

# Higher spin fields from indefinite Kac–Moody algebras

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## Abstract

The emergence of higher spin fields in the Kac–Moody theoretic approach to M-theory is studied. This is based on work done by Schnakenburg, West and the second author. We then study the relation of higher spin fields in this approach to other results in different constructions of higher spin field dynamics. Of particular interest is the construction of space-time in the present set-up and we comment on the various existing proposals.

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# 1 Introduction and Motivation

M-theory is usually thought to comprise the different string theories and is therefore related to higher spin (HS) field theories in the sense that string theories contain HS. Indeed, the perturbative spectra of the different string theories characteristically contain massive HS fields if the tension of the string is finite and massless HS fields in the limit of vanishing tension. The tensionless limit  $\alpha' \rightarrow \infty$ , in which all perturbative states become massless, is of particular interest because of the restoration of very large symmetries [1]. One possible approach to M-theory is based on such infinite-dimensional symmetries; work along these lines can be found in [2, 3, 4, 5, 6] and references therein. Our focus here will be the Kac–Moody theoretical approach of [3] and [5, 6]. The aim of the present discussion is to show how higher spin fields arise naturally as objects in this algebraic formulation and speculate how a dynamical scheme for these fields might emerge. Before studying the details of the HS fields, let us briefly review the origin of the Kac–Moody algebras we are going to consider.

It is a longstanding conjecture that the reduction of certain (super-)gravity systems to very low space-time dimensions exhibits infinite-dimensional symmetry algebras [7]. As these sectors are normally thought of as low energy limits of M-theory, these conjectures extend to M-theory, albeit with certain modifications [8]. The precise statement for the (super-)gravity systems is that the scalar sector (with dualisation of fields to scalar fields whenever possible) is described by a non-linear sigma model  $G/K(G)$  where  $G$  is some group and  $K(G)$  its maximal compact subgroup [7].

As an example, the chain of these so-called ‘hidden symmetries’ in the case of  $D_{max} = 11$ ,  $N = 1$  supergravity is displayed in the table below. The scalar sector of this theory reduced to  $D$  space-time dimensions is described by a scalar coset model  $G/K(G)$  ( $G$  is in the split form) determined by

$D$	$G$	$K$
9	$SL(2, R) \times SO(1, 1)$	$SO(2)$
8	$S(3, R) \times SL(2, R)$	$U(2)$
7	$SL(5, R)$	$USp(4)$
6	$SO(5, 5)$	$USp(4) \times USp(4)$
5	$E_{6(6)}$	$USp(8)$
4	$E_{7(7)}$	$SU(8)$
3	$E_{8(8)}$	$Spin(16)/Z_2$
2	$E_{9(9)}$	$K(E_9)$
1	$E_{10(10)}$	$K(E_{10})$
(0	$E_{11(11)}$	$K(E_{11})$

After the reduction to three dimensions one obtains Kac–Moody theoretic extensions of the exceptional  $E$ -series. The two-dimensional symmetry  $E_9$  is an affine symmetry [7, 9], below two dimensions the expected symmetry is the hyperbolic extension  $E_{10}$  and in one dimension one formally finds the Lorentzian algebra  $E_{11}$ . Three dimensions are special since there all physical (bosonic) degrees of freedom can be converted into scalars. The procedure inverse to dimensional reduction, called oxidation, of a theory has also been studied [10, 11, 12]. There one starts with a scalar coset model  $G/K$  in three space-time dimensions for each simple  $G$  and asks for a higher-dimensional theory whose reduction yields this hidden symmetry. The answer is known for all semi-simple  $G$  and was presented in [11] and will be recovered in table

1 below. For conciseness of notation we denote the oxidized theory in the maximal dimension by  $\mathcal{O}_G$ . In this language, maximal eleven-dimensional supergravity is denoted by  $\mathcal{O}_{E_8}$ .

A different motivation for studying infinite-dimensional Kac–Moody algebras comes from cosmological billiards. Indeed, it has been shown recently that the dynamics of the gravitational scale factors becomes equivalent, in the vicinity of a spacelike singularity, to that of a relativistic particle moving freely on an hyperbolic billiard and bouncing off its walls [13, 14, 15, 16]. A criterion for the gravitational dynamics to be chaotic is that the billiard has a finite volume [17]. This in turn stems from the remarkable property that the billiard walls can be identified with the walls of the fundamental Weyl chamber of a hyperbolic Kac–Moody algebra.<sup>1</sup> Building on these observations it has been shown that a null geodesic motion of a relativistic particle in the coset space  $E_{10}/K(E_{10})$  can be mapped to the bosonic dynamics of  $D = 11$  supergravity reduced to one time dimension [5, 15].

It is at the heart of the proposal of [3] that the hidden symmetries of the reduced theory are already present in the unreduced theory. Furthermore, the symmetry groups actually get extended from the finite-dimensional  $G$  to the Lorentzian triple extension  $G^{+++}$  (also called very-extension).<sup>2</sup> To every finite-dimensional  $G$  we wish to associate a model possessing a non-linearly realized  $G^{+++}$  symmetry which might be called the M-theory corresponding to  $G$ , and denoted by  $\mathcal{V}_G$ . To try to construct M-theory in the sense given above, the basic tool is a non-linear sigma model based on Kac–Moody algebras in the maximal number of dimensions (corresponding to the oxidized  $\mathcal{O}_G$ ). Basic features about non-linear sigma models associated with the coset  $G/K(G)$  are recalled in appendix A. M-theories constructed in this way possess an infinite-dimensional symmetry structure and contain infinitely many fields. Some of the infinitely many fields may be auxiliary, however.<sup>3</sup> In addition one can hope to relate some of the infinitely many other fields to the perturbative string spectrum. Since the underlying Kac–Moody algebra is not well-understood the analysis is complicated and incomplete, and this identification of states has not been carried out. Below we will present the result for the supergravity fields. In addition, the precise dynamical theory for all fields is mostly elusive to date. There remain a number of important challenges in the full understanding of these M-theories, which will be highlighted at the relevant points.

Since  $E_{11}$  will be our guiding example, we briefly review the evidence for the conjecture that M-theory possesses an  $E_{11}$  symmetry.

- It has been shown that the bosonic sector of  $D = 11$  supergravity can be formulated as the simultaneous non-linear realization of two finite-dimensional Lie algebras [21]. The two corresponding groups, whose closure is taken, are the eleven-dimensional conformal group and a group called  $G_{11}$  in [21]. The generators of  $G_{11}$  contain the generators  $P_a$  and  $K^a_b$  of the group of affine coordinate transformations  $IGL(11)$  in eleven dimensions and the closure with the conformal group will therefore generate infinitesimal general coordinate transformations [22]. The precise structure of the Lie algebra of  $G_{11}$  is given by

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<sup>1</sup>Recall that an algebra is hyperbolic if upon the deletion of any node of the Dynkin diagram the remaining algebra is a direct sum of simple or affine Lie algebras.

<sup>2</sup>For the process of very-extension see also [18, 19, 20].

<sup>3</sup>In order to determine which fields are auxiliary one would require a (still missing) full dynamical understanding of the theory.

$$\begin{aligned}
[K^a{}_b, K^c{}_d] &= \delta_b^c K^a{}_d - \delta_d^a K^c{}_b \\
[K^a{}_b, P_c] &= -\delta_c^a P_b \\
[K^a{}_b, R^{c_1 \dots c_6}] &= -6\delta_b^{[c_1} R^{c_2 \dots c_6]a}, \\
[K^a{}_b, R^{c_1 \dots c_3}] &= 3\delta_b^{[c_1} R^{c_2 c_3]a}, \\
[R^{c_1 \dots c_3}, R^{c_4 \dots c_6}] &= 2R^{c_1 \dots c_6}.
\end{aligned} \tag{1.1}$$

The additional totally anti-symmetric generators  $R^{c_1 \dots c_3}$  and  $R^{c_1 \dots c_6}$  are obviously related to the three-form gauge potential of  $N = 1$ ,  $D = 11$  supergravity and its (magnetic) dual. To write the equations of motion of the bosonic sector of  $D = 11$  supergravity one should first consider the following coset element of  $G_{11}$  over the Lorentz group  $H_{11}$ ,

$$v(x) = e^{x^a P_a} e^{h_a{}^b K^a{}_b} \exp\left(\frac{A_{c_1 \dots c_3} R^{c_1 \dots c_3}}{3!} + \frac{A_{c_1 \dots c_6} R^{c_1 \dots c_6}}{6!}\right) \in G_{11}/H_{11} \tag{1.2}$$

The fields  $h^a{}_b$ ,  $A_{c_1 \dots c_3}$  and  $A_{c_1 \dots c_6}$  depend on the space-time coordinate  $x^\mu$ . Notice that the non-linear sigma model  $G_{11}/H_{11}$  does not produce the bosonic equations of motion of  $D = 11$  supergravity. The Cartan forms  $\partial_\mu v v^{-1}$  constructed from this coset element for instance do not give rise to the antisymmetrized field strengths. Only after the non-linear realization with respect to the coset of the conformal group with the *same* Lorentz group is also taken does one obtain the correct field strengths and curvature terms of the supergravity theory [21]. Then it is natural to write down the equations of motion in terms of these fields, which then are the full non-linear field equations. Similar calculations were done for the type IIA, IIB and massive IIA supergravity [21, 23, 24], and for the closed bosonic string [25].

- It has been conjectured that an extension of this theory has the rank eleven Kac–Moody algebra called  $E_{11}$  as a symmetry [3].  $E_{11}$  is the very-extension of  $E_8$  and is therefore also sometimes called  $E_8^{+++}$ . This extension is suggested because, if one drops the momentum generators  $P_a$ , the coset  $G_{11}/H_{11}$  is a truncation of the coset  $E_{11}/K(E_{11})$ , as we will see below. Note that dropping the translation generators  $P_a$  in a sense corresponds to forgetting about space-time since the field conjugate to  $P_a$  are the space-time coordinates  $x^a$ . The translation operators  $P_a$  will be re-introduced in section 4 where we will also discuss the rôle of space-time in more detail.
- The algebraic structure encoded in  $E_{11}$  has been shown to control transformations of Kasner solutions [19] and intersection rules of extremal brane solutions [26]. Furthermore, the relation between brane tensions in IIA and IIB string theory (including the D8 and D9 brane) can be deduced [27, 28].

As mentioned above, similar constructions can be done for all very-extended  $G^{+++}$ . In the sequel, the attention is focussed on these models.

The paper is organized as follow. Chapter 2 is devoted to studying the field content of the Kac–Moody models  $\mathcal{V}_G$ . A brief introduction to Kac–Moody algebras is given in section

2.1. A level decomposition of the generators of a Kac-Moody algebra with respect to a  $\mathfrak{gl}(D)$  subalgebra is explained in section 2.2. Such decompositions are of interest since the  $\mathfrak{gl}(D)$  subalgebra represents the gravitational degrees of freedom; the other generators are in tensorial representations of  $\mathfrak{gl}(D)$  and mostly provide higher spin fields. In section 2.3, the field content of the model  $\mathcal{V}_G$  is given for all  $G^{+++}$  and the (super-)gravity fields of  $\mathcal{O}_G$  are identified. Another remarkable feature of the algebras is how their Dynkin diagrams support different interpretations, in particular T-dualities. This is exemplified in section 2.4. In chapter 3, we offer a few remarks on the relation of the KM fields to higher spin theories. The emergence of space-time in the context of very-extended algebras is envisaged in chapter 4, where we also explain some diagrammatic tricks. The conclusions are given in the last chapter.

## 2 Field content of Kac–Moody algebras

### 2.1 Brief definition of Kac–Moody algebra

Let us define a Kac–Moody algebra  $\mathfrak{g}$  via its Dynkin diagram with  $n$  nodes and links between these nodes. The algebra is a Lie algebra with Chevalley generators  $h_i, e_i, f_i$  ( $i = 1, \dots, n$ ) obeying the following relations [29]

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= A_{ij} e_j \\ [h_i, f_j] &= -A_{ij} f_j \\ [h_i, h_j] &= 0 \end{aligned}$$

where  $A_{ii} = 2$  and  $-A_{ij}$  ( $i \neq j$ ) is a non-negative integer related to the number of links between the  $i^{\text{th}}$  and  $j^{\text{th}}$  nodes. The so-called Cartan matrix  $A$  in addition satisfies  $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$ . The generators must also obey the Serre relations,

$$\begin{aligned} (\text{ad } e_i)^{1-A_{ij}} e_j &= 0 \\ (\text{ad } f_i)^{1-A_{ij}} f_j &= 0 \end{aligned}$$

A root  $\alpha$  of the algebra is a non-zero linear form on the Cartan subalgebra  $\mathfrak{h}$  (= the subalgebra generated by the  $\{h_i \mid i = 1, \dots, n\}$ )<sup>4</sup> such that

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

is not empty.  $\mathfrak{g}$  can be decomposed in the following triangular form,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

or according to the root spaces,

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus \mathfrak{h}$$

where  $\mathfrak{n}_-$  is the direct sum of the negative roots spaces,  $\mathfrak{n}_+$  of the positive ones and  $\mathfrak{h}$  is the Cartan subalgebra. The dimension of  $\mathfrak{g}_\alpha$  is called the multiplicity of  $\alpha$ . These multiplicities obey the Weyl–Kac character formula

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<sup>4</sup>Here we assume the Cartan matrix  $A$  to be non-degenerate.

$$\prod_{\alpha \in \Delta_+} (1 - e^\alpha)^{\text{mult}\alpha} = \sum_{w \in W} \epsilon(w) e^{w(\rho) - \rho}$$

The sum is over the Weyl group which in the case of interest here is infinite.  $\epsilon(w)$  is the parity of  $w$  and  $\rho$  is the Weyl vector. This formula cannot be solved in closed form in general. We will normally denote by  $G$  the (simply-connected, formal) group associated to  $\mathfrak{g}$ .

Our interest is focussed here on a class of Kac–Moody algebras called *very-extensions* of simple Lie algebras. They are the natural extension of the over-extended algebras which are themselves extensions of the affine algebras. The procedure for constructing an affine Lie algebra  $\mathfrak{g}^+$  from a simple one  $\mathfrak{g}$  consist in the addition of a node to the Dynkin diagram in a certain way which is related to the properties of the highest root of  $\mathfrak{g}$ . One may also further increase by one the rank of the algebra  $\mathfrak{g}^+$  by adding to the Dynkin diagram a further node that is attached to the affine node by a single line. The resulting algebra  $\mathfrak{g}^{++}$  is called the over-extension of  $\mathfrak{g}$ . The very-extension, denoted  $\mathfrak{g}^{+++}$ , is found by adding yet another node to the Dynkin diagram that is attached to the over-extended node by one line [18].

A further important concept in this context is that of the *compact form*  $K(\mathfrak{g})$  of  $\mathfrak{g}$  [29], which we here define as the fixed point set under the compact involution  $\phi$  mapping

$$\phi(e_i) = -f_i, \quad \phi(f_i) = -e_i, \quad \phi(h_i) = -h_i.$$

An element in the coset space of the formal groups  $G/K(G)$  then can be parametrized as

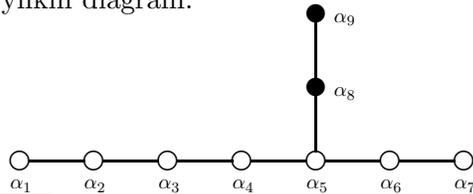
$$v = \exp\left(\sum_i \phi_i h_i\right) \exp\left(\sum_{\alpha > 0} A_\alpha E_\alpha\right), \quad (2.1)$$

where the (infinitely many) positive step operators  $E_\alpha$  can have multiplicities greater than one for imaginary roots  $\alpha$ .

## 2.2 Decomposition of a KM algebra under the action of a regular subalgebra

In order to construct a non-linear sigma model associated with a Kac–Moody algebra, e.g.  $E_8^{+++}$ , one needs to consider infinitely many step operators  $E_\alpha$  and therefore infinitely many corresponding fields  $A_\alpha$  because the algebra is infinite-dimensional.<sup>5</sup> To date, only truncations of this model can be constructed. A convenient organization of the data is to decompose the set of generators of a given Kac–Moody algebra  $\mathfrak{g}$  with respect to a finite regular subalgebra  $\mathfrak{s}$  and a corresponding *level decomposition*.<sup>6</sup> The ‘level’ provides a gradation on  $\mathfrak{g}$  and the assignment is as follows: generators in  $\mathfrak{h}$  are at level  $\ell = 0$  and elements in  $\mathfrak{g}_\alpha$  derive their level from the root  $\alpha$ , usually as a subset of the root labels. The following example is taken to illustrate the notion of level.

*Level of a root:* Consider the algebra  $\mathfrak{g} = E_6^{+++}$  with simple roots  $\alpha_i$  ( $i = 1, \dots, 9$ ) labelled according to the Dynkin diagram:



<sup>5</sup>See also appendix A for the notation.

<sup>6</sup>We restrict to finite-dimensional subalgebras  $\mathfrak{s}$  in order to have only a finite number of elements for each level [5, 30].

We want to make a level decomposition of  $E_6^{+++}$  under its  $\mathfrak{s} = A_7$  subalgebra corresponding to the sub-Dynkin diagram with nodes from 1 to 7. This singles out the nodes 8 and 9 which do not belong to the  $A_7$  subdiagram. Any positive root  $\alpha$  of  $E_6^{+++}$  can be written as  $\alpha = \sum_{i=1}^9 m_i \alpha_i = \sum_{s=1}^7 m_s \alpha_s + \sum_{g=8,9} \ell_g \alpha_g$  with  $m_s$  and  $\ell_g$  non-negative integers. Here,  $\ell_8$  and  $\ell_9$  are called respectively the  $\alpha_8$  level and the  $\alpha_9$  level of  $\alpha$ .

The truncation consists of considering only the generators of the lowest levels; the coset element (2.1) is then calculated by using only the fields corresponding to these generators. The question of which cut-off to take for the truncation will be answered universally below.

The adjoint action of  $\mathfrak{s}$  on  $\mathfrak{g}$  preserves the level, therefore the space of generators on a given level is a (finite-dimensional) representation space for a representation of  $\mathfrak{s}$  and hence completely reducible. The fields  $A_\alpha$  associated with roots of a given level  $\{\ell_g\}$  will be written as representations of  $\mathfrak{s}$ .

In order to obtain an interpretation of the fields associated with these generators as space-time fields, it is convenient to choose the regular subalgebra to be  $\mathfrak{s} = \mathfrak{sl}(D)$  for some  $D$ ; these are always enhanced to  $\mathfrak{gl}(D)$  by Cartan subalgebra generators associated with the nodes which were singled out. The reason for the choice of an  $A$ -type subalgebra is that one then knows how the resulting tensors transform under  $\mathfrak{so}(D) = K(\mathfrak{s}) \subset K(\mathfrak{g})$ . In the spirit of the non-linear realization explained in the introduction, this  $\mathfrak{so}(D)$  plays the rôle of the local Lorentz group.<sup>7</sup> In particular, the level  $\ell = 0$  sector will give rise to the coset space  $GL(D)/SO(D)$  since it contains the adjoint of the regular  $\mathfrak{gl}(D)$  subalgebra. This is the right coset for the gravitational vielbein which is an invertible matrix defined up to a Lorentz transformation.<sup>8</sup> If the rank of  $\mathfrak{g}$  is  $r$  then there will also be  $r - D$  additional dilatonic scalars at level  $\ell = 0$  from the remaining Cartan subalgebra generators.

Let us study in more detail which representations of  $\mathfrak{s}$  can occur at a given level. The necessary condition for a positive root  $\alpha = \sum_s m_s \alpha_s + \sum_g \ell_g \alpha_g$  of  $\mathfrak{g}$  to generate a lowest weight representation of  $\mathfrak{s}$  are that

$$p_s = - \sum_t A_{st} m_t - \sum_g A_{sg} \ell_g \geq 0$$

where  $A$  is the Cartan matrix of  $\mathfrak{g}$ , the  $\{p_s\}$  are the Dynkin labels of the corresponding lowest weight representation.<sup>9</sup> We can also rephrase this condition by realizing that a representation with labels  $\{p_s\}$  at level  $\{\ell_g\}$  corresponds to a root with coefficients

$$m_s = - \sum_t (A_{\text{sub}}^{-1})_{st} p_t \sum_{t,g} -(A_{\text{sub}}^{-1})_{st} A_{tg} \ell_g \geq 0$$

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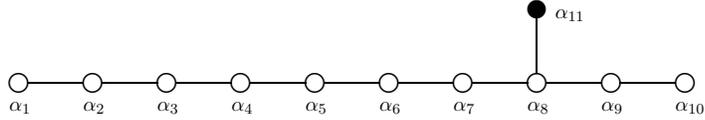
<sup>7</sup>The real form  $\mathfrak{so}(D)$  is actually not the correct one for an interpretation as Lorentz group. In order to obtain the correct  $\mathfrak{so}(D-1, 1)$  one needs to modify the compact involution to a so-called ‘temporal involution’ [6]. As discussed in [31, 32] this leads to an ambiguity in the signature of space-time since Weyl-equivalent choices of involution result in inequivalent space-time signatures on the ‘compact’ part of the subalgebra  $\mathfrak{s}$ . We will not be concerned with this important subtlety here as it does not affect the higher spin field content.

<sup>8</sup>For a recent ‘stringy’ decomposition of  $E_{10}$  under a subalgebra of type  $D$  see [33].

<sup>9</sup>Actually the lowest weight then has Dynkin labels  $\{-p_s\}$  since we have introduced an additional minus sign in the conversion between the different labels which is convenient for the class of algebras we are considering here.

and these are non-negative integers if they are to belong to a root of  $\mathfrak{g}$ . Here,  $A_{\text{sub}}$  is the Cartan matrix of  $\mathfrak{s}$ . Besides this necessary condition one has to check that there are elements in the root space of  $\alpha$  that can serve as highest weight vectors. This requires calculating the multiplicity of  $\alpha$  as a root of  $\mathfrak{g}$  and its weight multiplicity in other (lower) representations on the same level. The number of independent highest weight vectors is called the *outer multiplicity* of the representation with labels  $\{p_s\}$  and usually denoted by  $\mu$ .

*Example:*  $\mathfrak{g} = E_8^{+++}$ , which is the very extension of  $E_8$ , can be decomposed w.r.t its subalgebra  $\mathfrak{s} = A_{10} = \mathfrak{sl}(11)$  associated with the sub-Dynkin diagram with nodes from 1 to 10,



One finds that the level  $\ell \equiv \ell_{11} = 0, 1, 2$  and 3 generators are in a lowest weight representation of  $A_{11}$  with Dynkin labels given in the following table,

$\ell$	Dynkin labels	Tensor
0	$[1, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$	$K_a^b$
1	$[0, 0, 0, 0, 0, 0, 0, 1, 0, 0]$	$R^{a_1 a_2 a_3}$
2	$[0, 0, 0, 0, 1, 0, 0, 0, 0, 0]$	$R^{a_1 a_2 a_3 a_4 a_5 a_6}$
3	$[0, 0, 1, 0, 0, 0, 0, 0, 0, 1]$	$R^{a_1 \dots a_8, b}$
...		

The level zero generators are associated with the gravitational degrees of freedom as explained above. The level one field can be recognized as the three form potential of the supergravity in  $D = 11$  and the level two field correspond to its dual. Notice that up to level 2 the generators are just the fields of  $G_{11}$ , mentioned in the introduction, except for the momentum generators  $P_a$ . The level three tensor corresponds to a Young tableau of the form

$$\begin{array}{|c|c|} \hline a_1 & b \\ \hline a_2 & \\ \hline \vdots & \\ \hline a_8 & \\ \hline \end{array}$$

and is associated with the dual of the graviton [34, 35, 36].<sup>10</sup> The dualization of Einstein's field equations only works at the linearized level however [36]. One might speculate that a dualization of the non-linear equations will probably require some of the remaining fields in the infinite list of tensors above level 3. In summary, the lowest level generators ( $\ell = 0, 1, 2, 3$ ) of the  $E_8^{+++}$  algebra correspond to the degrees of freedom (and their duals) of  $D = 11$  supergravity, which is  $\mathcal{O}_{E_8^{+++}}$ .

<sup>10</sup>There is a subtlety here since the  $\mathfrak{g}(11)$  tensor associated with this mixed symmetry vanishes after antisymmetrization over all indices. This means that one cannot accommodate the trace of the spin connection in this dual picture.

$\mathfrak{g}^{+++}$	oxidized theory (maximal)
$E_8^{+++}$	$N = 1, D = 11$ SUGRA
$E_7^{+++}$	$N = IIB, D = 10$ SUGRA, truncated non-supersymmetrically to dilaton and 4-form potential with self-dual field strength
$E_6^{+++}$	$N = 2, D = 8$ SUGRA, truncated to dilaton, axion and 3-form potential
$D_n^{+++}$	Massless sector of the closed bosonic string in $D = n + 2$
$A_n^{+++}$	Gravity in $D = n + 3$
$G_2^{+++}$	Einstein-Maxwell $N = 2, D = 5$ SUGRA
$F_4^{+++}$	$N = (0, 1), D = 6$ chiral SUGRA
$B_n^{+++}$	massless ‘heterotic’ string in $D = n + 2 \equiv$ closed bosonic string coupled to a massless abelian vector potential
$C_n^{+++}$	$D = 4$ theory, $(n - 1)^2$ scalars and $2(n - 1)$ vector potentials transforming under $C_n$

Table 1: The list of oxidized theories as recovered by very-extended Kac–Moody algebras  $\mathfrak{g}^{+++}$ .

### 2.3 Results

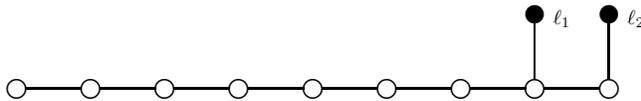
The analysis above can be repeated for all very-extended algebras  $\mathfrak{g}^{+++}$ . There is a natural maximal choice for the  $\mathfrak{sl}(D)$  algebra [19, 20]. This is obtained by starting at the very-extended node and following the line of long roots as far as possible.

The decomposition under this maximal gravity subalgebra, truncated at the level of the affine root of  $\mathfrak{g}^+$ , corresponds precisely to the bosonic fields of the oxidized theory  $\mathcal{O}_G$ , if one also includes *all* dual fields for form matter and gravity [20]. Instead of repeating the full analysis we summarize the fields by their oxidized theories in table 1; the Kac–Moody algebra  $\mathfrak{g}^{+++}$  captures only the bosonic fields. The truncation criterion of the affine root seems natural since one knows that the finite-dimensional algebras  $\mathfrak{g}$  correspond to the oxidized theories and generators in the ‘true Kac–Moody sector’ are likely to have a different rôle.

All oxidized theories are theories containing gravity and pure gravity itself is associated with  $A$  type algebras. It can be shown that on the level of very-extended algebras the relevant maximal  $A_{D-3}^{+++}$  algebra is contained in the M-theory algebra  $\mathfrak{g}^{+++}$  [20].

### 2.4 Different embeddings

We can analyse the field content of a theory with respect to different embeddings, i.e. different choices of  $A_n$  subalgebras. For example, the type IIA supergravity theory is associated with the following decomposition of  $E_8^{+++}$  (we now only mark in black the nodes to which we assign a level  $\ell_i$ ),



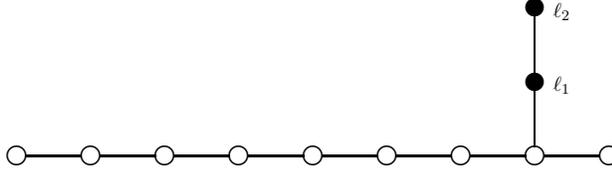
The level  $(\ell_1, \ell_2)$  generators of  $E_8^{+++}$  corresponding to the roots of the form  $\alpha = \sum_s m_s \alpha_s + \ell_1 \alpha_{\ell_1} + \ell_2 \alpha_{\ell_2}$  are in representations of  $A_9$  characterized by the Dynkin labels  $\{p_s\} = [p_1, \dots, p_9]$ . The following table gives, up to the level of the affine root, the Dynkin labels of the repre-

representations occurring in  $E_8^{+++}$  and their outer multiplicities  $\mu$ , as well as the interpretation of the fields.

$(\ell_1, \ell_2)$	$[p_1, \dots, p_9]$	$\mu$	Interpretation
(0,0)	$[1,0,0,0,0,0,0,0,1] \oplus [0,0,0,0,0,0,0,0,0]$	1	$h_a^b$
(0,0)	$[0,0,0,0,0,0,0,0,0]$	2	$\phi$
(1,0)	$[0,0,0,0,0,0,0,1,0]$	1	$B_{(2)}$
(0,1)	$[0,0,0,0,0,0,0,0,1]$	1	$A_{(1)}$
(1,1)	$[0,0,0,0,0,0,1,0,0]$	1	$A_{(3)}$
(2,1)	$[0,0,0,0,1,0,0,0,0]$	1	$\tilde{A}_{(5)}$
(2,2)	$[0,0,0,1,0,0,0,0,0]$	1	$\tilde{B}_{(6)}$
(3,1)	$[0,0,1,0,0,0,0,0,0]$	1	$\tilde{A}_{(7)}$
(3,2)	$[0,0,1,0,0,0,0,0,1]$	1	$\tilde{A}_{(7,1)}$
(3,2)	$[0,1,0,0,0,0,0,0,0]$	1	$\tilde{A}_{(8)}^\phi$
...			

These fields match the field content of type IIA supergravity. Indeed, in addition to the level zero gravitational fields and their dual fields, one recognizes the NS-NS two form  $B_{(2)}$ , the RR one form  $A_{(1)}$ , the RR three form  $A_{(3)}$  and the dual  $\tilde{B}_{(6)}$ ,  $\tilde{A}_{(7)}$  and  $\tilde{A}_{(5)}$ , respectively.

For type IIB supergravity, the choice of the  $A_9$  subalgebra can be depicted by redrawing the  $E_8^{+++}$  Dynkin diagram in the following way.



The decomposition produces the following table [20].

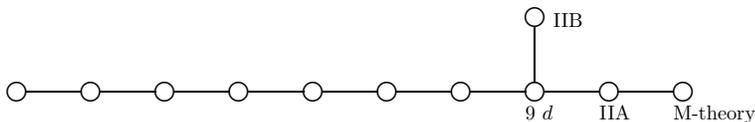
$(\ell_1, \ell_2)$	$[p_1, \dots, p_9]$	$\mu$	Interpretation
(0,0)	$[1,0,0,0,0,0,0,0,1] \oplus [0,0,0,0,0,0,0,0,0]$	1	$h_a^b$
(0,0)	$[0,0,0,0,0,0,0,0,0]$	2	$\phi$
(1,0)	$[0,0,0,0,0,0,0,0,0]$	1	$\chi$
(0,1)	$[0,0,0,0,0,0,0,1,0]$	1	$B_{(2)}$
(1,1)	$[0,0,0,0,0,0,0,1,0]$	1	$A_{(2)}$
(1,2)	$[0,0,0,0,0,1,0,0,0]$	1	$A_{(4)}$
(1,3)	$[0,0,0,1,0,0,0,0,0]$	1	$\tilde{B}_{(6)}$
(2,3)	$[0,0,0,1,0,0,0,0,0]$	1	$\tilde{A}_{(6)}$
(2,4)	$[0,0,1,0,0,0,0,0,1]$	1	$\tilde{A}_{(7,1)}$
(1,4)	$[0,1,0,0,0,0,0,0,0]$	1	$\tilde{A}_{(8)}^\chi$
(2,4)	$[0,1,0,0,0,0,0,0,0]$	1	$\tilde{A}_{(8)}^\phi$
...			

The fields in the table correspond to the gravitational fields  $h_a^b$  and the dilaton  $\phi$ , the RR zero form (axion)  $\chi$ , the NS-NS two form  $B_{(2)}$ , the RR two form  $A_{(2)}$  and their duals  $\tilde{A}_{(7,1)}$ ,  $\tilde{A}_{(8)}^\phi$ ,  $\tilde{A}_{(8)}^\chi$ ,  $\tilde{B}_{(6)}$ ,  $\tilde{A}_{(6)}$ , respectively. The RR four form  $A_{(4)}$  will have self-dual field strength. Remember that the field content corresponds only to the few lowest level generators. The

tables continue infinitely but there is no interpretation to date for the additional fields, most of which have mixed Young symmetry type.<sup>11</sup>

As string theories, type IIA and IIB string theories are related by ‘T-duality’. Here we notice a diagrammatic reflection of this fact through the ‘T-junction’ in the Dynkin diagram of  $E_8^{+++}$ . Therefore, the diagrams of the very-extended Kac–Moody algebras nicely encode the field content of the oxidized theories and also relations between different theories.

Moreover, Kaluza–Klein reduction corresponds to moving nodes out of the ‘gravity line’ (the  $A$ -type subalgebra). For example, we saw that the bosonic field content of  $D = 11$  supergravity can be retrieved by considering the Dynkin diagram of  $E_8^{+++}$  with a  $\mathfrak{sl}(11)$  subalgebra so that the endpoint of the gravity line is the ‘M-theory’ node in the diagram below. A reduction of  $D = 11$  supergravity produces IIA supergravity and the endpoint of the corresponding gravity line is also marked in the diagram in agreement with the analysis above. Finally, the ten-dimensional IIB theory is not a reduction of  $D = 11$  supergravity but agrees with it after reduction to nine dimensions. Roughly speaking this means taking a different decompactification after  $D = 9$  not leading to M-theory. This is precisely the structure of the  $E_8^{+++}$  diagram.



### 3 Relation to higher spin fields

We now turn to the question in what sense the fields contained in the Kac–Moody algebras are true higher spin fields. Our minimal criterion will be that they transform under the *space-time* Lorentz group and that their dynamics is consistent. These requirements will be discussed for the two familiar classes of Kac–Moody models.

**1. West’s proposal:** As explained in the introduction, in [21, 3] the Kac–Moody algebra is a symmetry of the unreduced theory. Therefore, the fields transform under local space-time  $SO(D) \subset K(G^{+++})$  transformations and therefore satisfy the first requirement. But as we have also stressed, the derivation of the dynamical equations requires the introduction of additional translation generators and the closure with the conformal algebra to obtain the correct curvatures, at least for gravity and the anti-symmetric matter fields. It is not known what this procedure yields for the mixed symmetry fields at higher levels. When writing down the dynamical equations there is also an ambiguity in numerical coefficients which should ultimately be fixed from the algebraic structure alone, maybe together with supersymmetry. At the present stage the consistency of the higher spin dynamics cannot be determined.

However, it might be that known higher spin formulations play a rôle in the resolution of these difficulties. In particular, the ‘unfolded dynamics’ of [37, 38]<sup>12</sup> could provide a dynamical scheme for the KM fields.

<sup>11</sup>We remark that there are two exceptions of fields which have a reasonable possible interpretation [20]. These occur in the IIA and IIB decomposition of  $E_8^{+++}$  and correspond to the nine-form potential in massive IIA supergravity and to the ten-form potential in IIB theory, which act as sources of the D8 in and D9 brane in IIA and IIB superstring theory, respectively.

<sup>12</sup>See also M. Vasiliev’s contribution to the proceedings of this workshop.

**2. Null geodesic world-lines on Kac–Moody coset spaces:** This is the approach taken in [5, 6], where the idea is to map a one-dimensional world-line, and not space-time itself, into the coset space  $G^{+++}/K(G^{+++})$ . The advantage of this approach is that one does not require translation operators or the closure with the conformal group. Rather space-time is conjectured to re-emerge from a kind of Taylor expansion via gradient fields of the oxidized fields present in the decomposition tables. For the details see [5] or the discussion in [40]. Therefore space-time is thought of as a Kac–Moody intrinsic concept in the world-line approach. The dynamical equations in this context are derived from a lagrangian formulation for the particle motion.

From the higher spin point-of-view the fields no longer are true higher spin fields under the space-time Lorentz group but rather under some internal Lorentz group. In order to transform the KM fields into honest higher spin fields the recovery of space-time from the KM algebra needs to be made precise. On the other hand, the dynamical scheme here is already consistent and therefore our second requirement for higher spin dynamics is fulfilled automatically.

## 4 Space-time concepts

Minkowski space-time can be seen as the quotient of the Poincaré group by the Lorentz group. The Poincaré group itself is the semi-direct product of the Lorentz group with the (abelian) group of translations and the translations form a vector representation of the Lorentz group:

$$[M^{ab}, M^{cd}] = \eta^{ac}M^{bd} - \eta^{ad}M^{bc} + \eta^{bd}M^{ac} - \eta^{bc}M^{ad}, \quad (4.2)$$

$$[P_a, P_b] = 0, \quad (4.3)$$

$$[M^{ab}, P_c] = \delta_c^a \eta^{bd} P_d - \delta_c^b \eta^{ad} P_d. \quad (4.4)$$

In the Kac–Moody context, the Lorentz group is replaced by  $K(G^{+++})$  so one needs a ‘vector representation’ of  $K(G^{+++})$ . By this we mean a representation graded as a vector space by level  $\ell \geq 0$  with a Lorentz vector as bottom component at  $\ell = 0$ , corresponding to the vector space decomposition of the compact subalgebra

$$K(\mathfrak{g}^{+++}) = \mathfrak{so}(D) \oplus \dots$$

Unfortunately, very little is known about  $K(\mathfrak{g}^{+++})$ , except that it is not a Kac–Moody algebra. However, it is evidently a subalgebra of  $\mathfrak{g}^{+++}$ . One can therefore construct representations of  $K(\mathfrak{g}^{+++})$  by taking a representation of  $\mathfrak{g}^{+++}$  and then view it as a  $K(\mathfrak{g}^{+++})$  module. Irreducibility (or complete reducibility) of such representations are interesting open questions. If we take a (unitary) lowest weight representation of  $\mathfrak{g}^{+++}$  then we can write it as a tower of  $\mathfrak{gl}(D)$ -modules in a fashion analogous to the decomposition of the adjoint. The idea in reference [39] was to take a representation of  $\mathfrak{g}^{+++}$  whose bottom component is a  $\mathfrak{gl}(D)$  vector for any choice of gravity subalgebra. This is naturally provided for by the  $\mathfrak{g}^{+++}$  representation with lowest weight Dynkin labels

$$[1, 0, \dots, 0]_{\mathfrak{g}^{+++}}.$$

We denote this irreducible  $\mathfrak{g}^{+++}$  representation by  $L(\lambda_1)$  since the lowest weight is just the fundamental weight of the first (very-extended) node.<sup>13</sup> Decomposed w.r.t. the gravity subalgebra  $A_{D-1}$ , this yields

$$\underbrace{[1, 0, \dots, 0]_{\mathfrak{g}^{+++}}}_{r \text{ labels}} \rightarrow \underbrace{[1, 0, \dots, 0]_{gravity}}_{(D-1) \text{ labels}} \oplus \dots$$

<sup>13</sup>Therefore, the representation is integrable and unitarizable.

where the "...” denote the infinitely many other representations of the gravity subalgebra at higher levels ( $r$  is the rank of  $\mathfrak{g}^{+++}$ ). These fields can be computed in principle with the help of a character formula. For example, for  $E_8^{+++}$ , one finds the following decomposition of the vectorial representation w.r.t. the M-theory gravity subalgebra  $A_{10}$  [39]:

$\ell$	$[p_1, \dots, p_{10}]$	Field
0	[1, 0, 0, 0, 0, 0, 0, 0, 0, 0]	$P_a$
1	[0, 0, 0, 0, 0, 0, 0, 0, 1, 0]	$Z^{ab}$
2	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0]	$Z^{a_1 \dots a_5}$
3	[0, 0, 0, 1, 0, 0, 0, 0, 0, 1]	$Z^{a_1 \dots a_7, b}$
3	[0, 1, 0, 0, 0, 0, 0, 0, 0, 0]	$Z^{a_1 \dots a_8}$
...	...	...

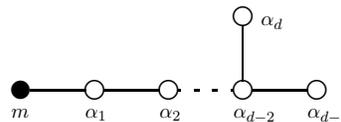
In addition to the desired vector representation  $P_a$  of  $\mathfrak{gl}(11)$ , one finds a 2-form and a 5-form at levels  $\ell = 1, 2$ . It was noted in [39] that these are related to the  $D = 11$  superalgebra

$$\{Q_\alpha, Q_\beta\} = ((\Gamma^a C)P_a + \frac{1}{2}(\Gamma_{ab} C)Z^{ab} + \frac{1}{5!}(\Gamma_{a_1 \dots a_5} C)Z^{a_1 \dots a_5})_{\alpha\beta} \quad (4.5)$$

in an obvious way. ( $C$  is the charge conjugation matrix.) The branes of M-theory couple as  $\frac{1}{2}$ -BPS states to these central charges. Therefore the vector representation also encodes information about the topological charges of the oxidized theory. We also note that the charges on  $\ell = 3$  suggest that they might be carried by solitonic solutions associated with the dual graviton field, but no such solutions are known.<sup>14</sup> Therefore, we will refer to the tensors in the  $\mathfrak{g}^{+++}$  representation  $L(l_1)$  as ‘generalized central charges’ [40].<sup>15</sup>

The result carries over to all algebras  $\mathfrak{g}^{+++}$ : To any field we found in the decomposition for the oxidized theory there is an associated generalized charge [40] in the ‘momentum representation’  $L(\lambda_1)$ . For instance, for an anti-symmetric  $p$ -form this is just a  $(p - 1)$ -form. It is remarkable that this result holds for all  $\mathfrak{g}^{+++}$  and not only for those where one has a supersymmetric extension of the oxidized theory.

To demonstrate this result we use a trick [40] which turns the statement into a simple corollary of the field content analysis. Instead of using the Weyl–Kac or the Freudenthal character formulae, we embed the semi-direct sum of the algebra with its vectorial representation in a larger algebra [40]. This can be illustrated nicely for the Poincaré algebra. The aim is to find a minimal algebra which contains (as a truncation) both the original Lorentz algebra and the vector representation by Dynkin diagram extensions. We consider the case of even space-time dimension for simplicity. The  $\mathfrak{so}(2d)$  vector has Dynkin labels  $[1, 0, \dots, 0]$  in the standard choice of labelling the simple nodes. Hence, adding a node (marked in black in the diagram below, with label  $m$  for momentum) at the position 1 below with a single line will give a vector in the decomposition under the Lorentz algebra at  $\ell = 1$ .



<sup>14</sup>This idea has appeared before in the context of U-duality where the mixed symmetry ‘charge’ was associated with a Taub-NUT solution [42].

<sup>15</sup>These are generalized since it is not clear in what algebra they should be central and the name is just by analogy. Note, however, that these charges will still all commute among themselves.

The resulting embedding algebra is the conformal algebra as the full decomposition shows<sup>16</sup>

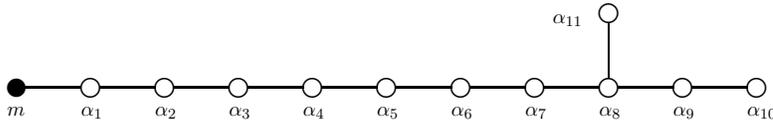
$$\{M^{ab}, P^c\} \xrightarrow{\text{embed}} M^{AB} \xrightarrow{\text{decompose}} \{M^{ab}, P^c, K^c, D\}, \quad (4.6)$$

where  $M^{ab}$  ( $a = 1, \dots, 2d$ ),  $P^a$ ,  $K^a$  and  $D$  are respectively the generators of the Lorentz group, the translations generators, the special conformal transformations generators and the dilatation operator in  $D$  dimensions.  $M^{AB}$  ( $A = 1, \dots, 2d + 2$ ) are the generators of the conformal algebra  $\mathfrak{so}(2d + 1, 1)$ . (Note that if one allows a change in the original diagram then there is smaller embedding in the AdS algebra  $\mathfrak{so}(2d, 1)$  [41]; a fact that has also been exploited in the higher spin literature.)

Returning to the KM case, one embeds the semi-direct product of  $\mathfrak{g}^{+++}$  with its vector representation in a larger algebra in a similar fashion. The resulting algebra is obtained by adding one more node to the Dynkin diagram of  $\mathfrak{g}^{+++}$  at the very-extended node and we denote this fourth extension of  $\mathfrak{g}$  by  $\mathfrak{g}^{++++}$ :

$$\mathfrak{g}^{+++} \times L(\lambda_1) \hookrightarrow \mathfrak{g}^{++++}. \quad (4.7)$$

Note that the embedding is only valid at the level of vector spaces; the Lie algebra relations in the two spaces are different, since not all the central charges will commute any longer in  $\mathfrak{g}^{+++}$ . (However, the momenta  $P_a$  will commute.) The Dynkin diagram of  $\mathfrak{g}^{++++}$ , for the very-extension  $E_8^{+++}$ , is the following:



By using this kind of embedding, it follows that one obtains all generalized central charges for the KM fields of the oxidized theories. This is a consequence of a Kaluza–Klein reduction of the oxidized fields: A field in  $D$  space-time dimensions gets lifted to one in  $D + 1$  dimensions by the diagram extension and subsequent reduction then results in the field in  $D$  dimensions and its corresponding charge. We remark that in this construction of space-time there are infinitely many space-time generators, rendering space-time infinite-dimensional. (The idea of associating space-time directions to central charges goes back to [43].)

## 5 Conclusions

We have demonstrated that indefinite Kac–Moody algebras are a rich natural source for higher spin fields with mixed symmetry type. In all existing proposals, however, there are important open problems in deriving true and consistent higher spin field theories from this algebraic approach to M-theory. As we indicated, known results from higher spin theory might be used to solve some of these problems. Further progress might be derived from an understanding of the KM structure to all levels which appears to be a very hard problem.

All the models we discussed so far are for bosonic fields, an extension to fermionic degrees of freedom relies on an understanding of the compact subalgebra  $K(\mathfrak{g})$ . The next step then would be to relate these to the fermionic (or supersymmetric) HS theories *à la* Fang–Fronsdal.

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<sup>16</sup>The diagram only shows this for the complex algebras but one can actually check it at the level of real forms as well.

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## A Coset Models

Let us recall some basic facts about non-linear sigma models associated with the coset model  $G/K(G)$ . We use the Cartan decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$ ,

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$$

where  $\mathfrak{k}$  is the Lie algebra corresponding to  $K(G)$  and is fixed by an involution  $\phi$  of  $\mathfrak{g}$ :

$$\begin{aligned}\mathfrak{k} &= \{x \in \mathfrak{g} \mid \phi(x) = x\}, \\ \mathfrak{p} &= \{x \in \mathfrak{g} \mid \phi(x) = -x\}.\end{aligned}$$

We will here take  $\phi$  to be the compact involution, i.e. a map from  $\mathfrak{g}$  to  $\mathfrak{g}$  such that

$$\phi(e_i) = -f_i ; \phi(f_i) = -e_i ; \phi(h_i) = -h_i$$

on the  $r$  Chevalley generators of  $\mathfrak{g}$ . Let us consider the coset element  $v$  in Iwasawa parametrization

$$v(x) = \exp(\vec{\phi}(x) \cdot \vec{h}) \exp\left(\sum_{\alpha \in \Delta_+} A_\alpha(x) E^\alpha\right) \in G/K(G)$$

where the  $\vec{h}$  are the generators of the Cartan subalgebra and the  $E^\alpha$ 's are the step operators associated with the positive roots. The coordinates on  $G/K(G)$  namely  $\vec{\phi}$  and  $A_\alpha$  depend on the space-time coordinates.  $dvv^{-1}$  is in  $\mathfrak{g}$ , and hence we can decompose it as  $dvv^{-1} = \mathcal{P} + \mathcal{Q} = (\mathcal{P}_\mu + \mathcal{Q}_\mu)dx^\mu$ . A natural lagrangian associated to this model is

$$L = \frac{1}{4n} (\mathcal{P}_\mu | \mathcal{P}^\mu)$$

where the inner product  $(\cdot | \cdot)$  is the invariant metric on  $\mathfrak{g}$ . Such a lagrangian is invariant under local  $K(G)$  transformations and global  $G$  transformations,

$$v(x) \rightarrow k(x)v(x)g$$

where  $k(x)$  is in  $K(G)$  and  $g$  is in  $G$ . The lagrangian equations of motion are

$$D^\mu(n^{-1}\mathcal{P}_\mu) = (\partial^\mu - \text{ad } \mathcal{Q}^\mu)(n^{-1}\mathcal{P}_\mu) = 0 \tag{A.1}$$

Furthermore, the Lagrange multiplier  $n$  constrains the motion to be null, which is covariantly constant along the coset by (A.1). An introduction to scalar coset models can be found in [44].

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