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Do Rotations Beyond the Cosmological Horizon Affect the Local Inertial Frame?

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If perturbations beyond the horizon have the velocities prescribed everywhere then the dragging of inertial frames near the origin is suppressed by an exponential factor. However if perturbations are prescribed in terms of their angular momenta there is no such suppression.

We resolve this paradox and in doing so give new explicit results on the dragging of inertial frames in closed, flat and open universe with and without a cosmological constant.

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I. INTRODUCTION

In a clearly written paper Schmid [1] describes how rotational perturbations of a spatially flat universe influence the inertial frames. He finds that such influences are attenuated by an exponential Yukawa factor whenever the perturbation lies beyond a 'horizon'. He expressed his results in terms of a quantity that he calls the energy current $\vec{J}_\epsilon$. What corresponds to Schmid's energy current $\vec{J}_\epsilon$ in our calculation is a quantity $\vec{J}_s$ with the dimensions of angular momentum. However his result appears to disagree with our earlier result [2] (hereafter LKB) that in a flat universe the rotation of the inertial frame, ω, due to any system of spheres with small rotations about a center is given by

$$\vec{\omega}(\overline{r},t) = \frac{2G}{c^2}\left[\frac{1}{\overline{r}^3}\vec{J}(t,<\overline{r}) + \int_{\overline{r}}^{\infty}\frac{1}{\overline{r}'^3}\frac{\mathrm{d}\vec{J}}{\mathrm{d}\overline{r}'}\,\mathrm{d}\overline{r}'\right], \quad (1.1)$$

where $\vec{J}(t,<\overline{r})$ is the total angular momentum within the sphere of proper radius [3] $\overline{r} = a(t)r$. This expression demonstrates how angular momenta at all distances contribute and shows no exponential cut-off and no influence of any horizon.

Both results agree, however, that inertial 'influences' may be expressed instantaneously i.e. with no light travel-time delay. This is because they follow from the constraint equations of General Relativity with an appropriate mapping onto an unperturbed universe to provide a suitable gauge.

The results are in *apparent* contradiction. However, when the details of both calculations are examined it is evident that the contradiction lies in the attribution of different causes for the effect. Schmid's 'energy current' is considered by him as the source of the rotational dragging of inertial frames ω. Schmid's quantity does not obey a conservation law but for a rotating sphere it can be directly expressed in terms of its angular velocity Ω, density ρ, pressure p and proper radius $\overline{r}$ as

$$J_\epsilon = 2\pi\iint(\rho+p)\Omega\sin^3\theta\,\overline{r}^4\,\mathrm{d}\theta\,\mathrm{d}\overline{r}, \quad (1.2)$$

Ω is the 'coordinate' angular velocity, not that measured relative to the local inertial frame which, as we show below, is the quantity needed in the angular momentum conservation law. The contribution to the conserved angular momentum is

$$2\pi(\rho+p)(\Omega-\omega)\sin^3\theta\,\overline{r}^4\,\mathrm{d}\theta\,\mathrm{d}\overline{r}. \quad (1.3)$$

The difference comes about mathematically because the perturbed metric is not diagonal but in spherical polar coordinates is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\Big\{\mathrm{d}r^2 + r^2\Big[\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi^2 - 2\omega\,\mathrm{d}\varphi\,\mathrm{d}t)\Big]\Big\}, \quad (1.4)$$

where $\omega(t,r)$ gives the small rotation of the inertial frame due to the rotational perturbations everywhere.

For a perturbed 3-flat universe, angular momentum conservation is given by the equation

$$\frac{\partial}{\partial x^\mu}\left(-\sqrt{-g}\,T^\mu_\nu\eta^\nu\right) = 0, \quad (1.5)$$

where η^ν is the angular Killing vector of the background (flat) space corresponding to the particular component of angular momentum considered.

Thus the conserved quantity is (the minus sign comes from the signature we use in the metric)

$$J = \int -T^0_\varphi\sqrt{-g}\,\mathrm{d}^3x. \quad (1.6)$$

Since the φ component of $T^{\mu\nu}$ is brought down in this expression, it is *not* merely the motion of the fluid that is involved in T^0_φ via its contribution $u^\mu u^\nu$ but also the off-diagonal metric component $g_{0\varphi}$ which depends on ω at the position of the source [see our metric (1.4)]. As we see from equation (1.1), we regard the conserved angular momentum J as the source of the dragging of inertial frames, and this was the quantity we used in LKB.

Schmid's work for a spatially flat universe is more general than our work published so far, since he considers *all* vector perturbations, nevertheless we treated closed and open universes as well as flat ones and indeed from a Machian viewpoint it is the closed universes that are more interesting by far. We also considered all spherical but inhomogeneous Lemaître-Tolman-Bondi universes with rotational perturbations that were constant on spheres. Finally we looked into the problem of the rotation of inertial frames induced by spheres of given angular velocities, rather than given angular momentum. This is a special case of Schmid's problem but generalized to closed and open universes. In our discussion we wrote down the equations governing $\omega(t, r)$ when $\Omega(t, r)$ was given and showed how they could be solved. We carried out the detailed solution only for the static closed Einstein universe (LKB Appendix A).

Schmid's beautiful result that the dragging is exponentially suppressed when a sphere of given angular velocity is outside his horizon, has stimulated us to work out all our solutions in detail for all FRW universes. Barring factors of $a(t)$ that Schmid seems to have omitted in error, we fully confirm his result for a flat universe. Thus we have the fascinating paradox that *while spheres of given angular velocity have their dragging exponentially suppressed if they are outside the horizon, nevertheless the dragging of spheres of given angular momentum suffers no such suppression!* How can this be!

In the prescribed angular momentum problem one may consider (for an open or 3-flat universe) having only one spherical shell of finite thickness having angular momentum. The gravity of this source will induce a rotation of inertial frames everywhere. The fluid at all other places will respond inertially and start to rotate so that $\Omega = \omega$ everywhere except on the original shell. Thus in the prescribed angular momentum problem we give one shell angular momentum, sit back and watch. We see how the inertial frames are affected everywhere else merely by watching the rotations of all other spheres.

The prescribed angular velocity problem needs more organization in the creation of the initial state. If we start one thick shell rotating at the prescribed rate then all the other will start moving so as to keep up with the induced rotation of the inertial frames. If the prescription is to have just the one thick shell rotating and none of the others we shall have to stop them. In doing so we have to give them negative angular momentum to keep Ω zero even though the inertial frame is rotating at ω. When in Schmid's problem the perturbation in angular velocity is considered as confined beyond his horizon he shows that the rotation of inertial frames is exponentially suppressed near the origin. The prime reason is that in order to keep the motion confined, the intervening spheres have to be given backward angular momentum to stop them from following their inertial tendency of rotating at the inertial rate ω. The influence of all the backward angular momentum (of non-rotating spheres!) rather effectively cancels most of the rotation of the inertial frames induced by the original shell. Hence the suppression of the effect is due to all the negative angular momentum that was supplied to keep the other spheres from rotating! The remaining suppression is due to the rotation of the inertial frame at the original shell itself: ω there is a fraction of Ω so that $\Omega - \omega$, on which the source depends, is less.

There is a long history of treating dragging effects within spheres starting with Einstein's treatment using an early version of his gravitational theories. Within General Relativity the early works of Thirring [4] and Lense and Thirring [5] were later generalized to deep potential wells by Brill and Cohen [6]. This raised questions as to whether the dragging would be perfect within a black-hole's horizon. We believe that the first paper to remark on the apparent instantaneity if inertial frames is the pioneering paper of Lindblom and Brill [7] on inertia in a sphere that falls through its horizon. More recently we explored observational effects seen within such a sphere [8] and gave an example of strong linear dragging in a rapidly accelerated charged sphere [9]. Strong cosmological perturbations in a weakly rotating sphere surrounding a void were treated by Klein [10], and in greater detail by Doležel, Bičák, and Deruelle [11] who also discussed how an observer within such a cosmological shell views the world outside.

We owe a debt to Schmid as his work stimulated us to work out the consequences of our solutions [2] in much greater detail and, without that, we would never have raised, let alone understood, the delightful paradox emphasized above. In particular we have now investigated thoroughly the problem when ω is to be solved for with the angular velocities given everywhere at one cosmic time. Previously we had concentrated on the problem with the angular momenta given. While both are important problems we strongly believe that it is the latter that is of dynamical importance in formulating Mach's Principle. It can nevertheless be argued that the apparent agreement between the angular positions of quasars at different epochs and the inertial frame defined by using the solar system as a giant gyroscope stimulates Machian ideas. While it is the angular momentum that is important for the physics it is the apparent kinematical agreement between the *angular* velocity of the sphere and the inertial frame that is observed. In this sense the problem with given angular velocities may be closer to Mach's original and it is unclear how distant observations could measure the true angular momentum of a sphere including its dragging term, while its angular velocity is more directly observable. However, see [11] for the complications of light bending. Unfortunately the problem of the observed agreement of frames is not that either Schmid or we have addressed since both our treatments relate instantaneous quantities at the same cosmic time whereas observers use no such world map (except in the solar system) but a world picture in which distant objects are seen as they were long ago on the backward light cone.

It seems unlikely that an *exact* causal relationship exists between proper motions of masses on our past light cone and our local inertial frame, since, *at any cosmic time the inertial frame's rotation has contributions from objects that were never in our past light cone.* Of course such objects will no doubt have been seen by some alien and the Copernican principle would suggest that the apparent agreement of the kinematic and inertial frames here will be repeated there. What is under discussion above is the influence of distant bodies on the local inertial frame. This is quite distinct from a comparison of the dynamics of the solar system with its kinematics relative to distant quasars (as seen on a hundred years of past light cones), from which the rotation of the inertial frame is computed.

Beside the resolution of the apparent contradiction with Schmid the main contributions of this paper are the following.

Section 2. The derivation of the equations governing general perturbations and a brief introduction to Machian gauge conditions which allow the separation of the (h_{0k}) vector perturbations equations from the others. The discussion of the equations of motion that must be obeyed if the contracted Bianchi identities are to be satisfied. As a consequence when axial symmetry is imposed each ring of fluid preserves its angular momentum. This section concludes with basic equations for odd parity axially symmetrical perturbations from which the remainder of the paper is derived.

Section 3 derives the explicit expressions for rotation of inertial frames in terms of the angular momentum distribution at any one time. This is done for all FRW universes with $k = \pm 1$ or 0 but the simplest case is solved in this section with Ω constant on spheres at the time considered. This corresponds to odd-parity vector $l = 1$ perturbations with Ω independent of θ. In the following paper [12] (Paper II) we allow for general θ dependence. With the integrals evaluated at fixed cosmic time and with the constants c and G restored we have the following results for $\vec{\omega}(r)$ at fixed time (for the derivation of the vector forms below see [2]):

For $k = 0$, $\overline{r} = a(t)r$,

$$\vec{\omega} = \frac{2G}{c^2a^3}\left[\vec{J}(<r)r^{-3} + \int_{\overline{r}}^{\infty}\frac{\mathrm{d}\vec{J}}{\mathrm{d}r}r^{-3}\,\mathrm{d}r\right]. \qquad (1.7)$$

Notice that $\vec{\omega} \propto [a(t)]^{-3}$ since $\vec{J}$ is conserved.

For $k = 1$,

$$\vec{\omega} = \frac{2G}{c^2a^3}\left[\vec{J}(<\chi)W(\chi) + \int_{\chi}^{\pi}\frac{\mathrm{d}\vec{J}}{\mathrm{d}\chi'}W(\chi')\,\mathrm{d}\chi'\right] + \vec{\omega}_0(t), \qquad (1.8)$$

here $\vec{\omega}_0(t)$ is undetermined, $r = \sin\chi$ and $W(\chi) = \cot^3\chi + 3\cot\chi$. The arbitrariness of $\vec{\omega}_0(t)$ is intimately connected with Mach's principle. The physical $\vec{J}$ involves $(\vec{\Omega} - \vec{\omega})$ and does not change for rotating axes as it involves a difference, see [2] and below.

For $k = -1$,

$$\vec{\omega} = \frac{2G}{c^2a^3}\left[\vec{J}(<\chi)\overline{W}(\chi) + \int_{\chi}^{\infty}\frac{\mathrm{d}\vec{J}}{\mathrm{d}\chi'}\overline{W}(\chi')\,\mathrm{d}\chi'\right], \qquad (1.9)$$

where $\overline{W}(\chi) = \coth^3\chi - 3\coth\chi + 2$, and $\overline{W}$ has an extra 2 so it tends to zero at $\chi \to \infty$ thus ensuring that the boundary condition $\omega \to 0$ is obeyed. When contributions from a θ dependence of Ω are included these results are supplemented by θ dependent terms that average to zero on spheres. More general results are given in the accompanying Paper II.

Section 4 gives explicit solutions for the rotations of inertial frames for the same special forms of perturbations as in Section 3 but now it is the angular velocities of the different spheres that are given rather than their angular momenta (this is *closer* to what might be observed but cf. earlier discussion). We define λ by

$$\lambda^2 = 2\kappa a^2(\rho + p) = 4(k - a^2\dot{H}), \qquad (1.10)$$

$\kappa = 8\pi G/c^4$, $\kappa = 8\pi$ in geometrical units used in the following, the dot denotes $\partial/\partial t$, $H = \dot{a}/a$ is the Hubble constant. The second relation in (1.10) follows from the combination of the background Einstein's equations for any ρ, p, k and also for any value of the cosmological constant Λ. The rotation of inertial frames near the origin due to an Ω distribution at large $z' = \lambda r$ is for $k = 0$

$$\omega(r) = \frac{1}{3}\left(1 + \frac{1}{10}\lambda^2r^2\right)\int_0^{\infty} z'^2e^{-z'}\Omega(z')\,\mathrm{d}z', \qquad (1.11)$$

which shows Schmid's exponential attenuation $e^{-z'}$. At the perturbation itself, close to z_0, we find for z' large:

$$\omega(r_0) = \frac{1}{2}\int_0^{\infty}\left(\frac{z'}{z_0}\right)^2 e^{-|z'-z_0|}\Omega(z')\,\mathrm{d}z'. \qquad (1.12)$$

For $k = 1$ we give the results near the origin and at the perturbation when $\lambda^2 > 4$. When $\lambda^2 < 4$, which can occur when a Λ-term is present, there is no exponential in the expression. It is assumed that $\exp(\sqrt{\lambda^2 - 4}\,\chi)$ is large at the source. With $r = \sin\chi$

$$\omega(\chi) = \frac{1}{3}\left(1 + \frac{\lambda^2\chi^2}{10}\right)\times \qquad (1.13)$$
$$\int_0^{\infty}\lambda^2\sqrt{\lambda^2 - 4}\;e^{-\sqrt{\lambda^2-4}\chi'}\sin^2(\chi')\Omega(\chi')\,\mathrm{d}\chi'.$$

We have assumed $\exp(\sqrt{\lambda^2 - 4}\,x) \gg 1$ for $x = \pi$, χ', and $\pi - \chi'$. At the 'source'

$$\omega(\chi_0) = \qquad (1.14)$$
$$\frac{1}{2}\int_0^{\infty}\lambda^2\frac{\lambda^2 - 3}{\sqrt{\lambda^2 - 4}}\left(\frac{\sin\chi'}{\sin\chi_0}\right)^2 e^{-\sqrt{\lambda^2-4}|\chi'-\chi_0|}\Omega(\chi')\,\mathrm{d}\chi'.$$

Similarly for $k = -1$: $r = \sinh\chi$,

$$\omega(\chi) = \frac{1}{3}\left(1 + \frac{\lambda^2\chi^2}{10}\right)\times \qquad (1.15)$$
$$\int_0^{\infty}\lambda^2\sqrt{\lambda^2 + 4}\;e^{-\sqrt{\lambda^2+4}\chi'}\sinh^2(\chi')\Omega(\chi')\,\mathrm{d}\chi',$$

and at the source

$$\omega(\chi_0) = \qquad (1.16)$$
$$\frac{1}{2}\int_0^\infty \frac{\lambda}{\sqrt{\lambda^2+4}}\left(\frac{\sinh\chi'}{\sinh\chi_0}\right)^2 e^{-\sqrt{\lambda^2+4}|\chi'-\chi_0|}\Omega(\chi')\,\mathrm{d}\chi'.$$

We emphasize that all of the above relationships are true at any given instant, but that both the angular momentum distribution and the angular velocity distribution at later instants are related to those at earlier times, so can not be given *independently* of those given at an earlier epoch. In axial symmetry the angular momentum distribution follows the motion of the perfect fluid but, as the angular momentum is first order and the movement across the background is of first order, the product can be neglected. Thus to first order the angular momentum density can be considered as painted on the background. This is not true of $\vec{J}_s$ which is not conserved and nor is it true of the angular velocity Ω. In both cases to find the time evolution one must appeal to the equations of motion which, in axial symmetry, leads back to local conservation of angular momentum density. Only by use of its conservation can one find how Ω and $\vec{J}_s$ can evolve consistently with Einstein equations (i.e. with the contracted Bianchi identities). In this sense the given angular momentum problem is far more physical than either Schmid's problem or the given Ω problem to which it is equivalent. The time evolution of ω and Ω are derived and discussed in Section 5.

In a paper that has long been in gestation we give a discussion of those gauges in which the Machian relations of the local inertial frames to the motions of distant masses can be expressed instantaneously at constant cosmic time. In that paper we derive all equations that govern all perturbations. All can be solved using harmonics in the 3-space of constant time. However harmonics are not as informative as Green's functions so in the following paper [12] we integrate the relationships between the rotations of the inertial frames and the angular momentum density for all axially symmetrical odd-parity vector perturbations, called usually "toroidal" perturbations in astrophysical and geophysical literature. These results allow $\Omega - \omega$, which enter the angular momentum density, to be any function of (r,θ) but independent of φ. However, since the background is spherically symmetric, non-axisymmetric perturbations can be generated by re-expanding axisymmetric perturbations around a new axis, and taking the component with the new $e^{im\varphi}$ as the component with general m.

II. THE EQUATIONS TO BE SOLVED

We write the perturbed FRW metric in the form

$$\begin{aligned}\mathrm{d}s^2 &= (\overline{g}_{\mu\nu} + h_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu \qquad (2.1)\\ &= \mathrm{d}t^2 - a^2(t) f_{ij}\mathrm{d}x^i\mathrm{d}x^j + h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu,\end{aligned}$$

where the background metric $\overline{g}_{\mu\nu}$ is used to move indices and the time-independent part of the spatial background metric f_{ij} $[i,j,k=1,2,3]$ is used to define the 3-covariant derivative ∇_k and $\nabla^k = f^{kl}\nabla_l$.

In one of the standard coordinate systems the background FRW metric reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right], \quad (2.2)$$

where in positive curvature (closed) universe ($k=+1$) $r \in \langle 0,1\rangle$, in flat ($k=0$) and negative curvature ($k=-1$) open universes $r\in\langle 0,\infty\rangle$, and $\theta\in(0,\pi)$, $\varphi\in(0,2\pi)$. We shall also employ hyperspherical coordinates

$$ds^2 = \mathrm{d}t^2 - a^2\left[\mathrm{d}\chi^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)\right], \qquad (2.3)$$

with $r=\sin\chi$, χ, $\sinh\chi$ for $k=1,\,0,\,-1$.

In a completely general gauge for general perturbations $h_{\mu\nu}$, the (momentum) constraint equation, $\delta G^0_k = \kappa\delta T^0_k$, turns out to be

$$\begin{aligned}\frac{1}{2}\nabla^2 h_{0k} \;+\; kh_{0k} - \frac{1}{6}a^2\nabla_k\nabla_j h^j_0 + \frac{2}{3}a\nabla_k\mathcal{K} - \frac{1}{2}a^2\dot{\mathcal{T}}_k&\\ = a^2\kappa\delta T^0_k, \qquad (2.4)&\end{aligned}$$

where the dot denotes $\partial/\partial t$,

$$\mathcal{K} = a\left[\frac{3}{2}Hh_{00} - \frac{1}{2}(h^j_j)^{\cdot} + \nabla_j h^j_0\right] \qquad (2.5)$$

is the perturbed mean external curvature of $t=$ constant slices, $H=\dot{a}/a$ is the Hubble constant,

$$\mathcal{T}_k = -\nabla_j\left(h^j_k - \frac{1}{3}\delta^j_k h^i_i\right). \qquad (2.6)$$

Notice that equation (2.4) is independent of the choice of the cosmological constant Λ because we perturbed "mixed" components of G^0_k. Other perturbed Einstein's equations will not be needed here. Since, however, we are interested primarily in perfect fluid perturbations we shall also consider the perturbed fluid equations of motion, i.e. the perturbed Bianchi identities

$$\begin{aligned}&(\delta\rho)^{\cdot} + 3H(\delta\rho+\delta p) \;+ \qquad (2.7)\\ &(\rho+p)\nabla_k(h^k_0+V^k) \;+\; (\rho+p)\left(\frac{3}{2}Hh_{00} - \frac{1}{a}\mathcal{K}\right) = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{a^3}\left[a^3(\rho+p)(a^2 f_{km}V^m - h_{0k})\right]^{\cdot} \;+ \qquad (2.8)\\ \nabla_k\delta p + \frac{1}{2}(\rho+p)\nabla_k h_{00} \;=\; 0,\end{aligned}$$

where $V^k = \frac{\mathrm{d}x^k}{\mathrm{d}t} \simeq \delta U^k$ and $V_k = -a^2 f_{kj}V^j$ is the fluid (small) velocity. The perturbed fluid energy-momentum tensor components entering the constraint equations (2.4) read

$$\delta T^0_k = (\rho+p)(h_{0k}+V_k) = (\rho+p)(h_{0k} - a^2 f_{km}V^m). \quad (2.9)$$

There have been various choices of gauges used in the literature, in particular the synchronous gauge ($h_{00} = h_{0k} = 0$). In order to understand the effect of dragging of inertial frames, in particular its 'instantaneous' character, it is convenient to use gauges — we call them 'Machian' — in which the constraint equations, and still another (combination of) the perturbed field equations are explicitly the elliptic equations. In order to achieve this it is first useful to choose coordinates on $t =$ constant slices such that the *spatial* harmonic gauge conditions are satisfied, i.e. $\mathcal{T}_k = 0$, where $\mathcal{T}_k$ is given in (2.6) (in numerical relativity $\dot{\mathcal{T}}_k = 0$ is frequently called the 'minimal distortion' shift vector gauge condition). Next, it is convenient to choose the time slices so that, for example, the perturbation of their external curvature vanishes: $\mathcal{K} = 0$, $\mathcal{K}$ given by (2.5) (so called 'uniform Hubble expansion gauge'). Under these gauge conditions (which determine the coordinates in a substantially more restrictive way than e.g. the synchronous gauge) the constraint field equations (2.4) become the elliptic equations for just the components h_{0k}, no other $h_{\mu\nu}$ enter.

Until now we considered general perturbations in the chosen gauge. Hereafter, we assume the vectors h_{0k}, V^k to be transverse,

$$\nabla^k h_{0k} = 0, \qquad \nabla_k V^k = 0, \tag{2.10}$$

so that also $\nabla^k \delta T^0_k = 0$. If (2.10) is not satisfied, we can apply ∇^k to equation (2.4), find the elliptic equation for the scalar $\nabla^k h_{0k}$, solve it and substitute back into (2.4) where the third term on the left hand side could be viewed as the source together with δT^0_k. Since, however, the longitudinal parts do not contribute to the dragging of inertial frames, we assume equations (2.10) to be satisfied.

The constraint field equations (2.4) with our choice of gauge $\mathcal{K} = \mathcal{T}_k = 0$ [cf. equations (2.5) and (2.6)] thus become

$$\nabla^2 h_{0k} + 2k h_{0k} = 2a^2\kappa\delta T^0_k = 2a^2\kappa(\rho+p)(h_{0k} - a^2 f_{km} V^m), \tag{2.11}$$

where for the perfect fluid δT^0_k is substituted from equation (2.9). This is our basic equation to be solved at a given time $t =$ constant, with either δT^0_k or V^k given. The Bianchi identities (fluid equations of motion) determine the time evolution of perturbations, the scalar equation (2.8) for $\delta\rho$, whereas the vector equation (2.9) governs the evolution of the term

$$j_k \equiv a^3(\rho + p)(a^2 f_{km} V^m - h_{0k}) = -a^3 \delta T^0_k. \tag{2.12}$$

In the following we shall often express the background time dependent term $a^2(\rho + p)$ by using equation (1.10).

Consider first the flat universe ($k = 0$). In Cartesian coordinates x^k used by Schmid [1], the 3-metric $f_{kl} = \delta_{kl}$, and (2.11) becomes

$$\nabla^2 h_{0k} = 2a^2\kappa\delta T^0_k = 2a^2\kappa(\rho + p)(h_{0k} + V_k), \tag{2.13}$$

where ∇^2 is the flat-space Laplacian. Substituting from equation (1.10) with $k = 0$ in the first term in the r.h.s. of equation (2.13), we get

$$\nabla^2 h_{0k} = -4a^2 \dot{H} h_{0k} - 2\kappa a^4(\rho + p)V_k. \tag{2.14}$$

Now comparing our general form of the perturbed FRW metric with the perturbed metric (5) in Schmid's work (and bewaring of the opposite signature), we see that $h_{0k} = -a\beta_k$(Schmid). Considering $(\rho + p)V_k$ (denoted by $\vec{J}_\epsilon$ in Schmid) as the source, the equation (2.14) written for Schmid's β_k becomes

$$-\nabla^2\beta_k - 4a^2\dot{H}\beta_k = -2\kappa a^3(\rho + p)V_k. \tag{2.15}$$

This is Schmid's basic equation (14), up to the factors a^2 and a^3 which in Schmid's equation (14) are missing but this does not change significantly Schmid's conclusions.

When δT^0_k is given, the solution of equation (2.13) is given as the Poisson integral over the source. If, however, the matter current is given, equation (2.14) can be written as

$$\nabla^2 h_{0k} - \lambda^2(t) h_{0k} = -2\kappa a^4(\rho + p)V_k, \tag{2.16}$$

with ($k = 0$)

$$\lambda^2 = -4a^2\dot{H}. \tag{2.17}$$

Usually (e.g. in the standard Friedmann models) $\dot{H} < 0$, so λ is real. The three equations (2.16) are, as emphasized by Schmid, of the Yukawa-type. The Green's functions are given by

$$G(x, x') = -\frac{1}{4\pi}\frac{e^{\mp\lambda|x-x'|}}{|x - x'|}, \tag{2.18}$$

the well-behaved solution of equation (2.18) is thus

$$h_{0k} = -\frac{1}{2\pi}\kappa a^4(\rho + p)\int V_k(x')\frac{e^{-\lambda|x-x'|}}{|x - x'|}\,\mathrm{d}x'. \tag{2.19}$$

Clearly *if* the perturbation $V_k(x')$ is located at $|x - x'| \gtrsim \lambda^{-1} = 1/2a\sqrt{-\dot{H}}$, i.e. beyond the '$\dot{H}$ radius' $R_{\dot{H}} = 2(-\dot{H})^{-\frac{1}{2}}$ in Schmid's terminology, the vector h_{0k} which determines the dragging of inertial frames is exponentially suppressed around the origin. Although we thus verified the interesting conclusion of Schmid, we do not resonate with his view that " because of the exponential cut-off... there is no need to impose 'appropriate boundary conditions of some kind'....". The Green's function in (2.18) with the '+' sign in the exponential is also the solution of equation (2.16) with a δ-function source but one discards it by demanding 'reasonable' boundary conditions at infinity.

From the Machian viewpoint the closed universes are of course preferable. There is, however, no vector Green's function available for equation (2.11) with either δT^0_k or V_k considered as a source. In order to understand how Schmid's conclusions get modified in curved universes and to generalize our previous work [2] which analyzed perturbations corresponding to rigid rotating spheres in the FRW universes, we shall study all axisymmetric, odd-parity l-pole perturbations corresponding to differentially rotating 'spheres'. We now derive the basic equations for such "toroidal" perturbations. Their solutions, in particular for $l \geq 2$ and closed universes, require special treatment. These solutions are analyzed in the following Paper II.

In spherical coordinates [as in the FRW metrics (2.2), (2.2)], the only non-vanishing vector components are $h_{0\varphi}(t, r, \theta)$ and $V_\varphi(t, r, \theta)$. [For the general axisymmetric even-parity vector fields $V_\varphi = 0$, whereas $V_r(t, r, \theta)$ and $V_\theta(t, r, \theta)$ are non-vanishing, the same being true for $h_{0r}, h_{0\theta}$]. There is now just one non-trivial constraint equation in (2.11) to be satisfied:

$$\nabla^2 h_{0\varphi} + 2k h_{0\varphi} = 2a^2 \kappa \delta T^0_\varphi, \tag{2.20}$$

in which $\nabla^2 = f^{kl}\nabla_k\nabla_l$, with f^{kl} being the inverse to f_{kl} given by FRW metric (2.2) (recall – see (2.2) – that f_{kl} is positive definite, without factor a^2). Calculating $\nabla^2 h_{0\varphi}$ explicitly, we find equation (2.20) to take the form

$$\left[(1-kr^2)\frac{\partial^2}{\partial r^2} - kr\frac{\partial}{\partial r}\right] h_{0\varphi} + \tag{2.21}$$
$$+ \ \frac{1}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) h_{0\varphi} + 4k h_{0\varphi} = 2a^2\kappa\delta T^0_\varphi.$$

Before solving this constraint equation it is interesting to notice what the perturbed equations of motion (Bianchi identities) say for axisymmetric odd-parity perturbations. Equation (2.8) in our gauge choice (with $\mathcal{K} = 0$) and transverse character of h_{0k}, V_k is a simple evolution equation for $\delta\rho$. The vector equation (2.9) for indices $1, 2$ ($x^1 = r, x^2 = \theta$) turns into the well known relativistic equilibrium conditions for perfect fluids, $\nabla_k \delta p = -(\rho + p)\nabla_k(\frac{1}{2}h_{00})$ (see e.g. [13]). In the following the crucial role is played by equation (2.9) for index 3 ($x^3 = \varphi$). Since in axisymmetric case $\nabla_\varphi \delta p = 0, \nabla_\varphi h_{00} = 0$, it becomes

$$\left[a^3(\rho+p)\left(a^2 r^2 \sin^2\theta V^\varphi - h_{0\varphi}\right)\right]^{\cdot} = 0, \tag{2.22}$$

or

$$\left[a^3 \delta T^0_\varphi\right]^{\cdot} = 0. \tag{2.23}$$

This is the conservation of angular momentum of each element of each axially symmetrical ring of fluid. The total angular momentum in a spherical layer $\langle \chi_1, \chi_2 \rangle$ is given by

$$J(\chi_1, \chi_2) = -\int_{\chi_1}^{\chi_2} \mathrm{d}\chi \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \sqrt{-\overline{g}}\, \delta T^0_\mu \eta^\mu, \tag{2.24}$$

where $\eta^\mu = (0, 0, 0, 1)$ is the rotational Killing vector, the background metric determinant $\overline{g} = \overline{g}^{(3)} = -a^6 r^4 \sin^2\theta$, $r = \sin\chi, \chi, \sinh\chi$ for respectively $k = +1, 0, -1$ as in equation (2.3). Integrating over φ we have

$$\begin{aligned} J(\chi_1, \chi_2) &= -2\pi \int_{\chi_1}^{\chi_2} \mathrm{d}\chi \int_0^\pi \mathrm{d}\theta\ a^3 r^2 \sin\theta \delta T^0_\varphi \\ &= 2\pi \int_{\chi_1}^{\chi_2} \mathrm{d}\chi \int_0^\pi \mathrm{d}\theta\ j(\theta, \chi, t), \end{aligned} \tag{2.25}$$

where $j(\theta, \chi, t)$ is the (coordinate) angular momentum density. Hence, the Bianchi identity (2.23) can be written as

$$\left[j(\theta, \chi)\right]^{\cdot} = 0. \tag{2.26}$$

This is important for studying the time evolution of the h_{0k} and V_k perturbations.

Defining the fluid angular velocity

$$\Omega = V^\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}t}, \tag{2.27}$$

we get

$$V_\varphi = -a^2 f_{\varphi\varphi} V^\varphi = -a^2 r^2 \sin^2\theta\ \Omega(t, r, \theta). \tag{2.28}$$

Writing similarly

$$h_{0\varphi} = a^2 r^2 \sin^2\theta\ \omega(t, r, \theta), \tag{2.29}$$

the only non-vanishing component of δT^0_k becomes

$$\delta T^0_\varphi = (\rho + p) a^2 r^2 \sin^2\theta (\omega - \Omega). \tag{2.30}$$

The angular momentum density conservation law (2.23), resp. (2.26), turns then into the simple evolution equation

$$\left[a^5(\rho+p)(\omega - \Omega)\right]^{\cdot} = 0. \tag{2.31}$$

Let us now return back to the constraint equation (2.22). The second term on its left hand side suggests the decomposition into the vector spherical harmonics. It should be emphasized that, in contrast to standard practice in the cosmological perturbation theory where perturbations are decomposed into harmonics in all three spatial dimensions (see e.g. [14]), we decompose in the usual coordinates θ, φ on spheres only, and assume axial symmetry (spherical functions Y_{lm} having $m = 0$). Thus, we write ($Y_{l0,\theta} \equiv \partial_\theta Y_{l0}$)

$$h_{0\varphi} = a^2 r^2 \sum_{l=1}^\infty \omega_l(t, r) \sin\theta Y_{l0,\theta}, \tag{2.32}$$

$$V_\varphi = -a^2 r^2 \sum_{l=1}^\infty \Omega_l(t, r) \sin\theta Y_{l0,\theta}, \tag{2.33}$$

and

$$\begin{aligned} \delta T^0_\varphi &= a^2(\rho+p) r^2 \sum_{l=1}^\infty (\omega_l - \Omega_l) \sin\theta\ Y_{l0,\theta} \\ &= \sum_{l=1}^\infty [\delta T^0_\varphi(t, r)]_l \sin\theta Y_{l0,\theta}. \end{aligned} \tag{2.34}$$

Substituting these expansions into equation (2.22) and using the orthogonality of functions $\sin\theta\; Y_{l0,\theta}$ for different l's, we obtain the 'radial' equation for each l:

$$\left[(1-kr^2)\frac{\partial^2}{\partial r^2}-kr\frac{\partial}{\partial r}\right](r^2\omega_l)-l(l+1)\omega_l+4kr^2\omega_l$$
$$=2a^2\kappa(\rho+p)r^2(\omega_l-\Omega_l)=\lambda^2r^2(\omega_l-\Omega_l), \qquad (2.35)$$

where we used equation (1.10). It is easy to convert the last equation into the form

$$-\sqrt{1-kr^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left[\sqrt{1-kr^2}\frac{\partial}{\partial r}(r^2\omega_l)\right]+$$
$$\frac{l(l+1)}{r^2}\omega_l-4k\omega_l=\lambda^2(\Omega_l-\omega_l). \qquad (2.36)$$

For $l=1$ (and the background pressure $p=0$) this equation coincides exactly with equation (4.32) in LKB. In the language of the present paper, in LKB we analyzed dipole ($l=1$) axisymmetric odd-parity perturbations. With $l=1$, $Y_{10,\theta}=-\sqrt{3/4\pi}\sin\theta$, so that putting $\omega=-\sqrt{3/4\pi}\;\omega_{l=1}$, $\Omega=-\sqrt{3/4\pi}\;\Omega_{l=1}$, we recover

$$h_{0\varphi} \;=\; a^2r^2\sin^2\theta\;\omega(t,r), \quad V_\varphi=-a^2r^2\sin^2\theta\;\Omega(t,r),$$
$$\delta T^0_\varphi \;=\; a^2(\rho+p)r^2\sin^2\theta(\omega-\Omega), \qquad (2.37)$$

which corresponds to the *rigidly* rotating spheres in the FRW universes considered in Section 4.4 in LKB, and, for $\Omega(t,r)$ given, analyzed in detail in Section 4 in the following.

Consider first the case $k=0$. Equation (2.36) can be written with the angular momentum density $(\delta T^0_\varphi)_l$ as a source,

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial\omega_l}{\partial r}\right)-\frac{l(l+1)-2}{r^2}\omega_l=\lambda^2(\omega_l-\Omega_l)=\frac{2\kappa}{r^2}(\delta T^0_\varphi)_l. \qquad (2.38)$$

If the fluid angular velocity is taken as a source, the equation reads

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial\omega_l}{\partial r}\right)-\left[\lambda^2+\frac{l(l+1)-2}{r^2}\right]\omega_l=-\lambda^2\Omega_l, \qquad (2.39)$$

where $\lambda^2=-4a^2\dot{H}=2\kappa a^2(\rho+p)$ by using equation (1.10) with $k=0$.

In the case of spatially curved ($k\neq 0$) backgrounds it is advantageous to write $r^2=k(1-\mu^2)$, i.e. $\mu=\sqrt{1-kr^2}$ to obtain

$$\frac{1}{[k(1-\mu^2)]^{3/2}}\frac{\partial}{\partial\mu}\left\{[k(1-\mu^2)]^{5/2}\frac{\partial\omega_l}{\partial\mu}\right\}-\frac{l(l+1)-2}{k(1-\mu^2)}\omega_l$$
$$=\frac{2\kappa}{k(1-\mu^2)}(\delta T^0_\varphi)_l. \qquad (2.40)$$

The substitution

$$\omega_l=[k(1-\mu^2)]^{-3/4}\varpi_l \qquad (2.41)$$

turns equation (2.40) into the Legendre equation for ϖ_l with $(\delta T^0_\varphi)_l$ as the source:

$$\frac{\partial}{\partial\mu}\left[k(1-\mu^2)\frac{\partial\varpi_l}{\partial\mu}\right]+\left[k\frac{3}{2}(\frac{3}{2}+1)-\frac{(l+\frac{1}{2})^2}{k(1-\mu^2)}\right]\varpi_l$$
$$=\frac{2\kappa}{[k(1-\mu^2)]^{1/4}}(\delta T^0_\varphi)_l. \qquad (2.42)$$

Finally, considering the fluid angular velocity as the source, we can write the last equation again as the Legendre equation with a more complicated degree:

$$\frac{\partial}{\partial\mu}\left[k(1-\mu^2)\frac{\partial\varpi_l}{\partial\mu}\right]+\left[k\nu(\nu+1)-\frac{(l+\frac{1}{2})^2}{k(1-\mu^2)}\right]\varpi_l$$
$$=-K_l\equiv-\lambda^2\Omega_l[k(1-\mu^2)]^{3/4}, \qquad (2.43)$$

where

$$\left(\nu+\frac{1}{2}\right)^2=4-2k\kappa a^2(\rho+p)=4-k\lambda^2=4ka^2\dot{H}. \qquad (2.44)$$

The degree ν of the Legendre equation does not depend on l. For $l=1$, equation (2.43) goes over into equation 4.35 in LKB [15].

III. SOLUTIONS FOR ω WITH GIVEN ANGULAR MOMENTUM DISTRIBUTION

We shall start by making more explicit the solutions obtained in LKB which are the $l=1$ odd-parity vector solutions of the general problem. In such modes each sphere rotates with no shear but it expands (or contracts) with the background and as it does so its angular velocity changes (see Section 5).

$k=0$

The equation to be solved is (2.38) with $l=1$, this is 4.33 LKB

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial\omega}{\partial r}\right)=-\lambda^2(\Omega-\omega)=\frac{2\kappa}{r^2}\delta T^0_\varphi, \qquad (3.1)$$

multiplying up by r^4 this takes the form

$$\frac{\partial}{\partial r}\left(r^4\frac{\partial\omega}{\partial r}\right)=-\frac{6}{a^3}\frac{\mathrm{d}J(<r)}{\mathrm{d}r}, \qquad (3.2)$$

so

$$\frac{\partial\omega}{\partial r}=-\frac{6J}{a^3r^4}, \qquad (3.3)$$

the constant of integration is zero since $J(<r)$ is zero at $r=0$ where $\partial\omega/\partial r$ must vanish. Integrating again and insisting that $\omega\to 0$ at ∞ we find

$$\omega \;=\; a^{-3}\int_r^\infty\frac{6J}{r'^4}\,\mathrm{d}r'=2a^{-3}\left[\frac{J(<r)}{r^3}+\int_r^\infty\frac{\mathrm{d}J}{\mathrm{d}r'}r'^{-3}\,\mathrm{d}r'\right]$$
$$=\;\frac{2}{r^3}\int_0^r\!\!\int_0^\pi 2\pi r'^2\sin\theta(-\delta T^0_\varphi)\,\mathrm{d}\theta\,\mathrm{d}r'$$
$$+2\int_r^\infty\!\!\int_0^\pi 2\pi r'^{-1}\sin\theta(-\delta T^0_\varphi)\,\mathrm{d}\theta\,\mathrm{d}r', \qquad (3.4)$$

where we have used (2.25) to define $J(<r)$ in terms of δT^0_φ.

$\underline{k=1}$

The equation to be solved is (2.40) with $l=1$ which is 4.34 LKB [16]

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)^{5/2}\frac{\partial\omega}{\partial\mu}\right\} \;=\; 2\kappa(1-\mu^2)^{\frac{1}{2}}\delta T^0_\varphi = \frac{6}{a^3}\frac{\mathrm{d}J}{\mathrm{d}\mu}$$

so that

$$(1-\mu^2)^{\frac{1}{2}}\frac{\partial\omega}{\partial\mu} \;=\; \frac{6J}{a^3(1-\mu^2)^2}. \tag{3.5}$$

As before there is no integration constant for the same reason. We now write $\mu=\cos\chi$, then χ is the normal cosmic radial angle and

$$\frac{\partial\omega}{\partial\chi} = -\frac{6J}{a^3\sin^4\chi}. \tag{3.6}$$

Now

$$\int^\chi \frac{\mathrm{d}\chi}{\sin^4\chi} = -\frac{1}{3}\left(\cot^3\chi+3\cot\chi\right) = -\frac{1}{3}W(\chi). \tag{3.7}$$

Hence

$$\omega = 2a^{-3}\left[WJ(<\chi)+\int_\chi^\pi W\frac{\mathrm{d}J}{\mathrm{d}\chi'}\,\mathrm{d}\chi'\right]+\omega_0, \tag{3.8}$$

where

$$J=\int_0^\chi\int_0^\pi 2\pi a^3[r(\chi')]^2\sin^2\theta(-\delta T^0_\varphi)\,\mathrm{d}\theta\,\mathrm{d}\chi'. \tag{3.9}$$

Just as in the last case W diverges at $\chi=0$ like χ^{-3}, however, the angular momentum of spheres near the origin is sufficiently small to make the WJ tend to a constant as χ tends to zero. It is shown in LKB that the condition of convergence of the second integral at $\chi=\pi$ is that the total angular momentum of the universe is zero. If that condition is fulfilled and $\Omega-\omega$ is regular near $\chi=\pi$ then the integral converges. If the total angular momentum is not zero then the integral for ω diverges at $\chi=\pi$. Thus for ω to be finite at $\chi=\pi$ the total angular momentum must be zero in the closed universe. There is no way of fixing ω_0 because there is no standard of zero rotation, as there is for the infinite universes. Indeed, according to Mach a description of the world in rotating axes is just as good in principle as a description in non-rotating ones. Note that the source $\Omega-\omega$ does not change when the axes are rotating since Ω and ω acquire the same constant ω_0. An absolute rotation can arise only from spatial boundary conditions which do not occur for closed universes.

$\underline{k=-1}$

The equation to be solved is (2.40) with $l=1$. Multiplying through by $(\mu^2-1)^{3/2}$ we obtain

$$\frac{\partial}{\partial\mu}\left\{(\mu^2-1)^{5/2}\frac{\partial\omega}{\partial\mu}\right\} = -2\kappa\sqrt{\mu^2-1}\delta T^0_\varphi = -\frac{6}{a^3}\frac{\mathrm{d}J}{\mathrm{d}\mu}, \tag{3.10}$$

so on integration and division

$$(\mu^2-1)^{1/2}\frac{\partial\omega}{\partial\mu} = -\frac{6}{a^3}(\mu^2-1)J. \tag{3.11}$$

Writing $\mu=\cosh\chi$ to introduce the natural radial variable of hyperbolic space, this becomes

$$\frac{\partial\omega}{\partial\chi} = -\frac{6}{a^3}\frac{J}{\sinh^4\chi}. \tag{3.12}$$

Integrating again and insisting that $\omega\to 0$ at infinity we use the integral

$$\int^\chi\frac{\mathrm{d}\chi'}{\sinh^4\chi'} = -\frac{1}{3}\left(\coth^3\chi-3\coth\chi+2\right)\equiv -\frac{1}{3}\overline{W}(\chi), \tag{3.13}$$

and on integrating by parts we obtain

$$\omega = 2a^{-3}\left[\overline{W}J(<\chi)+\int_\chi^\infty\overline{W}(\chi')\frac{dJ}{\mathrm{d}\chi'}\mathrm{d}\chi'\right], \tag{3.14}$$

where J is the same as in (3.9) with $r=\sinh\chi$. We have chosen the above definition of $\overline{W}$ so that $\overline{W}\to 0$ at infinity; so no constant of integration is needed to incorporate the boundary condition that $\omega\to 0$.

IV. SOLUTIONS FOR ω WITH GIVEN Ω

The method of solution was outlined in LKB but here we work through all the details starting with the simplest case.

$\underline{k=0}$

The relevant equation to be solved is (2.39) with $l=1$, equation 4.33 in LKB, rewritten as

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial\omega}{\partial r}\right)-\lambda^2\omega = -\lambda^2\Omega. \tag{4.1}$$

Here $\lambda^2 = 2a^2\kappa(\rho+p)>0$. $\lambda^{-1}a$ has the units of a length and we shall call it, following Schmid [1], the distance to the horizon. In dimensionless comoving coordinates this corresponds to $r=\lambda^{-1}$. We write $z=\lambda r$ and $\partial\omega/\partial z=\omega'$. Then equation (4.1) reduces to

$$\omega''+\frac{4}{z}\omega-\omega=-\Omega. \tag{4.2}$$

The corresponding homogeneous equation is Bessel's equation for $z^{-3/2}J_{3/2}(iz)$, which has real solutions $\omega=\overline{\mathcal{I}}$ and $\omega=\overline{\mathcal{K}}$, where $\overline{\mathcal{I}}=z^{-3/2}I_{3/2}(z)$ and $\overline{\mathcal{K}}=z^{-3/2}K_{3/2}(z)$. For small z, $\overline{\mathcal{I}}\to\frac{1}{3}\sqrt{2/\pi}(1+z^2/10)$; $\overline{\mathcal{K}}\to\sqrt{\pi/2}\;z^{-3}$. For large z, $\overline{\mathcal{I}}\to(1/\sqrt{2\pi})z^{-2}e^z$; $\overline{\mathcal{K}}\to\sqrt{\pi/2}\;z^{-2}e^{-z}$.

We use the method of variation of parameters to solve the inhomogeneous equation with boundary conditions that ω tends to zero at infinity and to a constant at the origin. We thus obtain

$$\omega(z) = \overline{\mathcal{K}}(z)\int_0^z (z')^4\overline{\mathcal{I}}(z')\Omega(z')\,\mathrm{d}z' + \overline{\mathcal{I}}(z)\int_z^\infty (z')^4\overline{\mathcal{K}}(z')\Omega(z')\,\mathrm{d}z'. \quad (4.3)$$

For the solutions near the origin with sources that are not so close, we may neglect the first term and then for small z,

$$\omega(z) = \frac{1}{3}\sqrt{\frac{2}{\pi}}\left(1+\frac{z^2}{10}\right)\int_z^\infty (z')^4\overline{\mathcal{K}}(z')\Omega(z')\,\mathrm{d}z'. \quad (4.4)$$

When the source Ω is beyond the horizon $z = 1$, i.e. $z' \gg 1$,

$$\omega(z) = \frac{1}{3}\left(1+\frac{z^2}{10}\right)\int_z^\infty (z')^2 e^{-z'}\Omega(z')\,\mathrm{d}z'; \quad (4.5)$$

so for a source localized in $r_0(1\pm\Delta)$ with $\Delta \ll 1/\lambda$,

$$\omega(z) = \frac{1}{3}\left(1+\frac{\lambda^2 r^2}{10}\right)(\lambda r_0)^3 e^{-\lambda r_0}\overline{\Omega}2\Delta, \quad (4.6)$$

which clearly shows the exponential decline of influence remarked on by Schmid [1]. When Ω is concentrated near z_0, in $z_0 \pm \lambda\Delta$, then with $z_0 \gg 1$ and $\Omega = \overline{\Omega}$ we get

$$\omega(z_0) = \frac{1}{2}\int_0^\infty \left(\frac{z'}{z_0}\right)^2 e^{-|z'-z_0|}\Omega(z')\,\mathrm{d}z' \simeq \lambda\Delta\overline{\Omega}. \quad (4.7)$$

Thus *at* the source the inertial frame rotates at $\lambda\Delta\overline{\Omega}$ and $\overline{\Omega} - \omega = (1-\lambda\Delta)\overline{\Omega}$.

We now turn to the solutions for a closed universe.

$k = 1$

The relevant equation is Legendre's equation for $\overline{\varpi} = (1-\mu^2)^{3/4}\omega$ with an inhomogeneous term written below. This is LBK equation 4.35 and the same as the equation (2.43) of Section 2 of this paper specialized for $l = 1$:

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\overline{\varpi}}{\partial\mu}\right\} + \left\{\nu(\nu+1) - \frac{(\frac{3}{2})^2}{(1-\mu^2)}\right\}\overline{\varpi} = -K, \quad K \equiv \lambda^2\Omega(1-\mu^2)^{3/4}, \quad (4.8)$$

where $(\nu+\frac{1}{2})^2 = 4-\lambda^2$ as in (2.44). Since $k = +1$ the space is hyperspherical and the convention is to write $\mu = \cos\chi$ so that χ becomes the radial variable. The solutions of the homogeneous equation are the Legendre functions $P_\nu^{3/2}(\mu)$ and $Q_\nu^{3/2}(\mu)$ and a recurrence relation that generates $P_\nu^{\mu+1}$ from P_ν^μ and $P_{\nu-1}^\mu$. (Here the order μ of the Legendre function has nothing to do with the variable $\mu = \sqrt{1-kr^2}$.) Thus

$$P_\nu^{1/2}(\cos\chi) = \left(\frac{\pi}{2}\right)^{-\frac{1}{2}}(\sin\chi)^{-\frac{1}{2}}\cos\left[\left(\nu+\frac{1}{2}\right)\chi\right],$$
$$Q_\nu^{1/2}(\cos\chi) = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\sin\chi)^{-\frac{1}{2}}\sin\left[\left(\nu+\frac{1}{2}\right)\chi\right]. \quad (4.9)$$

To keep $P_\nu^{3/2}(\cos\chi)$ and $Q_\nu^{3/2}(\cos\chi)$ real, we use $(1-\mu^2)^{-\frac{1}{2}} = (\sin\chi)^{-1}$ in place of $(\mu^2-1)^{-\frac{1}{2}}$ in the recurrence relation 8.5.1 of Abramowitz and Stegun [17] (this merely multiplies the results by $-i$).

$$P_\nu^{3/2}(\cos\chi) = \frac{1}{\sin\chi}\left[\left(\nu-\frac{1}{2}\right)P_\nu^{\frac{1}{2}}\cos\chi + \left(\nu+\frac{1}{2}\right)P_{\nu-1}^{\frac{1}{2}}\right], \quad (4.10)$$

the same relation holds for the $Q_\nu^{3/2}$. It turns out to be convenient to write $n = \nu + \frac{1}{2}$. We note that (2.44) and (4.8) involve this quantity and that n can be real but is often imaginary. Thus

$$P_{n-\frac{1}{2}}^{3/2}(\cos\chi) = -\left(\frac{\pi}{2}\right)^{-\frac{1}{2}}\frac{1}{\sin^{3/2}\chi}\left[\cos\chi\cos(n\chi) - n\sin\chi\sin(n\chi)\right], \quad (4.11)$$

similarly writing n when it is real but $n = iN$ when it is imaginary:

$$Q_{n-\frac{1}{2}}^{3/2}(\cos\chi) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sin^{3/2}\chi}\left[\cos\chi\sin(n\chi) - n\sin\chi\cos(n\chi)\right], \quad (4.12)$$
$$Q_{iN-\frac{1}{2}}^{3/2}(\cos\chi) = i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sin^{3/2}\chi}\left[\cos\chi\sinh(N\chi) - N\sin\chi\cosh(N\chi)\right].$$

We shall be concerned to have functions which, after multiplication by another $(\sin\chi)^{-3/2}$, are nevertheless still finite at the origin $\chi = 0$. A little expansion around $\chi = 0$ shows that the P function diverges but Q function satisfies this stringent test. Our next job is to find a solution that satisfies this stringent convergence not at $\chi = 0$ but at the 'other' $r = 0$ at $\chi = \pi$. Since that is an alternative origin it is clear that $Q_{n-\frac{1}{2}}^{3/2}[\cos(\pi-\chi)]$ passes that test. A little work shows that it is indeed the linear combination $(2/\pi)\sin(n\pi)P_{n-\frac{1}{2}}^{3/2}(\chi) - \cos(n\pi)Q_{n-\frac{1}{2}}^{3/2}(\chi)$. Finally we notice that $n = 0$, which is needed in some of our solutions, gives $Q_{-1/2}^{3/2} \equiv 0$. This is not a solution at all! However $\lim_{n\to 0}[(1/n)Q_{n-1/2}^{3/2}]$ gives the finite limit

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sin^{3/2}\chi}\left[\chi\cos\chi - \sin\chi\right]. \quad (4.13)$$

We shall therefore use the functions

$$q_n = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sin^{3/2}\chi}S_n(\chi), \quad (4.14)$$
$$p_n = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sin^{3/2}\chi}S_n(\pi-\chi)$$

as our independent solutions of the Legendre equation. These functions have the added advantage that they remain real when $n = iN$:

$$S_n(\chi) = -\cos\chi\frac{\sin(n\chi)}{n} + \sin\chi\cos(n\chi), \quad (4.15)$$
$$S_{iN}(\chi) = -\cos\chi\frac{\sinh(N\chi)}{N} + \sin\chi\cosh(N\chi).$$

The Wronskian may be shown to be

$$p_n\frac{\mathrm{d}q_n}{\mathrm{d}\mu} - q_n\frac{\mathrm{d}p_n}{\mathrm{d}\mu} = \frac{\pi}{2}\frac{\sin(n\pi)}{n}\frac{n^2-1}{1-\mu^2} = \frac{\mathcal{W}}{1-\mu^2}. \quad (4.16)$$

Having formed solutions p and q each of which satisfy *one* of the boundary conditions we look for solutions of the inhomogeneous equation of the form

$$\overline{\varpi} = A(\mu)p + B(\mu)q. \quad (4.17)$$

We choose $A'p+B'q = 0$, and then the equation demands that

$$(1-\mu^2)[A'p' + B'q'] = -K, \quad (4.18)$$

where a dash denotes $\partial/\partial\mu$. Solving for A' and B' we have, using the Wronskian $\mathcal{W}/(1-\mu^2)$ defined earlier, $A' = Kq/\mathcal{W}$. Now p does not satisfy the boundary conditions at $\chi = 0$, so A must be zero there; hence

$$A = -\int_\mu^1 \frac{Kq}{\mathcal{W}}\,\mathrm{d}\mu = -\int_0^\chi \frac{Kq}{\mathcal{W}}\sin\chi\ \mathrm{d}\chi. \quad (4.19)$$

Similarly $B' = -Kp/\mathcal{W}$ and to satisfy the boundary conditions at $\mu = -1, \chi = \pi$,

$$B = -\int_{-1}^\mu \frac{Kp}{\mathcal{W}}\,\mathrm{d}\mu = -\int_\chi^\pi \frac{Kp}{\mathcal{W}}\sin\chi\,\mathrm{d}\chi. \quad (4.20)$$

Thus the solution by variation of the parameters is

$$\overline{\varpi} = -\left[p(\chi)\int_0^\chi \frac{Kq}{\mathcal{W}}\sin\chi'\,\mathrm{d}\chi' + q(\chi)\int_\chi^\pi \frac{Kp}{\mathcal{W}}\sin\chi'\,\mathrm{d}\chi'\right], \quad (4.21)$$

which gives our solution for $\omega(\chi) = (\sin\chi)^{-3/2}\overline{\varpi}$:

$$\omega(\chi) = -\frac{\pi/2}{\mathcal{W}\sin^3\chi}\left[S_n(\pi-\chi)\int_0^\chi \lambda^2\Omega S_n(\chi')\sin\chi'\,\mathrm{d}\chi' + S_n(\chi)\int_\chi^\pi \lambda^2\Omega S_n(\pi-\chi')\sin\chi'\,\mathrm{d}\chi'\right]. \quad (4.22)$$

For χ small,

$$S_n \to \frac{(1-n^2)}{3}\chi^3\left[1 - \frac{(1+n^2)\chi^2}{10}\right], \quad \text{i.e.}$$
$$\frac{1}{\sin^3\chi}S_n \to \frac{(1-n^2)}{3}\left[1 + \frac{(4-n^2)}{10}\chi^2\right], \quad (4.23)$$

and for $n = iN$,

$$\frac{1}{\sin^3\chi}S_{iN} \to \frac{(1+N^2)}{3}\left[1 + \frac{(4+N^2)}{10}\chi^2\right]. \quad (4.24)$$

We note that with $k = +1$, $4 + N^2 = \lambda^2$, and

$$\mathcal{W} = \frac{\pi}{2}(n^2-1)\frac{\sin(n\pi)}{n} = -\frac{\pi}{2}(1+N^2)\frac{\sinh(N\pi)}{N}. \quad (4.25)$$

For N large and χ small

$$\frac{S_{iN}}{\mathcal{W}\sin^3\chi} \to -\frac{4}{3\pi}Ne^{-N\pi}\left(1 + \frac{\lambda^2\chi^2}{10}\right). \quad (4.26)$$

For N large and χ not small nor near π,

$$S_{iN}(\chi) = \frac{1}{2}\sin\chi\, e^{N\chi}, \qquad S_{iN}(\pi-\chi) = \frac{1}{2}\sin\chi\, e^{N(\pi-\chi)}. \quad (4.27)$$

Hence our solution near the origin is

$$\omega(\chi) = \frac{1}{3}\left(1 + \frac{\lambda^2\chi^2}{10}\right)N\int_\chi^\pi \lambda^2\Omega(\chi')\sin^2\chi' e^{-N\chi'}\,\mathrm{d}\chi', \quad (4.28)$$

and near the perturbation

$$\omega(\chi_0) = \frac{1}{2}\int_0^\pi \frac{\lambda^2 N}{N^2+1}\left(\frac{\sin\chi'}{\sin\chi_0}\right)^2 e^{-N|\chi'-\chi_0|}\Omega(\chi')\,\mathrm{d}\chi', \quad (4.29)$$

where at the last line we consider a perturbation with a mean Ω of $\overline{\Omega}$ in $r_0 \pm \Delta$ with $N\Delta \ll 1$.

$\underline{k = -1}$

The equation to be solved is (2.43) with $k = -1$ and $l = 1$. Now we write $\mu = \cosh\chi$, $(\nu + \frac{1}{2})^2 = \lambda^2 + 4$. Space is now hyperbolic and μ runs from 1 to ∞. The relevant solutions of the homogeneous equation are

$$p = -\left(P_\nu^{3/2} + \frac{2}{\pi}iQ_\nu^{3/2}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{-\frac{1}{2}}\frac{1}{\sinh^{3/2}\chi}S_e(\chi),$$
$$q = iQ_\nu^{3/2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sinh^{3/2}\chi}E(\chi), \quad (4.30)$$

where $n = (\nu + \frac{1}{2})$,

$$E(\chi) = -(n-1)e^{-(n+1)\chi} + (n+1)e^{-(n-1)\chi},$$
$$S_e(\chi) = \frac{1}{2}\left[E(\chi) - E(-\chi)\right]. \quad (4.31)$$

The Wronskian

$$p\frac{\mathrm{d}q}{\mathrm{d}\mu} - q\frac{\mathrm{d}p}{\mathrm{d}\mu} = -\frac{(n^2-1)n}{\mu^2-1}. \quad (4.32)$$

The solution by variation of parameters is

$$\overline{\varpi} = -\frac{1}{n(n^2-1)}\left[p\int_\mu^\infty qK\,\mathrm{d}\mu + q\int_1^\mu pK\,\mathrm{d}\mu\right], \quad (4.33)$$

hence, changing the integrations from μ to χ and $\overline{\varpi}$ to ω, we have

$$\omega = \frac{(\sinh\chi)^{-3}}{4(n^2-1)n}\left[E(\chi)\int_0^\chi \lambda^2\Omega(\chi')S_e(\chi')\sinh\chi'\,\mathrm{d}\chi' + S_e(\chi)\int_\chi^\infty \lambda^2\Omega(\chi')E(\chi')\sinh\chi'\,\mathrm{d}\chi'\right]. \quad (4.34)$$

For small χ

$$E(\chi) = 2-(n^2-1)\chi^2 \times \left[1-\frac{2n}{3}\chi+\frac{3n^2+1}{12}\chi^2-\frac{n(n^2+1)}{15}\chi^3+...\right], \tag{4.35}$$

so

$$S_e(\chi) = \frac{2n}{3}(n^2-1)\chi^3\left[1-\frac{(n^2+1)}{15}\chi^2\right]. \tag{4.36}$$

At large χ

$$E(\chi) = (n+1)e^{-(n-1)\chi} = 2(n+1)e^{-n\chi}\sinh\chi,$$
$$S_e(\chi) = \frac{1}{2}(n-1)e^{(n+1)\chi} = (n-1)e^{n\chi}\sinh\chi. \tag{4.37}$$

Near the origin

$$\omega = \frac{1}{3}\left[1-\frac{(4-n^2)\chi^2}{10}\right]\times \int_\chi^\infty (n^2-4)(n+1)\sinh^2\chi' e^{-n\chi'}\Omega(\chi')\,\mathrm{d}\chi'. \tag{4.38}$$

At the perturbation

$$\omega(\chi_0) = \frac{1}{2}\frac{n^2-4}{n}\int_0^\infty\left(\frac{\sinh\chi'}{\sinh\chi_0}\right)^2 e^{-n|\chi'-\chi_0|}\Omega(\chi')\,\mathrm{d}\chi'. \tag{4.39}$$

V. THE TIME EVOLUTION OF THE DRAGGING

The evolution of ω and Ω as functions of cosmic time is governed by the equations of motion (contracted Bianchi identities) (2.9). For axisymmetric, odd-parity perturbations these become the angular momentum density conservation law, as discussed in equations (2.22)–(2.26) in Section 2. In terms of $\omega(t,r,\theta)$ and $\Omega(t,r,\theta)$ the conservation law simply becomes (2.31), i.e.

$$\left[a^5(\rho+p)(\omega-\Omega)\right]^{\cdot} = 0 \tag{5.1}$$

or, in terms of the angular momentum density, we get

$$\Omega-\omega = \frac{1}{a^5(\rho+p)}\cdot\frac{j(\chi,\theta)}{r^4\sin^3\theta}. \tag{5.2}$$

In this formula the first factor singles out the time dependence of $\Omega-\omega$. Notice that we have already obtained $\omega(t,r,\theta)$ as a function of the angular momentum within χ, $J(<\chi)$, in all three cases $k=+1,0,-1$ [see equations (3.4), (3.8), (3.14)]. We found ω to depend on the time as $1/a^3(t)$. Equation (5.2) then can be regarded as a solution $\Omega(t,r,\theta)$ implied by the equations of motion.

On the other hand, for $\Omega-\omega$ given at some time $t=t_0$ as a function of χ,θ, equation (5.2) determines the density $j(\chi,\theta)$ which in turn gives $J(<\chi)$ and $\omega(t,\chi,\theta)$ is then obtained from equations (3.4), (3.8), (3.14). Angular velocity of matter, $\Omega(t,\chi,\theta)$, is then given again by equation (5.2).

If we are interested in proper azimuthal velocities, we can write

$$V = ar\sin\theta\;\Omega, \qquad v = ar\sin\theta\;\omega, \tag{5.3}$$

and rewrite (5.2) as

$$V-v = \frac{1}{a^4(\rho+p)}\cdot\frac{j(\chi,\theta)}{r^3\sin^2\theta}. \tag{5.4}$$

Since $|\Omega r|, |\omega r| \ll 1$, we have also $|V|, |v| \ll 1$. In the case of the dust universes ($p=0$) the density obeys the conservation law $\rho a^3 = \text{constant} \equiv C$. Equation (5.4) then implies

$$V-v = \frac{j(\chi,\theta)}{Cr^3\sin^2\theta}\cdot\frac{1}{a}. \tag{5.5}$$

This is not valid near $t\sim 0$ when $a\to 0$ due to our approximation. For $a\to\infty$, $V-v\to 0$ — the dragging becomes perfect.

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