

Eleven dimensional supergravity and the $E_{10}/K(E_{10})$ σ -model at low A_9 levels ¹

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Abstract. Recently, the concept of a nonlinear σ -model over a coset space G/H was generalized to the case where the group G is an infinite-dimensional Kac-Moody group, and H its (formal) ‘maximal compact subgroup’. Here, we study in detail the one-dimensional (geodesic) σ -model with $G = E_{10}$ and $H = K(E_{10})$. We re-examine the construction of this σ -model and its relation to the bosonic sector of eleven-dimensional supergravity, up to height 30, by using a new formulation of the equations of motion. Specifically, we make systematic use of $K(E_{10})$ -orthonormal local frames, in the sense that we decompose the ‘velocity’ on $E_{10}/K(E_{10})$ in terms of objects which are representations of the compact subgroup $K(E_{10})$. This new perspective may help in extending the correspondence between the $E_{10}/K(E_{10})$ σ -model and supergravity beyond the level currently checked.

1. Introduction

In this contribution we explain in detail the recent construction of the $E_{10}/K(E_{10})$ non-linear σ -model of [1] and its relation to the bosonic sector of $D = 11$ supergravity [2]. The present approach differs from [1] in some important technical aspects, in particular the use of ‘ $K(E_{10})$ -orthonormal frames’, rather than ‘coordinate frames’. We hope that this new perspective will allow one to extend the results of [1] to higher-level σ -model degrees of freedom, and to more general spatial dependences of the supergravity fields than those previously taken into account. For other, and in part competing, approaches to the search for symmetries that might underly M theory we refer readers to [3, 4, 5, 7, 8] and references therein. An analysis very similar to the present one, but based on the decomposition of E_{10} under its $D_9 \equiv SO(9,9)$ subgroup has been carried out recently in [9].

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The present construction grew out of an attempt to extend the well known Belinskii-Khalatnikov-Lifshitz (BKL) analysis of spacelike (cosmological) singularities [10] in Einstein's theory (possibly with additional massless fields) and to deepen the surprising discovery of a profound relation between this analysis and the theory of indefinite Kac Moody algebras [11, 12, 13, 1, 14]. As described in detail in [14], one considers a big-bang-like spacetime with an initial singular spacelike hypersurface 'located' at time $t = 0$, on which some (but not all) components of the metric become singular. According to the BKL hypothesis spatial gradients should become less and less important in comparison with time derivatives as $t \rightarrow 0$. This suggests that the resulting theory should be effectively describable in terms of a one dimensional reduction. Ref. [14], which generalized previous results by BKL and others, made this idea more precise by ²

(i) proving in full generality that, except for a finite number of them, the infinite number of degrees of freedom encoded in the spatially inhomogeneous metric, and in other fields, *freeze* in the sense that they tend to some finite limits as $t \rightarrow 0$;

(ii) showing that the dynamics of the remaining 'active' degrees of freedom (corresponding to the diagonal components of the metric, and to the dilaton(s)) could be asymptotically described in terms of a simple 'billiard dynamics';

(iii) proving that in many interesting physical theories the 'billiard table' encoding the dynamics of the active degrees of freedom could be identified with the Weyl chamber of some Lorentzian Kac-Moody algebra; and

(iv) generalizing the concept of nonlinear σ -model on a coset space G/H to the case where G is a Lorentzian Kac-Moody group, and H its 'maximal compact subgroup', and showing that such (one-dimensional) σ -models are asymptotically (as $t \rightarrow 0$) equivalent to the billiard dynamics describing the active degrees of freedom.

The *correspondence* between 'cosmological billiards' and 'Kac-Moody σ -model billiards', *i.e.* geodesics on Kac-Moody cosets G/H , is relatively easy to establish when one considers only the *leading* terms in the dynamics near $t \rightarrow 0$. Going beyond this leading behaviour is much harder and has been attempted only for pure gravity (in which case the relevant Kac Moody algebra is AE_3 [13, 14]), and for the bosonic sector of $D = 11$ supergravity. We recall that this model includes, besides the metric field $g_{\mu\nu}(t, \mathbf{x})$, a 3-form $A_{\mu\nu\lambda}(t, \mathbf{x})$ having a specific Chern-Simons self-coupling FFF , where $F = dA$ [2]. Ref. [1] introduced a precise identification between the purely t -dependent σ -model quantities obtained from the geodesic action on the $E_{10}/K(E_{10})$ coset space on the one hand, and certain fields of $D = 11$ supergravity and their spatial gradients evaluated *at a given, but arbitrarily chosen spatial point* $\mathbf{x} = \mathbf{x}_0$ on the other. So far, this correspondence works for suitably truncated versions of both models, with $\ell \leq 3$ and height ≤ 29 on the σ -model side, and zeroth and first order spatial gradients on the supergravity side. There are, however, indications that it extends to higher levels and higher order spatial gradients: as shown in [1], the level expansion of E_{10} contains all the representations needed for the higher order spatial gradients (as well as many other representations, see [15]). This observation gave rise to the key conjecture of [1], according to which the full geometrical data of $D = 11$ supergravity, or some theory containing it, can be mapped onto a geodesic motion in the $E_{10}/K(E_{10})$ coset space. The hope is that the infinite number of degrees of freedom associated to the ten-dimensional spatial gradients of the metric and the

² We take this opportunity to correct three misprints in [14]: in eq. (6.24), the dilaton contribution should appear with a plus sign: $+(\lambda_p/2)\phi$, whereas it should come with a minus sign in eq. (6.29). The argument of the Θ function in eq. (6.31) should read $-2\tilde{m}_{a_1 \dots a_{d-p-1}}(\beta)$.

3-form (and possibly to other M-theoretic degrees of freedom) can be put in one-to-one correspondence with the infinite number of parameters of the $E_{10}/K(E_{10})$ coset space³. Another way to view this proposal is in terms of a ‘small tension expansion’ of the full theory, which in turn might be related to the zero tension limit of string theory [17].

We emphasize that no extra assumptions beyond the geodesic action are required in the present setup, and that our proposed geodesic action is *essentially unique*, as we will show here. In particular, the relative normalization of all terms in the equations of motion follow from that action, independently of the existence of a supersymmetric extension. For instance, the unique value of the coefficient of the Chern-Simons coupling $\propto AFF$ present in the 11-dimensional supergravity action [2] is found to match exactly a coefficient in the $E_{10}/K(E_{10})$ geodesic action. As further evidence of the correspondence between the two actions one might also count the fact that the geodesic action on $E_{10}/K(E_{10})$ is not compatible with the addition of a cosmological constant in the $D = 11$ supergravity action (an addition which has been proven to be incompatible with supersymmetry in [18]). Indeed, it is easily checked that such a term in the supergravity Hamiltonian cannot be matched to any E_{10} root.

In the present paper, we shall introduce a new formulation of the $E_{10}/K(E_{10})$ σ -model. In our previous work [1] we had studied the dynamics defined by the $E_{10}/K(E_{10})$ action in terms of what were, essentially, some global coordinates $A^{(\ell)}$ on the coset space $E_{10}/K(E_{10})$. These coordinates are defined by considering a Borel-type triangular exponential parametrization of a general coset element in the form

$$\mathcal{V}(A(t)) = \exp\left(\sum_{\ell=0}^{\infty} A^{(\ell)}(t) * E^{(\ell)}\right) \quad (1.1)$$

The notation here is very schematic and will be further explained below. Suffice it to say here that $E^{(0)}$ denote the Cartan and positive-root generators of the $GL(10)$ subalgebra of E_{10} , while $E^{(\ell)}$, $\ell > 0$ denote all the remaining raising (positive root) generators of E_{10} . Here ℓ denotes the $GL(10)$ level, and all the $GL(10)$ (and degeneracy) indices needed to specify the representations appearing among the $E^{(\ell)}$'s are suppressed. The infinite sequence of real numbers $A^{(\ell)}$ for $\ell \geq 0$ explicitly defines the specific triangular coordinates of any coset class $\mathcal{V} \in E_{10}/K(E_{10})$. The coordinates $A^{(\ell)}$ are globally defined on $E_{10}/K(E_{10})$, and they were explicitly used in [1] to write the σ -model action in the second order coordinate form $S = \int dt n^{-1} \mathcal{L}(A, \partial_t A)$. This form leads to Euler-Lagrange equations of motion which contain two time derivatives of the coordinates $A^{(\ell)}(t)$.

By contrast to this second order explicit coordinate form, $\ddot{A} = F(A, \dot{A})$, we shall work in this paper with a (formal) first-order form, $\dot{\mathcal{P}} = F(\mathcal{P}, \mathcal{Q})$ based on what one might call ‘local $K(E_{10})$ -orthonormal frames’, by analogy with the geodesic motion on, say, the Lobachevskii plane viewed as the coset $SL(2)/SO(2)$. Indeed, the objects \mathcal{P} and \mathcal{Q} will be defined by decomposing the ‘velocity’ $\mathcal{V}^{-1} \partial_t \mathcal{V}$ on E_{10} in terms of objects which are representations of the subgroup $K(E_{10})$. In the Lobachevskii plane analogy, one would say that the quantities \mathcal{P} and \mathcal{Q} carry ‘flat indices’ ($SO(2)$ indices). In most of our developments, we shall not need any explicit representation of \mathcal{P} and \mathcal{Q} in terms of local coordinates on $E_{10}/K(E_{10})$. Note, however, that, in any choice of parametrization of $E_{10}/K(E_{10})$, such as the triangular one mentioned above, \mathcal{P} and \mathcal{Q} become some functions of A and \dot{A} .

³ The residual *spatial dependence*, which on the σ -model side is supposed ‘to be spread’ over the E_{10} Lie algebra, is the main difference of the present scheme with a *bona fide* reduction of $D = 11$ supergravity to one dimension, for which the appearance of E_{10} had first been conjectured [16].

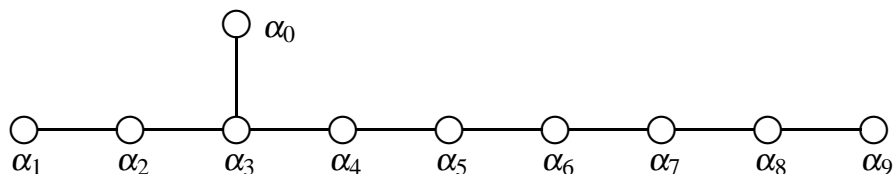
We hope that this new perspective can help in extending the correspondence between the $E_{10}/K(E_{10})$ σ -model and supergravity beyond the level where it is currently checked. Indeed, as in [1], we shall be able here to verify this correspondence up to height 29, included. Beyond this height, there appear terms in both versions of the equations of motion that we will exhibit explicitly, but that we do not know how to match. The more streamlined form of the equations of motion used here might help in guessing how to extend the ‘dictionary’ between the supergravity variables and the $E_{10}/K(E_{10})$ ones. In [1] some guesses were proposed to relate higher order spatial gradients of supergravity variables and higher-level coset variables $A^{(\ell)}$. One needs to make these guesses fully concrete, and to check that, together with a suitably extended dictionary, they extend the match between the two actions, to confirm the conjecture that E_{10} is a hidden symmetry of 11-dimensional supergravity (and M-theory). In a separate publication [19], we shall report some recent progress in this direction based on the consideration of the higher order (in Planck length) corrections to the low-energy supergravity Lagrangian.

Our use of flat indices here also paves the way for the introduction of fermionic couplings. The fermions of the theory will transform under *local* $K(E_{10})$, which in a supersymmetric extension of the $E_{10}/K(E_{10})$ coset model would become the ‘R-symmetry’. $K(E_{10})$ contains the spatial Lorentz group $SO(10)$ at level zero, and this symmetry is only manifest when one formulates the equations of motion entirely in terms of $K(E_{10})$ objects on the σ -model side. The use of flat indices was also found to be more convenient for the D_9 decomposition of E_{10} in [9], where the present analysis was extended to fermionic degrees of freedom and the compatibility of a Romans type mass term (for IIA supergravity) with E_{10} was established.

2. Basic facts about E_{10}

2.1. Basic definitions

The only known way to define the Kac Moody (KM) Lie algebra $\mathfrak{e}_{10} \equiv \text{Lie}(E_{10})$ is via its Chevalley-Serre presentation in terms of generators and relations and its Dynkin diagram [20, 21], which we give below with our labeling conventions for the simple roots $\{\alpha_i | i = 0, 1, \dots, 9\}$:



The nine simple roots $\alpha_1, \dots, \alpha_9$ along the horizontal line generate an $A_9 \equiv \mathfrak{sl}(10)$ subalgebra of \mathfrak{e}_{10} ⁴. The simple root α_0 , which connects to α_3 will be referred to as the ‘exceptional’ simple root. Its dual Cartan subalgebra (CSA) element h_0 enlarges $\mathfrak{sl}(10)$ to the Lie algebra $\mathfrak{gl}(10)$.

⁴ We will only be concerned with these Lie algebras as algebras over the real numbers, *i.e.* $\mathfrak{sl}(10) \equiv \mathfrak{sl}(10, \mathbb{R})$ and $\mathfrak{e}_{10} \equiv \mathfrak{e}_{10}(\mathbb{R})$, *etc.*

The Lie algebra \mathfrak{e}_{10} is built in terms of multiple commutators of a set of basic triples $\{e_i, f_i, h_i\}$, where $i, j = 0, 1, \dots, 9$, and each triple generates an $A_1 \equiv \mathfrak{sl}(2)$ subalgebra. The CSA is spanned by the generators $\{h_i\}$, i.e. $[h_i, h_j] = 0$; the remaining bilinear relations are

$$[h_i, e_j] = A_{ij}e_j, \quad [h_i, f_j] = -A_{ij}f_j, \quad [e_i, f_j] = \delta_{ij}h_i, \quad (2.2)$$

where A_{ij} is the E_{10} Cartan matrix. In addition, we have the multilinear Serre relations

$$(\text{ad } e_i)^{1-A_{ij}}(e_j) = 0, \quad (\text{ad } f_i)^{1-A_{ij}}(f_j) = 0 \quad (2.3)$$

We will also need the standard bilinear form

$$\langle e_i | f_j \rangle = \delta_{ij}, \quad \langle h_i | h_j \rangle = A_{ij} \quad (2.4)$$

It extends to the full Lie algebra \mathfrak{e}_{10} by its invariance property $\langle [x, y] | z \rangle = \langle x | [y, z] \rangle$.

An important role will be played by the ‘maximal compact subalgebra’ $\text{Lie}(K(E_{10})) =: \mathfrak{k}\mathfrak{e}_{10} \subset \mathfrak{e}_{10}$. It is defined as the invariant Lie subalgebra of \mathfrak{e}_{10} under the Chevalley involution

$$\theta(h_i) = -h_i, \quad \theta(e_i) = -f_i, \quad \theta(f_i) = -e_i \quad (2.5)$$

This involution extends to all of the Kac Moody Lie algebra by means of $\theta([x, y]) := [\theta(x), \theta(y)]$. The associated ‘maximal compact subgroup’ will be designated by $K(E_{10})$ (we put quotation marks because $K(E_{10})$ is not compact in the topological sense). It is not difficult to see that $\mathfrak{k}\mathfrak{e}_{10}$ is generated by all multiple commutators of the elements $(e_i - f_i)$. The corresponding real form of \mathfrak{e}_{10} is the analog of the split forms $E_{n(n)}$ for $n \leq 8$, which is the reason for sometimes denoting E_{10} as $E_{10(10)}$. For later use we also introduce the notion of the ‘transposed element’: for any element $x \in \mathfrak{e}_{10}$ we define

$$x^T := -\theta(x) \quad (2.6)$$

In this sense $\mathfrak{k}\mathfrak{e}_{10}$ consists of all ‘antisymmetric’ elements $x = -x^T$ of \mathfrak{e}_{10} , in the same way that $\mathfrak{so}(10)$ consists of all antisymmetric matrices in $\mathfrak{sl}(10)$.

2.2. Level decomposition w.r.t. $\mathfrak{sl}(10) \subset \mathfrak{e}_{10}$

Because no closed form construction exists for the Lie algebra elements $x \in \mathfrak{e}_{10}$, nor their invariant scalar products, we will rely on a recursive approach based on the decomposition of \mathfrak{e}_{10} into irreducible representations of its $\mathfrak{sl}(10)$ subalgebra⁵. Any positive root of E_{10} can be written as

$$\alpha = \ell\alpha_0 + \sum_{j=1}^9 m^j \alpha_j \quad (2.7)$$

with $\ell, m^j \geq 0$. The integer $\ell \equiv \ell(\alpha)$ is called the ‘ A_9 level’, or simply the ‘level’ of the root α ; below, we will, however, switch conventions by associating positive levels with multiple commutators of f ’s, i.e. negative roots. The decomposition (2.7) corresponds to a slicing (or ‘grading’) of the forward lightcone in the root lattice by spacelike hyperplanes, with only finitely many roots in each slice (slicings by lightlike or timelike hyperplanes would produce gradings w.r.t. affine or indefinite KM subalgebras, with each slice containing infinitely many roots).

⁵ A similar analysis of \mathfrak{e}_{10} in terms of its $D_9 \equiv \mathfrak{so}(9, 9)$ subalgebra is given in [9]. The decomposition of \mathfrak{e}_{10} under its affine \mathfrak{e}_9 subalgebra had already been studied in [22].

Every positive root α is associated with a set of ‘raising operators’ $E_{\alpha,s}$, where $s = 1, \dots, \text{mult } \alpha$ counts the number of independent such elements of \mathfrak{e}_{10} , and $\text{mult } \alpha$ is the ‘multiplicity’ of the root in question; similarly, the ‘lowering operators’ are associated with negative roots.

The adjoint action of the $\mathfrak{sl}(10)$ subalgebra leaves the level $\ell(\alpha)$ invariant. The set of \mathfrak{e}_{10} elements corresponding to a given level ℓ can therefore be decomposed into a (finite) number of irreducible representations of $\mathfrak{sl}(10)$. Because of the recursive definition of \mathfrak{e}_{10} in terms of multiple commutators, all representations occurring at level $\ell + 1$ are contained in the product of the level- ℓ representations with the $\ell = 1$ representation. The multiplicity of α as a root of \mathfrak{e}_{10} is equal to the sum of its multiplicities as a weight occurring in the $\mathfrak{sl}(10)$ representations. Each irreducible representation of A_9 can be characterized by its highest weight Λ , or equivalently by its Dynkin labels (p_1, \dots, p_9) where $p_k := (\alpha_k, \Lambda) \geq 0$ is the number of columns with k boxes in the associated Young tableau. For instance, the Dynkin labels (001000000) correspond to a Young tableau consisting of one column with three boxes, *i.e.* the antisymmetric tensor with three indices. The Dynkin labels are related to the 9-tuple of integers (m^1, \dots, m^9) appearing in (2.7) (for the highest weight $\Lambda \equiv -\alpha$) by

$$S^{i3}\ell - \sum_{j=1}^9 S^{ij}p_j = m^i \geq 0 \quad (2.8)$$

where S^{ij} is the inverse Cartan matrix of A_9 . This relation strongly constrains the representations that can appear at level ℓ , because the entries of S^{ij} are all positive, and the 9-tuples (p_1, \dots, p_9) and (m^1, \dots, m^9) must both consist of non-negative integers. In addition to satisfying the Diophantine equations (2.8), the highest weights must be roots of E_{10} , which implies the inequality

$$\Lambda^2 = \alpha^2 = \sum_{i,j=1}^9 p_i S^{ij} p_j - \frac{1}{10}\ell^2 \leq 2 \quad (2.9)$$

The problem of finding an explicit representation of \mathfrak{e}_{10} in terms of an infinite tower of $\mathfrak{sl}(10)$ representations can thus be reformulated as the problem of identifying all $\mathfrak{sl}(10)$ representations compatible with the Diophantine inequalities (2.8), (2.9). The more difficult task is to determine their outer multiplicities, *i.e.* the number of times each representation appears at a given level ℓ . Making use of the known root multiplicities of \mathfrak{e}_{10} it is possible to determine the level decomposition and the outer multiplicities of all representations to rather high levels (up to $\ell = 28$ so far [15]; analogous tables for the D_9 decomposition of \mathfrak{e}_{10} are given in [9]).

Let us now describe the lowest levels of this decomposition in detail ⁶. For E_{10} , the level $\ell = 0$ sector is just the $\mathfrak{gl}(10)$ subalgebra spanned by A_9 and the exceptional CSA generator h_0 . In ‘physicists’ notation’, this algebra is written as

$$[K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b \quad (2.10)$$

with indices $a, b, \dots \in \{1, \dots, 10\}$. Note that $(K^a_b)^T = K^b_a$.

The level $\ell = 1$ elements transform in the (001000000) representation of $\mathfrak{sl}(10)$, *i.e.* as a 3-form; they are thus represented by the $\mathfrak{gl}(10)$ tensor E^{abc} . The Chevalley conjugate elements at level $\ell = -1$ are

$$F_{abc} = (E^{abc})^T \quad (2.11)$$

⁶ Modulo dimensions, the representations at the first three levels are actually the same for all E_{n+1} in the decomposition w.r.t. A_n , see [3] for information concerning E_{11} .

and transform in the contragredient representation; thus

$$[K^a_b, E^{cde}] = 3\delta_b^{[c} E^{de]a} \quad , \quad [K^a_b, F_{cde}] = -3\delta_c^a F_{de]b} \quad (2.12)$$

The remaining level $|\ell| \leq \pm 1$ commutators are

$$[F_{abc}, E^{def}] = -18\delta_{[ab}^{[de} K^f]_c] + 2\delta_{abc}^{def} K \quad (2.13)$$

where $K := K^a_a$. We normalize all antisymmetric objects with weight one, so that, say, $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$, and $\delta_{abc}^{def} = \frac{1}{6}(\delta_a^d \delta_b^e \delta_c^f + 5 \text{ terms})$. Hence, the normalization of the last equation is such that, e.g.

$$[F_{123}, E^{123}] = -(K^1_1 + K^2_2 + K^3_3) + \frac{1}{3}K$$

The above elements are already sufficient to identify the Chevalley generators of \mathfrak{e}_{10} : we have

$$e_0 = F_{123} \quad , \quad f_0 = E^{123} \quad , \quad h_0 = -K^1_1 - K^2_2 - K^3_3 + \frac{1}{3}K \quad (2.14)$$

for the exceptional node, and

$$e_i = K^i_{i+1} \quad , \quad f_i = K^{i+1}_i \quad , \quad h_i = K^i_i - K^{i+1}_{i+1} \quad (2.15)$$

for the remaining nodes with $1 \leq i \leq 9$ which generate the the $\mathfrak{sl}(10)$ subalgebra. With the scalar products

$$\langle K^a_b | K^c_d \rangle = \delta_a^c \delta_b^d - \delta_b^c \delta_a^d \quad , \quad \langle F_{abc} | E^{def} \rangle = 3! \delta_{abc}^{def} \quad (2.16)$$

it is straightforward to recover the bilinear form (2.4) above.

There are two elements of the CSA which play a distinguished role: the *central charge* c of the affine subalgebra $\mathfrak{e}_9 \subset \mathfrak{e}_{10}$ is given by

$$c = 2h_1 + 4h_2 + 6h_3 + 5h_4 + 4h_5 + 3h_6 + 2h_7 + h_8 + 3h_0 = K^{10}_{10} \quad (2.17)$$

The affine subalgebra \mathfrak{e}_9 must commute with the central charge; its Chevalley generators are obtained from the above set by omitting the triple $\{e_9, f_9, h_9\}$. The affine algebra is thus generated from the level $|\ell| \leq 1$ elements by restricting the indices a, b, \dots to the values $\in \{1, \dots, 9\}$. The affine level counting (alias mode counting) operator is

$$d = c + h_9 = K^9_9 \quad (2.18)$$

It tells us that the affine mode number of a given affine element is equal to the difference of the number of upper and lower indices equal to 9⁷.

Similarly to the decomposition of \mathfrak{e}_{10} in terms of $\mathfrak{sl}(10)$ representations, we can analyze its invariant subalgebra $\mathfrak{k}_{\mathfrak{e}_{10}}$ in terms of its $\mathfrak{so}(10)$ subalgebra, the invariant subalgebra of $\mathfrak{sl}(10)$. At lowest order, we have

$$L_{ab} := \frac{1}{2}(K^a_b - (K^a_b)^T) \equiv \frac{1}{2}(K^a_b - K^b_a) \quad , \quad L_{abc} := \frac{1}{2}(E^{abc} - F_{abc}) \quad (2.19)$$

Note that the $\mathfrak{k}_{\mathfrak{e}_{10}}$ elements combine level ℓ with level $-\ell$ elements. For them, the (upper or lower) position of indices no longer matters, as they have to be regarded as $SO(10) \equiv K(GL(10))$ rather than $GL(10)$ indices. We will also make use of the coset space generators

$$S_{ab} := \frac{1}{2}(K^a_b + (K^a_b)^T) \equiv \frac{1}{2}(K^a_b + K^b_a) \quad , \quad S_{abc} := \frac{1}{2}(E^{abc} + F_{abc}) \quad (2.20)$$

⁷ In the $d = 2$ reduction, where one is left with the dependence on the time and one space coordinate $x^a \equiv x^{10}$, and maximal supergravity is known to admit an affine E_9 symmetry, these generators are realized as follows: c acts on the conformal factor [23, 24], whereas d acts as a dilatation operator [25, 26, 27].

2.3. Levels $\ell = \pm 2, \pm 3$

At levels 2 and 3 we have the representations (000001000) and (100000010), which are respectively generated by

$$E^{a_1 \dots a_6} := [E^{a_1 a_2 a_3}, E^{a_4 a_5 a_6}] \quad , \quad E^{[a_0 | a_1 a_2] a_3 \dots a_8} := [E^{a_0 a_1 a_2}, E^{a_3 \dots a_8}] \quad (2.21)$$

Making use of $E^{a_0 | a_1 \dots a_8} = -8E^{[a_1 | a_2 \dots a_8] a_0}$ the last relation can be rewritten in the form

$$E^{a_0 | a_1 \dots a_8} = 4[E^{a_0 [a_1 a_2}, E^{a_3 \dots a_8}]] \quad (2.22)$$

The adjoint ($\ell = -2, -3$) elements are

$$F_{a_1 \dots a_6} := (E^{a_1 \dots a_6})^T = -[F_{a_1 a_2 a_3}, F_{a_4 a_5 a_6}] \quad (2.23)$$

$$F_{a_0 | a_1 \dots a_8} := (E^{a_0 | a_1 \dots a_8})^T = -4[F_{a_0 [a_1 a_2}, F_{a_3 \dots a_8}]] \quad (2.24)$$

Further commutation yields

$$[F_{a_1 a_2 a_3}, E^{b_1 \dots b_6}] = 5! \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} E^{b_4 b_5 b_6]} \quad (2.25)$$

$$[F_{a_1 \dots a_6}, E^{b_1 \dots b_6}] = -6 \cdot 6! \delta_{[a_1 \dots a_5}^{[b_1 \dots b_5} K^{b_6]}_{a_6}] + \frac{2}{3} \cdot 6! \delta_{a_1 \dots a_6}^{b_1 \dots b_6} K \quad (2.26)$$

and

$$\begin{aligned} [F_{a_1 a_2 a_3}, E^{b_0 | b_1 \dots b_8}] &= 7 \cdot 48 \left(\delta_{a_1 a_2 a_3}^{b_0 [b_1 b_2} E^{b_3 \dots b_8]} - \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} E^{b_4 \dots b_8] b_0} \right) \\ [F_{a_1 \dots a_6}, E^{b_0 | b_1 \dots b_8}] &= 8! \left(\delta_{a_1 a_2 \dots a_6}^{b_0 [b_1 \dots b_5} E^{b_6 b_7 b_8]} - \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} E^{b_7 b_8] b_0} \right) \end{aligned} \quad (2.27)$$

The conjugate relations are easily obtained by taking the transpose of these commutators (not forgetting minus signs). The above relations can be conveniently restated after multiplication of the level $\ell = 3$ generators by a dummy tensor $X_{a|b_0 \dots b_8}$, which gives (after some reshuffling of indices by means of Schouten's identity)

$$\begin{aligned} [F_{cde}, X_{a|b_1 \dots b_8} E^{a|b_1 \dots b_8}] &= 7 \cdot 72 X_{[c|de] b_1 \dots b_6} E^{b_1 \dots b_6} \\ [F_{c_1 \dots c_6}, X_{a|b_1 \dots b_8} E^{a|b_1 \dots b_8}] &= 3 \cdot 8! X_{[c_1 | c_2 \dots c_6] def} E^{def} \end{aligned} \quad (2.28)$$

The remaining commutation relation between levels $\ell = 3$ and $\ell = -3$ is also most easily written in this way

$$\begin{aligned} [F_{c|d_1 \dots d_8}, X_{a|b_1 \dots b_8} E^{a|b_1 \dots b_8}] &= 8 \cdot 9! \left(-X_{c|e[d_1 \dots d_7} K^e_{d_8]} - \frac{1}{9} X_{c|d_1 \dots d_8} K \right. \\ &\quad \left. - X_{[d_1 | d_2 \dots d_8] e} K^e_c + \frac{1}{9} X_{d_1 | d_2 \dots d_8] c} K \right) \end{aligned} \quad (2.29)$$

The standard bilinear form on these elements is evaluated by making use of the invariance property $\langle [x, y] | z \rangle = \langle x | [y, z] \rangle$, with the result

$$\begin{aligned} \langle F_{a_1 \dots a_6} | E^{b_1 \dots b_6} \rangle &= 6! \delta_{a_1 \dots a_6}^{b_1 \dots b_6} \\ \langle F_{a_0 | a_1 \dots a_8} | E^{b_0 | b_1 \dots b_8} \rangle &= 8 \cdot 8! \left(\delta_{b_0}^{a_0} \delta_{b_1 \dots b_8}^{a_1 \dots a_8} - \delta_{b_0}^{[a_1} \delta_{b_1 \dots b_7 b_8]}^{a_2 \dots a_8] a_0} \right) \end{aligned} \quad (2.30)$$

such that e.g. $\langle F_{123456} | E^{123456} \rangle = 1$ and $\langle F_{1|1 \dots 8} | E^{1|1 \dots 8} \rangle = 9$.

3. The $E_{10}/K(E_{10})$ σ -model for $\ell \leq 3$

3.1. General remarks

In this section we will set up the general formalism for σ -models in one (time) dimension. The geodesic Lagrangian \mathcal{L} on $E_{10}/K(E_{10})$ is defined by generalizing the standard Lagrangian on a finite dimensional coset space $G/K(G)$, where $K(G)$ is the maximal compact subgroup of the Lie group G (for a given real form of G). Despite the formal replacement of the finite dimensional groups G and $K(G)$ by the infinite dimensional groups E_{10} and $K(E_{10})$, all elements entering the construction of \mathcal{L} have natural generalizations to the case where G is the group obtained by (formal) exponentiation of an indefinite or hyperbolic KM algebra. In particular, our expansion in terms of levels provides us with an algorithmic scheme which is completely well defined and computable to any given finite order, and which in principle can be carried to arbitrarily high levels. An essential ingredient in this construction is the so-called triangular gauge, which we will explain below.

One important difference between the finite dimensional coset spaces $G/K(G)$ and the infinite dimensional space $E_{10}/K(E_{10})$ is the following. For $K(G)$ the maximal compact subgroup of G , the space $G/K(G)$ is always Euclidean (i.e. endowed with a positive definite metric). This not so for the space $E_{10}/K(E_{10})$: even though $K(E_{10})$ is ‘compact’ in the algebraic sense, the metric on $E_{10}/K(E_{10})$ has precisely *one negative eigenvalue* coming from the negative norm CSA generator. It is for this reason that we can define null (= lightlike) geodesics on $E_{10}/K(E_{10})$ which do not exist in the finite dimensional case.

3.2. σ -model and level expansions

Following the standard formulation of nonlinear σ -models for $G/K(G)$ coset spaces with $K(G)$ the maximal compact subgroup of G , we assume the bosonic degrees of freedom are described by a ‘matrix’ $\mathcal{V} \in E_{10}$, which itself can be parametrized in terms of coordinates (‘fields’) $A^{(\ell)} = A^{(\ell)}(t)$ depending on the affine parameter t , the time coordinate, for all $\ell \in \mathbb{Z}$. Being an element of the coset space $G/K(G)$, the ‘matrix’ \mathcal{V} is subject to rigid and local transformations acting from the left and the right, respectively:

$$\mathcal{V}(A(t)) \longrightarrow g\mathcal{V}(A(t))k(t) \quad \text{with } g \in E_{10} \text{ , } k(t) \in K(E_{10}) \quad (3.31)$$

The local $K(E_{10})$ invariance allows us to choose a convenient gauge. For our calculations we will always adopt the triangular gauge where by definition all fields corresponding to negative levels are set to zero⁸. In other words, in triangular gauge \mathcal{V} is (formally) obtained by exponentiating the Borel subalgebra consisting of the level $\ell \geq 0$ elements of \mathfrak{e}_{10} , viz.

$$\mathcal{V}(A(t)) = \exp\left(\sum_{\ell=0}^{\infty} A^{(\ell)}(t) * E^{(\ell)}\right) \quad (3.32)$$

The notation here is slightly schematic: the symbol ‘*’ includes a sum over all the irreducible representations appearing at level ℓ (whose number grows very rapidly with ℓ , see [15]) as well as all indices labeling a particular representation (we will be more specific below). With this choice of gauge – as well as in any other gauge – there remains only the rigid E_{10} symmetry,

⁸ By abuse of language, we will use this terminology even if \mathcal{V} is triangular only with regard to levels $\ell \geq 1$, but not necessarily for $\ell = 0$.

which is now realized *non-linearly*: those E_{10} transformations in (3.31) which violate the chosen gauge must be compensated by field dependent $K(E_{10})$ transformations.

In order to write down the geodesic action, we decompose the E_{10} -valued ‘velocity’ $v \equiv \mathcal{V}^{-1} \partial_t \mathcal{V}$ into its symmetric and antisymmetric parts, *i.e.* $\mathcal{P}_t := \frac{1}{2}(v + v^T)$, and $\mathcal{Q}_t := \frac{1}{2}(v - v^T)$, respectively. Thus we write

$$\mathcal{V}^{-1} \partial_t \mathcal{V} = \mathcal{Q}_t + \mathcal{P}_t \in \mathfrak{e}_{10} \quad , \quad \mathcal{Q}_t^T = -\mathcal{Q}_t \quad , \quad \mathcal{P}_t^T = +\mathcal{P}_t \quad (3.33)$$

(see (2.6) for the definition of ‘transposition’). Hence $\mathcal{Q}_t \in \mathfrak{k}_{\mathfrak{e}_{10}}$, and \mathcal{P}_t belongs to the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}_{\mathfrak{e}_{10}}$. For convenience of notation we will omit the subscript t in the remainder, such that $\mathcal{Q} \equiv \mathcal{Q}_t$ and $\mathcal{P} \equiv \mathcal{P}_t$ will be understood to be ‘world tensors’ under reparametrizations of the time coordinate t . From (3.31) it follows immediately that the quantities on the r.h.s. are $K(E_{10})$ objects, *i.e.* they transform as

$$\mathcal{Q} \longrightarrow k^{-1}(\partial_t + \mathcal{Q})k \quad \mathcal{P} \longrightarrow k^{-1} \mathcal{P}k \quad (3.34)$$

Thus \mathcal{Q} plays the role of a $\mathfrak{k}_{\mathfrak{e}_{10}}$ gauge connection; however, its full significance will become apparent only in a supersymmetric extension of the theory, where \mathcal{Q} is the quantity through which the bosonic degrees of freedom couple to the fermions (supposed to belong to a *spinorial* representation of $K(E_{10})$). The \mathfrak{e}_{10} -invariant bosonic Lagrangian is the standard one for a point particle moving on the coset manifold $E_{10}/K(E_{10})$:

$$\mathcal{L} \equiv \mathcal{L}(n, A, \partial_t A) := \frac{1}{2} n^{-1} \langle \mathcal{P} | \mathcal{P} \rangle \quad (3.35)$$

where $\langle \cdot | \cdot \rangle$ is the standard invariant bilinear form (2.4), and $n(t)$ is the lapse function required for the invariance of the theory under reparametrizations of the time coordinate t , and whose variation yields the Hamiltonian constraint, which in turn ensures that the motion is along a null geodesic. Unlike finite dimensional simple Lie algebras, for which the number of independent polynomial Casimir invariants grows linearly with the rank, the bilinear form (2.4) is the *only polynomial invariant* for infinite dimensional KM algebras[20]. For this reason, the Lagrangian (3.35) is *essentially unique*: its replacement by

$$\mathcal{L}' = n f \left(n^{-2} \langle \mathcal{P} | \mathcal{P} \rangle \right) \quad (3.36)$$

with $f(\xi) = \frac{1}{2} \xi + \mathcal{O}(\xi^2)$ yields the *same* null geodesic solutions as (3.35). As a consequence *all couplings are already fixed by E_{10} , and there is no need to invoke supersymmetry or some other extraneous argument for this purpose*. For completeness we note that there are non-polynomial invariants [28], which might be relevant for non-perturbative effects and the (conjectured) breaking of E_{10} to $E_{10}(\mathbb{Z})$, but these are not explicitly known.

From (3.35) we obtain the equation of motion

$$n \mathcal{D}(n^{-1} \mathcal{P}) = 0 \quad (3.37)$$

where \mathcal{D} denotes the $K(E_{10})$ covariant derivative whose action is defined as

$$\mathcal{D} \mathcal{P} := \partial_t \mathcal{P} + [\mathcal{Q}, \mathcal{P}] \quad (3.38)$$

Here we omit the subscript t on the covariant derivative \mathcal{D} , and we omit also to recall the fact that the covariant derivative \mathcal{D} depends on the solution of the geodesic equation we are writing down, through its dependence on the value of \mathcal{Q} . The simple looking compact form (3.37) of

the σ -model equations of motion is formally valid for any choice of gauge on the $E_{10}/K(E_{10})$ coset space. On the other hand, (3.37) by itself does not constitute a well-defined, autonomous set of evolution equations. It must be completed by some (gauge-dependent) supplementary information telling us how \mathcal{P} and \mathcal{Q} both depend on some basic coordinates, $A^{(\ell)}$, and their time derivatives.

The equations of motion (3.37) are equivalent to the conservation of the E_{10} Noether charges

$$\mathcal{J} = n^{-1} \mathcal{V} \mathcal{P} \mathcal{V}^{-1} \quad (3.39)$$

which transform under rigid E_{10} as

$$\mathcal{J} \longrightarrow g \mathcal{J} g^{-1} \quad \text{for } g \in E_{10} \quad (3.40)$$

The main advantage of the triangular gauge is that (3.35) and (3.37) are both well defined and computable if one analyzes the resulting equations level by level. We can thus expand (3.33) in non-negative levels according to

$$\mathcal{V}^{-1} \partial_t \mathcal{V} = Q^{(0)} * L + P^{(0)} * S + P^{(1)} * E^{(1)} + P^{(2)} * E^{(2)} + \dots \quad (3.41)$$

Inspection shows that

$$P^{(\ell)} = \partial_t A^{(\ell)} + F_\ell(A^{(1)}, \partial_t A^{(1)}, \dots, A^{(\ell-1)}, \partial_t A^{(\ell-1)}) \quad (3.42)$$

where each F_ℓ is *polynomial* (of ascending order) and depends only on fields $A^{(n)}$ of *lower* level $n < \ell$. With $F^{(1)} \equiv (E^{(1)})^T$ etc. we next perform the required split into compact and non-compact elements

$$\begin{aligned} \mathcal{Q} &= Q^{(0)} * L + \frac{1}{2} P^{(1)} * (E^{(1)} - F^{(1)}) + \frac{1}{2} P^{(2)} * (E^{(2)} - F^{(2)}) + \dots \\ \mathcal{P} &= P^{(0)} * S + \frac{1}{2} P^{(1)} * (E^{(1)} + F^{(1)}) + \frac{1}{2} P^{(2)} * (E^{(2)} + F^{(2)}) + \dots \end{aligned} \quad (3.43)$$

where, say, $Q^{(0)} * L \equiv Q_{ab}^{(0)} L_{ab}$, with L and S from (2.19) and (2.20) above.

To write out the equations of motion we define a new ‘covariant derivative’ operator, $\mathcal{D}^{(0)}$, associated to rotations under the $SO(10)$ subgroup by

$$\mathcal{D}^{(0)}(P^{(0)} * S) := \partial_t(P^{(0)} * S) + [Q^{(0)} * L, P^{(0)} * S] \quad (3.44)$$

when this operator acts on the level $\ell = 0$ fields, and by

$$\begin{aligned} \mathcal{D}^{(0)}(P^{(\ell)} * (E^{(\ell)} + F^{(\ell)})) &:= \partial_t P^{(\ell)} * (E^{(\ell)} + F^{(\ell)}) + [Q^{(0)} * L, P^{(\ell)} * (E^{(\ell)} + F^{(\ell)})] \\ &\quad - [P^{(0)} * S, P^{(\ell)} * (E^{(\ell)} - F^{(\ell)})] \end{aligned} \quad (3.45)$$

for $\ell \geq 1$. The second term on the r.h.s. is the expected covariantization w.r.t. the $SO(10)$ subgroup, whereas the third term results from the covariantization w.r.t. the remaining generators of \mathfrak{e}_{10} . With this notation the equation of motion at level $\ell = 0$ reads

$$n \mathcal{D}^{(0)}(n^{-1} P^{(0)} * S) = -\frac{1}{2} \sum_{k=1}^{\infty} [P^{(k)} * E^{(k)}, P^{(k)} * F^{(k)}] \quad (3.46)$$

At levels $\ell \geq 1$ we similarly obtain

$$\begin{aligned} n\mathcal{D}^{(0)}(n^{-1}P^{(\ell)} * (E^{(\ell)} + F^{(\ell)})) &= - \sum_{k=1}^{\infty} \left[P^{(\ell+k)} * E^{(\ell+k)}, P^{(k)} * F^{(k)} \right] \\ &+ \sum_{k=1}^{\infty} \left[P^{(\ell+k)} * F^{(\ell+k)}, P^{(k)} * E^{(k)} \right] \end{aligned} \quad (3.47)$$

The equations of motion thus consist of a derivative term and a term which is always quadratic in the ‘momenta’ $P^{(k)}$. It takes only a little algebra to verify that (3.46) and (3.47) together are indeed equivalent to the standard geodesic equations on a coset manifold. Because we are here working with $K(E_{10})$ tensors, the ‘coordinates’ $A^{(\ell)}$ do not appear explicitly in the above equations.

3.3. Equations of motion for levels $\ell \leq 3$

To spell out the equations in more detail up to level $\ell = 3$ we write

$$\begin{aligned} Q^{(0)} * L &\equiv Q_{ab}^{(0)} L_{ab} \quad , \quad P^{(0)} * S \equiv P_{ab}^{(0)} S_{ab} \\ P^{(1)} * E^{(1)} &\equiv \frac{1}{3!} P_{abc}^{(1)} E^{abc} \quad , \quad P^{(2)} * E^{(2)} \equiv \frac{1}{6!} P_{a_1 \dots a_6}^{(2)} E^{a_1 \dots a_6} \\ P^{(3)} * E^{(3)} &\equiv \frac{1}{9!} P_{a_0 | a_1 \dots a_8}^{(3)} E^{a_0 | a_1 \dots a_8} \end{aligned} \quad (3.48)$$

One easily checks that the covariant derivative $\mathcal{D}^{(0)}$ introduced in (3.45) acts as

$$\mathcal{D}^{(0)} P_{ab}^{(0)} = \partial_t P_{ab}^{(0)} + Q_{ac}^{(0)} P_{cb}^{(0)} + Q_{bc}^{(0)} P_{ac}^{(0)} \quad (3.49)$$

at level $\ell = 0$, and like

$$\mathcal{D}^{(0)} V_a = \partial_t V_a + Q_{ab}^{(0)} V_b - P_{ab}^{(0)} V_b \quad (3.50)$$

on the vector indices of the higher level fields. We emphasize that all indices a, b, \dots are to be treated as $SO(10)$ (‘flat’) indices, and therefore *all index contractions are performed with the flat metric δ_{ab}* . This property will be shown below to reflect the fact that, under the $E_{10}/K(E_{10}) \leftrightarrow$ supergravity correspondence, the $SO(10)$ subgroup of E_{10} can be identified with the $SO(10)$ subgroup of the local Lorentz group $SO(1, 10)$ in eleven dimensions. The link with the anholonomic frames used in [1] will be clarified in the following section.

For the comparison with the appropriately truncated equations of motion of $D = 11$ supergravity, we will impose the σ -model truncation

$$0 = P^{(4)} = P^{(5)} = P^{(6)} = \dots \quad (3.51)$$

To see that this is a *consistent truncation*, we note that each term on the r.h.s. of (3.47) contains a field of higher level than the l.h.s. of this equation. However, the vanishing of the infinite tower of ‘momenta’ $P^{(\ell)}$ (3.51) does not imply the vanishing or constancy of the associated infinite tower of coordinates $A^{(\ell)}$. Indeed, in view of (3.42), we find that the time evolution of the higher level component fields $A^{(\ell)}$ for $\ell \geq 4$ is generically non-trivial and determined (up to constants of integration) by the conditions (3.51). Remarkably, the truncation to a finite

number of low level ‘momenta’ requires the excitation of the whole tower of E_{10} fields for its consistency! Making use of the commutators for the first three levels (cf. section 2), we obtain

$$\begin{aligned}
n\mathcal{D}^{(0)}(n^{-1}P_{ab}^{(0)}) &= -\frac{1}{4}\left(P_{acd}^{(1)}P_{bcd}^{(1)} - \frac{1}{9}\delta_{ab}P_{cde}^{(1)}P_{cde}^{(1)}\right) \\
&- \frac{1}{2\cdot 5!}\left(P_{ac_2\dots c_6}^{(2)}P_{bc_2\dots c_6}^{(2)} - \frac{1}{9}\delta_{ab}P_{c_1\dots c_6}^{(2)}P_{c_1\dots c_6}^{(2)}\right) \\
&+ \frac{4}{9!}\left(P_{c_1|c_2\dots c_8a}^{(3)}P_{c_1|c_2\dots c_8b}^{(3)} + \frac{1}{8}P_{a|c_1\dots c_8}^{(3)}P_{b|c_1\dots c_8}^{(3)} - \frac{1}{8}\delta_{ab}P_{c_1|c_2\dots c_9}^{(3)}P_{c_1|c_2\dots c_9}^{(3)}\right)
\end{aligned} \tag{3.52}$$

at level $\ell = 0$. At levels $\ell = 1$ and $\ell = 2$, we have, respectively,

$$n\mathcal{D}^{(0)}(n^{-1}P_{abc}^{(1)}) = -\frac{1}{6}P_{abcdef}^{(2)}P_{def}^{(1)} + \frac{1}{3\cdot 5!}P_{d_1|d_2\dots d_6abc}^{(3)}P_{d_1\dots d_6}^{(2)} \tag{3.53}$$

and

$$n\mathcal{D}^{(0)}(n^{-1}P_{a_1\dots a_6}^{(2)}) = \frac{1}{6}P_{b|cda_1\dots a_6}^{(3)}P_{bcd}^{(1)} \tag{3.54}$$

where higher level contributions have been suppressed in accordance with the cutoff (3.51). Finally, at level $\ell = 3$

$$n\mathcal{D}^{(0)}(n^{-1}P_{a_0|a_1\dots a_8}^{(3)}) = 0 \tag{3.55}$$

More generally, it is easy to see that truncating at some higher level, the highest non-vanishing component of \mathcal{P} is always covariantly constant w.r.t. (3.45).

4. Comparison with $D = 11$ supergravity

We will now exhibit the relation between the σ -model equations of motion derived in the foregoing section and the appropriately truncated bosonic equations of motion of $D = 11$ supergravity. This relation can be obtained in two steps. First, we can formally identify the various objects P, Q entering the compact, first-order form (3.37) of the σ -model equations of motion with some corresponding objects entering the supergravity equations of motion written in orthonormal frames. This preliminary identification will be explicitly performed in this section. However, because, as we said above, the equations (3.37) do not constitute an autonomous evolution system, it remains to check that the objects P, Q defined in the first step, can indeed be derived from a consistent set of evolving coordinates $(A^{(\ell)}, \dot{A}^{(\ell)})$ on the tangent bundle to the coset space $E_{10}/K(E_{10})$. We will not explicitly perform this second step here, but show instead how the results of the first step match with the previous results of [1]. As the identifications obtained in [1] were directly done for the autonomous second order form of the equations of motion, $\ddot{A} = F(A, \dot{A})$, the fact that our identifications for \mathcal{P}, \mathcal{Q} can be matched to those of [1] suffices to show that, indeed, \mathcal{P}, \mathcal{Q} derive from a consistent set of $(A^{(\ell)}, \dot{A}^{(\ell)})$ on the tangent bundle to the coset space $E_{10}/K(E_{10})$.

The key point here is that we relate the time evolution of the σ -model quantities, which depend only on the affine (time) parameter t to the time evolution of the spin connection and the field strengths and their first order spatial gradients *at an arbitrary, but fixed spatial point* $\mathbf{x} = \mathbf{x}_0$. Taking into account successively higher order spatial gradients will certainly require relaxing the cutoff conditions (3.51). We do not know at present how the matching works at higher levels. We will display below some of the terms which seem, within the present ‘dictionary’, problematic for the extension of the matching to higher levels.

For the bosonic equations of motion of $D = 11$ supergravity [2] we adopt the same conventions as in [1], except that we will systematically project all quantities on a *field of orthonormal frames*, *i.e.* on an elfbein E_M^A . Here, M, N, \dots denote coordinate (world) indices, and A, B, \dots flat indices in $D = 11$ with the metric $\eta^{AB} = (- + \dots +)$. Using flat indices throughout, the supergravity equations of motion read

$$\begin{aligned} D_A F^{ABCD} &= \frac{1}{8 \cdot 144} \varepsilon^{BCDE_1 \dots E_4 F_1 \dots F_4} F_{E_1 \dots E_4} F_{F_1 \dots F_4} \\ R_{AB} &= \frac{1}{12} F_{ACDE} F_B{}^{CDE} - \frac{1}{144} \eta_{AB} F_{CDEF} F^{CDEF} \end{aligned} \quad (4.56)$$

In addition we have the Bianchi identity

$$D_{[A} F_{BCDE]} = 0 \quad (4.57)$$

Here D_A is the Lorentz covariant derivative

$$D_A V^B := \partial_A V^B + \omega_A{}^B{}_C V^C \quad (4.58)$$

and the spin connection $\omega_{ABC} \equiv \eta_{BB'} \omega_A{}^{B'}{}_C$ (which is antisymmetric in BC) is given by the standard formula in terms of the coefficients of anholonomity Ω_{ABC} (which are antisymmetric in the first pair of indices AB):

$$\omega_{ABC} = \frac{1}{2} \left(\Omega_{ABC} + \Omega_{CAB} - \Omega_{BCA} \right) \quad , \quad \Omega_{AB}{}^C := E_A{}^M E_B{}^N (\partial_M E_N{}^C - \partial_N E_M{}^C) \quad (4.59)$$

In flat indices, the Riemann tensor is

$$R_{ABCD} = \partial_A \omega_{BCD} - \partial_B \omega_{ACD} + \Omega_{AB}{}^E \omega_{ECD} + \omega_{AC}{}^E \omega_{BED} - \omega_{BC}{}^E \omega_{AED} \quad (4.60)$$

Next, we perform a 1+10 split of the elfbein, setting the shift $N^a = 0$,

$$E_M^A = \begin{pmatrix} N & 0 \\ 0 & e_m{}^a \end{pmatrix} \quad (4.61)$$

with the spatial zehnbain $e_m{}^a$. With this split, the coefficients of anholonomity become

$$\Omega_{abc} = 2e_{[a}{}^m e_b]{}^n \partial_m e_{nc} \quad , \quad \Omega_{0bc} = N^{-1} e_b{}^n \partial_t e_{nc} \quad , \quad \Omega_{a00} = \omega_{00a} = e_a{}^m N^{-1} \partial_m N \quad (4.62)$$

with all other coefficients of anholonomity vanishing.

The purely spatial components Ω_{abc} can be separated into a trace $\Omega_a \equiv \Omega_{abb} = \omega_{bba}$ and a traceless part $\tilde{\Omega}_{abc}$ (hence $\Omega_{abb} = 0$)

$$\Omega_{abc} = \tilde{\Omega}_{abc} + \frac{2}{9} \Omega_{[a} \delta_{b]c} \quad (4.63)$$

Below we will see that the respective equations of motion can only be matched if we set

$$\Omega_a = 0 \quad (4.64)$$

Because our analysis is local, *i.e.* takes place in some neighborhood of a given spatial point $\mathbf{x} = \mathbf{x}_0$, it is easy to see that this condition can always be satisfied by a suitable choice of gauge for the spatial zehnbain.

Next we write out (4.56) and (4.57) with the (1+10) split of indices. To this aim, we separate the spatial components of the Ricci tensor in (4.56) as

$$R_{ab} =: R_{ab}^{(0)} + R_{ab}^{(3)} \quad (4.65)$$

where the first term contains only time derivatives, and the second only spatial gradients (the superscripts are to indicate at which E_{10} levels these contributions become relevant). For the first term we obtain, with $\partial_0 \equiv N^{-1} \partial_t$, and remembering that $\eta^{00} = -1$ in our conventions,

$$R_{ab}^{(0)} = \partial_0 \omega_{ab0} + \omega_{cc0} \omega_{ab0} + 2\omega_{0c(a} \omega_{b)c0} = N^{-1} e^{-1} \partial_t (e N^{-1} \omega_{abt}) + 2N^{-2} \omega_{tc(a} \omega_{b)ct} \quad (4.66)$$

where $e \equiv \det e_m^a$. Recalling (3.49) we see that $R_{ab}^{(0)}$ matches the structure of the l.h.s. of the $\ell = 0$ equation of motion (3.52) up to an overall factor N^{-2} if we equate

$$P_{ab}^{(0)}(t) = \omega_{abt}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0} \quad , \quad Q_{ab}^{(0)}(t) = \omega_{tab}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0} \quad (4.67)$$

where (cf. (4.59) and (4.62))

$$\omega_{tbc} = e_{[b}^n \partial_t e_{nc]} \quad , \quad \omega_{abt} = e_{(a}^n \partial_t e_{nb)} \quad (4.68)$$

and identify the σ -model lapse function n appearing in (3.35) and the lapse N in (4.61) via

$$n = N e^{-1} \quad (4.69)$$

(this quantity was called \tilde{N} in [14]). As we will see more explicitly below this identification does not determine the time-independent (but space dependent) part of the spatial zehnbain, but it shows that the $\ell = 0$ sector of the σ -model correctly reproduces the dimensional reduction of Einstein's equations to one time dimension.

As we will now show the remaining components in (3.48) can be consistently related to the $D = 11$ supergravity fields by the identification (with n from (4.69))

$$\begin{aligned} P_{abc}^{(1)}(t) &= F_{tabc}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0} \quad (\equiv N F_{0abc}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0}) \\ P_{a_1 \dots a_6}^{(2)}(t) &= -\frac{1}{24} n e \varepsilon_{a_1 \dots a_6 bcde} F_{bcde}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0} \\ P_{a_0 | a_1 \dots a_8}^{(3)}(t) &= \frac{3}{2} n e \varepsilon_{a_1 \dots a_8 bc} \tilde{\Omega}_{bc|a_0}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0} \end{aligned} \quad (4.70)$$

This identification implies in particular that only the traceless part of Ω_{abc} can appear on the r.h.s. of the formula for $P^{(3)}$ because ⁹

$$P_{[a_0 | a_1 \dots a_8]}^{(3)} = 0 \quad \iff \quad \tilde{\Omega}_{abb} = 0 \quad (4.71)$$

The first two lines in (4.70) follow already by matching the r.h.s. of Einstein's equations (4.56) with $\ell = 1, 2$ terms on the r.h.s. of (3.52). To see this we write out

$$\begin{aligned} &\frac{1}{12} F_{aCDE} F_b{}^{CDE} - \frac{1}{144} \delta_{ab} F^{CDEF} F_{CDEF} \\ &= -\frac{1}{4} F_{acd0} F_{bcd0} + \frac{1}{36} \delta_{ab} F_{cde0} F_{cde0} + \frac{1}{12} F_{acde} F_{bcde} - \frac{1}{144} \delta_{ab} F_{cdef} F_{cdef} \end{aligned} \quad (4.72)$$

⁹ Inspection of the tables [15] shows that also at higher levels there is no natural place for the trace Ω_a .

After multiplication with N^2 this agrees indeed with

$$-\frac{1}{4}\left(P_{acd}^{(1)}P_{bcd}^{(1)} - \frac{1}{9}\delta_{ab}P_{cde}^{(1)}P_{cde}^{(1)}\right) - \frac{1}{2 \cdot 5!}\left(P_{ac_2\dots c_6}^{(2)}P_{bc_2\dots c_6}^{(2)} - \frac{1}{9}\delta_{ab}P_{c_1\dots c_6}^{(2)}P_{c_1\dots c_6}^{(2)}\right) \quad (4.73)$$

upon substitution of the $P^{(1)}$ and $P^{(2)}$ from (4.70). The level $\ell = 3$ contribution to Einstein's equation will be discussed below.

At this point the identifications at levels $\ell = 1, 2$ are fixed, and the equations of motion and the Bianchi identity for F_{ABCD} merely provide consistency checks on the identification (4.70). The abc component of the 3-form equation of motion yields

$$D_0F^{0abc} + D_eF^{eabc} = -\frac{1}{144}\varepsilon^{abcd_1d_2d_3e_1\dots e_4}F_{0d_1d_2d_3}F_{e_1\dots e_4} \quad (4.74)$$

Writing out the l.h.s. we obtain

$$\begin{aligned} D_0F^{0abc} + D_eF^{eabc} &= -\partial_0F_{0abc} - \omega_{00e}F_{eabc} + 3\omega_{0e[a}F_{bc]0e} \\ &+ \partial_eF_{eabc} - \omega_{ee0}F_{0abc} + \omega_{eef}F_{fabc} - 3\omega_{de[a}F_{bc]de} + 3\omega_{e0[a}F_{bc]e0} \end{aligned} \quad (4.75)$$

With the identification (4.69) and (4.70), we get

$$\partial_0F_{0abc} + \omega_{ee0}F_{0abc} + 3\omega_{0e[a}F_{bc]0e} - 3\omega_{e[a0}F_{bc]e0} = N^{-2}n\mathcal{D}^{(0)}(n^{-1}P_{abc}^{(0)}) \quad (4.76)$$

The remaining terms can be worked out to be

$$\partial_eF_{eabc} + \omega_{eef}F_{fabc} - 3\omega_{de[a}F_{bc]de} - \omega_{00e}F_{eabc} = -\frac{3}{2}\tilde{\Omega}_{de[a}F_{bc]de} + N\partial_e(N^{-1}F_{eabc}) \quad (4.77)$$

where we made use again of (4.64). Relating the first term on the r.h.s. to the $P^{(3)}P^{(2)}$ term in (3.53) implies the last formula in (4.70). The second term $N\partial_e(N^{-1}F_{eabc})$ cannot be accounted for with the present truncation, and will require a proper analysis of the higher level contributions. As mentioned in [1] this term formally corresponds to a term in the Hamiltonian which is of 'height' (with respect to the simple roots that it would contain as exponent) higher or equal to the level 30. A similar assertion holds for the other spatial gradients that we shall neglect below.

To check the Bianchi identities (4.57) with the $\ell = 2$ equation (3.54) we write out

$$\begin{aligned} D_0F_{abcd} + 4D_{[a}F_{bcd]0} &= \partial_0F_{abcd} + 4\omega_{0e[a}F_{bcd]e} - 4\omega_{00[a}F_{bcd]0} \\ &+ 4\partial_{[a}F_{bcd]0} + 12\omega_{[abe}F_{cd]e0} + 4\omega_{[a0e}F_{bcd]e} \end{aligned} \quad (4.78)$$

As before, we recognize that

$$\partial_0F_{abcd} + 4\omega_{0e[a}F_{bcd]e} + 4\omega_{[a0e}F_{bcd]e} = N^{-1}\mathcal{D}^{(0)}F_{abcd} \quad (4.79)$$

Disregarding again space derivatives in the second term, as appropriate for our approximation, and using

$$12\omega_{[ab}{}^eF_{cd]e0} = 6\Omega_{[ab}{}^eF_{cd]e0} \quad (4.80)$$

together with the formula for $P^{(3)}$ from (4.70) we again obtain perfect agreement.

Let us now return to the $\ell = 3$ contributions. The corresponding terms on the r.h.s. of (3.52) must be checked against the remaining term in (4.65), which is $(\partial_a \equiv e_a^m \partial_m)$

$$R_{ab}^{(3)} = \frac{1}{4}\tilde{\Omega}_{cda}\tilde{\Omega}_{cdb} - \frac{1}{2}\tilde{\Omega}_{acd}\tilde{\Omega}_{bcd} - \frac{1}{2}\tilde{\Omega}_{acd}\tilde{\Omega}_{bdc} - \frac{1}{2}\partial_b\tilde{\Omega}_{bac} - \frac{1}{2}\partial_b\tilde{\Omega}_{bca} \quad (4.81)$$

where we have again dropped all trace terms in accordance with the gauge choice (4.64). On the other hand, substituting $P^{(3)}$ from (4.70) we get

$$\begin{aligned} & \frac{4}{9!} \left(P_{c_1|c_2\dots c_8 a}^{(3)} P_{c_1|c_2\dots c_8 b}^{(3)} + \frac{1}{8} P_{a|c_1\dots c_8}^{(3)} P_{b|c_1\dots c_8}^{(3)} - \frac{1}{8} \delta_{ab} P_{c_1|c_2\dots c_9}^{(3)} P_{c_1|c_2\dots c_9}^{(3)} \right) \\ &= \frac{1}{4} \tilde{\Omega}_{cd|a} \tilde{\Omega}_{cd|b} - \frac{1}{2} \tilde{\Omega}_{ac|d} \tilde{\Omega}_{bc|d} \end{aligned} \quad (4.82)$$

We thus see that the first two terms agree, but that the matching fails for the other terms. However, as mentioned in [1], the terms that do not match formally correspond to terms in the Hamiltonian that would involve the exponentiation of E_{10} roots of height 30 or more.

The final equation to be checked against (4.70) is (3.55). With (4.67) and (4.68) we have

$$\mathcal{D}^{(0)}(e\mathcal{E}_{a_1\dots a_{10}}) \propto e_c{}^m \partial e_{m[c} \mathcal{E}_{a_1\dots a_{10}]} = 0 \quad (4.83)$$

Noticing that n drops out, and making use of (4.70) again, a little algebra shows that the $\ell = 3$ equation of motion reduces to

$$\partial_t \tilde{\Omega}_{bc|a} - \tilde{\Omega}_{bc|d} e_d{}^m \partial_t e_{ma} + e_b{}^m \partial_t e_{md} \tilde{\Omega}_{dc|a} + e_c{}^m \partial_t e_{md} \tilde{\Omega}_{bd|a} = 0 \quad (4.84)$$

Factorizing the spatial zehnbain as

$$e_m{}^a(t, \mathbf{x}) = \theta_m{}^{\bar{a}}(\mathbf{x}) S_{\bar{a}}{}^a(t) \quad (4.85)$$

the space dependent matrix $\theta_m{}^{\bar{a}}$ drops out from the expressions for $Q^{(0)}$ and $P^{(0)}$ in (4.67) and (4.68), and hence is left undetermined by the $\ell = 0$ equation of motion. It is, however, fixed by the $\ell = 1, 2$ equations of motion, as we saw. Introducing the structure constants

$$C_{\bar{a}\bar{b}}{}^{\bar{c}} := 2\theta_{[\bar{a}}{}^m \theta_{\bar{b}]}{}^n \partial_m \theta_n{}^{\bar{c}} \quad (4.86)$$

and substituting the ansatz (4.85) into (4.84) we obtain

$$\tilde{\Omega}_{abc}(t) = (S^{-1})_a{}^{\bar{a}}(t) (S^{-1})_b{}^{\bar{b}}(t) C_{\bar{a}\bar{b}}{}^{\bar{c}} S_{\bar{c}}{}^c(t) \implies \partial_t C_{\bar{a}\bar{b}}{}^{\bar{c}} = 0 \quad (4.87)$$

This indeed solves (4.84) for constant and traceless $C_{\bar{a}\bar{b}}{}^{\bar{c}}$.

In retrospect we recognize the factorization of the spatial zehnbain that was introduced in [1], following an earlier study of homogeneous cosmological solutions to $D = 11$ supergravity in [29]. The background geometry is described by a purely spatial background frame

$$\theta^{\bar{a}} = dx^m \theta_m{}^{\bar{a}}(\mathbf{x}) \quad (4.88)$$

whereas the time dependence of the zehnbain is entirely contained in the factor $S(t)$ and governed by (3.52). Consequently, in [1], all tensors were referred to the anholonomic frame $\theta^{\bar{a}}$, and contracted with the purely time dependent metric

$$g_{\bar{a}\bar{b}}(t) = S_{\bar{a}}{}^a(t) S_{\bar{b}}{}^b(t) \quad (4.89)$$

We can thus directly relate the σ -model fields used here and the quantities $\mathcal{D}A$ used in [1]; for instance,

$$\mathcal{D}A_{\bar{a}\bar{b}\bar{c}} = \theta_{\bar{a}}{}^m \theta_{\bar{b}}{}^n \theta_{\bar{c}}{}^p \partial_t A_{mnp} = S_{\bar{a}}{}^a S_{\bar{b}}{}^b S_{\bar{c}}{}^c P_{abc}^{(1)} \quad (4.90)$$

In contrast to the Lorentz tensors used here, the quantities $\mathcal{D}A_{\bar{a}\bar{b}\bar{c}}$ etc. possess finite limits on the singular initial hypersurface, defining various ‘walls’ as explained in [14]. The frame metric $g_{\bar{a}\bar{b}}(t)$, on the other hand, has no limit, but exhibits a singular behavior with chaotic oscillations as $t \rightarrow 0$.

We thus see that the truncated σ -model equations of motion imply the factorization on which the analysis of [1] was based. Furthermore, the matching up to level $\ell = 3$ with the cutoff (3.51) restricts the spatial geometry to frames with spatially constant $C_{\bar{a}\bar{b}}^{\bar{c}}$. Neither of these statements remains true if we relax (3.51). For instance, including level four, the $\ell = 3$ equations become, schematically,

$$\mathcal{D}^{(0)}P^{(3)} \sim P^{(4)}P^{(1)} \neq 0 \quad (4.91)$$

with similar corrections for the $\ell \leq 3$ equations. Therefore, the split into a space-dependent background frame, and a purely time dependent part S no longer works. Moreover, when switching on higher levels, we expect that we will have to modify the identifications (4.70) which were found to work when only the first three levels were turned on; in other words, the ‘dictionary’ is probably sensitive to the level at which we truncate. The challenge is now to find the (spatially) non-local and level-dependent dictionary between supergravity objects and σ -model ones, that will resolve the remaining discrepancies between $D = 11$ supergravity and the $E_{10}/K(E_{10})$ σ -model.

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