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Spinning membranes

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Abstract

We present new solutions of the classical equations of motion of bosonic (matrix-)membranes. Those relating to minimal surfaces in spheres provide spinning membrane solutions in $AdS_p \times S^q$, as well as in flat space–time. Nontrivial reductions of the BMN matrix model equations are also given.

1. Introduction

Starting from the premise that 'membranes are to M-theory what strings are to string theory' the search for classical solutions of membrane dynamics needs almost no justification. Given the additional fact that promising approaches to M-theory are within the context of matrix mechanics, solutions to its equations of motion are equally relevant. The observation that a discretized formulation of membrane dynamics is matrix mechanics [1] links the two.

In the context of string theory, the study of classical solutions was recently revived in [2] (see [3] for a review of further interesting subsequent developments). Relating time-dependent classical solutions of the string sigma-model in an $AdS_5 \times S^5$ target space–time to the dual conformal field theory, extends the testable features of the duality between string theory and $\mathcal{N} = 4$ SYM, i.e., of the AdS/CFT correspondence.

A likely extension of these ideas to M-theory is to consider their motion on maximally supersymmetric backgrounds which, aside from eleven-dimensional Minkowski space, are $AdS_7 \times S^4$ and $AdS_4 \times S^7$. The former is the near-horizon limit of a stack of N coincident M5 branes with $\frac{1}{2}R_{\mathrm{AdS}} = R_S = l_p(\pi N)^{1/3}$ and the latter is the near-horizon limit of a stack of N M2 branes with $2R_{AdS} = R_S = l_P(32\pi^2 N)^{1/6}$. The dualities between classical supergravity on these background and the conformal field theories on the world-volume of the branes which create

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them has been studied. In particular for the $AdS_7 \times S^4$ case, if the duality holds, nontrivial information about the $(0,2)$ conformal field theory of N interacting tensor multiplets in six dimensions has been obtained, e.g., its conformal anomaly has been computed [4,5]. Direct verifications have, however, so far been impossible, mainly due to the lack of knowledge of the interacting $(0,2)$ theory.

One of the open problems in string theory is its quantization in nontrivial backgrounds, such as $AdS_5 \times S^5$. An exception is the gravitational plane wave background which is obtained as the Penrose limit of the $AdS_5 \times S^5$ vacuum of type IIB string theory. In this background light-cone quantization leads to a free theory on the worldsheet whose spectrum is easily computed [6]. This opens the way to the duality between string theory and another sector of large-N SYM, which is characterized by large R-charge ($\sim \sqrt{N}$) and conformal weight ($\sim \sqrt{N}$). The extensive activity to which this has led was initiated in [7].

The difficulties related to quantization are much more severe in M-theory where quantization on any background is still elusive. The semiclassical analysis, which in the case of string theory provides valuable nontrivial information about the dual conformal field theory, can, however, be extended to M-theory. While the equations of motion of strings on $AdS_5 \times S^5$ reduce, for special symmetric configurations, to classical integrable systems [8,9], this is not as simple for membranes. Also, the integrable spin-chains which appear in the discussion of the dual gauge theory [10,11], have so far no known analogue in the $(0,2)$ tensor theory. However, the matrix model of the discrete light cone description of M-theory on plane waves obtained as Penrose limits of $AdS_4 \times S^7$ and $AdS_7 \times S^4$ is known [7] and has been studied (see, e.g., [12]).

In this Letter we present new solutions to bosonic matrix model equations (in Minkowski space, and of the BMN matrix model), as well as make a first step towards the semi-classical analysis of M-theory in $AdS_p \times S^q$ backgrounds, where we will find that the equations of motion, upon imposing a suitable ansatz, may be reduced to the equations describing minimal embeddings of 2-surfaces into higher spheres (as well as generalizations thereof).

2. The bosonic matrix model equations

The time evolution of spatially constant $SU(N)$ gauge fields in $\mathbb{R}^{1,d}$ as well as of regularized membranes in $\mathbb{R}^{1,d+1}$ [1] is governed by equations of motion

$$\ddot{X}_i = -\sum_{j=1}^{d} \big[[X_i, X_j], X_j\big] \tag{1}$$

involving d Hermitean traceless $N \times N$ time-dependent matrices, with the constraint (‘Gauss law’, respectively, reflecting a residual diffeomorphism invariance in a light cone orthonormal gauge description of relativistic membranes)

$$\sum_{i=1}^{d} [X_i, \dot{X}_i] = 0. \tag{2}$$

As shown in [13], solutions of these equations may be found by making the ansatz

$$X_i(t) = x(t)\mathcal{R}_{ij}(t)M_j, \tag{3}$$

with $\mathcal{R}(t) = e^{\mathcal{A}\varphi(t)}$ a real, orthogonal $d \times d$ matrix and $\{M_j\}_{j=1}^{d}$ time-independent $N \times N$ matrices. Define $\vec{M} := (M_1, M_2, \ldots, M_d)$ and require $\mathcal{A}^2\vec{M} = -\vec{M}$. Imposing that no component of both $\vec{M}$ and $\mathcal{A}\vec{M}$ vanishes, restricts d to be even. By a suitable change of basis one can always cast $\mathcal{A}$ into the form

$$\mathcal{A} = \mathrm{diag}(J, \ldots, J) \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4}$$

or, alternatively,

$$\mathcal{A}=\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}. \tag{5}$$

Inserting the ansatz (3) into (1) yields, under the assumption that φ and x are related through $\dot{\varphi}x^2 = L$ $(=\text{const})$,

$$\frac{1}{2}\dot{x}^2+\frac{\lambda}{4}x^4+\frac{L^2}{2x^2}=\text{const}, \tag{6}$$

$$\sum_{j=1}^{d}\big[[\vec{M},M_j],M_j\big]=\lambda\vec{M}, \tag{7}$$

and the constraint (2) becomes

$$\sum_{i=1}^{d}\big[M_i,(\mathcal{A}\vec{M})_i\big]=0. \tag{8}$$

Before we turn to the construction of solutions of the matrix equations, let us note that given any solution of (7) there are always trivial ways to solve the contraint (8). Given a solution $\vec{M}'$ of (7) one can define $\vec{M} := (\vec{M}', \vec{0})$ (by adding d zeroes) and choose $\mathcal{A}$ such that $\mathcal{A}\vec{M} = (\vec{0}, -\vec{M}')$. In this way each term in the sum (8) will be identically zero. Clearly, $\vec{M}'$ is a solution of (7) with $d' = 2d$. Another way to satisfy (8) is by letting $\vec{M} = (\vec{M}', \vec{M}')$. Below we will find solutions which do not rely on this “doubling mechanism”.

3. Solutions of the matrix equation for $d = 8$

A very simple way to solve (7) is in terms of the Hermitian generators T^a of any semi-simple Lie algebra

$$[T_a,T_b]=if_{ab}{}^{c}T_c. \tag{9}$$

If we choose the basis such that the Cartan–Killing metric is $\kappa_{ab} = c_2\delta_{ab}$, $\vec{M} = (T_a)$ solves (8) with $\lambda = c_2$.

If we require $d \leqslant 9$ and even, the only physically interesting case, apart from $SU(2)$ (the ‘fuzzy sphere’) is $SU(3)$ with $d = 8$. However, since the discussion can be easily generalized to any $SU(N)$ with N odd, we will give the solution of (8) for the general case.

To solve (8) with $\mathcal{A}$ as given in (4), we choose a particular basis for the $su(N)$ Lie algrebra. A standard basis is $((E_{ij})_{kl} = \delta_{ik}\delta_{jl})$

$$H_k=\frac{1}{\sqrt{k(k+1)}}\left(\sum_{j=1}^{k}E_{jj}-kE_{k+1,k+1}\right),\qquad k=1,\dots,N-1,$$
$$E^{+}_{kl}=\frac{1}{\sqrt{2}}(E_{kl}+E_{lk}),\qquad E^{-}_{kl}=\frac{i}{\sqrt{2}}(E_{kl}-E_{lk}),\quad k<l. \tag{10}$$

It is not difficult to verify that

$$\vec{M}=\big(H_1,\dots,H_{N-1},E^{+}_{12},-E^{-}_{12},\dots,E^{+}_{kl},(-)^{k+l}E^{-}_{kl},\dots,E^{+}_{N-1,N},-E^{-}_{N-1,N}\big) \tag{11}$$

solves (7) and (8) and satisfies $\vec{M}^2 = \frac{N^2-1}{N}$.

This being a consequence of the algebra, not its particular representation means, that higher-dimensional representations of $SU(N)$ yield higher-dimensional solutions of (7) and (8). In particticluar we obtain a solution for $d = 8$ for any representation of $SU(3)$.

Another way to present solutions related to $SU(N)$, which has the advantage of allowing to pass to a continuum limit, is as follows. For arbitrary odd $N > 1$, define N^2 independent $N \times N$ matrices

$$U_{\mathbf{m}}^{(N)} := \frac{N}{4\pi M(N)} \omega^{\frac{1}{2}m_1 m_2} g^{m_1} h^{m_2}, \tag{12}$$

where $\omega := e^{\frac{4\pi i M(N)}{N}}$ is a primitive Nth root of unity, $\mathbf{m} = (m_1, m_2)$ and

$$g_{ij} = \omega^{i-1}\delta_{ij}, \qquad h_{ij} = \delta_{i,j-1} \quad (j + N \equiv j) \tag{13}$$

providing a basis of the Lie algebra $gl(N, \mathbb{C})$, with [14]

$$\left[U_{\mathbf{m}}^{(N)}, U_{\mathbf{n}}^{(N)}\right] = -\frac{iN}{2\pi M(N)} \sin\left(\frac{2\pi M(N)}{N}(\mathbf{m} \times \mathbf{n})\right) U_{\mathbf{m}+\mathbf{n}}^{(N)} \tag{14}$$

(for the moment, we will put $M(N) = 1$, as only when $N \to \infty$, $\frac{M(N)}{N} \to \Lambda \in \mathbb{R}$, this "degree of freedom" is relevant). Using (14), it is easy to see that

$$\left[\left[U_{\mathbf{m}}^{(N)}, U_{\mathbf{n}}^{(N)}\right], U_{-\mathbf{n}}^{(N)}\right] = \frac{N^2}{4\pi^2} \sin^2 \frac{2\pi}{N} (\mathbf{m} \times \mathbf{n}) U_{\mathbf{m}}^{(N)}. \tag{15}$$

Let now $N = 3$ and

$$\begin{aligned}\vec{M} = \frac{2\pi}{3}\bigg(&\frac{U_{1,0} + U_{-1,0}}{2}, \frac{U_{1,0} - U_{-1,0}}{2i}, \frac{U_{0,1} + U_{0,-1}}{2}, \frac{U_{0,1} - U_{0,-1}}{2i},\\ &\frac{U_{1,1} + U_{-1,-1}}{2}, \frac{U_{1,1} - U_{-1,-1}}{2i}, \frac{U_{-1,1} + U_{1,-1}}{2}, \frac{U_{-1,1} - U_{1,-1}}{2i}\bigg).\end{aligned} \tag{16}$$

The components of $\vec{M}$ form a basis of hermitian 3×3 matrices, and thus of the Lie algebra $su(3)$. It is straightforward to relate this basis to the basis (10) but perhaps one should note that (10) is *not* invariant under general linear transformations.

It is also easy to check that (16) satisfies (7) (for $N = 3$, $\sin^2(2\pi/N) = \sin^2(4\pi/N)$), $\vec{M}^2 =$ and $[M_{2i-1}, M_{2i}] = 0$. Thus, with $\mathcal{A}$ as in (4), (8) is also satisfied. We therefore obtain a solution of (1), satisfying the constraint (2) for $N = 3$ and $d = 8$, by letting

$$(X_i) := x(t)\left[\vec{M} \cos\varphi(t) + \mathcal{A}\vec{M} \sin\varphi(t)\right] \tag{17}$$

with $x(t)$ and $\varphi(t)$ satisfying (6).

The above construction can be generalized to yield other solutions with $d = 8$. It is straightforward to verify that

$$\begin{aligned}\vec{M} = \frac{2\pi}{3}\bigg(&\frac{U_{\mathbf{m}} + U_{-\mathbf{m}}}{2}, \frac{U_{\mathbf{m}} - U_{-\mathbf{m}}}{2i}, \frac{U_{\mathbf{m}'} + U_{-\mathbf{m}'}}{2}, \frac{U_{\mathbf{m}'} - U_{-\mathbf{m}'}}{2i},\\ &\frac{U_{\mathbf{n}} + U_{-\mathbf{n}}}{2}, \frac{U_{\mathbf{n}} - U_{-\mathbf{n}}}{2i}, \frac{U_{\mathbf{n}'} + U_{-\mathbf{n}'}}{2}, \frac{U_{\mathbf{n}'} - U_{-\mathbf{n}'}}{2i}\bigg),\end{aligned} \tag{18}$$

with

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \qquad \mathbf{m}' = \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix}, \qquad \mathbf{n}' = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}, \tag{19}$$

is a solution of (7) and (8) if $\mathbf{m}^2 = \mathbf{n}^2$ with N arbitrary. The reason is that, by using (15) the "discrete Laplace operator"

$$\Delta_{\vec{M}}^{(N)} := \sum_{j=1}^{d} \left[[\,\cdot\,, M_j], M_j\right], \tag{20}$$

when acting on any of the components of $\vec{M}$, in each case yields the same scalar factor ("eigenvalue")

$$\frac{N^2}{4\pi^2}\left(\sin^2\frac{2\pi}{N}(\mathbf{m}\times\mathbf{n})+\sin^2\frac{2\pi}{N}\mathbf{m}^2+\sin^2\frac{2\pi}{N}(\mathbf{m}\cdot\mathbf{n})\right). \tag{21}$$

In the general case (18) is a solution for fixed $N=\mathbf{m}^2+\mathbf{n}^2$, which we assume to be odd. Higher-dimensonal representations can be obtained if we expand the eight $N\times N$ matrices in terms of a basis of $gl(N,\mathbb{C})$

$$M_j^{(N)}=\sum_{a=1}^{N^2-1}\mu_j^a(N)T_a^{(N)}\quad\text{with}\quad\left[T_a^{(N)},T_b^{(N)}\right]=if_{ab}{}^cT_c^{(N)} \tag{22}$$

and then define

$$M_j^{(N')}:=\sum_{a=1}^{N^2-1}\mu_j^a(N)T_a^{(N')} \tag{23}$$

with $T_a^{(N')}$ be a $N'>N$-dimensional representation of (22).

We want to stress that these generalizations of (16) are not higher-dimensional representations of $SU(3)$; the set of matrices $\vec{M}$ does not form a closed commutator algebra.

4. The continuum limit of matrix solutions as minimal surfaces in S^7

As mentioned in [13], (7) (with $\vec{M}^2=$) is a discrete version of the equations for a minimal surface in a (higher-dimensional) sphere. In [15], such surfaces in S^3 were proven to exist for arbitrary genus.

Equations for a minimal surface $\vec{m}(\varphi^1,\varphi^2)$ in a sphere can be obtained by varying the integral

$$\int\left(\sqrt{g}-\mu\left(\vec{m}^2-1\right)\right)d\varphi^1\,d\varphi^2$$

with $g=\det(g_{rs})$ and $g_{rs}=\partial_r\vec{m}\cdot\partial_s\vec{m}$. One obtains the equations

$$\Delta\vec{m}=-2\vec{m},\qquad \vec{m}^2=1, \tag{24}$$

where Δ is the *Laplace–Beltrami* operator on scalar functions

$$\Delta:=\frac{1}{\sqrt{g}}\partial_r\sqrt{g}g^{rs}\partial_s.$$

The $N\to\infty$ limit of (18),

$$\vec{m}\left(\varphi^1,\varphi^2\right)=\frac{1}{2}(\cos\mathbf{m}\boldsymbol{\varphi},\sin\mathbf{m}\boldsymbol{\varphi},\cos\mathbf{m}'\boldsymbol{\varphi},\sin\mathbf{m}'\boldsymbol{\varphi},\cos\mathbf{n}\boldsymbol{\varphi},\sin\mathbf{n}\boldsymbol{\varphi},\cos\mathbf{n}'\boldsymbol{\varphi},\sin\mathbf{n}'\boldsymbol{\varphi}), \tag{25}$$

(where $\boldsymbol{\varphi}:=(\varphi^1,\varphi^2)$) gives a solution of (24), which for each choice (19) with $\mathbf{m}^2=\mathbf{n}^2$ describes a minimal torus in S^7.

Interestingly, the $N\to\infty$ limit, (25), allows for nontrivial deformations (apart from the arbitrary constant that can be added to each of the 4 different arguments), namely

$$\begin{aligned}\vec{m}_\gamma=\frac{1}{\sqrt{2}}(&\cos\gamma\cos\mathbf{m}\boldsymbol{\varphi},\cos\gamma\sin\mathbf{m}\boldsymbol{\varphi},\cos\gamma\cos\mathbf{m}'\boldsymbol{\varphi},\cos\gamma\sin\mathbf{m}'\boldsymbol{\varphi},\\&\sin\gamma\cos\mathbf{n}\boldsymbol{\varphi},\sin\gamma\sin\mathbf{n}\boldsymbol{\varphi},\sin\gamma\cos\mathbf{n}'\boldsymbol{\varphi},\sin\gamma\sin\mathbf{n}'\boldsymbol{\varphi}).\end{aligned} \tag{26}$$

It is easy to check that (26) solves (24) and, equivalently,

$$\sum_{j=1}^{d}\big\{\{m_i, m_j\}, m_j\big\} = -2m_i \quad \text{with} \quad \vec{m}^2 = 1, \tag{27}$$

where $\{f, h\} = \frac{2}{\mathbf{m}^2}(\partial_1 f \partial_2 h - \partial_2 f \partial_1 h)$ (cf. below). When checking (27) via the $N \to \infty$ limit of (14), the γ-dependence of the m_j at first looks as if leading to a "contradicition" (it *would*, in the finite N-case), but the rationality of the structure-constants ($\mathbf{m} \times \mathbf{n}$ instead of $\frac{N}{2\pi}\sin\frac{2\pi}{N}(\mathbf{m} \times \mathbf{n})$) comes at rescue.

Finally, rewrite (26) as

$$\vec{m}_\gamma = \frac{1}{\sqrt{2}}\vec{x}_+^{[\gamma]} + \frac{1}{\sqrt{2}}\vec{x}_-^{[\gamma]} \tag{28}$$

with

$$\begin{aligned}\vec{x}_\pm^{[\gamma]} = \frac{1}{2}\big(&\cos(\mathbf{m}\boldsymbol{\varphi} \pm \gamma), \sin(\mathbf{m}\boldsymbol{\varphi} \pm \gamma), \cos(\mathbf{m}'\boldsymbol{\varphi} \pm \gamma), \sin(\mathbf{m}'\boldsymbol{\varphi} \pm \gamma),\\ &\pm\sin(\mathbf{n}\boldsymbol{\varphi} \pm \gamma), \mp\cos(\mathbf{n}\boldsymbol{\varphi} \pm \gamma), \pm\sin(\mathbf{n}'\boldsymbol{\varphi} \pm \gamma), \mp\cos(\mathbf{n}'\boldsymbol{\varphi} \pm \gamma)\big).\end{aligned} \tag{29}$$

While γ, in this form, becomes irrelevant (insofar each of the 4 arguments in $\vec{x}_+ := \vec{x}_+^{[0]}$, as well as those in $\vec{x}_- := \vec{x}_-^{[0]}$ can have an arbitrary phase-constant), not only their sum, (28), but (due to the mutual orthogonality of $\vec{x}_+$, $\partial_1\vec{x}_+$, $\partial_2\vec{x}_+$, $\vec{x}_-$, $\partial_1\vec{x}_-$ and $\partial_2\vec{x}_-$) *both* $\vec{x}_+$ and $\vec{x}_-$ *separately*, in fact any linear combination

$$\vec{x}_\theta = \cos\theta\,\vec{x}_+ + \sin\theta\,\vec{x}_- \tag{30}$$

gives a minimal torus in S^7.

5. Bosonic membranes on $AdS_p \times S^q$

Let us consider closed bosonic membranes in $AdS_p \times S^q$ (the action for the super-membrane in these backgrounds was constructed in [16]). Their dynamics is derived from the action

$$S = \int d^3\varphi\,\big(\sqrt{G} + \lambda(\vec{x}^2 - 1) + \tilde{\lambda}(y^2 - 1)\big), \tag{31}$$

where $y^\mu(\varphi^\alpha)$ ($\mu = 1, \ldots, p$; $\alpha = 0, 1, 2$) and $x_k(\varphi^\alpha)$ ($k = 1, \ldots, q+1$) are the embedding coordinates, $\vec{x}^2 = \sum_{k=1}^{q+1} x_k x_k$, $y^2 = y^\mu y^\nu \eta_{\mu\nu} = y_0^2 + y_p^2 - \sum_{\mu'=1}^{p-1}(y_{\mu'})^2$ and

$$G_{\alpha\beta} = \partial_\alpha y^\mu \partial_\beta y^\nu \eta_{\mu\nu} - \partial_\alpha\vec{x} \cdot \partial_\beta\vec{x}. \tag{32}$$

The constraints $y^2 = 1 = \vec{x}^2$ follow by varying (31) w.r.t. the Lagrange multipliers λ and $\tilde{\lambda}$ while variation w.r.t. y^μ and x_k yields the equations of motion

$$\partial_\alpha\big(\sqrt{G}\,G^{\alpha\beta}\partial_\beta y^\mu\big) = 2\tilde{\lambda} y^\mu, \tag{33}$$

$$\partial_\alpha\big(\sqrt{G}G^{\alpha\beta}\partial_\beta\vec{x}\big) = -2\lambda\vec{x}. \tag{34}$$

Note that we take the radii of the AdS spaces and the sphere to be equal. It is straightforward to generalize the discussion to the case of unequal radii, which is the situation in the M-theory context. Contracting (33) with y^μ and (34) with $\vec{x}$, respectively, and using the constraints $y^2 = \vec{x}^2 = 1$, one finds that

$$\begin{aligned}2\tilde{\lambda} &= -\sqrt{G}G^{\alpha\beta}\partial_\alpha y^\mu \partial_\beta y^\nu \eta_{\mu\nu},\\ 2\lambda &= +\sqrt{G}G^{\alpha\beta}\partial_\alpha\vec{x} \cdot \partial_\beta\vec{x},\end{aligned} \tag{35}$$

implying

$$\lambda+\tilde{\lambda}=-\frac{1}{2}\sqrt{G}G^{\alpha\beta}\left(\partial_\alpha y^\mu\partial_\beta y_\mu-\partial_\alpha\vec{x}\cdot\partial_\beta\vec{x}\right)=-\frac{3}{2}\sqrt{G}. \tag{36}$$

Denoting φ^0 by t, let us make the ansatz (analogous to the corresponding string case, and similar to [17]),

$$\begin{aligned}&y_0=\sin(\omega_0 t),\qquad y_p=\cos(\omega_0 t),\qquad y_{\mu'}=0\quad(\mu'=1,\ldots,p-1),\\&\vec{x}\left(t,\varphi^1,\varphi^2\right)=\mathcal{R}(t)\vec{m}\left(\varphi^1,\varphi^2\right)\end{aligned} \tag{37}$$

with

$$\mathcal{R}(t)=\begin{pmatrix}\cos(\omega_1 t)&-\sin(\omega_1 t)&&&\\\sin(\omega_1 t)&\cos(\omega_1 t)&&&\\&&\cos(\omega_2 t)&-\sin(\omega_2 t)&\\&&\sin(\omega_2 t)&\cos(\omega_2 t)&\\&&&&\ddots\end{pmatrix}. \tag{38}$$

Let us further demand $\dot{\vec{x}}\cdot\partial_1\vec{x}=0=\dot{\vec{x}}\cdot\partial_2\vec{x}$, which, writing $\vec{m}^{\mathrm{T}}=(r_1\cos\theta_1,r_1\sin\theta_1,r_2\cos\theta_2,r_2\sin\theta_2,\ldots)$ reads

$$\sum_{a=1}^{d\equiv[\frac{1}{2}(q+1)]}\omega_a r_a^2\partial_1\theta_a=0=\sum_{a=1}^{d}\omega_a r_a^2\partial_2\theta_a. \tag{39}$$

The world-volume metric is then block-diagonal

$$G_{\alpha\beta}=\mathrm{diag}\left(\omega_0^2-\dot{\vec{x}},-g_{rs}\right) \tag{40}$$

with $g_{rs}=\partial_r\vec{x}\cdot\partial_s\vec{x}=\partial_r\vec{m}\cdot\partial_s\vec{m}$ $(r,s=1,2)$ and $\dot{\vec{x}}^2=\sum_{a=1}^d\omega_a^2 r_a^2$. As is not difficult to see, (33) implies that

$$\rho:=\sqrt{G}G^{00}=\frac{\sqrt{g}}{\sqrt{\omega_0^2-\sum_{a=1}^d\omega_a^2 r_a^2}}=\frac{g}{\sqrt{G}} \tag{41}$$

is (a) *time-independent* (density). In any case,

$$\sum_{a=1}^{d}\omega_a^2 r_a^2+\frac{g}{\rho^2}=\omega_0^2 \tag{42}$$

has to hold and $\tilde{\lambda}$ is determined as $-\rho\omega_0^2/2$.

Let us now turn to the equation for $\vec{x}$ which determines $\vec{m}(\varphi^1,\varphi^2)$, i.e., the shape of the membrane that is being rotated inside S^q by the orthogonal matrix $\mathcal{R}(t)$ (cf. (38)), in order to yield an extremal three-manifold in $AdS_p\times S^q$. With (40), (34) becomes

$$\frac{1}{\rho}\partial_r\left(g\frac{g^{rs}}{\rho}\partial_s\vec{x}\right)=\ddot{\vec{x}}+\frac{2\lambda\vec{x}}{\rho}. \tag{43}$$

Due to Eqs. (37), (38) and (35), implying $\ddot{\vec{x}}=\ddot{\mathcal{R}}(t)\vec{m}$,

$$\frac{2\lambda}{\rho}=\dot{\vec{x}}^2-\frac{\sqrt{G}}{\rho}g^{rs}\partial_r\vec{m}\cdot\partial_s\vec{m}=\sum_{a=1}^{d}\omega_a^2 r_a^2-\frac{2g}{\rho^2} \tag{44}$$

(43) reduces to

$$\left\{\{m_i,m_j\},m_j\right\}=\left(-\omega_{(i)}^2+\sum\omega_a^2 r_a^2-\frac{2g}{\rho^2}\right)m_i, \tag{45}$$

where $\omega_{(1)} = \omega_{(2)} := \omega_1$, $\omega_{(3)} = \omega_{(4)} := \omega_2$, etc.,

$$g = \det(\partial_r \vec{x} \cdot \partial_s \vec{x}) = \det(\partial_r \vec{m} \cdot \partial_s \vec{m}) = \rho^2 \sum_{i<j} \{m_i, m_j\}^2$$

and the (Poisson) bracket is defined as ($\epsilon^{12} = -\epsilon^{21} = 1$)

$$\{f, g\} = \frac{1}{\rho} \epsilon^{rs} \partial_r f \partial_s g \tag{46}$$

for any two differentiable functions on the two-dimensional parameter manifold. The density ρ, though time-independent, was defined in (41) in a 'dynamical' way, i.e., depending on $\vec{x}(t, \varphi^1, \varphi^2)$. However, due to [18] we may assume it to be any given 'nondynamical' density having the same 'volume' $\int \rho(\varphi^1, \varphi^2)\, d^2\varphi$. This frees (46) from its seeming $\vec{x}$-dependence while reducing the original (φ^1, φ^2)-diffeomorphism invariance to those preserving ρ.

Confining ourselves (for the time being) to solving (39) in a trivial way by letting the $\theta_a(\varphi^1, \varphi^2)$ be constants, i.e., independent of $\varphi^{1,2}$, the equations to be solved are

$$\bigl\{\{r_a, r_b\}, r_b\bigr\} = \left(-\omega_a^2 + \sum \omega_c^2 r_c^2 - \frac{2g}{\rho^2}\right) r_a, \quad a = 1, \dots, d \tag{47}$$

subject to (42) and to $\sum r_a^2 = 1$. In the case of the string, rather than the membrane, this equation becomes [8], for $d = 3$, the equation of motion of the Neumann system, namely the constrained motion of a three-dimensional harmonic oscillator on the surface of a two-sphere.

If the 'spatial' frequencies ω_a are chosen to be all equal, it follows that $\sum \omega_c^2 r_c^2 = \omega^2 = \text{const}$ as well as (from (42)) $g/\rho^2 = \omega_0^2 - \omega^2 = \text{const}$. This simplifies (47) to

$$\bigl\{\{r_a, r_b\}, r_b\bigr\} = -2\bigl(\omega_0^2 - \omega^2\bigr) r_a \tag{48}$$

which can be explicitly solved by (known) minimal embeddings of two-surfaces into $d = [\frac{1}{2}(q+1)]$-dimensional unit spheres.

To see this, one could recall (41), which shows that (48), rewritten as

$$\frac{1}{\rho} \partial_s \left(g \frac{g^{su}}{\rho} \partial_u \vec{r} \right) = -2\bigl(\omega_0^2 - \omega^2\bigr) \vec{r}, \tag{49}$$

is identical to the standard 'minimal surface' equation

$$\frac{1}{\sqrt{g}} \partial_s \bigl(\sqrt{g} g^{su} \partial_u \vec{r}\bigr) = -2\vec{r}. \tag{50}$$

This, incidentally, justifies calling (20) 'discrete Laplace operator'. Eq. (50) is the Euler–Lagrange equation which one obtains if one varies

$$\int d^2\varphi \left(\sqrt{g} - \mu(\varphi)\bigl(\vec{r}^2 - 1\bigr)\right) \tag{51}$$

w.r.t. the embedding coordinates $r_a(\varphi^1, \varphi^2)$ and the local Lagrange multiplier $\mu(\varphi)$ (which guarantees $\vec{r}^2 = 1$).

Another way to show the equivalence of (49) (hence (48)) to (50) is as follows: the results of Ref. [18] allow one to choose the coordinates φ^s in the diffeomorphism invariant equation (50) such that $\sqrt{g/(\omega_0^2 - \omega^2)}$ is equal to any given density with the same 'volume' (i.e., integral over $d^2\varphi$). Choosing it to be ρ shows that solutions of (50) give solutions of (49). To show the converse, one notes that (49) automatically implies that $\frac{g}{\rho^2} = \omega_0^2 - \omega^2$ (multiply (49) by $\vec{r}$, and use $\vec{r}^2 = 1$ three times: once on the r.h.s., once for $\vec{r} \cdot \partial_u \vec{r} = 0$ and, finally, to write $\vec{r} \cdot \partial_s \partial_u \vec{r}$ as $-g_{su}$).

Concerning explicit solutions of (48), respectively, (50) (from now on we put $\omega_0^2 - \omega^2 = 1$ by rescaling ρ) let us only mention the two simplest ones:

$$r_1 = \sin\theta\cos\varphi, \qquad r_2 = \sin\theta\sin\varphi, \qquad r_3 = \cos\theta, \qquad r_{a>3} = 0 \tag{52}$$

(the equator 2-sphere in $S^{d-1\geqslant 2}$, $\varphi^1 = \theta \in [0,\pi]$, $\varphi^2 = \varphi \in [0, 2\pi]$, $\rho = \sin\theta$) and

$$\vec{r} = \frac{1}{\sqrt{2}}(\cos\varphi_1, \sin\varphi_1, \cos\varphi_2, \sin\varphi_2, 0, \ldots, 0) \tag{53}$$

(the Clifford-torus in $S^{d-1\geqslant 3}$). Lawson [15] proved that there exist minimal embeddings into S^3 of any topological type.

6. Non-trivial reductions of the bosonic BMN matrix model equations

Consider the bosonic BMN [7] matrix model equations

$$\begin{aligned}
\ddot{X}_a &= -\sum_{i=1}^{9}\big[[X_a, X_i], X_i\big] - 4m^2 X_a - 3im\epsilon_{abc}[X_b, X_c],\\
\ddot{X}_\mu &= -\sum_{i=1}^{9}\big[[X_\mu, X_i], X_i\big] - m^2 X_\mu, \qquad \sum_{i=1}^{9}[X_i, \dot{X}_i] = 0,
\end{aligned} \tag{54}$$

where $a, b, c = 1, \ldots, 3$, $\mu = 4, 5, \ldots, 9$ and $i = 1, 2, \ldots, 9$. We want to find nontrivial time-dependent solutions of these equations by using similar techniques as for (1).

One of the reasons for making the ansatz (3) was to find solutions that do not collapse to zero. In (54) we have mass-terms and hence, we are not forced to only consider "rotating" solutions, as we did for (1).

Consider the following nine traceless Hermitean 3×3 matrices ($a = 1, 2, 3$; $a' = a + 3$; $a'' = a + 6$, $(E_{ab})_{cd} = \delta_{ac}\delta_{bd}$)

$$\hat{M}_a = -i\epsilon_{abc}E_{bc}, \qquad \hat{M}_{a'} = E_{aa} - \frac{1}{3}, \qquad \hat{M}_{a''} = |\epsilon_{abc}|E_{bc}, \tag{55}$$

which are antisymmetric, diagonal and symmetric, respectively and which satisfy $\sum \hat{M}_a^2 = 3\sum \hat{M}_{a'}^2 = \sum \hat{M}_{a''}^2 = 2\cdot$. The corresponding discrete Laplace operators are

$$\begin{aligned}
\Delta_- &:= \big[[\,\cdot\,, \hat{M}_a], \hat{M}_a\big] = -\sum_{b<c}\big[[\,\cdot\,, E_{bc}^-], E_{bc}^-\big],\\
\Delta_\parallel &:= \big[[\,\cdot\,, \hat{M}_{a'}], \hat{M}_{a'}\big] = \sum_a\big[[\,\cdot\,, E_{aa}], E_{aa}\big],\\
\Delta_+ &:= \big[[\,\cdot\,, \hat{M}_{a''}], \hat{M}_{a''}\big] = \sum_{b<c}\big[[\,\cdot\,, E_{bc}^+], E_{bc}^+\big],
\end{aligned} \tag{56}$$

where $E_{ab}^\pm := E_{ab} \pm E_{ba}$. As is easy to check, the action of (56) on (55) is purely diagonal, with eigenvalues

$$\Delta_- = \operatorname{diag}(222\,666\,666), \qquad \Delta_\parallel = \operatorname{diag}(222\,000\,222), \qquad \Delta_+ = \operatorname{diag}(666\,666\,222). \tag{57}$$

As an aside we want to mention that this structure generalizes to traceless hermitian $N\times N$ matrices for any N. The eigenvalues of the three Laplacians are $(2(N-2), 2N, 2N)$ for Δ_-, $(2, 0, 2)$ for $\Delta_\parallel$ and $(2N, 2N, 2(N-2))$ for Δ_+, where the multiplicities of the entries are $(\frac{1}{2}N(N-1), N, \frac{1}{2}N(N-1))$.

As (56) only involves commutators, (57) extends to the action of (56) on the 9 $N \times N$ matrices corresponding to (55) in an arbitrary N-dimensional representation of $su(3)$. This way one can find nine Hermitean $N \times N$ matrices $M_{i=1,\dots,9}$ with eigenvalues (57). Letting, e.g.,

$$X_a(t) = x(t) M_a, \qquad X_{a'}(t) = \sqrt{3}\, y(t) M_{a'}, \qquad X_{a''}(t) = z(t) M_{a''}, \tag{58}$$

with $[M_a, M_b] = i\epsilon_{abc} M_c$ reduces (54), for arbitrary N, to differential equations involving only 3 scalar functions (x, y and z):

$$\begin{aligned} &\ddot{x} + x\big(4m^2 + 2x^2 + 6y^2 + 6z^2 - 6mx\big) = 0, \\ &\ddot{y} + y\big(m^2 + 6x^2 + 6z^2\big) = 0, \qquad \ddot{z} + z\big(m^2 + 6x^2 + 6y^2 + 2z^2\big) = 0. \end{aligned} \tag{59}$$

Another reduction can be obtained by considering rotating solutions

$$\begin{aligned} &X_a(t) = x(t) M_a, \qquad X_\mu = \sqrt{\frac{3}{5}} z(t) \mathcal{R}_{\mu\nu}(t) \tilde{M}_\nu, \\ &(R_{\mu\nu}) = e^{\mathcal{A}\varphi(t)} \in SO(6), \qquad z^2 \dot{\varphi}(t) = L = \text{const}, \qquad \mathcal{A}^2 = -\,\text{id}, \\ &\tilde{M}_{a'} := \sqrt{2} M_{a'}, \qquad \tilde{M}_{a''} := M_{a''} \end{aligned} \tag{60}$$

(note that $[\tilde{M}_{a'}, \tilde{M}_{a''}] = 0$), yielding

$$\ddot{x} + x\big(4m^2 + 2x^2 + 6z^2 - 6mx\big) = 0, \qquad \ddot{z} + z\left(m^2 + 6x^2 + \frac{18}{5} z^2 - \frac{L^2}{z^4}\right) = 0. \tag{61}$$

For the ansatz (60) to work it is important that all six $N \times N$ matrices $\tilde{M}_\nu$ have the same eigenvalue under the action of both Δ_- and $\tilde{\Delta}_+ + 2\tilde{\Delta}_\parallel$.

Various other choices and combinations are possible, e.g., $\tilde{M}_{a''} = 0$, $\tilde{M}_{a'} = M_a$, i.e.,

$$\begin{aligned} &X_a = x(t) M_a, \\ &X_\mu = z(t)(\cos\varphi M_1, \cos\varphi M_2, \cos\varphi M_3, \sin\varphi M_1, \sin\varphi M_2, \sin\varphi M_3), \\ &z^2 \dot{\varphi} = L = \text{const}, \end{aligned} \tag{62}$$

giving

$$\ddot{x} + x\big(2x^2 + 2z^2 + 4m^2 - 6mx\big) = 0, \qquad \ddot{z} + z\left(2x^2 + 2z^2 + m^2 - \frac{L^2}{z^4}\right) = 0. \tag{63}$$

Apart from the trivial static (known) solutions, ($L = 0, z = 0; x = 0, m$ or $2m$), and genuinely timedependent solutions of (63), there are several "intermediate" solutions, for which z is constant, but nonzero (making $\varphi(t)$ linear in t): 2 for which $x = 0$, $z = \pm z_0$, as well as those corresponding to the roots of the quintic equation obtained via $z^2 = 3mx - x^2 - 2m^2$.

Replacing M_a by $M_{a'}$, respectively, $M_{a''}$, in the second part of (62) leads to yet other solutions. One can consider both the $m \to 0$ ($m \to \infty$) limit of these solutions as well as their $N \to \infty$ continuum limit.

Finally note that one can also let both X_μ and X_a rotate, letting e.g.,

$$\begin{aligned} &X_a(t) = \sqrt{6}\, x(t)(\cos\theta M_4 - \sin\theta M_5, \sin\theta M_4 + \cos\theta M_5, M_6), \qquad X_\mu(t) = y(t) \mathcal{R}_{\mu\nu} \tilde{M}_\nu, \\ &\tilde{M}_{a'} = M_a, \qquad \tilde{M}_{a''} = M_{a''}, \qquad x^2 \dot{\theta} = K, \qquad y^2 \dot{\varphi} = L \end{aligned} \tag{64}$$

(as before, $\mathcal{R} = e^{\mathcal{A}\varphi(t)}, \dots$) which results in equations of motion,

$$\ddot{x} + x\left(4m^2 + 12y^2 - \frac{K^2}{x^4}\right) = 0, \qquad \ddot{y} + y\left(m^2 + 12x^2 + 8y^2 - \frac{L^2}{y^4}\right) = 0. \tag{65}$$

It is easy to show that all four reductions lead to systems of ordinary differential equations which are in a canonical way Hamiltonian, e.g., for (65) w.r.t.

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{L^2}{2y^2} + \frac{K^2}{2x^2} + \frac{m^2}{2}\left(y^2 + 4x^2\right) + 6x^2y^2 + 2y^4. \tag{66}$$

Even though exact solutions of these systems of equations are as yet unknown and probably may not exist in terms of known functions, they can be easily solved numerically.

Note added

After this paper was submitted, we became aware of Refs. [19,20] where simple solutions to the membrane equations on $AdS_7 \times S^4$ were found.

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