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The superfluid two-stream instability

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ABSTRACT

This paper provides the first study of a new dynamical instability in superfluids. This instability is similar to the two-stream instability known to operate in plasmas. It is analogous to the Kelvin–Helmholtz instability, but has the distinguishing feature that the two fluids are interpenetrating. The instability sets in once the relative flow between the two components of the system reaches a critical level. Our analysis is based on the two-fluid equations that have been used to model the dynamics of the outer core of a neutron star, where superfluid neutrons are expected to coexist with superconducting protons and relativistic electrons. These equations are analogous to the standard Landau model for superfluid helium. We study this instability for two different model problems. First we analyse a local dispersion relation for waves in a system where one fluid is at rest while the other flows at a constant rate. This provides a proof of principle of the existence of the two-stream instability for superfluids. Our second model problem concerns two rotating fluids confined within an infinitesimally thin spherical shell. The two model scenarios are physically distinct: in the first model the two fluids are coupled 'chemically' and the instability sets in through acoustic waves, while in the second problem the fluids are only coupled via the entrainment effect and the instability is associated with the superfluid r modes. The two scenarios illustrate that the instability mechanism is generic, and that it may set in through various modes of oscillation. We briefly discuss whether there are conditions, e.g. in the inner crust of a mature neutron star, where the coupling between the two fluids is sufficiently strong that the instability sets in at a relative flow small enough to be astrophysically plausible.

Key words: instabilities – stars: neutron.

1 INTRODUCTION

In this paper we describe a new dynamical instability in superfluids. This two-stream instability is analogous to the Kelvin–Helmholtz instability (Drazin & Reid 1981). The key distinguishing feature of the *two-stream* instability is that the two fluids are interpenetrating rather than in contact across an interface as in the standard Kelvin–Helmholtz scenario. The two-stream instability is well known in plasma physics [where it is sometimes referred to as the 'Farley–Buneman' instability (Farley 1963; Buneman 1963, 1959)], and it has also been discussed in various astrophysical contexts like merging galaxies (Lovelace, Jore & Haynes 1997) and pulsar magnetospheres (Cheng & Ruderman 1977; Weatherall 1994; Lyubarsky 2002), but as far as we are aware it has not been previously considered for superfluids. In fact, the 'standard' Kelvin–Helmholtz instability was only recently observed in the context of two superfluid phases separated by an interface (Blaauwgeers et al. 2002). A theoretical description can be found in Volovik (2002) and references therein.

The similarity of the equations used in plasma physics [a nice pedagogical description of the plasma two-stream instability can be found in Anderson, Fedele & Lisak (2001)] to the ones that have been extensively used for two-fluid superfluid models [see, for instance, Mendell (1991a,b), Langlois, Sedrakian & Carter (1998), Andersson & Comer (2001), Comer (2002) and Prix (2004)] inspired us to ask whether an analogous instability could be relevant for superfluids. That this ought to be the case seemed inevitable. To prove the veracity of this expectation, we have adapted the arguments from the plasma problem to the superfluid case, and discuss various aspects of the two-stream instability in this paper.

Of particular interest to us is the possibility that the two-stream instability may operate in rotating superfluid neutron stars. Mature neutron stars are expected to be sufficiently cold (e.g. below 10^9 K) that their interiors contain several superfluid/superconducting components. Such loosely coupled components are usually invoked to explain the enigmatic pulsar glitches, sudden spin-up events where the observed spin rate jumps by as much as one part in 10^6 (Lyne,

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Shemar & Graham Smith 2000). Theoretical models for glitches (Baym et al. 1969; Ruderman 1969; Anderson & Itoh 1975) have been discussed ever since the first Vela pulsar glitch was observed in 1969 (Radakrishnan & Manchester 1969; Reichley & Downs 1969), but these events are still not well understood. After three decades of theoretical effort it is generally accepted that the glitches arise because a superfluid component can rotate at a rate different from that of the bulk of the star. A sudden transfer of angular momentum from the superfluid to the crust of the star could lead to the observed phenomenon. The relaxation following the glitch is well explained in terms of vortex creep [see for example Cheng et al. (1988)], but the mechanism that triggers the glitch event remains elusive. In this context, it seems plausible that the superfluid two-stream instability may turn out to be relevant.

Any realistic neutron star model must cover a diverse collection of physical phenomena. This obviously makes the modelling problem extremely difficult. Fortunately, if one is mainly interested in the dynamics of the superfluid constituents one can make several, reasonably justified, simplifications. The problem becomes particularly tractable in the outer core region where the density is such that one expects to find superfluid neutrons, superconducting protons and a normal fluid of highly degenerate electrons. The analysis in this paper is based on equations that are expected to be valid in this region. The problem can be discussed in terms of two coupled fluids if one assumes that the electromagnetic interaction forces the electrons to track the protons very closely. This coupling acts on very short timescales, while superfluidity allows the neutrons to be decoupled and thus function as an independent fluid. Furthermore, since the average distance between vortices (by means of which the superfluid mimics bulk rotation) is small one can perform a smooth-averaging over them for both stationary and pulsating configurations (Langlois et al. 1998; Comer, Langlois & Lin 1999). This means that, despite being superfluid, the neutrons can be treated as an ordinary fluid. Mendell (1991a) has determined that this two-fluid model, which neglects other effects like magnetic fields, vortex pinning, etc., can be reliably applied when the core matter oscillations are of suitably high frequency.

Although the neutrons and protons can be considered as independent fluids they are still affected by the strong force. As a result the bare neutrons (or protons) are 'dressed' by a polarization cloud of nucleons comprising both neutrons and protons. Since both types of nucleon contribute to the cloud the momentum of the neutrons, say, is modified so that it is a linear combination of the neutron and proton particle number density currents (the same is true for the proton momentum). Thus the fluids exhibit the 'entrainment' effect, which means that when one of the fluids begins to flow it will induce a momentum in the other. Because of entrainment a portion of the protons (and electrons) will be pulled along with the superfluid neutrons that surround the vortices. This motion leads to magnetic fields being attached to the vortices and dissipative scattering of electrons off these magnetic fields. This 'mutual friction' is expected to provide one of the main dissipative mechanisms in superfluid neutron star cores (Mendell 1991b).

In the next section, we will use a local analysis of the linearized two-fluid equations to provide a proof of principle for the two-stream instability. The analysed oscillation modes have a predominately acoustic nature and depend mainly on the local equation of state of the matter. To gain some insight into the possibility that this instability may act in neutron stars, we consider a suitably realistic neutron-star equation of state. In Section 3, we consider a qualitatively different situation by performing a global analysis of modes of two rotating fluids confined within an infinitesimally thin spherical shell. In this case, the considered modes are inertial rather than acoustic. In fact, they are closely related to the r modes of rotating single-fluid objects (Papaloizou & Pringle 1978; Provost, Berthomieu & Rocca 1981; Andersson & Comer 2001). Because of their inertial character, entrainment is the main coupling agent that leads to the two-stream instability operating for these modes.

2 PROOF OF PRINCIPLE: A LOCAL ANALYSIS

2.1 Superfluid dispersion relation

We take as our starting point the two-fluid equations that have been used to model superfluid neutron stars in a variety of contexts (Mendell 1991a; Andersson & Comer 2001; Prix 2004). Hence, we consider superfluid neutrons (index n) coexisting with a conglomerate of charged components (index p). The corresponding equations are (Andersson & Comer 2001; Prix 2004)

$$\partial_t n_X + \nabla \cdot (n_X \boldsymbol{v}_X) = 0, \tag{1}$$

where n_X represent the respective number densities and $\boldsymbol{v}_X$ are the two velocities. Here, and in the following, we use the constituent index X which can be either n or p. (Note that a repeated constituent index never implies summation.) The respective mass-densities are obviously given by $\rho_X = m_X n_X$ and we further introduce the relative velocity $\boldsymbol{\Delta}$ between the two fluids as

$$\boldsymbol{\Delta} \equiv \boldsymbol{v}_{\mathrm{p}} - \boldsymbol{v}_{\mathrm{n}}. \tag{2}$$

The first law of thermodynamics is defined by the differential of the energy density or 'equation of state', $\mathcal{E} = \mathcal{E}(n_{\mathrm{n}}, n_{\mathrm{p}}, \Delta^2)$, namely

$$\mathrm{d}\mathcal{E} = \mu^{\mathrm{n}}\, \mathrm{d}n_{\mathrm{n}} + \mu^{\mathrm{p}}\, \mathrm{d}n_{\mathrm{p}} + \alpha\, \mathrm{d}\Delta^2, \tag{3}$$

which defines the chemical potentials μ^X and the 'entrainment' function α as the thermodynamic conjugates to the densities and the relative velocity. With these definitions we can write the two Euler-type equations:

$$(\partial_t + \boldsymbol{v}_{\mathrm{n}} \cdot \nabla)(\boldsymbol{v}_{\mathrm{n}} + \varepsilon_{\mathrm{n}} \boldsymbol{\Delta}) + \nabla(\Phi + \widetilde{\mu}^{\mathrm{n}}) + \varepsilon_{\mathrm{n}} \Delta_j \nabla v_{\mathrm{n}}^j = 0 \tag{4}$$

$$(\partial_t + \boldsymbol{v}_{\mathrm{p}} \cdot \nabla)(\boldsymbol{v}_{\mathrm{p}} - \varepsilon_{\mathrm{p}} \boldsymbol{\Delta}) + \nabla(\Phi + \widetilde{\mu}^{\mathrm{p}}) - \varepsilon_{\mathrm{p}} \Delta_j \nabla v_{\mathrm{p}}^j = 0 \tag{5}$$

where we have defined $\widetilde{\mu}^X = \mu^X / m_X$ and introduced the dimensionless entrainment parameters

$$\varepsilon_X \equiv \frac{2\alpha}{\rho_X}. \tag{6}$$

In the following we will assume that $m_{\mathrm{n}} = m_{\mathrm{p}} = m$. The equation for the gravitational potential Φ is

$$\nabla^2 \Phi = 4\pi G \rho, \tag{7}$$

where $\rho = \rho_{\mathrm{n}} + \rho_{\mathrm{p}}$. When $\alpha \neq 0$ these equations make manifest the entrainment effect.

In order to establish the existence of the superfluid two-stream instability we consider the following model problem. Let the unperturbed configuration be such that the 'protons' remain at rest, while the neutrons flow with a constant velocity v_0. For simplicity, we neglect the coupling through entrainment (even though it should be noted that it may have a significant effect), i.e. we take $\alpha = 0$, and we also neglect perturbations in the gravitational potential. Under these assumptions, the two fluids are only coupled 'chemically' through the equation of state.

Writing the two velocities as $\boldsymbol{v}_{\rm n} = [v_0 + \delta v_{\rm n}(t,x)]\hat{x}$ and $\boldsymbol{v}_{\rm p} = \delta v_{\rm p}(t,x)\hat{x}$ where $\delta v_{\rm n}$ and $\delta v_{\rm p}$ are taken to be suitably small, we get the perturbation equations

$$\partial_t \delta n_{\rm n} + v_0 \partial_x \delta n_{\rm n} + n_{\rm n} \partial_x \delta v_{\rm n} = 0, \tag{8}$$

$$\partial_t \delta n_{\rm p} + n_{\rm p} \partial_x \delta v_{\rm p} = 0, \tag{9}$$

and

$$\partial_t \delta v_{\rm n} + v_0 \partial_x \delta v_{\rm n} + \partial_x \delta \widetilde{\mu}^{\rm n} = 0, \tag{10}$$

$$\partial_t \delta v_{\rm p} + \partial_x \delta \widetilde{\mu}^{\rm p} = 0. \tag{11}$$

Next, we assume harmonic dependence on both t and x, i.e. we use the Fourier decomposition $\delta v_X(t,x) = \delta v_X \exp[{\rm i}(\omega t - kx)]$. This leads to the four equations

$${\rm i}(\omega - k v_0)\delta n_{\rm n} - {\rm i}k n_{\rm n} \delta v_{\rm n} = 0, \tag{12}$$

$${\rm i}\omega \delta n_{\rm p} - {\rm i}k n_{\rm p} \delta v_{\rm p} = 0, \tag{13}$$

$${\rm i}(\omega - k v_0)\delta v_{\rm n} - {\rm i}k \delta \widetilde{\mu}^{\rm n} = 0, \tag{14}$$

$${\rm i}\omega \delta v_{\rm p} - {\rm i}k \delta \widetilde{\mu}^{\rm p} = 0. \tag{15}$$

We thus have four equations relating the six unknown variables δv_X, δn_X and $\delta \widetilde{\mu}^X$. To close the system we need to provide an equation of state. Given an energy functional $\mathcal{E} = \mathcal{E}(n_{\rm n}, n_{\rm p})$ (and letting $m = m_{\rm n} = m_{\rm p}$) we have

$$\begin{aligned} m\delta\widetilde{\mu}^{\rm n} &= \left(\frac{\partial \widetilde{\mu}^{\rm n}}{\partial n_{\rm n}}\right)\bigg|_{n_{\rm p}} \delta n_{\rm n} + \left(\frac{\partial \widetilde{\mu}^{\rm n}}{\partial n_{\rm p}}\right)\bigg|_{n_{\rm n}} \delta n_{\rm p} \\ &= \frac{\partial^2 \mathcal{E}}{\partial n_{\rm n}^2}\delta n_{\rm n} + \frac{\partial^2 \mathcal{E}}{\partial n_{\rm p} \partial n_{\rm n}}\delta n_{\rm p} \end{aligned} \tag{16}$$

and similarly

$$m\delta\widetilde{\mu}^{\rm p} = \frac{\partial^2 \mathcal{E}}{\partial n_{\rm p} \partial n_{\rm n}}\delta n_{\rm n} + \frac{\partial^2 \mathcal{E}}{\partial n_{\rm p}^2}\delta n_{\rm p}. \tag{17}$$

Finally, we define the two sound speeds by, cf. Andersson & Comer (2001),

$$c_{\rm n}^2 = n_{\rm n} \left.\frac{\partial \widetilde{\mu}^{\rm n}}{\partial n_{\rm n}}\right|_{n_{\rm p}} = \frac{n_{\rm n}}{m}\frac{\mathcal{S}_{\rm pp}}{\det \mathcal{S}}, \tag{18}$$

$$c_{\rm p}^2 = n_{\rm p} \left.\frac{\partial \widetilde{\mu}^{\rm p}}{\partial n_{\rm p}}\right|_{n_{\rm p}} = \frac{n_{\rm p}}{m}\frac{\mathcal{S}_{\rm nn}}{\det \mathcal{S}}, \tag{19}$$

and introduce the 'coupling parameter'

$$\mathcal{C} = n_{\rm n} \left.\frac{\partial \widetilde{\mu}^{\rm n}}{\partial n_{\rm p}}\right|_{n_{\rm n}} = n_{\rm n} \left.\frac{\partial \widetilde{\mu}^{\rm p}}{\partial n_{\rm n}}\right|_{n_{\rm p}} = -\frac{n_{\rm n}}{m}\frac{\mathcal{S}_{\rm np}}{\det \mathcal{S}} \tag{20}$$

which also has the dimension of a velocity squared. For later convenience we have given the relation to the coefficients of the 'structure matrix' $\mathcal{S}_{XY}$ used by Prix, Comer & Andersson (2002).

With these definitions we get

$$n_{\rm n}\delta\widetilde{\mu}^{\rm n} = c_{\rm n}^2 \delta n_{\rm n} + \mathcal{C}\delta n_{\rm p}, \tag{21}$$

$$n_{\rm p}\delta\widetilde{\mu}^{\rm p} = \frac{n_{\rm p}}{n_{\rm n}}\mathcal{C}\delta n_{\rm n} + c_{\rm p}^2 \delta n_{\rm p}, \tag{22}$$

and we can rewrite our set of equations as

$$\begin{aligned} n_{\rm n}\delta v_{\rm n} &= \left(\frac{\omega}{k} - v_0\right)\delta n_{\rm n} \\ &= \left(\frac{\omega}{k} - v_0\right)^{-1}\left(c_{\rm n}^2 \delta n_{\rm n} + \mathcal{C}\delta n_{\rm p}\right), \end{aligned} \tag{23}$$

$$n_{\rm p}\delta v_{\rm p} = \frac{\omega}{k}\delta n_{\rm p} = \frac{k}{\omega}\left(\frac{n_{\rm p}}{n_{\rm n}}\mathcal{C}\delta n_{\rm n} + c_{\rm p}^2 \delta n_{\rm p}\right). \tag{24}$$

Reshuffling we get

$$\left[\left(\frac{\omega}{k} - v_0\right)^2 - c_{\rm n}^2\right]\delta n_{\rm n} = \mathcal{C}\delta n_{\rm p}, \tag{25}$$

$$\left[\left(\frac{\omega}{k}\right)^2 - c_{\rm p}^2\right]\delta n_{\rm p} = \frac{n_{\rm p}}{n_{\rm n}}\mathcal{C}\delta n_{\rm n}, \tag{26}$$

and a dispersion relation

$$\left[\left(\frac{\omega}{k} - v_0\right)^2 - c_{\rm n}^2\right]\left[\left(\frac{\omega}{k}\right)^2 - c_{\rm p}^2\right] = \frac{n_{\rm p}}{n_{\rm n}}\mathcal{C}^2. \tag{27}$$

Introducing the 'pattern speed' (the phase velocity) $\sigma_p = \omega/k$ we have

$$\left[(\sigma_p - v_0)^2 - c_{\rm n}^2\right]\left(\sigma_p^2 - c_{\rm p}^2\right) = \frac{n_{\rm p}}{n_{\rm n}}\mathcal{C}^2. \tag{28}$$

Not surprisingly, this local dispersion relation is qualitatively similar to the one for the plasma problem (Anderson et al. 2001). We will now use it to investigate under what circumstances we can have complex roots for σ_p, i.e. a dynamical instability.

2.2 The superfluid two-stream instability

First of all, it is easy to see that (28) leads to the simple roots

$$\sigma_p = \begin{cases} \pm c_{\rm p} \\ v_0 \pm c_{\rm n} \end{cases} \tag{29}$$

in the uncoupled case, when $\mathcal{C} = 0$. This establishes the interpretation of c_X as the sound speeds.

To investigate the coupled case, we introduce new variables $x = \sigma_p/c_{\rm n}$ and $y = v_0/c_{\rm n}$. Then we get

$$f(x,y) = \frac{1}{a^2}[(x-y)^2 - 1](x^2 - b^2) = 1 \tag{30}$$

where we have defined

$$a^2 \equiv \frac{n_{\rm p}}{n_{\rm n}}\frac{\mathcal{C}^2}{c_{\rm n}^4} \quad \text{and} \quad b^2 \equiv \frac{c_{\rm p}^2}{c_{\rm n}^2}. \tag{31}$$

The onset of dynamical instability typically corresponds to the merger of two real-frequency modes. If this is the case, a marginally stable configuration will be such that equation (30) has a double root. This happens when an inflexion point of $f(x,y)$ coincides with $f(x,y) = 1$. This is a useful criterion for searching for the marginally stable modes of our system.

As a 'proof of principle' we consider the particular case of $a^2 = 0.0249$ and $b^2 = 0.0379$ (we will motivate this particular choice in Section 2.5). The real and imaginary parts of the mode-frequencies for these parameter values are shown as functions of y in Fig. 1. We have complex roots (an instability) in the range $0.6 < y < 1.5$. The corresponding mode frequencies lie in the range $0.03 < x_0 < 0.36$. The fastest growing instability occurs for $y \approx 1.1$ for which we find that ${\rm Im}\, x \approx 0.17$. In other words, in this particular case we encounter the two-stream instability once the rate of the background flow is increased beyond

$$v_0 = c_{\rm n} y \approx 0.6 c_{\rm n}. \tag{32}$$

The corresponding frequency is given by

$$\omega = k c_{\rm n} x_0 \approx 0.1 k c_{\rm n}. \tag{33}$$

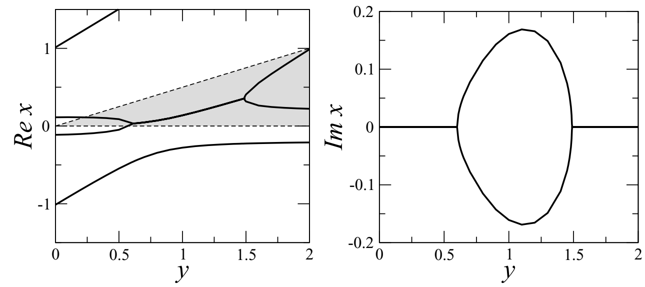

Figure 1. Real (left-hand panel) and imaginary (right-hand panel) parts of the four roots of the dispersion relation (30) for model parameters $a^2 = 0.0249$ and $b^2 = 0.0379$. For these parameters the quartic dispersion relation has four real roots for both $y = 0$ and $y = 2$, while it has two real roots and a complex conjugate pair for y in the range $0.6 < y < 1.5$. In this range, the two-stream instability is operating. The grey area corresponds to $0 \leqslant \mathrm{Re}\, x \leqslant 1/2$ which is contained in the range of the instability criteria discussed in Section 2.3.

From this we see that the instability is present well before the neutron flow becomes 'supersonic'. This is crucial since one would expect the superfluidity to be destroyed for supersonic flows.

Even though the indicated velocity scale for the onset of the instability is large, the above example clearly establishes that the two-stream instability may, in principle, operate in superfluids. Our example indicates the existence of a lower limit of the background flow for the instability. This turns out to be a generic feature. In contrast, the plasma two-stream instability can operate at arbitrarily slow flows. An ideal plasma is unstable to sufficiently long wavelengths for any given v_0. In reality, however, one must also account for dissipative mechanisms. In the case of real plasmas one finds that so-called Landau damping stabilizes the longest wavelengths (Anderson et al. 2001). Thus the two-stream instability sets in below a critical wavelength in more realistic plasma models, and there is typically (just like in the present case) a range of flows for which the instability is present. We will discuss the effects of dissipation on the superfluid two-stream instability briefly in Section 4.

2.3 Necessary criteria for instability

It is useful to consider whether we can derive a necessary condition for the two-stream instability. To approach this problem in full generality would likely be quite complicated, but we can make good progress for the simple one-dimensional toy problem discussed above.

We begin by multiplying the Euler equation (14) for the neutrons by the complex conjugate δv_{n}^*. This leads to (after also using the continuity equations to replace the perturbed number densities)

$$\left(\omega - k v_0 - \frac{k^2 c_{\mathrm{n}}^2}{\omega - k v_0}\right) |\delta v_{\mathrm{n}}|^2 = \mathcal{C} \frac{k^2 n_{\mathrm{p}}}{n_{\mathrm{n}} \omega} \delta v_{\mathrm{n}}^* \delta v_{\mathrm{p}}. \quad (34)$$

Similarly, we obtain from the second Euler equation (15)

$$\left(\omega - \frac{k^2 c_{\mathrm{p}}^2}{\omega}\right) |\delta v_{\mathrm{p}}|^2 = \mathcal{C} \frac{k^2}{\omega - k v_0} \delta v_{\mathrm{p}}^* \delta v_{\mathrm{n}}. \quad (35)$$

Next we combine these two equations to get

$$\mathcal{L} = \frac{n_{\mathrm{n}}}{n_{\mathrm{p}}} \sigma_p \left(\sigma_p - v_0 - \frac{c_{\mathrm{n}}^2}{\sigma_p - v_0}\right) |\delta v_{\mathrm{n}}|^2 + (\sigma_p - v_0)\left(\sigma_p - \frac{c_{\mathrm{p}}^2}{\sigma_p}\right) |\delta v_{\mathrm{p}}|^2 = \mathcal{C}\left(\delta v_{\mathrm{n}}^* \delta v_{\mathrm{p}} + \delta v_{\mathrm{p}}^* \delta v_{\mathrm{n}}\right) \quad (36)$$

where we have introduced the pattern speed σ_p. From this expression we see that the imaginary part of the left-hand-side must vanish, so we should have $\mathrm{Im}\, \mathcal{L} = 0$. Allowing the pattern speed to be complex, i.e. using $\sigma_p = \sigma_{\mathrm{R}} + i\sigma_{\mathrm{I}}$ we find that the following condition must be satisfied:

$$\mathrm{Im}\, \mathcal{L} = \sigma_{\mathrm{R}} \sigma_{\mathrm{I}} \left\{ \frac{n_{\mathrm{n}}}{n_{\mathrm{p}}} \left[2 - \frac{v_0}{\sigma_{\mathrm{R}}}\left(1 - \frac{c_{\mathrm{n}}^2}{|\sigma_p - v_0|^2}\right)\right] |\delta v_{\mathrm{n}}|^2 + \left[2 - \frac{v_0}{\sigma_{\mathrm{R}}}\left(1 + \frac{c_{\mathrm{p}}^2}{|\sigma_p|^2}\right)\right] |\delta v_{\mathrm{p}}|^2 \right\} = 0. \quad (37)$$

If we are to have an unstable mode, $\sigma_{\mathrm{I}} \neq 0$, the frequency clearly must be such that the factors multiplying the absolute values of the two velocities have different signs.

Let us first consider the case when the factor multiplying $|\delta v_{\mathrm{n}}|^2$ is negative. Then we find that an instability is only possible if $\sigma_{\mathrm{R}}/v_0 < 0$, and the following condition is satisfied:

$$0 < \left|\frac{\sigma_{\mathrm{R}}}{v_0}\right| < \frac{1}{2}\left(\frac{c_{\mathrm{n}}^2}{|\sigma_p - v_0|^2} - 1\right). \quad (38)$$

This shows that we must have

$$\frac{c_{\mathrm{n}}}{|v_0|} > \left|\frac{\sigma_p}{v_0} - 1\right| > \left|\frac{\sigma_{\mathrm{R}}}{v_0} - 1\right| = \left|\frac{\sigma_{\mathrm{R}}}{v_0}\right| + 1 > 1 \quad (39)$$

which constrains the permissible frequencies to the range $|\sigma_{\mathrm{R}}| < c_{\mathrm{n}} - |v_0|$. Thus we see that the flow must be subsonic, i.e. $|v_0| < c_{\mathrm{n}}$.

In the case when the factor multiplying $|\delta v_{\mathrm{p}}|^2$ is negative we can only have an instability if $\sigma_{\mathrm{R}}/v_0 > 0$. We also require

$$0 < \frac{\sigma_{\mathrm{R}}}{v_0} < \frac{1}{2}\left(1 + \frac{c_{\mathrm{p}}^2}{|\sigma_p|^2}\right) < \frac{1}{2}\left(1 + \frac{c_{\mathrm{p}}^2}{\sigma_{\mathrm{R}}^2}\right) \quad \text{if } |\sigma_p - v_0|^2 < c_{\mathrm{n}}^2 \quad (40)$$

or

$$\frac{1}{2}\left(1 - \frac{c_{\mathrm{n}}^2}{|\sigma_{\mathrm{R}} - v_0|^2}\right) < \frac{1}{2}\left(1 - \frac{c_{\mathrm{n}}^2}{|\sigma_p - v_0|^2}\right) < \frac{\sigma_{\mathrm{R}}}{v_0} < \frac{1}{2}\left(1 + \frac{c_{\mathrm{p}}^2}{\sigma_{\mathrm{R}}^2}\right) \quad \text{if } |\sigma_p - v_0|^2 > c_{\mathrm{n}}^2. \quad (41)$$

For the example illustrated in Fig. 1, the condition that must be satisfied is (40). It is useful to note two things about this criterion. First of all, any unstable mode for which $\sigma_{\mathrm{R}} > c_{\mathrm{p}}$ must lie in the range $0 < c_{\mathrm{p}} < \sigma_{\mathrm{R}} < v_0$. Secondly, when $\sigma_{\mathrm{R}} >> c_{\mathrm{p}}$ the permissible range will be well approximated by $0 < \sigma_{\mathrm{R}} < v_0/2$, cf. Fig. 2. As is clear from Fig. 1 the unstable modes satisfy this last, and most severe, criterion.

It is worth noting that the instability can be discussed in terms of a simple energy argument [see Casti et al. (1998) and Pierce (1974) for similar discussions in other contexts]. After averaging over several wavelengths, the kinetic energy of the protons is

$$E_{\mathrm{p}} \approx \frac{m n_{\mathrm{p}} \delta v_{\mathrm{p}}^2}{2} > 0. \quad (42)$$

Meanwhile we get for the neutrons

$$E_{\mathrm{n}} \approx \frac{m}{2}(n_{\mathrm{n}} + \delta n_{\mathrm{n}})(v_0 + \delta v_{\mathrm{n}})^2 \quad (43)$$

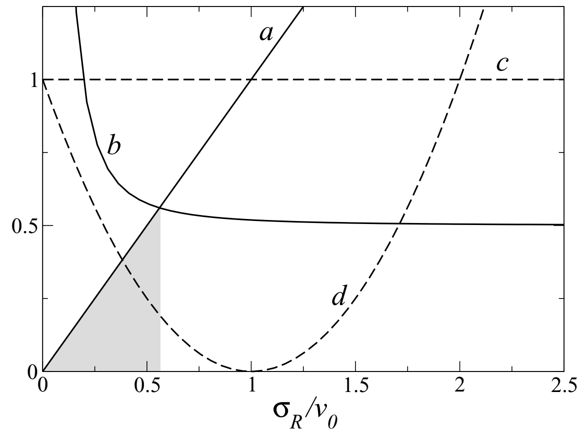

Figure 2. An illustration of the instability criterion (40) which is relevant for the example considered in Section 2.2. This example is constructed by introducing $z = \sigma_R/v_0$, and then showing the four curves: $a(z) = z$, $b(z) = (1 + \gamma^2/z^2)/2$, $c(z) = 1$ and $d(z) = (z - 1)^2$. Here we have taken $c_n^2/v_0^2 = 1$ and $\gamma^2 = c_p^2/c_n^2 = 0.0379$. Criterion (40) is satisfied when $d < c$ and $a < b$ (in the grey area). The corresponding range is well approximated by $0 < \sigma_R/v_0 < 1/2$.

which leads to

$$E_n \approx \frac{mn_n}{2}\left(v_0^2 + \frac{\omega + kv_0}{\omega - kv_0}\delta v_n^2\right) \tag{44}$$

from which we see that the energy in the perturbed flow is smaller than the energy in the unperturbed case, which means that we can associate the wave with a 'negative energy', when

$$-v_0 < \frac{\omega}{k} < v_0\,, \quad \text{i.e.} \quad -v_0 < \sigma_R < v_0. \tag{45}$$

A wave that satisfies $0 < \sigma_R < v_0$ moves forwards with respect to the protons but backwards according to an observer riding along with the unperturbed neutron flow. As we have seen above, the unstable modes in our problem satisfy this criterion and hence it is easy to explain the physical conditions required for the two-stream instability to be present.

2.4 Results for a simple model equation of state

Having established that the two-stream instability may be present in superfluids, we want to assess to what extent one should expect this mechanism to play a role for astrophysical neutron stars. To do this we will consider two particular equations of state. The results we obtain illustrate different facets of what we expect to be a rich problem.

We begin by making contact with our recent analysis of rotating superfluid models (Prix et al. 2002) as well as the study of oscillating non-rotating stars by Prix & Rieutord (2002). From the definitions above we have

$$a^2 = \frac{n_p}{n_n}\frac{\mathcal{S}_{np}^2}{\mathcal{S}_{pp}^2}, \tag{46}$$

$$b^2 = \frac{n_p}{n_n}\frac{\mathcal{S}_{nn}}{\mathcal{S}_{pp}}. \tag{47}$$

We combine these results with the explicit structure matrix given in equation (144) of Prix et al. (2002), which is based on a simple 'analytic' equation of state. This leads to

$$a^2 = \frac{n_p}{n_n}\sigma^2, \tag{48}$$

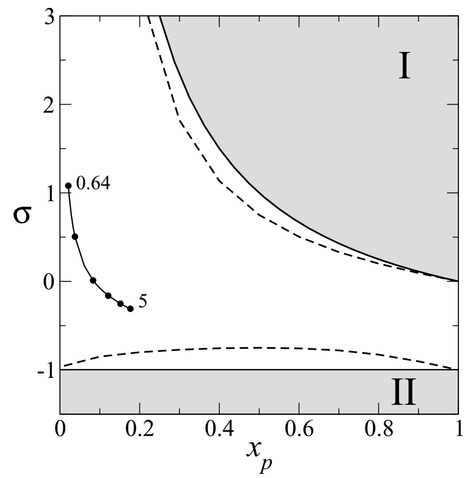

Figure 3. An illustration of the various domains of instability for the simple 'analytic' equation of state of Prix et al. (2002). An absolute instability (see discussion in the main text) is active in the grey areas (also labelled I and II). The two-stream instability is, in principle, relevant in the remaining parameter space. The dashed curves indicate the onset of instability when the relative flow is equal to the neutron sound speed ($y = 1$). For slower flows, these critical curves approach the absolute instability regions. The region where the two-stream instability may operate in physical flows therefore lies between each dashed curve and the nearest grey area. For comparison we also indicate the curve in the x_p–σ plane traced out by the PAL equation of state (discussed in Section 2.5) as the density is varied from that near the crust–core interface ($u = 0.64$) to five times that of nuclear saturation ($u = 5$).

$$b^2 = 1 + \frac{\sigma(1 - 2x_p)}{1 - x_p}, \tag{49}$$

where $x_p = n_p/(n_p + n_n)$ is the proton fraction, and σ is defined by

$$\sigma = \frac{\mathcal{S}_{np}}{\mathcal{S}_{pp}}\,. \tag{50}$$

As discussed by Prix et al. (2002), σ is related to the 'symmetry energy' of the equation of state, cf. Prakash, Ainsworth & Lattimer (1988).

The instability regions for this model equation of state are illustrated in Fig. 3. A key feature of this figure is the presence of regions of 'absolute instability'. This happens when $a^2 > b^2$. Then there exist unstable solutions already for vanishing background flow, $y = 0$. That this is the case is easy to see. Consider equation (30) in the limit $y = 0$. In the limit we can solve directly for x^2:

$$x^2 = \frac{1 + b^2}{2} \pm \sqrt{\left(\frac{1 + b^2}{2}\right)^2 - b^2 + a^2} \tag{51}$$

from which it is easy to see that one of the roots for x^2 will be negative if $a^2 > b^2$. Hence, one of the roots to the quartic (30) will be purely imaginary.

The physics of this instability is quite different from the two-stream instability that is the main focus of this paper. Yet it is an interesting phenomenon. From the above relations we find that $a^2 < b^2$ corresponds to

$$\mathcal{S}_{nn}\mathcal{S}_{pp} > \mathcal{S}_{np}^2. \tag{52}$$

In the discussion by Prix et al. (2002) it was assumed that 'reasonable' equations of state ought to satisfy this condition. We expected this to be the case since the structure matrix would not be invertible if its determinant were to vanish at some point. We now see that this constraint has a strong physical motivation: the condition is violated when $a^2 > b^2$, i.e. when we have an absolute instability. The regions where this instability is active are indicated by the grey areas in Fig. 3.

2.5 Results for the PAL equation of state

In order to strengthen the argument that the two-stream instability may operate in astrophysical neutron stars, we have considered a 'realistic' equation of state due to Prakash et al. (1988) (PAL). The advantage of this model is that it is relatively simple. In particular, it leads to analytical expressions for the various quantities needed in our analysis. The energy density of the baryons for the PAL equation of state can be written

$$\mathcal{E}(n_n, n_p) = (n_n + n_p)\left[E_0(u) + S(u)(1 - 2x_p)^2\right], \tag{53}$$

where E_0 corresponds to the energy per nucleon, S corresponds to the 'symmetry energy' (and is closely related to σ in the 'analytic' equation of state discussed above), and $u = (n_n + n_p)/n_0$ with $n_0 = 0.16\,\mathrm{fm}^{-3}$ the nuclear saturation density. E_0 takes the following form:

$$E_0(u) = A_0 u^{2/3} + B_0 u + C_0 u^\sigma + 3\sum_{i=1}^{2} C_i \alpha_i^{-3}\left[\alpha_i u^{1/3} - \arctan\left(\alpha_i u^{1/3}\right)\right], \tag{54}$$

with $A_0 = 22.11$ MeV, $B_0 = 220.47$ MeV, $C_0 = -213.41$ MeV, $\sigma = 0.927, C_1 = -83.84$ MeV, $C_2 = 23.0$ MeV, $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$. The symmetry term is

$$S(u) = A_S[u^{2/3} - F(u)] + S_0 F(u), \tag{55}$$

with $A_S = 12.99$ MeV and $S_0 = 30$ MeV. Here $F(u)$ is a function satisfying $F(1) = 1$ which is supposed to simulate the behaviour of the potentials used in theoretical calculations. In our study we have only considered $F(u) = u$, which is one of four possibilities discussed by Prakash et al. (1988).

We further need to account for the energy contribution of the electrons, which is important since the electrons are highly relativistic inside neutron stars. Hence, they can obtain high (local) Fermi energies which may be comparable with the proton (local) Fermi energies. Considering only the electrons, the leptonic contribution to the energy density is given – in units where the speed of light is unity – by (Shapiro & Teukolsky 1983)

$$\mathcal{E}_e = \frac{m_e}{\lambda_e^3}\chi\left(\chi_e^F\right), \tag{56}$$

where $m_e = m/1836$ is the electron mass (in terms of the baryon mass m), $\lambda_e = \hbar/m_e$ is the electron Compton wavelength, and

$$\chi(x) = \frac{1}{8\pi^2}\{x(1+x^2)^{1/2}(1+2x^2) - \ln[x + (1+x^2)^{1/2}]\} \tag{57}$$

$$\chi_e^F = 1836\left[3\pi^2\left(\frac{\hbar}{m}\right)^3\right]^{1/3} n_p^{1/3}. \tag{58}$$

In doing this calculation we have assumed local charge neutrality, i.e. $n_e = n_p$. The above energy term is added linearly in the equation of state.

Having obtained an expression for the total energy as a function of the density, we can derive explicit expressions for all quantities needed to discuss the two-stream instability. First we need to determine the proton fraction x_p. We do this by assuming that the star is in chemical equilibrium, i.e.

$$\mu^n = \mu^p + \mu^e. \tag{59}$$

Solving (59) for x_p provides us with the proton fraction as a function of u. Given this, and the relevant partial derivatives of $\mathcal{E} + \mathcal{E}_e$ we can readily evaluate the symmetry energy as well as the sound speeds c_n^2, c_p^2 and the chemical coupling parameter $\mathcal{C}$. With this data we can determine the two parameters a^2 and b^2 which are needed if we want to solve the local dispersion relation (30). The results we obtain for the proton fraction and the symmetry energy are indicated in Fig. 3. We consider the range from $u = 0.64$, presumed to correspond to the core–crust boundary, to $u = 5$ which represents the deep core of a realistic neutron star. The corresponding results for the two-stream instability are shown in Fig. 4. From this figure we can see that the two-stream instability may operate (albeit at comparatively large relative flows) in the region immediately below the crust. Finally, we find that the conditions at the core–crust interface are such that $a^2 = 0.0249$ and $b^2 = 0.0379$. These are the values we chose for the example in Section 2.2 and hence the results shown in Fig. 1 correspond to a physically realistic model.

3 THE ROTATING-SHELL PROBLEM

Our aim in this section is to construct a toy problem that probes a different aspect of the superfluid two-stream instability. In order to focus attention on the coupling of the two fluids caused by the entrainment effect, we consider two fluids, allowed to rotate at different rates, confined within an infinitesimally thin spherical shell. By assuming that the shell is infinitesimal we ignore radial motion, i.e. we restrict the permissible perturbations of this system in such a way that the perturbed velocities must take the form

$$\delta \boldsymbol{v}_X = -\frac{1}{R\sin\theta}U^X_{lm}(t)\partial_\varphi Y^m_l \hat{e}_\theta + \frac{1}{R}U^X_{lm}(t)\partial_\theta Y^m_l \hat{e}_\varphi \tag{60}$$

where $Y^m_l(\theta, \varphi)$ are the spherical harmonics and R is the radius of the shell. This means that the system only permits toroidal mode solutions. In other words, all oscillation modes of this shell model

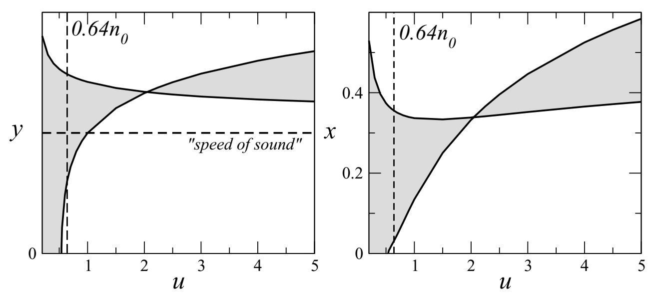

Figure 4. Two-stream instability results for the PAL equation of state. Left-hand panel: the region where the two-stream instability is present (grey area) is shown as a function of the density parameter u. We indicate the location of the core–crust boundary ($u \approx 0.64$) by a vertical dashed line. Our model is only relevant for the core fluid, i.e. to the right of the vertical line. Finally, the horizontal dashed line indicates when the relative flow is equal to the (neutron) sound speed. We expect the superfluid degeneracy to be broken beyond this level, so an instability located above this line is unlikely to have physical relevance. The results indicate that there may be a region of instability immediately below the crust. Right-hand panel: the corresponding oscillation frequencies. Particularly notable is the point near $u = 2$ where the two critical curves cross. At this point the symmetry energy σ changes sign, cf. Fig. 3, and there exists a particular density such that the two fluids are uncoupled, cf. equation (31).

are closely related to the inertial r modes of rotating single-fluid objects (Papaloizou & Pringle 1978; Provost et al. 1981).

The perturbation equations for the configuration we consider have been derived in a different context by Prix, Comer & Andersson (2004). Our primary interest here concerns whether the modes of this simple system may suffer the two-stream instability. The presence of the instability in this toy problem would be a strong indication that it will also be relevant when the shells are 'thick' and radial motion is possible; that is, when we consider a rotating star that contains a partially decoupled superfluid in the core (and perhaps also the inner crust). That this is, indeed, the case has recently been shown for the case of inertial modes of Newtonian superfluid stars, cf. Prix et al. (2004).

We can derive the required dispersion relation in a manner very similar to that which yields the ordinary fluid r modes. Having assumed that motion is confined to the shell, we take the curl of the linearized forms of equation (5). This removes the pure gradient terms containing the gravitational and chemical potentials from the analysis. It is also found that the particle number conservation equations are automatically satisfied for perturbed velocities taking the toroidal form written above (to leading order in a slow-rotation analysis). The net effect is that the equation of state only enters the problem through the entrainment parameter. In other words, the velocity that sets the scale for the shell problem, and the associated two-stream instability, is the relative rotation of the two fluids, and not the speeds of sound which was the case for the acoustic modes analysed earlier.

After a somewhat laborious calculation, see Prix et al. (2004) for details, one finds that the dispersion relation for the toroidal two-fluid modes of the shell problem is[1]

$$\begin{aligned}&[l(l+1)(1-\varepsilon_{\rm n})(\omega+m\Omega_{\rm n})-2m\tilde{\Omega}_{\rm n}]\\&\quad\times[l(l+1)(1-\varepsilon_{\rm p})(\omega+m\Omega_{\rm p})-2m\tilde{\Omega}_{\rm p}]\\&\quad-l^2(l+1)^2\varepsilon_{\rm n}\varepsilon_{\rm p}(\omega+m\Omega_{\rm n})(\omega+m\Omega_{\rm p})=0.\end{aligned} \tag{61}$$

where we use the shorthand notation

$$\tilde{\Omega}_X=\Omega_X-\varepsilon_X(\Omega_X-\Omega_Y)\qquad Y\neq X. \tag{62}$$

This equation should be valid for the conditions in the outer core of a mature neutron star, where superfluid neutrons are permeated by superconducting protons. Following Prix et al. (2004) we express the dispersion relation in terms of the entrainment parameter

$$\varepsilon=\varepsilon_{\rm p}=\frac{2\alpha}{\rho_{\rm p}}=\frac{\rho_{\rm p}}{\rho_{\rm n}}\varepsilon_{\rm n}, \tag{63}$$

the frequency as measured with respect to the rotation of the protons,

$$\kappa=\frac{\omega+m\,\Omega_{\rm p}}{\Omega_{\rm p}}, \tag{64}$$

and a dimensionless measure of the relative rotation,

$$\mathcal{R}\equiv\frac{\Omega_{\rm n}-\Omega_{\rm p}}{\Omega_{\rm p}}. \tag{65}$$

[1] This equation corrects the dispersion relation we used in an earlier discussion of the two-stream instability for the shell problem (Andersson, Comer & Prix 2003). Unfortunately, the error introduced in the earlier version of the dispersion relation significantly effects the nature of the instability. For example, the data discussed by Andersson et al. (2003) suggests that the instability acts mainly through short-wavelength waves. As we show in the present paper, the correct analysis leads to the instability mainly affecting the long-wavelength oscillations. This has repercussions for the inferred growth times of the unstable modes.

This leads to a quadratic equation which can be solved for the eigenvalue κ. The onset of the two-stream instability readily follows as the curves along which the discriminant of the quadratic vanishes.

As discussed in Section 2.3, the two-stream instability can be understood in terms of negative energy waves. In the current problem, a simple criterion for waves to carry negative energy according to one fluid but positive energy according to the other fluid is that the mode pattern speed [we are assuming a decomposition $\exp(\mathrm{i}\omega\,t+\mathrm{i}m\varphi)$], which is given by

$$\sigma_p=-\frac{\omega}{m} \tag{66}$$

lies between the two (uniform) rotation rates. In other words, one may expect a necessary condition for instability to be

$$(\sigma_p-\Omega_{\rm p})(\sigma_p-\Omega_n)<0. \tag{67}$$

In terms of our new variables a mode would satisfy this criterion (67) if κ is such that

$$\kappa(\kappa+m\mathcal{R})<0. \tag{68}$$

As we will now establish, there exist unstable modes that satisfy this criterion for 'reasonable' parameter values.

Having analysed the quadratic dispersion relation (61) we draw the following conclusions:

(i) We will not have any instabilities for $\varepsilon>0$ and a proton fraction in the physical range $0<x_{\rm p}<1$. Since the entrainment in the outer core of a neutron star is implied by Newtonian calculations to lie in the range $0.4\leqslant\varepsilon\leqslant0.7$ (Prix et al. 2002), and by relativistic results to lie in the larger (but still positive) range of $0.3\leqslant\varepsilon\leqslant1.2$ (Comer & Joynt 2003), this means that the two-stream instability is not likely to be relevant for the superfluid r modes in the core of a mature neutron star.

(ii) The situation is markedly different if we allow ε to assume negative values. Then the instability will set in provided that $|\mathcal{R}|$ is sufficiently large. Three examples of two-stream instability regions for quadrupole ($l=2$) modes and fixed values of ε are shown in Figs 5–7. The three figures correspond to $\varepsilon=-2,-10$ and -100, respectively. We show the results for $l=2$ since the instability sets in first (i.e. at smallest $|\mathcal{R}|$) for these modes. The results in Fig. 5 can be directly compared to the numerical results discussed by Prix et al. (2004), cf. fig. 7 in that paper.

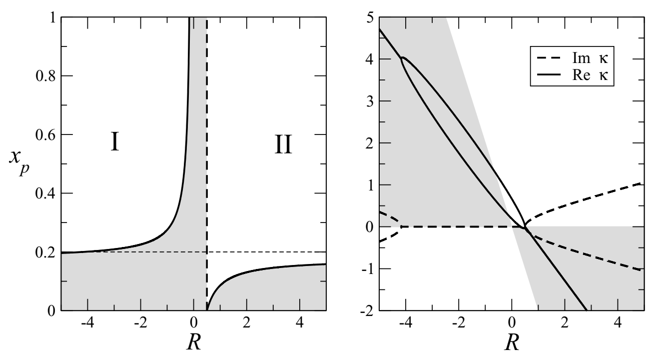

Figure 5. Instability regions for the shell problem. We show results for $l=2$ and $\varepsilon=-2$. The left-hand panel shows the regions of instability in the $x_{\rm p}$–$\mathcal{R}$ plane. The two-stream instability operates in regions I and II. The right-hand panel shows the real and imaginary parts of the frequency κ for the particular case of $x_{\rm p}=0.2$ (indicated by a dashed horizontal line in the left-hand panel. Here the grey regions indicate where a mode is prograde with respect to one of the fluids but retrograde with respect to the other, i.e. where the naive instability criterion (68) holds.

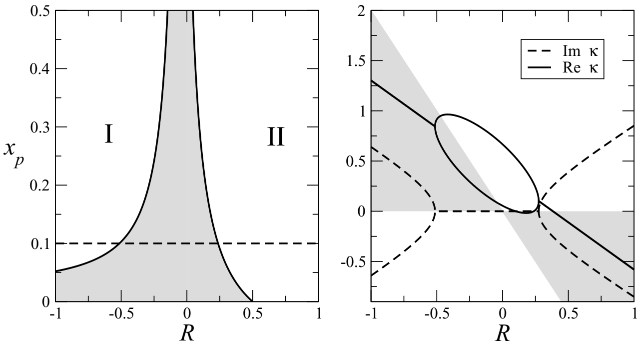

Figure 6. Instability regions for the shell problem. We show results for $l = 2$ and $\varepsilon = -10$. The left-hand panel shows the regions of instability in the x_p–$\mathcal{R}$ plane. The two-stream instability operates in regions I and II. The right-hand panel shows the real and imaginary parts of the frequency κ for the particular case of $x_p = 0.1$ (indicated by a dashed horizontal line in the left-hand panel. Here the grey regions indicate where a mode is prograde with respect to one of the fluids but retrograde with respect to the other, i.e. where the naive instability criterion (68) holds.

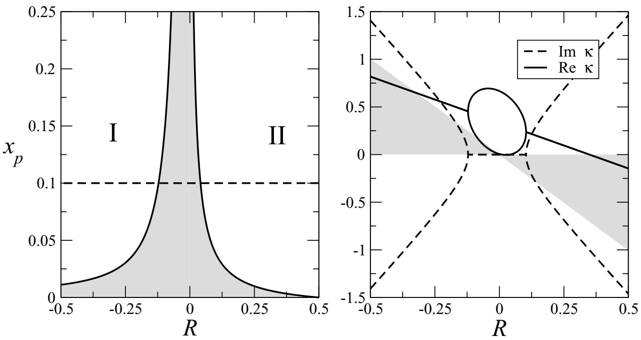

Figure 7. Same as Fig. 6 but for $\varepsilon = -100$.

(iii) From the mode frequencies shown in the right-hand panels of Figs 5–7 we deduce that the simple instability criterion (68) cannot be relied upon. While the criterion agrees reasonably with the onset of instability in the case $\varepsilon = -2$ (Fig. 5), it is clear that the two-stream instability operates outside the predicted domain for $\varepsilon = -100$ (Fig. 7). This shows that our simple argument is too naive, and emphasizes the need for a more detailed stability analysis for rotating superfluid systems. In this respect, extension of the recent work of Andersson et al. (2004) may lead to useful progress.

(iv) From the examples in Figs 5–7 it is clear that the instability is dynamical. Since $|\mathrm{Im}\,\kappa| \sim |\mathrm{Re}\,\kappa|$ the growth time is of the same order of magnitude as the dynamical time-scale.

(v) Finally, it is worth noting that for large $|\varepsilon|$ (and $l = 2$) the condition that the discriminant vanishes, i.e. the onset of the instability, is well approximated by the solution to

$$8\mathcal{R}^2 x_p \varepsilon + (\mathcal{R}x_p + 1 - \mathcal{R})^2 = 0. \tag{69}$$

For a 'typical' proton fraction of $x_p = 0.1$, this leads to the approximate relation

$$\varepsilon \approx -\frac{5}{4\mathcal{R}^2}. \tag{70}$$

The above examples show that the two-stream instability operates in the shell problem. In fact, the analysis goes beyond the local analysis of the plane parallel problem in Section 2.2 since we have now solved for the actual unstable modes (satisfying the relevant boundary conditions).

4 A BRIEF DISCUSSION OF PULSAR GLITCHES

The results we have discussed so far are interesting from a conceptual point of view, but it is not yet clear to what extent this new instability is astrophysically relevant. It is clear that, unless the entrainment parameter ε assumes very large negative values, the relative rotation rates required to make the superfluid quadrupole r modes unstable in the range $0 < x_p < 1$ will be too large to be physically attainable. Yet, it must be recognized that our current understanding of these parameters in astrophysical neutron stars is very poor. Recent results indicate that the entrainment parameter ε may, indeed, assume negative values in the inner crust (Carter, Chamel & Haensel 2004), but it is not clear how large $|\varepsilon|$ it may be reasonable to consider. Anyway, given the many uncertainties we do not feel that we can rule out the possibility that this instability may play a role in the spin evolution of neutron stars. It is a tantalizing possibility given that the mechanism underlying the enigmatic glitches observed in dozens of pulsars remains poorly understood. A speculative suggestion would be that the superfluid two-stream instability is relevant in this context: perhaps it serves as a trigger mechanism for large pulsar glitches? As we will discuss below, the notion that an instability sets in at a critical relative rotation in a two-component model would seem to agree well with the current observational data. While our current model is too simple for us to investigate this possibility in any detail, it is nevertheless useful to discuss the relevant issues. In particular, this may help identify promising directions for future research.

A quantity of key importance for this discussion is the rotational lag between the two components. In order to argue that the two-stream instability is relevant for pulsar glitches we need to consider lags that may actually occur in astrophysical neutron stars. To estimate the size of the rotational lag required to 'explain' the observed glitches we assume that a glitch corresponds to a transfer of angular momentum from a partially decoupled superfluid component (index n) to the bulk of the star (index p). Then we have

$$I_n|\Delta\Omega_n| \approx I_p\Delta\Omega_p \rightarrow \Delta\Omega_p \approx \frac{I_n}{I_p}|\Delta\Omega_n| \tag{71}$$

where I_X are the two moments of inertia. Now assume that the decoupled component corresponds to 1 per cent of the total moment of inertia, e.g. the superfluid neutrons in the inner crust or a corresponding amount of fluid in the core. This would mean that $I_n \sim 10^{-2} I_p$, and we have

$$\Delta\Omega_p \approx 10^{-2}|\Delta\Omega_n|. \tag{72}$$

Combine this with the observations of large Vela glitches to get

$$\frac{\Delta\Omega_p}{\Omega_p} \approx 10^{-2}\frac{|\Delta\Omega_n|}{\Omega_p} \sim 10^{-6}. \tag{73}$$

In other words, we must have

$$|\Delta\Omega_n| \approx 10^{-4}\Omega_p. \tag{74}$$

If we assume that the glitch brings the two fluids back into corotation, then we have $\Delta\Omega_n = \Omega_n - \Omega_p = \Delta\Omega$ and we see that the two rotation rates will maximally differ by one part in 10^4 or so, corresponding to $\mathcal{R} \sim 10^{-4}$. Rotational lags of this order of magnitude have often been discussed in the context of glitches. Even though the key quantity in models invoking catastrophic vortex unpinning in the inner crust – the pinning strength – is very uncertain, and there have been suggestions that the pinning force is too weak to allow a build up of the required rotational lag (Jones 1998), typical values considered are consistent with our rough estimate. In addition, frictional heating due to a difference in the rotation rates of the bulk of a neutron star and a superfluid component has been discussed as a possible explanation for the fact that old isolated pulsars seem to be somewhat hotter than expected from standard cooling models (Shibazaki & Lamb 1989; Larson & Link 1999). In particular, Larson & Link (1999) argue that a lag of

$$\frac{\Delta\Omega}{\Omega_\mathrm{p}} \approx (3.2 \times 10^{-4} - 9.5 \times 10^{-3}) \times \left(\frac{0.1\ \mathrm{s}}{P}\right) \tag{75}$$

could explain the observational data. Finally, the presence of rotational lags of the proposed magnitude is supported by a statistical analysis of 48 glitches in 18 pulsars (Lyne et al. 2000). This study suggests that the critical rotational lag at which a glitch occurs is

$$\frac{\Delta\Omega}{\Omega_\mathrm{p}} \approx 5 \times 10^{-4}. \tag{76}$$

In order for an instability to be relevant the unstable mode must grow faster than all dissipation time-scales. In the case of a superfluid neutron-star core the main dissipation mechanisms are likely to be mutual friction and 'standard' shear viscosity due to electron–electron scattering. A rough estimate of when mutual friction is likely to dominate the shear viscosity is, due to Mendell (1991b),

$$\Omega > 100 \left(\frac{10^6\ \mathrm{cm}}{\lambda}\right)^2 \left(\frac{T}{10^7\ \mathrm{K}}\right)^2 \mathrm{s}^{-1}$$
$$\approx 100 \left(\frac{l}{\pi}\right)^2 \left(\frac{T}{10^7\ \mathrm{K}}\right)^2 \mathrm{s}^{-1} \tag{77}$$

where we assume that the wavelength of the mode is $\lambda = \pi R/l$. We can write this as

$$P < 0.62 l^{-2} \left(\frac{T}{10^7\ \mathrm{K}}\right)^2 \ \mathrm{s} \tag{78}$$

which shows that mutual friction is likely to dominate over shear viscosity for quadrupole oscillations. For example, for a neutron star rotating with the period of the Vela pulsar mutual friction would dominate for $l < 15$ or so (assuming $T \approx 5 \times 10^7$ K).

From the estimates we obtained in the previous section we see that in order for the superfluid r modes to become unstable at $\mathcal{R} \approx 5 \times 10^{-4}$ we would need $\varepsilon \approx -5 \times 10^6$, cf. (70). This value is significantly larger than those indicated by the recent work of Carter et al. (2004), and may be too large to be attainable in a realistic description of the superfluid in the crust. There are, however, many remaining uncertainties concerning the coupling between the various core components. Another important coupling mechanism between neutrons and protons in the core might come from interactions between the neutron vortices and proton flux tubes if the protons form a type-II superconductor (Ruderman, Zhu & Chen 1998, see also Link 2003). Alternatively, if we consider the neutron superfluid in the crust, the most important dissipative mechanism would probably be vortex–crust friction (see Jones 1998 for more discussion). Considering the many uncertainties concerning this issue, it would seem premature to rule out the interesting possibility that the superfluid two-stream instability may be relevant in the context of pulsar glitches. Obviously, our shell model is too simplistic to be considered a representative model of a realistic neutron star. For example, a more detailed model should account for radial motion and internal stratification (as well as many other features). This will complicate the analysis significantly by bringing several classes of oscillation modes into the picture. In fact, the plane parallel toy problem considered in the first part of this paper indicates that the acoustic p modes may be susceptible to the instability.

Supposing that the two-stream instability is relevant for neutron stars, what affects can it have? The answer to this question obviously requires much further work, but it is nevertheless interesting to speculate about some possibilities. Most standard models for glitches are based on the idea of catastrophic vortex unpinning in the inner crust (Anderson & Itoh 1975). This is an attractive idea since the glitch relaxation (on a time-scale of days to months) would seem to be well described by vortex creep models (Cheng et al. 1988). An interesting scenario is provided by the thermally induced glitch model discussed by Link & Epstein (1996). They have shown that a deposit of 10^{42} erg of heat would be sufficient to induce a Vela-type glitch. The mechanism that leads to the unpinning of vortices, e.g. by the deposit of heat in the crust, is however not identified. Maybe the two-stream instability could fill this gap in the theory? It should, of course, be pointed out that glitches need not originate in the inner crust. In particular, Jones (1998) has argued that the vortex pinning is too weak to explain the recurrent Vela glitches. If this argument is correct then the glitches must be due to some mechanism operating in the core fluid. Since the model problems we have considered would be relevant for the conditions expected to prevail in the outer core of a mature neutron star, our results show that the two-stream instability may serve as a trigger for glitches originating there (although this would require a large negative entrainment parameter not predicted by the standard models). The key requirement for the instability to operate is the presence of a rotational lag. It is worth pointing out that such a lag will build up both when there is a strong coupling between the two fluids (i.e. when the vortices are pinned) and when this coupling is weak. One would generally expect the strength of this coupling to vary considerably at various depths in the star (Langlois et al. 1998), and it is not yet clear to what extent a rotational lag can build up in various regions. This is, of course, a key issue for future theoretical work on pulsar glitches.

One final relevant point concerns the recent observation of a Vela-size glitch in the anomalous X-ray pulsar 1RXS J170849.0–400910 (Kaspi, Lackey & Chakrabarty 2000). This object has a spin period of 11 s, which means that any feasible glitch model must not rely on the star being rapidly rotating. What does this mean for a model based on the notion of a critical relative rotation rate? Let us assume that the rotational lag builds up at the same rate as the electromagnetic spin-down of the main part of the star (i.e. that the superfluid component does not change its spin rate at all under normal circumstances). Then the lag would be $\Delta\Omega \approx t\dot{\Omega}$ after time t. If there is a critical value at which a glitch will happen (corresponding to $\Delta\Omega_\mathrm{crit}$) then the interglitch time t_g could be approximated by

$$t_\mathrm{g} \approx \frac{\Delta\Omega_\mathrm{crit}}{\Omega}\frac{\Omega}{\dot{\Omega}} = 2\tau\frac{\Delta\Omega_\mathrm{crit}}{\Omega}$$

where τ is the standard 'pulsar age'. This argument implies the following:

(i) For $\Delta\Omega_\mathrm{crit}/\Omega \approx 5 \times 10^{-4}$ we would get $t_\mathrm{g} \approx 10^{-3}\tau$. This (roughly) means that only pulsars younger than 10^4 yr would be seen to glitch during 30 yr of observation, which accords well with the fact that only young pulsars are active in this sense.

(ii) There is no restriction on the rotation rate in this scenario; a star spinning slowly may well exhibit a glitch as long as its spin-down rate is fast enough. This means that one should not be surprised to find glitches in stars with extreme magnetic fields (magnetars).

5 CONCLUDING REMARKS

In this paper we have introduced the superfluid two-stream instability: a dynamical instability analogous to that known to operate in plasmas (Anderson et al. 2001), which sets in once the relative flow between the two components of the system reaches a critical level. We have studied this instability for two model problems. First we analysed a local dispersion relation derived for the case of a background such that one fluid was at rest while the other had a constant flow rate. This provided a proof of principle of the existence of the two-stream instability for superfluids. Our analysis was based on the two-fluid equations that have been used to model the dynamics of the outer core of a neutron star, where superfluid neutrons are expected to coexist with superconducting protons and relativistic electrons. These equations are analogous to the Landau model for superfluid helium,[2] and should also (after suitable modifications to incorporate elasticity and possible vortex pinning) be relevant for the conditions in the inner crust of a mature neutron star. Thus we expect the two-stream instability to be generic in dynamical superfluids, possibly limiting the relative flow rates of any multifluid system.

Our second model problem concerned two fluids confined within an infinitesimally thin spherical shell. The aim of this model was to assess whether the two-stream instability may be relevant (perhaps as a trigger mechanism) for pulsar glitches. The results for this problem demonstrated that the entrainment effect could provide a sufficiently strong coupling for the instability to set in, although it is debatable whether sufficiently large negative values of the entrainment coefficient may be reached in realistic models. Incidentally, the modes that become dynamically unstable in this problem are the superfluid analogues of the inertial r modes of a rotating single-fluid star. This is interesting since the r modes are known to be secularly unstable due to the emission of gravitational radiation (Andersson & Kokkotas 2001). In fact, the connection between the two instabilities goes even deeper than this since the radiation-driven secular instability is also a variation of the Kelvin–Helmholtz instability. In that case, the two fluids are the stellar fluid and the radiation.

This paper is only a first probe into what promises to be a rich problem area. Future studies must address issues concerning the effects of different dissipation mechanisms, the non-linear evolution of the instability, possible experimental verification for superfluid helium, etc. These are all very interesting problems which we hope to investigate in the near future.

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[2] Even though we do not discuss this issue in detail here, it is exciting to contemplate possible experimental tests of the superfluid two-stream instability in, for example, superfluid ^{4}He.

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