, Stelle and West [10]. The theory was formulated in terms of $d = 6$ superfields by Tollsten [11] and Howe, Sierra and Townsend [12]. The free six-dimensional hypermultiplet was formulated in terms of four-dimensional $\mathcal{N} = 1$ superspace in references [13, 14].

2 $\mathcal{N} = 4$ Yang-Mills in Light-Cone Superspace

With the space-time metric $(-, +, +, +)$, the light-cone coordinates and their derivatives are

$$\begin{aligned} x^{\pm} = \;\; & \frac{1}{\sqrt{2}} \, (\, x^0 \pm x^3 \,) \; ; \qquad & \partial^{\pm} = \frac{1}{\sqrt{2}} \, (\, - \, \partial_0 \pm \partial_3 \,) \; , \\ x = \;\; & \frac{1}{\sqrt{2}} \, (\, x_1 \, + \, i \, x_2 \,) \; ; \qquad & \bar{\partial} = \frac{1}{\sqrt{2}} \, (\, \partial_1 \, - \, i \, \partial_2 \,) \; , \\ \bar{x} = \;\; & \frac{1}{\sqrt{2}} \, (\, x_1 \, - \, i \, x_2 \,) \; ; \qquad & \partial = \frac{1}{\sqrt{2}} \, (\, \partial_1 \, + \, i \, \partial_2 \,) \; . \end{aligned} \tag{1}$$

We introduce $SU(4) \sim SO(6)$ spinors θ^m and their conjugates $\bar{\theta}_n$ ($m\,,n\,=\,1,2,3,4$). The $d=4$ d'Alembertian reads

$$\Box_4 \,=\, 2\,(\,\partial\,\bar{\partial}\,-\,\partial^+\,\partial^-\,)\ . \tag{2}$$

All the physical degrees of freedom of the $\mathcal{N}=4$ theory are captured by a single complex superfield [15, 16]

$$\begin{aligned}\phi\,(y) = \;\; & \frac{1}{\partial^+}\,A\,(y)\,+\,\frac{i}{\sqrt{2}}\,\theta^m\,\theta^n\,\overline{C}_{mn}\,(y)\,+\,\frac{1}{12}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\epsilon_{mnpq}\,\partial^+\,\bar{A}\,(y) \\ & +\,\frac{i}{\partial^+}\,\theta^m\,\bar{\chi}_m\,(y)\,+\,\frac{\sqrt{2}}{6}\,\theta^m\,\theta^n\,\theta^p\,\epsilon_{mnpq}\,\chi^q\,(y)\ ,\end{aligned} \tag{3}$$

with the $\frac{1}{\partial^+}$ interpreted as [17]

$$\frac{1}{\partial^+}\,f\,(\,x^-\,)\,=\,\frac{1}{2}\int\epsilon\,(\,\xi\,-\,x^-\,)\,f\,(\,\xi\,)\,d\xi\ . \tag{4}$$

The gauge fields appear as

$$A\,=\,\frac{1}{\sqrt{2}}\,(A_1+i\,A_2)\ ;\qquad \bar{A}\,=\,\frac{1}{\sqrt{2}}\,(A_1-i\,A_2)\ , \tag{5}$$

the six scalars as $SU(4)$ bispinors

$$C^{m\,4}\,=\,\frac{1}{\sqrt{2}}\,(A_{m+3}\,+\,i\,A_{m+6})\ ;\qquad \overline{C}_{mn}\,=\,\frac{1}{2}\,\epsilon_{mnpq}\,C^{pq}\ , \tag{6}$$

(for $m\ \neq\,4$) and the fermi fields as χ^m and $\bar{\chi}_n$. All fields are local in the modified light-cone coordinates

$$y\,=\,(\,x,\,\bar{x},\,x^+,\,y^-\,\equiv\,x^-\,-\,\frac{i}{\sqrt{2}}\,\theta^m\,\bar{\theta}_m\,)\ . \tag{7}$$

The $\mathcal{N}=4$ Yang-Mills light-cone action is then simply

$$72\int d^4x\int d^4\theta\,d^4\bar{\theta}\,\mathcal{L}\ , \tag{8}$$

where

$$\begin{aligned}\mathcal{L}\ \ =\ \ & -\bar{\phi}\,\frac{\Box_4}{\partial^{+2}}\,\phi\,+\,\frac{4g}{3}\,f^{abc}\left(\frac{1}{\partial^+}\,\bar{\phi}^a\,\phi^b\,\bar{\partial}\,\phi^c\,+\,\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\partial\,\overline{\phi}^c\right) \\ & -g^2 f^{abc}\,f^{ade}\left(\frac{1}{\partial^+}(\phi^b\,\partial^+\phi^c)\frac{1}{\partial^+}\,(\bar{\phi}^d\,\partial^+\,\bar{\phi}^e)+\frac{1}{2}\,\phi^b\bar{\phi}^c\,\phi^d\,\bar{\phi}^e\right)\ .\end{aligned} \tag{9}$$

where the f^{abc} are the structure functions of the Lie algebra and Grassmann integration is normalized so that $\int d^4\theta\;\theta^1\theta^2\theta^3\theta^4=1$.

2.1 The $d=4$ SuperPoincaré Algebra

The action in equation (8) is left invariant by the $d=4$ SuperPoincaré algebra. We will simply write down the generators here and refer the reader to references [15, 16, 18] for further details.

The bosonic generators include the four-momenta

$$p^+ \;=\; -i\,\partial^+ \;, \qquad p \;=\; -\,i\,\partial \;, \qquad \bar{p} \;=\; -\,i\,\bar{\partial} \;, \qquad p^- \;=\; -\,i\frac{\partial\bar{\partial}}{\partial^+} \;, \tag{10}$$

the rotations

$$\begin{aligned} j \;&=\; x\,\bar{\partial} - \bar{x}\,\partial + \frac{1}{2}\,(\,\theta^m\,\bar{\partial}_m \;-\; \bar{\theta}_m\,\partial^m\,) \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^m\,\bar{d}_m \;-\; \bar{d}_m\,d^m\,) \\ j^+ \;&=\; i\,x\,\partial^+ \;, \qquad\qquad \bar{j}^+ \;=\; i\,\bar{x}\,\partial^+ \;, \\ j^{+-} \;&=\; i\,x^-\,\partial^+ - \frac{i}{2}\,(\,\theta^m\,\bar{\partial}_m \;+\; \bar{\theta}_m\,\partial^m\,) \;, \end{aligned} \tag{11}$$

and the boosts

$$\begin{aligned} j^- \;&=i\,x\,\frac{\partial\bar{\partial}}{\partial^+} \;-\; i\,x^-\,\partial \;+\; i\left(\theta^m\,\bar{\partial}_m \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^m\,\bar{d}_m \;-\; \bar{d}_m\,d^m\,)\right)\frac{\partial}{\partial^+} \;, \\ \bar{j}^- \;&=i\,\bar{x}\,\frac{\partial\bar{\partial}}{\partial^+} \;-\; i\,x^-\,\bar{\partial} \;+\; i\left(\bar{\theta}_n\,\partial^n \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^n\,\bar{d}_n \;-\; \bar{d}_n\,d^n\,)\right)\frac{\bar{\partial}}{\partial^+} \;. \end{aligned} \tag{12}$$

The fermionic operators are the chiral derivatives

$$d^{\,m} \;=\; -\,\partial^m \;-\; \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+ \;; \qquad \bar{d}_{\,n} \;=\; \bar{\partial}_n \;+\; \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \;, \tag{13}$$

the kinematical supersymmetries

$$q^m_{\,+} \;=\; -\,\partial^m \;+\; \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+ \;; \qquad \bar{q}_{\,+\,n} \;=\; \bar{\partial}_n \;-\; \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \;, \tag{14}$$

and the dynamical supersymmetries

$$q^m_{\,-} \;\equiv\; i\,[\,\bar{j}^-\,,\,q^m_{\,+}\,] \;=\; \frac{\partial}{\partial^+}\,q^m_{\,+} \;; \qquad \bar{q}_{\,-\,n} \;\equiv\; i\,[\,j^-\,,\,\bar{q}_{\,+\,n}\,] \;=\; \frac{\bar{\partial}}{\partial^+}\,\bar{q}_{\,+\,n} \;. \tag{15}$$

The superfield and its complex conjugate satisfy chiral constraints,

$$d^{\,m}\,\phi \;=\; 0\;; \qquad \bar{d}_{\,m}\,\bar{\phi} \;=\; 0 \;, \tag{16}$$

as well as “inside-out” constraints

$$\bar{d}_m\,\bar{d}_n\,\phi \;=\; \frac{1}{2}\,\epsilon_{mnpq}\,d^p\,d^q\,\bar{\phi}\ . \tag{17}$$

The next step is to enlarge this SuperPoincaré algebra to six dimensions.

3 Six Dimensions

The reduction from six to four dimensions involves the little-group decomposition

$$SO(4)\,\supset\,SO(2)\,\times\,SO(2)\ . \tag{18}$$

The first $SO(2)$ is described by the first generator in equation (11). In order to build the entire $d=6$ algebra, we need to introduce the second $SO(2)$ and the generators of the coset: $SO(4)/(SO(2){\times}SO(2))$.

In the bispinor language of equation (6) we may introduce upto six new coordinates as

$$x^{mn}\ ; \qquad \bar{x}_{mn}\;=\;\frac{1}{2}\,\epsilon_{mnpq}\,x^{pq}\ . \tag{19}$$

However, since we require only two additional directions, we choose to introduce only

$$\begin{aligned} x^{12} \;&=\; \frac{1}{\sqrt{2}}\,(\,x_6\,-\,i\,x_9\,)\ ; \qquad \bar{x}_{12}\;=\;\frac{1}{\sqrt{2}}\,(\,x_6\,+\,i\,x_9\,)\ ,\\ x^{34} \;&=\; \frac{1}{\sqrt{2}}\,(\,x_6\,+\,i\,x_9\,)\ ; \qquad \bar{x}_{34}\;=\;\frac{1}{\sqrt{2}}\,(\,x_6\,-\,i\,x_9\,)\ , \end{aligned} \tag{20}$$

and their derivatives

$$\begin{aligned} \partial^{12} \;&=\; \frac{1}{\sqrt{2}}\,(\,\partial_6\,-\,i\,\partial_9\,)\ ; \qquad \bar{\partial}_{12}\;=\;\frac{1}{\sqrt{2}}\,(\,\partial_6\,+\,i\,\partial_9\,)\ ,\\ \partial^{34} \;&=\; \frac{1}{\sqrt{2}}\,(\,\partial_6\,+\,i\,\partial_9\,)\ ; \qquad \bar{\partial}_{34}\;=\;\frac{1}{\sqrt{2}}\,(\,\partial_6\,-\,i\,\partial_9\,)\ . \end{aligned} \tag{21}$$

The corresponding (new) $SO(2)$ generator is

$$\begin{aligned} \mathcal{J}\;=&\frac{1}{2}\,(\,x^{12}\,\bar{\partial}_{12}\,-\,\bar{x}_{12}\,\partial^{12}\,)\,-\,\frac{1}{2}\,(\,x^{34}\,\bar{\partial}_{34}\,-\,\bar{x}_{34}\,\partial^{34}\,)\,+\,\frac{1}{2}\,(\,\theta^1\,\bar{\partial}_1\,+\,\theta^2\,\bar{\partial}_2\,-\,\bar{\theta}_1\,\partial^1\,-\,\bar{\theta}_2\,\partial^2\,)\\ &-\,\frac{1}{2}\,(\,\theta^3\,\bar{\partial}_3\,+\,\theta^4\,\bar{\partial}_4\,-\,\bar{\theta}_3\,\partial^3\,-\,\bar{\theta}_4\,\partial^4\,)\,+\,\frac{i}{4\,\sqrt{2}\,\partial^+}\,(\,d^1\,\bar{d}_1\,+\,d^2\,\bar{d}_2\,-\,\bar{d}_1\,d^1\,-\,\bar{d}_2\,d^2\,)\\ &-\,\frac{i}{4\,\sqrt{2}\,\partial^+}\,(\,d^3\,\bar{d}_3\,+\,d^4\,\bar{d}_4\,-\,\bar{d}_3\,d^3\,-\,\bar{d}_4\,d^4\,)\ . \end{aligned} \tag{22}$$

The four generators of the coset space $SO(4)/(SO(2)\times SO(2))$ are

$$
\begin{aligned}
J^{12} &= x\,\partial^{12}\,-\,x^{12}\,\partial\,+\,\frac{i}{\sqrt{2}}\,\partial^+\,\theta^1\,\theta^2\,-\,\frac{i\,\sqrt{2}}{\partial^+}\,\partial^1\,\partial^2\,+\,\frac{i}{\sqrt{2}\,\partial^+}\,d^1\,d^2\ ,\\
\bar{J}_{12} &= \bar{x}\,\bar{\partial}_{12}\,-\,\bar{x}_{12}\,\bar{\partial}\,+\,\frac{i}{\sqrt{2}}\,\partial^+\,\bar{\theta}_1\,\bar{\theta}_2\,-\,\frac{i\,\sqrt{2}}{\partial^+}\,\bar{\partial}_1\,\bar{\partial}_2\,+\,\frac{i}{\sqrt{2}\,\partial^+}\,\bar{d}_1\,\bar{d}_2\ ,\\
J^{34} &= x\,\partial^{34}\,-\,x^{34}\,\partial\,+\,\frac{i}{\sqrt{2}}\,\partial^+\,\theta^3\,\theta^4\,-\,\frac{i\,\sqrt{2}}{\partial^+}\,\partial^3\,\partial^4\,+\,\frac{i}{\sqrt{2}\,\partial^+}\,d^3\,d^4\ ,\\
\bar{J}_{34} &= \bar{x}\,\bar{\partial}_{34}\,-\,\bar{x}_{34}\,\bar{\partial}\,+\,\frac{i}{\sqrt{2}}\,\partial^+\,\bar{\theta}_3\,\bar{\theta}_4\,-\,\frac{i\,\sqrt{2}}{\partial^+}\,\bar{\partial}_3\,\bar{\partial}_4\,+\,\frac{i}{\sqrt{2}\,\partial^+}\,\bar{d}_3\,\bar{d}_4\ .
\end{aligned}
\tag{23}
$$

They satisfy the commutation relations

$$
\begin{aligned}
[\,J^{12}\,,\,\bar{J}_{12}\,] &= -j\,-\,\mathcal{J}\ ,\\
[\,J^{12}\,,J^{34}\,] &= [\,J^{12}\,,\,\bar{J}_{34}\,]\,=\,0\ ,
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
[\,J^{34}\,,\,\bar{J}_{34}\,] &= -j\,+\,\mathcal{J}\ ,\\
[\,J^{34}\,,J^{12}\,] &= [\,J^{34}\,,\,\bar{J}_{12}\,]\,=\,0\ .
\end{aligned}
\tag{25}
$$

The remaining generators are fairly straightforward to write down. The new "plus" rotations read

$$
J^{+\,12}\ =\ i\,x^{12}\,\partial^+\ ;\qquad \bar{J}^{+}{}_{12}\ =\ i\,\bar{x}_{12}\,\partial^+\ .
\tag{26}
$$

The dynamical boosts are

$$
\begin{aligned}
J^- \ =\ & i\,x\,\frac{\partial\bar{\partial}\,+\,\frac{1}{2}\,\partial^{12}\,\bar{\partial}_{12}\,+\,\frac{1}{2}\,\partial^{34}\,\bar{\partial}_{34}}{\partial^+}\,-\,i\,x^-\,\partial\,+\,i\,\frac{\partial}{\partial^+}\left\{\theta^m\,\bar{\partial}_m\,+\,\frac{i}{4\sqrt{2}\,\partial^+}\,(d^m\,\bar{d}_m\,-\,\bar{d}_m\,d^m)\right\}\\
&-\frac{1}{2}\,\frac{\bar{\partial}_{12}}{\partial^+}\left\{\frac{1}{\sqrt{2}}\,\partial^+\,\theta^1\,\theta^2\,-\,\frac{\sqrt{2}}{\partial^+}\,\partial^1\,\partial^2\,+\,\frac{1}{\sqrt{2}\,\partial^+}\,d^1\,d^2\right\}\\
&-\frac{1}{2}\,\frac{\bar{\partial}_{34}}{\partial^+}\left\{\frac{1}{\sqrt{2}}\,\partial^+\,\theta^3\,\theta^4\,-\,\frac{\sqrt{2}}{\partial^+}\,\partial^3\,\partial^4\,+\,\frac{1}{\sqrt{2}\,\partial^+}\,d^3\,d^4\right\}\ ,
\end{aligned}
\tag{27}
$$

and its complex conjugate

$$
\begin{aligned}
\bar{J}^- \ =\ & i\,\bar{x}\,\frac{\partial\bar{\partial}\,+\,\frac{1}{2}\,\partial^{12}\,\bar{\partial}_{12}\,+\,\frac{1}{2}\,\partial^{34}\,\bar{\partial}_{34}}{\partial^+}\,-\,i\,x^-\,\bar{\partial}\,+\,i\,\frac{\partial}{\partial^+}\left\{\theta^m\,\bar{\partial}_m\,+\,\frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^m\,\bar{d}_m\,-\,\bar{d}_m\,d^m\,)\right\}\\
&-\frac{1}{2}\,\frac{\partial^{12}}{\partial^+}\left\{\frac{1}{\sqrt{2}}\,\partial^+\,\bar{\theta}_1\,\bar{\theta}_2\,-\,\frac{\sqrt{2}}{\partial^+}\,\bar{\partial}_1\,\bar{\partial}_2\,+\,\frac{1}{\sqrt{2}\,\partial^+}\,\bar{d}_1\,\bar{d}_2\right\}\\
&-\frac{1}{2}\,\frac{\partial^{34}}{\partial^+}\left\{\frac{1}{\sqrt{2}}\,\partial^+\,\bar{\theta}_3\,\bar{\theta}_4\,-\,\frac{\sqrt{2}}{\partial^+}\,\bar{\partial}_3\,\bar{\partial}_4\,+\,\frac{1}{\sqrt{2}\,\partial^+}\,\bar{d}_3\,\bar{d}_4\right\}\ .
\end{aligned}
\tag{28}
$$

In addition, we have new dynamical boosts obtained using the coset generators

$$
\begin{aligned}
J^{-\,12} &= [\, J^- \,,\, J^{12} \,] \;; \qquad \bar{J}^-_{12} \,=\, [\, \bar{J}^- \,,\, \bar{J}_{12} \,] \;, \\
J^{-\,34} &= [\, J^- \,,\, J^{34} \,] \;; \qquad \bar{J}^-_{34} \,=\, [\, \bar{J}^- \,,\, \bar{J}_{34} \,] \;,
\end{aligned} \tag{29}
$$

(which are not explicitly shown here). The dynamical supersymmetries in six dimensions are simply obtained by boosting the kinematical supersymmetries.

$$
i\, [\, \bar{J}^- \,,\, q^m{}_+ \,] \,\equiv\, \mathcal{Q}^m \;; \qquad i\, [\, J^- \,,\, \bar{q}_{+\,m} \,] \,\equiv\, \bar{\mathcal{Q}}_m \;. \tag{30}
$$

For example, the dynamical supersymmetries carrying a "1" index read

$$
\mathcal{Q}^1 \,=\, \frac{\bar{\partial}}{\partial^+}\, q^1_+ \,+\, \frac{\partial^{12}}{\partial^+}\, \bar{q}_{+\,2} \;; \qquad \bar{\mathcal{Q}}_1 \,=\, \frac{\partial}{\partial^+}\, \bar{q}_{+\,1} \,+\, \frac{\bar{\partial}_{12}}{\partial^+}\, q^2_+ \;, \tag{31}
$$

and satisfy

$$
\{\, \mathcal{Q}^1 \,,\, \bar{\mathcal{Q}}_1 \,\} \,=\, i\, \sqrt{2}\, \frac{1}{\partial^+}\, (\, \partial\, \bar{\partial} \,+\, \bar{\partial}_{12}\, \partial^{12} \,) \;, \tag{32}
$$

permitting the introduction of central charges into the theory by setting ∂^{12} to a constant, Z^{12}.

4 Oxidation From $d = 4$ To $d = 6$

Having built the six-dimensional SuperPoincaré algebra, we now focus on obtaining an invariant action to describe the $\mathcal{N} = (1, 1)$ theory.

As outlined in our proposal, we permit the superfields, dependence on the two new directions

$$
\phi \,=\, \phi\, (\, x^+, x^-, x, \bar{x}, x^{12}, \bar{x}_{12}, x^{34}, \bar{x}_{34} \,) \;, \tag{33}
$$

and define the extended six-dimensional d'Alembertian

$$
\Box_6 \,=\, 2\, \partial\, \bar{\partial} \,+\, \partial^{12}\, \bar{\partial}_{12} \,+\, \partial^{34}\, \bar{\partial}_{34} \,-\, 2\, \partial^+\, \partial^- \;. \tag{34}
$$

The key step is the generalization of the transverse derivatives. We define

$$
\overline{\nabla} \,=\, \bar{\partial} \,+\, \sigma\, \bar{d}_1\, \bar{d}_2\, \frac{\partial^{12}}{\partial^+} \,+\, \sigma\, \bar{d}_3\, \bar{d}_4\, \frac{\partial^{34}}{\partial^+} \;. \tag{35}
$$

where σ is a parameter that will be determined based on invariance requirements. The conjugate derivative reads

$$\nabla \;=\; \partial \,+\, \sigma\, d^1\, d^2\, \frac{\bar{\partial}_{12}}{\partial^+} \,+\, \sigma\, d^3\, d^4\, \frac{\bar{\partial}_{34}}{\partial^+} \;. \tag{36}$$

Our proposal for the $\mathcal{N} = (1,1)$ Yang-Mills action in six dimensions is then simply

$$72 \int d^6x \int d^4\theta\, d^4\bar{\theta}\, \mathcal{L} \;, \tag{37}$$

where

$$\begin{aligned}\mathcal{L} \;\; = \;\; & -\bar{\phi}\, \frac{\Box_6}{\partial^{+2}}\, \phi \;+\; \frac{4g}{3}\, f^{abc} \Big(\frac{1}{\partial^+}\, \bar{\phi}^a\, \phi^b\, \overline{\nabla}\, \phi^c + \frac{1}{\partial^+}\, \phi^a\, \bar{\phi}^b\, \nabla\, \bar{\phi}^c \Big) \\ & -g^2 f^{abc}\, f^{ade} \Big(\frac{1}{\partial^+} (\phi^b\, \partial^+ \phi^c) \frac{1}{\partial^+}\, (\bar{\phi}^d\, \partial^+\, \bar{\phi}^e) + \frac{1}{2}\, \phi^b \bar{\phi}^c\, \phi^d\, \bar{\phi}^e \Big) \;.\end{aligned} \tag{38}$$

In the next section, we will explicitly show that this action is left invariant by the $d = 6$ SuperPoincaré algebra.

4.1 Invariance of the Action

We intend to prove that the action in equation (37) is $SO(4)-$invariant (Lorentz invariance in six dimensions follows once little group invariance has been established).

We start by noting that the kinetic term is trivially $SO(4)$-invariant thanks to the inclusion of the two new derivatives in the d'Alembertian. The quartic interactions are obviously invariant since they do not depend on the transverse derivatives. Hence we focus purely on the cubic vertex

$$\frac{4g}{3}\, f^{abc} \int d^{10}x \int d^4\theta\, d^4\bar{\theta} \Big(\frac{1}{\partial^+}\, \bar{\phi}^a\, \phi^b\, \overline{\nabla}\, \phi^c \,+\, \frac{1}{\partial^+}\, \phi^a\, \overline{\phi}^b\, \nabla\, \overline{\phi}^c \Big) \;. \tag{39}$$

Since this term is manifestly invariant under each $SO(2)$, we need consider only the coset variations. These coset generators vary both the superfields and the generalized derivatives. For example,

$$\delta_{J^{12}}\, \phi \,\equiv\, \overline{\omega}_{12}\, J^{12}\, \phi \;=\; i\, \sqrt{2}\, \overline{\omega}_{12}\, \partial^+\, \theta^1\, \theta^2\, \phi \;, \tag{40}$$

where the chiral constraint has been used.

$$\delta_{J^{12}}\,\phi\,\equiv\,\overline{\omega}_{12}\,J^{12}\,\phi\,=\,\overline{\omega}_{12}\left\{\,\frac{i}{\sqrt{2}}\,\partial^+\,\theta^1\,\theta^2\,-\,i\,\frac{\sqrt{2}}{\partial^+}\,\partial^1\,\partial^2\,+\,\frac{i}{\sqrt{2}\,\partial^+}\,d^1\,d^2\,\right\}\phi\ , \qquad (41)$$

and

$$\delta_{J^{12}}\,\overline{\nabla}\,=\,\overline{\omega}_{12}\,[\,J^{12}\,,\,\overline{\nabla}\,]\,=\,\overline{\omega}_{12}\left(\,-\,\partial^{12}\,+\,\sigma\,\bar{d}_3\,\bar{d}_4\,\frac{\partial}{\partial^+}\,\right)\,. \qquad (42)$$

Invariance under $SO(4)$ is verified by doing a δ_J variation on the entire cubic vertex.

4.2 The Variation

The proposed three-point function reads

$$\mathbf{T}\,+\,\mathbf{T}^*\,=\,f_{abc}\int\frac{1}{\partial^+}\,\overline{\phi}^a\,\;\phi^b\,\;\overline{\nabla}\,\phi^c\quad+\quad f_{abc}\int\frac{1}{\partial^+}\,\phi^a\,\;\overline{\phi}^b\,\;\nabla\,\overline{\phi}^c\ . \qquad (43)$$

There are four coset generators and the aim is to show that each of them leaves this three-point function invariant. We start with the coset generator J^{12} (the details of this calculation are presented in Appendix **A**):

$$\begin{aligned}\delta_{J^{12}}\,(\,\mathbf{T}\,)\,=\,-\,[\,1\,+\,i\sqrt{2}\,\sigma\,]\int\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\partial^{12}\,\phi^c\\ +\,\sigma\int\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\bar{d}_3\,\bar{d}_4\,\frac{\partial}{\partial^+}\,\phi^c\ ,\end{aligned} \qquad (44)$$

and

$$\begin{aligned}\delta_{J^{12}}\,(\,\mathbf{T}^*\,)\,=\,[\,-\,\frac{i}{\sqrt{2}}\,+\,\sigma\,]\int\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,d^1\,d^2\,\frac{\partial}{\partial^+}\,\overline{\phi}^c\\ i\,\sqrt{2}\,\sigma\int\phi^a\,\frac{1}{\partial^+}\,\overline{\phi}^b\,\partial^{12}\,\phi^c\ .\end{aligned} \qquad (45)$$

Choosing

$$\sigma\,=\,\frac{i}{2\,\sqrt{2}}\ , \qquad (46)$$

ensures that

$$\delta_{J^{12}}\,(\,\mathbf{T}\,+\,\mathbf{T}^*\,)\,=\,0\ . \qquad (47)$$

Thus the generalized derivative reads

$$\overline{\nabla} \;=\; \bar{\partial} \,+\, \frac{i}{2\,\sqrt{2}}\,\bar{d}_1\,\bar{d}_2\,\frac{\partial^{12}}{\partial^+} \;+\; \frac{i}{2\,\sqrt{2}}\,\bar{d}_3\,\bar{d}_4\,\frac{\partial^{34}}{\partial^+}\ . \tag{48}$$

In deriving the above results, use has also been made of the inside-out relations, the identities listed in Appendix **B** and numerous partial integrations with respect to ∂^+, $\bar{\partial}$ and ∂.

Having fixed σ we move to the other generators of the coset. Complex conjugation tells us that

$$\delta_{\overline{J}_{12}}\quad (\,\mathbf{T}^* \,+\, \mathbf{T}\,) \,=\, 0\ , \tag{49}$$

while the remaining two variations proceed along identical lines

$$\delta_{J^{34}}\,(\,\mathbf{T} \,+\, \mathbf{T}^*\,) \,=\, 0\ ; \qquad \delta_{\overline{J}_{34}}\,(\,\mathbf{T} \,+\, \mathbf{T}^*\,) \,=\, 0\ . \tag{50}$$

This completes the proof of $SO(4)$ invariance for the three-point function.

Lorentz invariance in six dimensions is a direct consequence of little group invariance. Thus the $\mathcal{N} = (1,1)$ SuperYang-Mills theory in six dimensions is described by the light-cone action

$$\int d^6x \int d^4\theta\, d^4\bar{\theta}\,\mathcal{L}\ , \tag{51}$$

where,

$$\begin{aligned}\mathcal{L} = \qquad & -\bar{\phi}^a\,\frac{\Box_6}{\partial^{+2}}\,\phi^a \;+\frac{4g}{3}\,f^{abc}\left(\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\overline{\nabla}\,\phi^c + \frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\nabla\,\overline{\phi}^c\right) \\ & -g^2 f^{abc}\,f^{ade}\left(\frac{1}{\partial^+}\,(\phi^b\,\partial^+\phi^c)\frac{1}{\partial^+}\,(\overline{\phi}^d\,\partial^+\,\overline{\phi}^e) + \frac{1}{2}\,\phi^b\overline{\phi}^c\,\phi^d\,\overline{\phi}^e\right)\ .\end{aligned} \tag{52}$$

5 $(\mathcal{N} = 1, d = 6)$ Yang-Mills through Truncation

Oxidation (using the generalized derivatives) thus allows us to “lift” a supersymmetric theory to its parent version. This procedure is however, supercharge-preserving and does not permit us access to theories with less supersymmetry. This is where supersymmetric truncation is useful.

Supersymmetric truncation reduces the supersymmetries in a theory, one step at a time. We start by noting that [9]

$$\int d^6x\,d^4\theta\,d^4\bar{\theta}\,\mathcal{L} \;=\; \frac{1}{16}\int d^6x\,d^3\theta\,d^3\bar{\theta}\,\bar{d}_4\,d^4\,\mathcal{L}\,|_{\theta^4\,=\,\bar{\theta}_4\,=0}\ . \tag{53}$$

This truncation, produces two kinds of superfields, the first being bosonic

$$\phi^{(3)} \;=\; \phi\,|_{\theta^4\,=\,\bar\theta_4\,=0}\;\; , \tag{54}$$

and the second fermionic

$$\psi_4 \;=\; \bar d_4\,\phi\,|_{\theta^4\,=\,\bar\theta_4\,=0}\;\; . \tag{55}$$

This fermionic superfield may be eliminated in favor of the bosonic one thanks to the 'inside-out" constraints in equation (17). Additional truncation involves rewriting

$$\int d^6x\,d^3\theta\,d^3\bar\theta\,\mathcal{L} \;=\; -\frac{1}{9}\int d^6x\,d^2\theta\,d^2\bar\theta\,\bar d_3\,d^3\,\mathcal{L}\,|_{\theta^3\,=\,\bar\theta_3\,=0}\;\; , \tag{56}$$

which generates two new and *independent* superfields,

$$\phi^{(2)} \;=\; \phi^{(3)}\,|_{\theta^3\,=\,\bar\theta_3\,=0}\;\; , \tag{57}$$

and

$$\psi_3 \;=\; \bar d_3\,\phi^{(3)}\,|_{\theta^3\,=\,\bar\theta_3\,=0}\;\; , \tag{58}$$

We set the fermionic superfield (which produces Wess-Zumino couplings [9]) to zero and focus exclusively on the bosonic superfield.

The doubly truncated bosonic superfield now reads

$$\phi^{(2)}\,(y) \;=\; \frac{1}{\partial^+}\,A\,(y) \;+\; i\,\sqrt{2}\,\theta^1\,\theta^2\,\overline{C}_{12}\,(y) \;+\; \frac{i}{\partial^+}\,\theta^1\,\bar\chi_1(y) \;+\; \frac{i}{\partial^+}\,\theta^2\,\bar\chi_2(y)\;\; , \tag{59}$$

and carries the degrees of freedom relevant to the $N=1$ theory in six-dimensions (and the $N=2$ theory in four dimensions, an issue we will return to shortly).

We now apply relations (53) and (56) to our six-dimensional $\mathcal{N}=(1,1)$ action (equation 51) to obtain the light-cone superspace description of ($\mathcal{N}=1, d=6$) Yang-Mills. The calculation is fairly straightforward and yields

$$\int d^6x \int d^2\theta\,d^2\bar\theta\,\mathcal{L}\;, \tag{60}$$

where,

$$\begin{aligned}\mathcal{L} = &-\,2\,\bar\phi^{(2)\,a}\,\Box\,\phi^{(2)\,a}\\ &+\,4\,g\,f^{abc}\Big\{\,\partial^+\,\phi^{(2)\,a}\,\overline{\phi}^{(2)\,b}\,\bar\partial\,\phi^{(2)\,c} \,+\, \frac{i}{2\,\sqrt{2}}\,\partial^+\,\overline{\phi}^{(2)\,a}\,\phi^{(2)\,b}\,\bar d_1\,\bar d_2\,\frac{\partial^{12}}{\partial^+}\,\phi^{(2)\,c}\\ &+\,\frac{i}{2\,\sqrt{2}}\,\partial^+\,\phi^{(2)\,a}\,\overline{\phi}^{(2)\,b}\,d^1\,d^2\,\frac{\bar\partial_{12}}{\partial^+}\,\overline{\phi}^{(2)\,c}\Big\} \,+\, 4\,g\,f^{abc}\Big\{\text{complex conjugate}\Big\}\\ &-\,\frac{g^2}{2}\,f^{abc}\,f^{ade}\,\frac{d^2}{\partial^+}\,(\partial^+\,\phi^{(2)\,b}\,\bar\phi^{(2)\,c})\,\frac{\bar d^2}{\partial^+}\,(\partial^+\,\bar\phi^{(2)\,d}\,\phi^{(2)\,e})\;.\end{aligned} \tag{61}$$

We note that the superfields in the above expression are no longer constrained (although they still satisfy the chirality relations).

As expected, this action can be reduced to four dimensions, producing the $(\mathcal{N}=2, d=4)$ theory. This is easily verified - we simply remove the superfield dependence on the new coordinates thus setting

$$\partial^{12} \;=\; \bar{\partial}_{12} \;\rightarrow\; 0 \;. \tag{62}$$

This results in the light-cone description of four-dimensional $\mathcal{N}=2$ Yang-Mills [19, 20]

$$\int d^4x \int d^2\theta\, d^2\bar{\theta}\, \mathcal{L} \;, \tag{63}$$

$$\begin{aligned}
\mathcal{L} = &- 2\, \bar{\phi}^{(2)\,a} \,\Box\, \phi^{(2)\,a} \\
&+ \frac{4}{3}\, g\, f^{abc} \left\{ \partial^+ \phi^{(2)\,a}\, \overline{\phi}^{(2)\,b}\, \bar{\partial}\, \phi^{(2)\,c} \;+\; \partial^+ \overline{\phi}^{(2)\,a}\, \phi^{(2)\,b}\, \partial\, \overline{\phi}^{(2)\,c} \right\} \\
&- \frac{g^2}{2}\, f^{abc}\, f^{ade} \frac{d^2}{\partial^+} \left(\partial^+ \phi^{(2)\,b}\, \bar{\phi}^{(2)\,c}\right) \frac{\bar{d}^2}{\partial^+} \left(\partial^+ \bar{\phi}^{(2)\,d}\, \phi^{(2)\,e}\right) \;.
\end{aligned} \tag{64}$$

6 Concluding Remarks

Light-cone superspace offers an excellent stage to build Lorentz-invariant interactions of massless particles with arbitrary helicities. In this language for example, the entire classical $PSU(2,2|4)$-invariant $(\mathcal{N}=4, d=4)$ action can be written as the square of a single fermionic superfield [2]. The techniques presented here (and in the quoted references) should prove extremely useful in building light-cone actions for theories whose form is still unknown.

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Appendix A

Variation: $\delta_{J^{12}}\,(\,\mathbf{T}\,)$

From varying the first superfield,

$$\begin{aligned}\delta_{J^{12}}\,(\,\frac{1}{\partial^+}\,\overline{\phi}^a\,)\,\phi^b\,\overline{\nabla}\,\phi^c \;=\;& \frac{i}{\sqrt{2}}\,\theta^1\,\theta^2\,\Big\{\,\overline{\phi}^a\,\phi^b\,\overline{\nabla}\,\phi^c \,+\, \frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\partial^{+2}\,\phi^b\,\overline{\nabla}\,\phi^c \\ &+\,2\,\frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\partial^+\,\phi^b\,\partial^+\,\overline{\nabla}\,\phi^c \,+\, \frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\phi^b\,\partial^{+2}\,\overline{\nabla}\,\phi^c\,\Big\} \\ &+\,i\,\sqrt{2}\,\sigma\,\theta^1\,(\,\frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\partial^+\,\phi^b\,\bar{d}_1\,\partial^{12}\,\phi^c \,+\, \frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\phi^b\,\bar{d}_1\,\partial^{12}\,\partial^+\,\phi^c\,) \\ &+\,i\,\sqrt{2}\,\sigma\,\theta^2\,(\,\frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\partial^+\,\phi^b\,\bar{d}_2\,\partial^{12}\,\phi^c \,+\, \frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\phi^b\,\bar{d}_2\,\partial^{12}\,\partial^+\,\phi^c\,)\;.\end{aligned}$$

The variation on the second superfield is

$$\frac{1}{\partial^+}\,\overline{\phi}^a\,(\,\delta_{J^{12}}\,\phi^b\,)\,\overline{\nabla}\,\phi^c \;=\; i\,\sqrt{2}\,\theta^1\,\theta^2\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\partial^+\,\phi^b\,\overline{\nabla}\,\phi^c\;, \tag{A-1}$$

The third contribution is from the newly introduced derivative and reads

$$\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,(\,\delta_J\,\overline{\nabla}\,)\,\phi^c \;=\; -\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\partial^{12}\,\phi^c \,+\, \sigma\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\bar{d}_3\,\bar{d}_4\,\frac{\partial}{\partial^+}\,\phi^c\;. \tag{A-2}$$

The final term in the variation is

$$\begin{aligned}\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\overline{\nabla}\,(\,\delta_{J^{12}}\,\phi^c\,) \;=\;& -\,i\sqrt{2}\,\sigma\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\partial^{12}\,\phi^c \,+\, i\,\sqrt{2}\,\sigma\,\theta^1\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\bar{d}_1\,\partial^{12}\,\phi^c \\ &+\,i\,\sqrt{2}\,\sigma\,\theta^2\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\bar{d}_2\,\partial^{12}\,\phi^c\;.\end{aligned} \tag{A-3}$$

These terms simplify greatly (after some partial integrations) to

$$\delta_{J^{12}}\,(\int\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\overline{\nabla}\,\phi^c\,) \;=\; -\,[\,1\,+\,i\sqrt{2}\,\sigma\,]\int\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\partial^{12}\,\phi^c \,+\, \sigma\int\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b\,\bar{d}_3\,\bar{d}_4\,\frac{\partial}{\partial^+}\,\phi^c \tag{A-4}$$

Variation: $\delta_{J^{12}}\,(\mathbf{T}^*\,)$

$$\delta_{J^{12}}\,(\,\frac{1}{\partial^+}\,\phi^a\,)\,\overline{\phi}^b\,\nabla\,\overline{\phi}^c \;=\; i\,\sqrt{2}\,\theta^1\,\theta^2\,\phi^a\,\overline{\phi}^b\,\nabla\,\overline{\phi}^c \tag{A-5}$$

$$\frac{1}{\partial^+}\,\phi^a\,(\,\delta_{J^{12}}\,\overline{\phi}^b\,)\,\nabla\,\overline{\phi}^c\;=\frac{i}{\sqrt{2}}\,\theta^1\,\theta^2\,\frac{1}{\partial^+}\,\phi^a\,\partial^+\,\overline{\phi}^b\,\nabla\,\overline{\phi}^c\;-\;\frac{i}{\sqrt{2}}\,\frac{1}{\partial^+}\,\phi^a\,\frac{\partial^1\,\partial^2}{\partial^+}\,\overline{\phi}^b\,\nabla\,\overline{\phi}^c$$
$$+\;\frac{i}{\sqrt{2}}\,\frac{1}{\partial^+}\,\phi^a\,\frac{d^1\,d^2}{\partial^+}\,\overline{\phi}^b\,\nabla\,\overline{\phi}^c\;. \qquad \text{(A-6)}$$

$$\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,(\,\delta_{J^{12}}\,\nabla\,)\,\overline{\phi}^c\;=\;\sigma\,\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,d^1\,d^2\,\frac{\partial}{\partial^+}\,\overline{\phi}^c\;. \qquad \text{(A-7)}$$

$$\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\nabla\,(\,\delta_{J^{12}}\,\overline{\phi}^c\,)\;=\frac{i}{\sqrt{2}}\,\theta^1\,\theta^2\,\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\nabla\,\partial^+\,\overline{\phi}^c\;-\;i\,\sqrt{2}\,\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\nabla\,\frac{\partial^1\,\partial^2}{\partial^+}\,\overline{\phi}^c$$
$$+\;\frac{i}{\sqrt{2}}\,\frac{1}{\partial^+}\,\phi^a\,\overline{\phi}^b\,\nabla\,\frac{d^1\,d^2}{\partial^+}\,\overline{\phi}^c\;. \qquad \text{(A-8)}$$

Appendix B

Useful Identities

The inside-out constraints read

$$\bar{d}_p\,\bar{d}_q\,\phi\;=\;\frac{1}{2}\,\epsilon_{pqmn}\,d^m\,d^n\,\overline{\phi} \qquad ; \qquad \bar{d}_p\,\bar{d}_q\,\bar{d}_m\,\bar{d}_n\,\phi\;=\;2\,\epsilon_{pqmn}\,{\partial^+}^2\,\overline{\phi} \qquad \text{(B-1)}$$

Consequence #1,

$$f_{abc}\int\,\frac{1}{{\partial^+}^2}\,\overline{\phi}^a\;\phi^b\;\bar{\partial}\,\phi^c\;\;=\;\;0 \qquad \text{(B-2)}$$

Proof:

$$\int\,\frac{1}{{\partial^+}^2}\,\overline{\phi}^a\;\phi^b\;\bar{\partial}\,\phi^c=\int\,\frac{1}{{\partial^+}^2}\,\overline{\phi}^a\;\frac{3\,d^4}{{\partial^+}^2}\,\overline{\phi}^b\;\bar{\partial}\,\phi^c\;=\;\int\,\frac{3\,d^4}{{\partial^+}^2}\,\overline{\phi}^a\;\frac{1}{{\partial^+}^2}\,\overline{\phi}^b\;\bar{\partial}\,\phi^c$$
$$=\int\,\phi^a\;\frac{1}{{\partial^+}^2}\,\overline{\phi}^b\;\bar{\partial}\,\phi^c\;=\;0\;\;\text{(due to symmetry between the } a \text{ and } b \text{ indices)} \qquad \text{(B-3)}$$

Consequence #2

$$f_{abc}\int \frac{1}{\partial^+}\,\overline{\phi}^a\ \frac{1}{\partial^+}\,\phi^b\ d^m\,d^n\,\partial\,\overline{\phi}^c\ =\ 0 \tag{B-4}$$

Proof:

$$\begin{aligned}\int \frac{1}{\partial^+}\,\overline{\phi}^a\ \frac{1}{\partial^+}\,\phi^b\ d^m\,d^n\,\partial\,\overline{\phi}^c\ &= \int \frac{1}{\partial^+}\,d^m\,d^n\,\overline{\phi}^a\ \frac{1}{\partial^+}\,\phi^b\ \partial\,\overline{\phi}^c \\ &= \frac{1}{2}\,\epsilon^{mnpq}\int \frac{1}{\partial^+}\,\bar{d}_p\,\bar{d}_q\,\phi^a\ \frac{1}{\partial^+}\,\phi^b\ \partial\,\overline{\phi}^c \\ &= \frac{1}{2}\,\epsilon^{mnpq}\ \int \frac{1}{\partial^+}\,\phi^a\ \frac{1}{\partial^+}\,\bar{d}_p\,\bar{d}_q\,\phi^b\ \partial\,\overline{\phi}^c\ , \\ &= 0 \quad (a-b \text{ symmetry})\end{aligned} \tag{B-5}$$

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