

# Brownian Motion, Chern-Simons Theory, and 2d Yang-Mills

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## Abstract

We point out a precise connection between Brownian motion, Chern-Simons theory on  $S^3$ , and 2d Yang-Mills theory on the cylinder. The probability of reunion for  $N$  vicious walkers on a line gives the partition function of Chern-Simons theory on  $S^3$  with gauge group  $U(N)$ . The probability of starting with an equal-spacing condition and ending up with a generic configuration of movers gives the expectation value of the unknot. The probability with arbitrary initial and final states corresponds to the expectation value of the Hopf link. We find that the matrix model calculation of the partition function is nothing but the additivity law of probabilities. We establish a correspondence between quantities in Brownian motion and the modular  $S$ - and  $T$ -matrices of the WZW model at finite  $k$  and  $N$ . Brownian motion probabilities in the affine chamber of a Lie group are shown to be related to the partition function of 2d Yang-Mills on the cylinder. Finally, the random-turns model of discrete random walks is related to Wilson's plaquette model of 2d QCD, and the latter provides an exact two-dimensional analog of the melting crystal corner. Brownian motion provides a useful unifying framework for understanding various low-dimensional gauge theories.

# 1 Introduction

Low-dimensional gauge theories are a useful playground for understanding quantum field theory. They also play an important role in string theory, and in particular in topological string theories.

Relations between statistical mechanical systems and quantum field theory are well-known and have proved useful on both sides (see, for example, [1]). A well-known example is provided by conformal field theories in two dimensions. For the case of higher-dimensional topological theories, however, such examples have until recently remained somewhat limited. An important recent breakthrough was the realization [2] that the closed topological A-model vertex discovered in [3] is related to a melting crystal corner [4]. In fact, D-branes can be included and have a natural interpretation as defects [5]. In [4] this melting crystal picture was used as a definition of ‘quantum Kähler gravity’, the quantum gravitational theory describing fluctuations of the Kähler structure and topology while keeping the complex structure fixed. In [6], certain types of random walks were used to describe the combinatorics of triangulations of 2+1 quantum gravity. A review of the manifold connections between random walks/Brownian motion and conformal field theory and two-dimensional quantum gravity can be found in [7].

In this letter we provide further examples of precise reformulations of statistical mechanical systems in terms of gauge theories that are topological or close to topological. Full details will be given elsewhere [8]. Since the gauge theories in question have string theory realizations [9, 10], it is our hope that this relation can be useful in string theory.

The first example relates Brownian motion of  $N$  vicious — that is, non-intersecting — walkers on a line [11] with certain boundary conditions – or, equivalently, one particle moving in the Weyl chamber of a simply-laced, compact Lie group  $G$  – to Chern-Simons theory [12] on  $S^3$  for the corresponding group. The correspondence works for the partition function, the expectation value of the unknot, and the expectation value of the Hopf link invariant, the basic reason being the correspondence between the WZW modular  $S$ - and  $T$ - matrices and Brownian motion quantities. The additivity law of probabilities can be reinterpreted as a matrix model derivation of the partition function of Chern-Simons theory on  $S^3$ , where the repulsive force of the Vandermonde interaction translates into an effective repulsion exerted by the walls of the Weyl chamber. The correspondence is at finite values of  $k$  and  $N$ , where  $N$  is the number of movers and  $g_s = \frac{2\pi i}{k+N} = -\frac{1}{t}$  is the inverse time. This is reminiscent of relation found in [2], where  $t$  played the role of the temperature.

The second example concerns a walker moving in the fundamental Weyl chamber of an affine Lie algebra. This can be reinterpreted as the partition function of 2d Yang-Mills on the cylinder, where the initial and final positions correspond to states at the two ends of the cylinder, and now time is related to the area. Our derivation reproduces formulas in the mathematical literature which make the structure of the partition function much more transparent than the usual sum over representations. In particular, the partition function is shown to be an affine character, and so the modular properties of 2d YM on the cylinder are most explicit in this formulation.

The third example is the case of discrete random walkers rather than continuous Brownian motion; we analyze the random turns model [11, 13]. Its relation to randomly growing Young tableaux and (by Poissonization) to lattice QCD<sub>2</sub> are known in the mathematical literature. We reinterpret the lattice QCD<sub>2</sub> as a two-dimensional melting crystal, where the energy cost for removing a particle is the (logarithm of) the gauge coupling.

## 2 Brownian motion and Chern-Simons theory

In this paper we will be concerned with Brownian motion of  $N$  movers on a line. We will study so-called *vicious walkers* [11]. These are random walkers whose trajectories are not allowed to intersect. As is well-known, the intersection properties of such a random walk process plays an important role in quantum field theory (as reviewed in detail in [7, 1]).

Let us first consider the case of a single free walker performing Brownian motion on a line. The probability for it to travel from  $y$  to  $x$  in time  $t$  is given by

$$p_t(x, y) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-y)^2/4Dt} . \quad (1)$$

Here,  $D$  is the diffusion coefficient of the medium, which from now on we will set equal to  $\frac{1}{2}$ . This probability, and its generalizations, is the basic quantity that we will study in this paper. It can be obtained by a variety of methods; for an overview see e. g. [1, 14]. Let us however mention some of its important properties. First of all, it can be obtained as the continuum limit of a discrete random walk, where at each tick of the clock the particle can travel a fixed finite distance left or right with equal probability (see also section 3). The probability is then a binomial distribution whose continuum limit is Gaussian. Further,  $p_t(x, y)$  satisfies the diffusion or heat equation

$$\frac{\partial}{\partial t} p_t(x, y) = D\Delta p_t(x, y), \quad (2)$$

where  $\Delta$  is the Laplacian in  $x$ . Finally,  $p_t(x, y)$  turns into a Dirac delta function of the position  $x - y$  when  $t$  tends to zero.

The above has an obvious higher-dimensional generalization:

$$p_{t,N}(x, y) = \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|x-y|^2}{2t}}, \quad (3)$$

and  $N$  is the dimension. Equivalently, this can be regarded as the product of the probabilities of  $N$  single movers on a line, i.e. as the probability for  $N$  movers on a line to start at positions  $y_1, \dots, y_N$  and end up at  $x_1, \dots, x_N$  after time  $t$ .

The particular process that we will relate to Chern-Simons theory is that of  $N$  vicious walkers on a line. Walkers are vicious [11] if they annihilate each other when they meet. Thus, we will impose a non-intersecting condition and compute a probability for these walkers to walk from one configuration to another without ever intersecting. If we denote their coordinates by  $\lambda_i$ ,  $i = 1, \dots, N$ , they satisfy  $\lambda_1 > \lambda_2 > \dots > \lambda_N$ . Alternatively, this process can be regarded as motion of a single particle in the fundamental Weyl chamber of  $U(N)$ . The particle starts moving at position  $\mu_i$  satisfying  $\mu_1 > \mu_2 > \dots > \mu_N$ , and is required to stay within the Weyl chamber. The process stops when the particle hits one of the walls. One then computes the probability of going from an initial position  $\mu_i$  to a final position  $\lambda_i$  staying always within the chamber. This is given by [11]:

$$p_{t,N}(\lambda, \mu) = \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \det |e^{\lambda_i \mu_j / t}|_{1 \leq i < j \leq N} . \quad (4)$$

Obviously, this probability is symmetric under interchange of initial and final boundary conditions.

Let us now evaluate this amplitude in a specific case. We take the same initial and final boundary conditions, i.e.  $\mu = \lambda$ , and further an equal spacing condition, that is,  $\lambda_{0j} - \lambda_{0,j+1} = a$ ,

where  $a$  is the initial and final spacing between two neighboring movers. We compute the so-called probability of reunion — the probability that the movers go back to their (almost coinciding) positions after time  $t$ . Of course, this is an exponentially vanishing probability. Notice that, since the  $\lambda$ 's also label highest weights of irreducible representations of  $U(N)$ , this boundary condition is labeled by the Weyl vector for a suitable choice of the overall scale. Now a straightforward computation yields

$$p_{t,N}(\lambda_0, \lambda_0) = \frac{1}{(2\pi t)^{N/2}} \prod_{k=1}^N (1 - e^{-ka^2/t})^{N-k} . \quad (5)$$

We are going to relate this probability to the partition function of Chern-Simons theory.

Recall that Chern-Simons theory is a topological quantum field theory whose action is built of a Chern-Simons term involving as gauge field a gauge connection associated to a group  $G$  on a three-manifold  $M$  [12] (see [15] for a recent review). The action is:

$$S(A) = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) , \quad (6)$$

with  $k$  an integer number. Now if we choose units where  $a^2 = 1$  and identify

$$-\frac{1}{t} = g_s = \frac{2\pi i}{k + N} , \quad (7)$$

with  $g_s$  the string coupling, equation (5) is the partition function of Chern-Simons on  $S^3$  with gauge group  $U(N)$  [12]. Observables in Chern-Simons theory always come with a choice of framing, corresponding to a choice of trivialization of the tangent bundle in the gravitational Chern-Simons term [12]. In our case, notice that the framing is the matrix model framing [16, 17], which is related to the canonical framing as follows:

$$Z_{\text{CS}}(S^3) = e^{\frac{\pi i}{2} N^2} q^{-\frac{1}{12} N(N^2-1)} p_{t,N}(\lambda_0, \lambda_0) , \quad (8)$$

where the label 0 refers to the Weyl vector  $\lambda_0 = \rho$ , and as usual  $q = e^{g_s}$ .

One way to understand why we get the partition function of Chern-Simons theory is to generalize the above to other compact groups [11, 18]. Using the method of images, one finds that the above probability generalizes to

$$p_{t,r}(\lambda, \mu) = \frac{1}{(2\pi t)^{r/2}} e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \sum_{w \in W} \epsilon(w) e^{(\lambda, w\mu)/t} , \quad (9)$$

where  $r$  is the rank of  $G$  and  $W$  the Weyl group. From here, using the Weyl denominator formula we can get amplitudes for more general boundary conditions:

$$p_{t,r}(\lambda, \rho) = \frac{1}{(2\pi t)^{r/2}} e^{-\frac{|\lambda|^2 + |\rho|^2}{2t}} \prod_{\alpha > 0} 2 \sinh \frac{(\alpha, \lambda)}{2t} , \quad (10)$$

where  $\alpha$  are the positive roots and  $\rho$  is the Weyl vector. This expression is the (unnormalized) expectation value of a Wilson loop around the unknot. The partition function is obtained by setting  $\lambda = \rho$ . Normalizing (10) by the partition function gives the quantum dimension.

Somewhat more generally, taking into account the framing and the central charge we can in fact write, dropping an overall sign,

$$p_{t,r}(\lambda, \mu) = e^{\frac{2\pi i}{12} \dim g} (TST)_{\lambda\mu} , \quad (11)$$

where

$$\begin{aligned} S_{\lambda\mu} &= \frac{i^{|\Delta_+|}}{(k+g)^{r/2}} |P/Q^\vee|^{-\frac{1}{2}} \sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{k+g}(\lambda, w \cdot \mu)} \\ T_{\lambda\mu} &= \delta_{\lambda\mu} e^{\frac{2\pi i C(\lambda)}{2(k+g)} - \frac{2\pi i c}{24}}. \end{aligned} \quad (12)$$

The central charge is  $c = k \dim g / (k + g)$ ,  $C(\lambda)$  is the Casimir of the representation  $\lambda$ ,  $\Delta_+$  is the set of positive roots,  $P$  is the weight lattice, and  $Q^\vee$  is the coroot lattice. Obviously,  $S$  is the Brownian motion probability (with the external factors of  $T$  amputated), and  $T$  is the Boltzmann factor. It is now also clear that  $p(\lambda, \mu)$  itself corresponds to the (unnormalized) expectation value of the Hopf link invariant with representations  $\lambda$  and  $\mu$ .  $TST$  is in fact the operator that performs the modular transformation  $\tau \rightarrow \tau / (\tau + 1)$  (see also [16]). By this particular surgery [12] one obtains  $S^3$  out of two solid tori.

To end this section, let us remark that there is a matrix model expression for the partition function of Chern-Simons on  $S^3$  [19, 17]:

$$Z_{\text{CS}}(S^3) = \frac{e^{-\frac{1}{12} N(N^2-1) g_s}}{N!} \int \prod_{i=1}^N \frac{d\lambda_i}{2\pi} e^{-|\lambda|^2/2g_s} \prod_{i < j} \left( 2 \sinh \frac{\lambda_i - \lambda_j}{2} \right)^2. \quad (13)$$

It is not hard to check that this is nothing but the extensivity property of probabilities:

$$p_{t+t', r}(\rho, \rho) = \int [d\lambda] p_{t, r}(\rho, \lambda) p_{t', r}(\lambda, \rho), \quad (14)$$

where the range of integration is the same as in the matrix model, but using symmetry can be restricted to the Weyl chamber. This can also be seen as a renormalization group property. It is also clear that the repulsive  $\sinh$ , coming from the Vandermonde determinant, is related to the fact that the walls of the Weyl chamber are effectively *repelling*. Indeed, all the paths that end on these walls are suppressed from the expression for the probability, and hence only paths will contribute for which the particle stays away from the walls.

We saw that the partition function comes out in the natural matrix model framing. It would be interesting to see if one can obtain more generic framings by rescalings and shifts of the boundary conditions. For the case of the partition function and the unknot this seems possible [8]. This results in real exponential factors, as expected from the analytic continuation of the phase factors.

### 3 Brownian motion and QCD<sub>2</sub>

It is now natural to ask what happens if we consider Brownian motion in the fundamental Weyl chamber of an affine Lie algebra. The affine Weyl group is  $\tilde{W} = W \ltimes T$ , where  $T$  is the group of translations in root space. In the case analyzed in the previous section, constraining the motion to the fundamental Weyl chamber was achieved by appropriately adding all images generated by the action of  $W$ . Now, in addition, we have to mod out by translations in the coroot lattice. We get the following probability

$$q_{t, r}(\lambda, \mu) = \frac{1}{(2\pi t)^{r/2}} \sum_{\gamma \in lQ^\vee} \sum_{w \in W} \epsilon(w) e^{-\frac{1}{2t} |\gamma + \lambda - w \cdot \mu|^2}, \quad (15)$$

where  $r$  is the rank and  $l = k + g$ . Standard manipulations yield

$$q_{t,r}(\lambda, \mu) = \frac{1}{(2\pi t)^{r/2}} \sum_{w \in \hat{W}} e^{-\frac{1}{2t} |\hat{\lambda} - w \cdot \hat{\mu}|^2}, \quad (16)$$

for affine vectors  $\hat{\lambda} = \lambda + l\hat{\omega}_0$ , following common notation  $\hat{\omega}_0 = (0; 1; 0)$ . In fact, the normalized probability is the affine character  $\text{ch}_{\hat{\lambda}}(\hat{\mu}/t)$ , in complete analogy with the finite case. With our boundary conditions, formula (16) agrees with the general result found in [20].

Using the results in [21], it is not hard to see that this probability is related to the partition function of 2d Yang-Mills theory on the cylinder. Two-dimensional Yang-Mills (see [22] for a review) is not a topological theory, but it is close to that in the sense that it has no local degrees of freedom. It is invariant under area-preserving diffeomorphisms. On the cylinder, the partition function is [23]:

$$Z_{2\text{dYM}}(g, g'; t) = \sum_{\lambda \in P_{++}} \chi_{\lambda}(g^{-1}) \chi_{\lambda}(g') e^{-\frac{t}{2} C(\lambda) - \frac{t}{2} |\rho|^2}, \quad (17)$$

where  $P_{++}$  are the dominant weights<sup>1</sup>. The gauge coupling  $g_{\text{YM}}$  and the area of the cylinder  $A$  always appear through the combination  $\frac{1}{2} t = g_{\text{YM}}^2 A$ .

From the cylinder one can easily obtain the partition function for other two-dimensional manifolds. A salient feature of the partition function on the cylinder is its extensivity. If we glue two cylinders of areas  $A_1$  and  $A_2$  along a common boundary, the partition function on the resulting cylinder of area  $A' = A_1 + A_2$  will have the same form, with new area  $A'$ . An analogous property also holds on the disk, and can be seen as an invariance under renormalization group transformations and indicates that the plaquette model for this theory is in fact exact. Notice that the area is linearly related to the Brownian motion time  $t$ , and so this is like the additivity property (14) that we encountered before.

As is well-known, the partition function of 2d Yang-Mills is a solution of the heat equation on a group manifold. It is the kernel of the heat equation defined such that it satisfies the heat equation for both  $g$  and  $g'$ , and in the limit  $t \rightarrow 0$  it tends to a Dirac delta function  $\delta(g - g')$  with respect to the Haar measure. From it one can construct the unique solution of the heat equation with specified boundary condition at  $t = 0$  [24].

Let us now state the precise relation between Brownian motion and 2d Yang-Mills. We consider the group elements  $g = e^{2\pi i \lambda / l}$ ,  $g' = e^{2\pi i \mu / l}$ , where  $\lambda' = 2\pi i \lambda / l$ . Using the manipulations in [21], we find

$$Z_{2\text{dYM}}(g, g'; t) = \frac{l^r \text{vol}(P/Q^{\vee})}{D_{\rho}(2\pi i \lambda / l) D_{\rho}(2\pi i \mu / l)} q_{t,r}(\lambda, \mu), \quad (18)$$

where  $q_{t,r}$  is given in (16). The normalization factors  $D_{\rho}$  come from the normalization of the characters that enter  $Z_{2\text{dYM}}$ , whereas  $q_{t,l}$  is always unnormalized, as we also saw in the previous section. In particular, the normalization is independent of  $t$  and thus the  $t$ -dependence is totally contained in  $q_{t,r}$ . The above relation is our main result in this section.

It is well-known that the partition function of 2d YM on the sphere does not have nice modular transformation properties. The reason is that it is proportional to the derivative of a modular form, rather than the modular form itself. For the case of the torus, see [26, 25]. The modular properties of the partition function on the cylinder are easy to work out from the above. It suffices to realize that  $q_{t,r}(\lambda, \mu)$  is an affine character, which is obtained from a theta function by summing over all images. We hope to come back to this issue [8]. Let us

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<sup>1</sup>For convenience we included a constant factor of  $|\rho|^2$ .

here remark that there is a special group element which one can insert, so that the partition function on the sphere does have nice modular properties. This is the special group element defined in [27, 24] such that the character has values  $0, \pm 1$ . Let us denote by  $|a\rangle$  the associated state. We can then interpret  $Z_{2\text{dYM}}(a; t)$  as a partition function on the sphere with insertion of a state  $|a\rangle$  at  $t = 0$ . The partition function can be written in terms of the Dedekind  $\eta$ -function by Macdonald's  $\eta$ -function formula [28]:

$$Z_{2\text{dYM}}(a; t) = e^{-\frac{t}{24} \dim G} \eta(t)^{\dim G} . \quad (19)$$

We will end this section with some comments on one-dimensional discrete random walks and their relation to growing Young tableaux and Wilson's plaquette model<sup>2</sup>. Brownian motion is a limiting case of this [11]. We will focus on the random-turns model [11, 13], where  $N$  movers are allowed to move on a (one-dimensional) lattice, but at each tick of the clock only one mover walks, and it can take a step left or right with equal probability. It is known [30, 13] that the probability of reunion after  $2n$  steps in this model is given by the expectation value of  $2n$  powers of a unitary matrix in a unitary ensemble of  $N \times N$  matrices. This probability is also proportional to the probability of the largest increasing subsequence of a random permutation of  $n$  objects of having length  $\leq N$ . By the Schensted correspondence between random permutations and Young diagrams, this is also the probability for the top row of a Young diagram with  $n$  boxes to have length  $\leq N$ . Let us call this probability  $P(l_n \leq N)$ . The Gross-Witten model is obtained by studying the Poissonized quantity, i.e. studying all possible (Poisson distributed) Young tableaux whose top row has length  $L_\lambda \leq N$ :

$$\begin{aligned} P(L_\lambda \leq N) &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n P(l_n \leq N) \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \sum_{\mu \vdash n, \mu_1 \leq N} \frac{1}{n!} \lambda^n \frac{d_\mu^2}{n!}, \end{aligned} \quad (20)$$

where  $L_\lambda$  is the length of the top row of a set of diagrams that are Poisson distributed, and  $\mu \vdash n$  means that  $\mu$  partitions  $n$ , and  $d_\mu$  is given by the hook formula of the diagram. But  $\frac{d_\mu^2}{n!}$  is the Plancherel measure, which is also the probability to pick a Young diagram of shape  $\mu$  among a random set. Thus, the above is the grand canonical ensemble distribution for tableaux with top rows of lengths  $L_\lambda \leq N$ , and  $\log \lambda$  is the chemical potential for adding a box anywhere in the diagram so that it remains a valid  $U(N)$  tableau. If we identify  $\lambda$  with the gauge coupling, we obtain Wilson's lattice version of QCD<sub>2</sub>

$$Z_{\text{GW}} = \int dU \exp\left[\frac{1}{g^2} \text{Tr}_F(U + U^\dagger)\right] = e^\lambda P(L_\lambda \leq N) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{1}{g^{4n}} Z_{2n}(\mu_j = j, \lambda_i = i) , \quad (21)$$

where  $\lambda = 1/g^4$ . The trace is taken in the fundamental representation of  $U(N)$ . The last equality relates the partition function to the random walks probability. The plaquette model was solved at finite  $N$  in [31]. In [32] it was found that the model has a third-order phase transition<sup>3</sup> at large  $N$ . Thus, lattice QCD<sub>2</sub> can be reformulated as a growing (or shrinking) Young tableau.

<sup>2</sup>There are various possible reformulations of the discrete random walks problem, that we will not consider here: as a Hammersley process, a growing PNG droplet, etc. See e.g. [29].

<sup>3</sup>It is not hard to see that the phase transition precisely comes from the restriction on the number of boxes in the top row to be  $\leq N$ .

The lattice model of  $\text{QCD}_2$  is celebrated for its phase transition. This occurs at  $\lambda = g^2 N = 2$  and is closely related to a similar phase transition in the probabilities of the random distribution. Indeed, [33] has proved a depoissonization lemma that bounds the value of the probability of the random distribution from above and from below with its Poissonized value. The phase transition for the random Young diagram occurs when the number of boxes grows as  $n \sim 2\sqrt{N}$ .

Finally, let us comment that also in the case of the growing Young tableau there is a limiting shape after appropriate rescaling with  $N$ . This limiting shape is given by a continuous (but non-differentiable) function, the discontinuity being of course at the point of the phase transition. This is a two-dimensional analogue of the limiting shape of the 3d partitions found in [2].

## 4 Discussion and outlook

In this letter we have pointed out three connections: between Brownian motion in the fundamental Weyl chamber of a simply-laced, compact Lie group, and Chern-Simons theory on  $S^3$  for the corresponding group; between Brownian motion in the Weyl chamber of an affine Lie group and 2d Yang-Mills theory; and between the random-turns model of discrete random walks, a two-dimensional melting crystal, and lattice  $\text{QCD}_2$ . The fact that the connections are quite general — in the case of Chern-Simons, we get a statistical mechanical realization of the modular matrices — and work for various gauge groups, might lead one to think that there may be more to the relation between random/diffusion walks and gauge theories than just a mathematical coincidence, and one might hope to find physical applications. Therefore these connections immediately raise several questions. First of all, how far does the correspondence go? In the case of Chern-Simons, we saw that the agreement can be understood from the representation of the modular  $S$ - and  $T$ -matrices as non-intersecting Brownian motion probabilities and Boltzmann factors, respectively. Taking into account the role of  $SL(2, \mathbb{Z})$  in the surgery approach of Chern-Simons theory [9], one may hope that for manifolds other than  $S^3$  at least the simplest cases may have an interpretation in terms of Brownian motion quantities. This is certainly worth exploring further. Also, the basic cases of the partition function, the unknot and the Hopf link can be easily obtained from Brownian motion. It would be interesting to see if more general knots can also be obtained.

In the affine case, where one finds the partition function of 2d Yang-Mills on the cylinder, it would be interesting to see if one can extend the connection to expectation values of Wilson lines, or whether there is a Brownian motion interpretation of the partition function on the three-punctured sphere.

A particularly interesting point would be to see if Brownian motion can give us further connections between these low-dimensional theories. For example, the fact that Brownian motion is a limit of a discrete random walk is very suggestive of a connection between two-dimensional and three-dimensional theories. Also, it would be interesting to see whether the lock-step model [11], which we have not considered in this paper, also has a reformulation in terms of a low-dimensional gauge theory. Notice also that the way 2d Yang-Mills arises from Brownian motion is by effectively compactifying the Cartan subalgebra to a torus. The resulting expression is an affine character, which makes the modular properties of the partition function completely explicit. A connection between 2d Yang-Mills on the torus and topological strings has recently been pointed out in [26].

We saw that the matrix model of Chern-Simons theory on  $S^3$  naturally arises in the composition law of probabilities. Indeed, since we are dealing with continuous paths it seems that the matrix model formulation of Chern-Simons theory is the most natural one for Brownian motion.

Yet from the point of view of the WZW model one naturally gets sums over representations rather than integrals. At the level of the intermediate states, there is a precise way in which both approaches are equivalent [8]. In Chern-Simons theory the representations are integral, and from the Brownian motion point of view these correspond to special points on the line. We can deal more generally with arbitrary points by using characters. It would also be interesting to explore the connection with the fermionic representation.

On the more mathematical side, the underlying principle allowing these connections seems to be the fact that all these models in one way or another satisfy the heat equation. It has been known for a long time that certain quantities in the WZW model satisfy the heat equation [34]. Also, the heat equation is closely related to modular invariance. We hope to report more on this in the future [8].

Perhaps one of the most interesting questions is whether Brownian motion can be used as a tool in string theory, in the spirit of [2, 4], for example. Chern-Simons theory is the effective gauge theory describing the topological A-model [9], and so a reformulation in terms of Brownian motion might be useful for string theory itself. Notice furthermore that the natural string theory coupling is related to the Brownian motion parameter  $t$  (which is actually the product of the time parameter and the diffusion coefficient) as  $-\frac{1}{t} = g_s$ , whereas the relation to the Chern-Simons coupling involves analytic continuation<sup>4</sup>. This suggests that the interpretation of topological strings in terms of a statistical mechanical system may in some ways be more natural than as a gauge theory. In particular, it would be extremely interesting to understand whether the heat equation plays a role in the topological A-model. One would also like to see if discrete random walks — of which Brownian motion is a limit — are related to topological strings. Notice that [35] have used random partitions — which, as pointed out, are equivalent to the random walks model and, after Poissonization, to the plaquette model of  $\text{QCD}_2$  — to compute the prepotential of  $\mathcal{N} = 2$  SYM theory.

Another interesting question is whether 2d Yang-Mills and lattice  $\text{QCD}_2$ , and their respective third-order phase transitions, have string theory interpretations. In [36] it was shown that the  $\text{QCD}_2$  plaquette model can be used to obtain the  $SU(2)$   $\mathcal{N} = 2$  Seiberg-Witten solution by taking a double scaling limit. In particular, a local Calabi-Yau geometry that engineers this curve was found. In this case, it was argued that the phase transition plays no role. In [37], 2d Yang-Mills on a Riemann surface  $\Sigma$  was obtained by wrapping D6 branes on  $S^1 \times \Sigma$ .

We saw that the Gross-Witten phase transition does play a role in the context of the shrinking two-dimensional Young tableau. It was related to the non-differentiability of the limiting shape. It would be interesting to see whether such phase transitions are present and play any role for the topological vertex.

Let us also mention that Cardy [38] has recently found a remarkable connection between a system of  $N$  non-intersecting Brownian motions (described through the celebrated SLE process) and boundary-bulk conformal field theory models and integrable models of Sutherland type.

Finally, recall that [39] pointed out connections between vertex models and Chern-Simons theory. The random walks that we have mentioned in this paper are special cases of vertex models. However, for us the connection with Chern-Simons theory appears in the continuous limit of Brownian motion rather than in the discrete case.

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<sup>4</sup>This analytic continuation is a subtle issue that we hope to get back to in the future.

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