Variational description of multi-fluid hydrodynamics: Uncharged fluids.

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We present a formalism for Newtonian multi-fluid hydrodynamics derived from an *unconstrained* variational principle. This approach provides a natural way of obtaining the general equations of motion for a wide range of hydrodynamic systems containing an arbitrary number of interacting fluids and superfluids. In addition to spatial variations we use "time shifts" in the variational principle, which allows us to describe dissipative processes with entropy creation, such as chemical reactions, friction or the effects of external non-conservative forces. The resulting framework incorporates the generalization of the *entrainment* effect originally discussed in the case of the mixture of two superfluids by Andreev and Bashkin. In addition to the conservation of energy and momentum, we derive the generalized conservation laws of vorticity and helicity, and the special case of Ertel's theorem for the single perfect fluid.

We explicitly discuss the application of this framework to thermally conducting fluids, superfluids, and superfluid neutron star matter. The equations governing thermally conducting fluids are found to be more general than the standard description, as the effect of entrainment usually seems to be overlooked in this context. In the case of superfluid ^{4}He we recover the Landau–Khalatnikov equations of the two-fluid model via a translation to the "orthodox" framework of superfluidity, which is based on a rather awkward choice of variables. Our two-fluid model for superfluid neutron star matter allows for dissipation via mutual friction and also "transfusion" via β-reactions between the neutron fluid and the proton-electron fluid.

I. INTRODUCTION

The main purpose of this work is to develop a formalism that allows one to derive the equations of motion for a general class of multi-constituent systems of interacting charged and uncharged fluids, such as conducting and non-conducting fluids, multi-fluid plasmas, superfluids and superconductors. For the sake of clarity of presentation we restrict ourselves here to uncharged fluids, while the case of charged fluids and their coupling to the electromagnetic field will be treated in a subsequent paper [1].

Long after the completion of classical Hamiltonian particle mechanics, the quest of finding a variational (or "Hamlitonian") description of hydrodynamics has surprisingly been a long-standing problem, which started only a few decades ago to be fully understood. The reason for this can be traced to the nature of the hydrodynamic equations, which are most commonly expressed in their Eulerian form in terms of the *density* ρ and *velocity* $\boldsymbol{v}$, where the information about the underlying flowlines has been hidden. Fluid particle trajectories, i.e. flowlines, can still be recovered by integrating the velocity field, but they are not independent quantities of the Eulerian description. However, it turns out that the "true" fundamental field variables of Hamiltonian hydrodynamics are the flowlines, which determine ρ and $\boldsymbol{v}$ as derived quantities.

Consider as an example the Lagrangian density Λ describing a barotropic perfect fluid, which in analogy to classical mechanics one would postulate to be

$$\Lambda(\rho, \boldsymbol{v}) = \frac{1}{2}\rho \boldsymbol{v}^2 - \mathcal{E}(\rho)\,,$$

where $\mathcal{E}(\rho)$ represents the internal energy density of the fluid. We note that the internal energy defines the chemical potential $\widetilde{\mu}$ and the pressure P as

$$d\mathcal{E} = \widetilde{\mu}\, d\rho\,, \quad \text{and} \quad P + \mathcal{E} = \rho\, \widetilde{\mu}\,.$$

The corresponding action is defined in the usual way as $\mathcal{I} \equiv \int \Lambda\, dV\, dt$, and the variation $\delta\Lambda$ of the Lagrangian density is

$$\delta\Lambda = \rho \boldsymbol{v} \cdot \delta\boldsymbol{v} + (\boldsymbol{v}^2/2 - \widetilde{\mu})\,\delta\rho\,.$$

Requiring the action $\mathcal{I}$ to be stationary with respect to *free variations* $\delta\rho$ and $\delta\boldsymbol{v}$ is immediately seen to be useless, as this leads to the over-constrained equations of motion $\rho\boldsymbol{v} = 0$ and $\widetilde{\mu} = \boldsymbol{v}^2/2$. In fact, it has been shown [2] that an unconstrained variational principle with ρ and $\boldsymbol{v}$ as the fundamental variables cannot produce the Eulerian hydrodynamic equations. The reason for this is rather intuitive, as it is evident that free variations of density and velocity probe configurations with different masses (i.e. different numbers of particles), which is not an actual degree of freedom of the dynamics of the system. Therefore the variational principle has to be constrained or reformulated in some way in order to restrict the variations to the physically meaningful degrees of freedom.

The historic approach to this problem in Newtonian physics has been to supplement the Lagrangian with appropriate constraints using Lagrange multipliers. This method was pioneered by Zilsel [3] in the context of the two-fluid model for superfluid ^{4}He, who used the constraints of conserved particles (i.e. mass) and entropy.

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However, as pointed out by Lin [4], this is generally insufficient, as it results in equations of motion restricted to *irrotational flow* in the case of uniform entropy. Lin showed that one has to add a further constraint, namely the "conservation of identity" of fluid particles in order to obtain the most general hydrodynamic equations. We can label particles by their initial positions $\boldsymbol{a}$, and so we can write their flowlines as $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$. The famous "Lin constraint" is $\partial_t \boldsymbol{a} + \boldsymbol{v} \cdot \nabla \boldsymbol{a} = 0$, i.e. the identity or label of a particle is conserved under its transport. For reviews of this approach and its relation to the "Clebsch representation" we refer the reader to [5–7], and references therein.

Although this method produces the correct equations of motion, it does not seem very natural due to the rather ad hoc introduction of constraints, and the need for unphysical auxiliary fields (the Lagrange multipliers). It was pointed out by Herivel [8] that the *Lagrangian* as opposed to Eulerian formulation of hydrodynamics results in a much more natural variational description, and this approach was further developed and clarified by Seliger and Whitham [5]. Instead of using ρ and $\boldsymbol{v}$ as fundamental variables, hydrodynamics can also be understood as a field theory in terms of the *flowlines* $\boldsymbol{x}(\boldsymbol{a}, t)$, or equivalently $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$. It turns out that this formulation allows for a perfectly natural *unconstrained* variational principle. This seems rather intuitive considering that hydrodynamics is a smooth-averaged description of a many-particle system, which is described by a variational principle based on the particle trajectories, i.e. $\boldsymbol{x}_N$ and $\dot{\boldsymbol{x}}_N$.

We can express the velocity and density in terms of the flowlines as $\boldsymbol{v} = \partial_t \boldsymbol{x}(\boldsymbol{a}, t)$ and $\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{a}) / \det(\mathcal{J}^i{}_j)$, where $\mathcal{J}^i{}_j = \partial x^i / \partial a^j$ is the Jacobian matrix corresponding to the map $\boldsymbol{a} \mapsto \boldsymbol{x}(\boldsymbol{a}, t)$ between the physical space $\boldsymbol{x}$ and the "material space" $\boldsymbol{a}$. Any further comoving quantities like the entropy s are determined in terms of their initial value $s_0(\boldsymbol{a})$. Substituting these expressions into the Lagrangian Λ, one obtains an unconstrained variational principle for the field $\boldsymbol{x}(\boldsymbol{a}, t)$, which results in the correct equations of motion. It is interesting to note that this approach implicitly satisfies Lin's constraint, as we are varying the particle trajectories $\boldsymbol{x}(\boldsymbol{a}, t)$, along which $\boldsymbol{a}$ is a constant by construction. Also, we do not need to impose an a priori constraint on the conservation of mass, as it is automatically satisfied by these "convective" variations: shifting around flowlines obviously conserves the number of flowlines, and therefore the number of particles. One can actually *derive* the Lin constraint by transforming this Lagrangian framework back into a purely Eulerian variational principle [5, 6], which shows that these two approaches are formally equivalent.

As pointed out by Bretherton [9], one can even more conveniently use a "hybrid" approach, in which the Lagrangian is expressed in terms of the Eulerian hydrodynamic quantities $\boldsymbol{v}$, ρ, s etc, but one consider them as functions of the underlying flowlines. Their variations are therefore naturally *induced* by variations $\boldsymbol{\xi}$ of the flowlines $\boldsymbol{x}(\boldsymbol{a}, t)$. In general relativity the same idea was pioneered by Taub [10], and has subsequently been largely developed and extended by Carter [11–13], who also coined the term "convective variational principle" for this approach. Carter and Khalatnikov [14] have further demonstrated the formal equivalence of the convective approach and the more common Clebsh formulation that results from an Eulerian variational approach. A "translation" of the covariant convective formalism into a Newtonian framework (albeit using a spacetime-covariant language close to general relativity) is also available [15, 16]. The convective approach in relativity has independently been developed by Kijowski [17], and Hamiltonian formulations have been constructed by Comer and Langlois [18] and Brown [19]. Here we are using the convective (or "hybrid") variational principle in order to derive the Newtonian multi-fluid equations, and our notation and formalism follows most closely the framework developed by Carter.

We conclude our example of the simple barotropic fluid by using the convective variational principle to derive the Euler equation. The expressions for (Eulerian) variations of density and velocity *induced* by infinitesimal spatial displacements $\boldsymbol{\xi}$ of the flowlines are well known[1] (e.g. see [20]), namely

$$\delta\rho = -\nabla\left(\rho\boldsymbol{\xi}\right)\,, \quad \text{and} \quad \delta\boldsymbol{v} = \partial_t\,\boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}\,.$$

Inserting these expressions into the variation of the action $\delta\mathcal{I} = \int \delta\Lambda\, dV\, dt$ with $\delta\Lambda$ given above, and after some integrations by parts and dropping total divergences and time derivatives (which vanish due to the boundary conditions), we find

$$\begin{aligned} \delta\mathcal{I} = \quad & -\int \boldsymbol{\xi}\cdot\Big[\rho(\partial_t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \rho\nabla\widetilde{\mu} \\ & \qquad + \boldsymbol{v}\left\{\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\right\}\Big]\, dV\, dt\,. \end{aligned}$$

If we assume conservation of mass[2], i.e. $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0$, then stationarity of the action (i.e. $\delta\mathcal{I} = 0$) under free variations $\boldsymbol{\xi}$ directly leads to Euler's equation, namely

$$\left(\partial_t + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} + \frac{1}{\rho}\nabla P = 0\,,$$

where we have used the thermodynamic identity $\rho\nabla\widetilde{\mu} = \nabla P$. This shows that an unconstrained convective variational principle produces to the correct hydrodynamic equations of motion in a surprisingly simple and straightforward way.

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[1] A generalization of these expressions to include time-shifts is derived in Appendix A

[2] This will be seen to be a consequence of the variational principle rather than an a-priori assumption when time-shift variations are included.

The spatial variations $\boldsymbol{\xi}$ have three degrees of freedom, resulting in one vector equation, which represents the conservation of momentum. In order to complete the description we will need a fourth variational degree of freedom to produce the missing energy equation. This can be achieved by considering time-shifts, which are a natural part of the covariant relativistic approach, but which we have to be considered explicitly in the conventional "3+1" language of Newtonian space-time. These time-shifts variations allow us to take this formalism to its full generality, as we can now describe even dissipative processes with entropy creation, particle transformations (i.e. chemical reactions), resistive frictional forces etc. These dissipative systems are of course still *conservative* as long as one includes entropy, which is why they can be described by an action principle. The second law of thermodynamics, however, is obviously not contained in the action principle and has to be imposed as an additional equation on the level of the equations of motion.

We note that the equations we derive here do not explicitly include shear- and bulk-viscosity effects. However, the current *form* of the equations is in principle general enough to allow for both of these effects: bulk viscosity is caused by heat flow or chemical reactions due to thermal or chemical disequilibrium, both of which can already be described in the current formulation. Shear viscosity on the other hand has to be introduced as an "external" force, the problem therefore consists in prescribing a physically reasonable model for a multi-fluid generalization of the shear stresses. Including viscosity should therefore not be a matter of actually *extending* the current framework but rather of appropriately applying it in order to describe such processes. An explicit discussion of this is postponed to future work. Further work is also necessary in order to extend this formalism to include elasticity (as pioneered in the relativistic framework [21]), and especially to allow for an elastic medium interpenetrated by fluids as encountered in the inner neutron star crust, or any type of conducting solid. As shown in [22], a Kalb-Ramond type extension is required for the macroscopic treatment of quantized vortices in superfluids. With the present formalism we can describe superfluids either on the local irrotational level, or on the smooth-averaged macroscopic level by neglecting the (generally small) anisotropy induced by the quantized vortices.

The plan of this paper is as follows: in Sect. II we derive the general form of the equations of motion for multi-constituent systems using the convective variational principle. In Sect. III we show the conservation of energy and momentum implied by these equations. In Sect. IV we derive conserved quantities under transport by the flow, namely the vorticity and helicity. We then give the explicit functional form of the Lagrangian density for hydrodynamic systems in Sect. V, and in Sect. VI we discuss several applications of the foregoing formalism to particular physical systems.

II. VARIATIONAL DESCRIPTION OF MULTI-CONSTITUENT SYSTEMS

A. Kinematics

We want to describe systems consisting of several constituents distinguished by suitably chosen labels, and we use capital letters X, Y, ... as indices which run over these constituents labels. As the fundamental quantities of the kinematic description we choose the constituent densities n_X and the associated transport currents $\boldsymbol{n}_X$, which are related to the respective velocities $\boldsymbol{v}_X$ as

$$\boldsymbol{n}_X = n_X \boldsymbol{v}_X \,, \quad \text{where} \quad X \in \{\text{constituent labels}\}\,. \quad (1)$$

Not all constituents can necessarily move independently from each other, i.e. not all velocities $\boldsymbol{v}_X$ have to be different: viscosity and friction due to particle collisions on the microscopic level can effectively bind constituents together on very short timescales. We therefore distinguish between the notions of *constituents* X, characterizing classes of microscopic particles, and *fluids*, which are sets of constituents with a common velocity.

We note that in this framework entropy can be described very naturally as a constituent for which we reserve the label $X = \mathrm{s}$, and we write

$$n_{\mathrm{s}} = s \,, \quad (2)$$

where s is the entropy density. In this context it is instructive to think of the entropy as a gas of particle-like thermal excitations (e.g. phonons, rotons etc.), which makes its description as a constituent on the same footing with particle number densities quite intuitive.

B. Dynamics

The dynamics of the system is governed by an action $\mathcal{I}$ defined as

$$\mathcal{I} = \int \Lambda_{\mathrm{H}}\, dV\, dt \,, \quad (3)$$

in terms of the hydrodynamic Lagrangian Λ_{H}. The Lagrangian density Λ_{H} depends on the kinematic variables, which are the densities n_X and the currents $\boldsymbol{n}_X$, i.e. $\Lambda_{\mathrm{H}} = \Lambda_{\mathrm{H}}(n_X, \boldsymbol{n}_X)$. The total differential of Λ_{H} defines the *dynamical* quantities p_0^X ("energy") and $\boldsymbol{p}^X$ ("momentum") per fluid particle as the canonically conjugate variables to n_X and $\boldsymbol{n}_X$, namely

$$d\Lambda_{\mathrm{H}} = \sum \left(p_0^X dn_X + \boldsymbol{p}^X \cdot d\boldsymbol{n}_X\right), \quad \text{so} \quad p_0^X = \frac{\partial \Lambda_{\mathrm{H}}}{\partial n_X}, \quad \boldsymbol{p}^X = \frac{\partial \Lambda_{\mathrm{H}}}{\partial \boldsymbol{n}_X}, \quad (4)$$

where here and in the following the sum over repeated constituent indices is explicitly indicated by a Σ, i.e. no automatic summation convention applies to constituent indices.

C. The convective variational principle

As we have seen in the introduction, one cannot apply the standard variational principle to $\Lambda_{\rm H}$ in terms of the Eulerian hydrodynamics variables n_X and $\boldsymbol{n}_X$. From (4) it is obvious that allowing *free* variations of densities δn_X and currents $\delta\boldsymbol{n}_X$ would lead to the trivial equations of motion $p_0^X = 0$ and $\boldsymbol{p}^X = 0$. Instead, we consider the Lagrangian to be a functional of the underlying *flowlines* $\boldsymbol{x}^X = \boldsymbol{x}^X(\boldsymbol{a}^X, t)$, and therefore admit only variations δn_X, $\delta\boldsymbol{n}_X$ that are *induced* by infinitesimal displacements of the flowlines. These "convective" variations naturally conserve the number of particles (i.e. the number of flowlines) and no constraints are required in the variational principle as was discussed in more detail in the introduction.

We apply infinitesimal spatial displacements $\boldsymbol{\xi}_X$ and time-shifts τ_X to the flowlines of the constituent X. The resulting induced variations of density and current have been derived in Appendix A, namely the density variation (A18) for constituent X is

$$\delta n_X = -\nabla\cdot[n_X\boldsymbol{\xi}_X] + [\boldsymbol{n}_X\cdot\nabla\tau_X - \tau_X\partial_t n_X]\,, \qquad (5)$$

while the current variation $\delta\boldsymbol{n}_X$ is given by (A20) and reads as

$$\begin{aligned}\delta\boldsymbol{n}_X &= n_X\partial_t\boldsymbol{\xi}_X + (\boldsymbol{n}_X\cdot\nabla)\,\boldsymbol{\xi}_X - (\boldsymbol{\xi}_X\cdot\nabla)\boldsymbol{n}_X \\ &\quad -\boldsymbol{n}_X(\nabla\cdot\boldsymbol{\xi}_X) - \partial_t\,(\boldsymbol{n}_X\tau_X)\,. \qquad (6)\end{aligned}$$

Inserting these expressions into the variation of the Lagrangian (4) and integrating by parts, we can rewrite the induced variation $\delta\Lambda_{\rm H}$ in the form

$$\delta\Lambda_{\rm H} = \sum\left(g^X\,\tau_X - \boldsymbol{f}^X\cdot\boldsymbol{\xi}_X\right) + \partial_t R + \nabla\cdot\boldsymbol{R}\,. \qquad (7)$$

The time derivative and divergence terms will vanish in the action integration (3) by the appropriate boundary conditions (i.e. $\boldsymbol{\xi} = 0$ and $\tau = 0$) and are irrelevant as far as the variational principle is concerned, but for completeness we note that their explicit expressions are

$$\begin{aligned} R &\equiv \sum\left(n_X\boldsymbol{p}^X\cdot\boldsymbol{\xi}_X - \boldsymbol{n}_X\cdot\boldsymbol{p}^X\tau_X\right)\,, \qquad (8)\\ \boldsymbol{R} &\equiv \sum[\boldsymbol{n}_X\,(p_0^X\,\tau_X + \boldsymbol{p}^X\cdot\boldsymbol{\xi}_X) \\ &\quad -\boldsymbol{\xi}_X(n_X p_0^X + \boldsymbol{n}_X\cdot\boldsymbol{p}^X)]\,. \qquad (9)\end{aligned}$$

The induced action variation therefore has the form

$$\delta\mathcal{I} = \sum\int\left(g^X\,\tau_X - \boldsymbol{f}^X\cdot\boldsymbol{\xi}_X\right)\,dV\,dt\,, \qquad (10)$$

where the force densities $\boldsymbol{f}^X$ (acting *on* the constituent) and the energy transfer rates g^X (*into* the constituent) are found explicitly as

$$\begin{aligned}\boldsymbol{f}^X &= n_X\big(\partial_t\boldsymbol{p}^X - \nabla p_0^X\big) - \boldsymbol{n}_X\times(\nabla\times\boldsymbol{p}^X) + \boldsymbol{p}^X\Gamma_X\,, \qquad (11)\\ g^X &= \boldsymbol{v}_X\cdot\big(\boldsymbol{f}^X - \boldsymbol{p}^X\Gamma_X\big) - p_0^X\,\Gamma_X\,, \qquad (12)\end{aligned}$$

where Γ_X is the particle creation rate for the constituent X, i.e.

$$\Gamma_X \equiv \partial_t n_X + \nabla\cdot\boldsymbol{n}_X\,. \qquad (13)$$

The force density $\boldsymbol{f}^X$ is the total momentum change rate of the constituent X, and we see that the last term in (11), i.e. the "rocket term" $\boldsymbol{p}^X\Gamma_X$, represents a contribution that is purely due to the change of the particle number. Therefore it will be convenient to define the purely "hydrodynamic force" $\boldsymbol{f}_{\rm H}^X$, as

$$\boldsymbol{f}_{\rm H}^X \equiv n_X\left(\partial_t\boldsymbol{p}^X - \nabla p_0^X\right) - \boldsymbol{n}_X\times(\nabla\times\boldsymbol{p}^X)\,. \qquad (14)$$

With this definition we can now write the force density (11) and energy transfer rate (12) in the form

$$\begin{aligned}\boldsymbol{f}^X &= \boldsymbol{f}_{\rm H}^X + \boldsymbol{p}^X\,\Gamma_X\,, \qquad (15)\\ g^X &= \boldsymbol{v}_X\cdot\boldsymbol{f}_{\rm H}^X - p_0^X\,\Gamma_X\,. \qquad (16)\end{aligned}$$

D. The equations of motion

Up to this point we have developed only purely mathematical identities without a specific physical content. The equations of motion are obtained by imposing which type of invariance the action $\mathcal{I}$ should satisfy under certain infinitesimal variations. The most general equations are obtained by requiring that a *common* displacement $\boldsymbol{\xi}_X = \boldsymbol{\xi}$ and time shift $\tau_X = \tau$ of all constituents should result in an action variation of the form

$$\delta\mathcal{I} = \int\left(g_{\rm ext}\,\tau - \boldsymbol{f}_{\rm ext}\cdot\boldsymbol{\xi}\right)\,dV\,dt\,, \qquad (17)$$

where $\boldsymbol{f}_{\rm ext}$ and $g_{\rm ext}$ are interpretable as the external force density and energy transfer rate. This generalizes the more common action principle of *isolated* systems, in which the external influences $\boldsymbol{f}_{\rm ext}$ and $g_{\rm ext}$ vanish and therefore the equations of motion are obtained by requiring the action to be *invariant* under small variations. "External" here is meant in the sense of not being included in the total Lagrangian, which could also mean, for example viscous or gravitational forces. The resulting minimal equations of motion obtained from comparing with (10) are therefore found as

$$\sum\boldsymbol{f}^X = \boldsymbol{f}_{\rm ext}\,,\ \ \text{and}\ \ \sum g^X = g_{\rm ext}\,. \qquad (18)$$

Together with (11) and (12) this represents the Euler-Lagrange equations associated with this variational principle. If all constituents X form a single fluid, namely all constituents have a common velocity, then only common displacements of all constituents make sense in the variational principle. For this class of *non–conducting* models, (18) represent the full equations of motion obtainable from the variational principle. In order to complete the model, one has to specify the hydrodynamic Lagrangian $\Lambda_{\rm H}$, the external interactions $\boldsymbol{f}_{\rm ext}$ and $g_{\rm ext}$, and the creation rates Γ_X as functions of the kinematic variables.

In the case of *conducting* models, at least some of the constituents are allowed to move independently, the system therefore consists of more than one fluid. This increases correspondingly the number of degrees of freedom, and more equations of motion are required. They are obtained very naturally from the variational principle, as independent displacements (in space and time) are permitted for *each fluid*. Therefore the resulting force acting on each fluid can be prescribed by the model, subject to the restriction only of satisfying the minimal equations of motion (18).

As an example, consider the case of a simple conducting model consisting of two fluids, where we use X and Y are constituent indices running only over the respective constituent labels, i.e. $X \in \{\text{fluid 1}\}$ and $Y \in \{\text{fluid 2}\}$. We then have the respective force densities acting on each of the two fluids as $\boldsymbol{f}_{(1)} = \sum_X \boldsymbol{f}^X$ and $\boldsymbol{f}_{(2)} = \sum_Y \boldsymbol{f}^Y$, which by (18) have to satisfy $\boldsymbol{f}_{(1)} + \boldsymbol{f}_{(2)} = \boldsymbol{f}_{\text{ext}}$. Therefore there are now exactly two force densities (e.g. $\boldsymbol{f}_{(1)}$ and $\boldsymbol{f}_{\text{ext}}$) freely specifiable in the model, corresponding to the additional degrees of freedom of two fluids. In this case $\boldsymbol{f}_{(1)}$ could for example represent a mutual force the two fluids exert on each other, e.g. a resistive friction force.

III. CONSERVATION OF ENERGY AND MOMENTUM

Using the explicit expression (11) for the force density $\boldsymbol{f}^X$, we can write

$$\sum \boldsymbol{f}^X = \partial_t \left(\sum n_X \, \boldsymbol{p}^X \right) + \nabla_j \left(\sum n_X^j \, \boldsymbol{p}^X \right) - \sum \left(n_X \nabla p_0^X + n_X^j \nabla p_j^X \right) . \qquad (19)$$

We define the "generalized pressure" Ψ via the total Legendre transformation of Λ_{H}, namely

$$\Psi \equiv \Lambda_{\text{H}} - \sum \left(n_X p_0^X + \boldsymbol{n}_X \cdot \boldsymbol{p}^X \right) , \qquad (20)$$

which is seen from (4) to result in the total differential

$$d\Psi = - \sum \left(n_X dp_0^X + \boldsymbol{n}_X \cdot d\boldsymbol{p}^X \right) , \qquad (21)$$

and therefore the last sum in (19) is simply $\nabla \Psi$. We can now cast the force equation (18) in the form of a conservation law for the total momentum, namely

$$\partial_t J^i_{\text{H}} + \nabla_j T^{ij}_{\text{H}} = f^i_{\text{ext}} , \qquad (22)$$

where the hydrodynamic momentum density $\boldsymbol{J}_{\text{H}}$ and stress tensor T^{ij}_{H} are defined as

$$\boldsymbol{J}_{\text{H}} \equiv \sum n_X \boldsymbol{p}^X , \quad \text{and} \quad T^{ij}_{\text{H}} \equiv \sum n^i_X p^{X\,j} + \Psi \, g^{ij} , \qquad (23)$$

and where g_{ij} are the components of the metric tensor determining the relation between physical distance dl and coordinate intervals dx^i, i.e. $dl^2 = g_{ij}\, dx^i \, dx^j$. In Cartesian coordinates this is simply $g_{ij} = \delta_{ij}$. A proof of the symmetry of the stress tensor T^{ij}_H together with a more elegant derivation of momentum conservation as a Noether identity of the variational principle is given in Appendix B.

Using expressions (11) and (12), we can further show that

$$\begin{aligned} \sum g^X &= \sum \left[\boldsymbol{n}_X \cdot \partial_t \boldsymbol{p}^X - \boldsymbol{n}_X \cdot \nabla p_0^X - \Gamma_X p_0^X \right] \\ &= \left(\partial_t \sum \boldsymbol{n}_X \cdot \boldsymbol{p}^X \right) - \nabla \cdot \left(\sum \boldsymbol{n}_X p_0^X \right) \\ &\quad - \sum (p_0^X \, \partial_t n_X + \boldsymbol{p}^X \cdot \partial_t \boldsymbol{n}_X) , \end{aligned} \qquad (24)$$

and we see from (4) that the last sum simply represents $\partial_t \Lambda_{\text{H}}$. We can therefore rewrite the energy equation (18) in the form of a conservation law, namely

$$\partial_t E_{\text{H}} + \nabla \cdot \boldsymbol{Q}_{\text{H}} = g_{\text{ext}} , \qquad (25)$$

where the hydrodynamic energy density E_{H} and energy flux $\boldsymbol{Q}_{\text{H}}$ are given by

$$E_{\text{H}} = \sum \boldsymbol{n}_X \cdot \boldsymbol{p}^X - \Lambda_{\text{H}} , \quad \text{and} \quad \boldsymbol{Q}_{\text{H}} = \sum (-p_0^X) \boldsymbol{n}_X . \qquad (26)$$

We see that the energy density E_{H} has quite naturally the form of a hamiltonian, i.e. $\mathcal{H}_{\text{H}}(n_X, \boldsymbol{p}^X) = E_{\text{H}}$, as it is the Legendre-transformed (with respect to the momenta) of the Lagrangian Λ_{H}.

IV. CONSERVATION ALONG FLOWLINES

In addition to the total energy-momentum conservation, derived in the previous section, we can find further conserved quantities for individual constituents, for which conservation holds under transport by the fluid flow. Because the following derivations apply to individual constituents instead of the sum over all constituents, we will omit the constituent index X in this section in order to simplify the notation.

Transport of a quantity by the fluid flow is closely related to the Lie derivative with respect to the fluid velocity, therefore these conservation laws are most easily derived using the language and theorems of differential forms instead of vectors. We will use this formalism in deriving the transport-conservation laws, but we also give the essential steps and results translated in the more common vector- and index-notation, so that familiarity with exterior calculus should not be necessary (albeit helpful) for reading this section.

A. Kelvin-Helmholtz vorticity conservation

We define the vorticity 2-form $\underline{w}$ (with components w_{ij}) as the exterior derivative (denoted by d) of the momentum 1-form $\underline{p}$ (with components p_i), namely

$$\underline{w} \equiv d\underline{p} , \quad \text{i.e.} \quad w_{ij} \equiv 2\nabla_{[i} p_{j]} , \qquad (27)$$

where $[ij]$ denotes antisymmetric averaging, i.e. $2A_{[i}B_{j]} = A_iB_j - A_jB_i$. In three dimensions we can define the more common vorticity *vector* $\boldsymbol{W}$ as the *dual* (with respect to the volume form ϵ_{ijk}) of the 2-form $\underline{w}$, namely

$$W^i \equiv \frac{1}{2}\epsilon^{ijk} w_{jk} = (\nabla \times \boldsymbol{p})^i \,. \tag{28}$$

The volume form is defined as

$$\epsilon_{ijk} = \sqrt{g}\,[i,j,k]\,, \tag{29}$$

where $g = \det(g_{ij})$ and $[i,j,k]$ is the sign of the permutation of $\{1,2,3\}$, which is zero if two indices are equal. The duality between $\underline{w}$ and $\boldsymbol{W}$ implies

$$w_{ij} = \epsilon_{ijk} W^k\,, \tag{30}$$

which is easily verified by inserting (28). We note that due to the Poincaré property (namely $dd = 0$), the exterior derivative of the vorticity 2-form vanishes identically, i.e.

$$d\underline{w} = 0 \quad \Longleftrightarrow \nabla\cdot\boldsymbol{W} = 0\,. \tag{31}$$

We can rewrite the expression (14) for the hydrodynamic force $\boldsymbol{f}_{\rm H}$ in the language of forms as

$$\partial_t \underline{p} + \boldsymbol{v}\rfloor d\underline{p} - dp_0 = \frac{1}{n}\underline{f}_{\rm H}\,, \tag{32}$$

where $\rfloor$ indicates summation over adjacent vector- and form- indices, i.e. in this case $(\boldsymbol{v}\rfloor d\underline{p})_i = 2v^j\nabla_{[j}p_{i]}$. In the following it will be convenient to separate the force per particle into its non-conservative part $\underline{\mathfrak{F}}$ and a conservative contribution $d\phi$, namely

$$\frac{1}{n}\underline{f}_{\rm H} = d\phi + \underline{\mathfrak{F}}\,. \tag{33}$$

The Cartan formula for the Lie derivative of a p-form applied to the 1-form $\underline{p}$ yields

$$\pounds_{\boldsymbol{v}}\,\underline{p} = \boldsymbol{v}\rfloor d\underline{p} + d(\boldsymbol{v}\rfloor\underline{p})\,, \tag{34}$$

which in explicit index notation reads as $\pounds_{\boldsymbol{v}} p_i = 2v^j\nabla_{[j}p_{i]} + \nabla_i(v^jp_j)$. Using this identity and (33) we rewrite the force equation (32) more conveniently as

$$(\partial_t + \pounds_{\boldsymbol{v}})\,\underline{p} = dQ + \underline{\mathfrak{F}}\,, \tag{35}$$

where the scalar Q is given by $Q = p_0 + \boldsymbol{v}\rfloor\underline{p}$. Lie derivatives and partial time derivatives commute with exterior derivatives, so we can apply an exterior derivative to (35) and obtain the Helmholtz equation of vorticity transport, namely

$$(\partial_t + \pounds_{\boldsymbol{v}})\,\underline{w} = d\underline{\mathfrak{F}}\,, \tag{36}$$

which shows that the vorticity is conserved under transport by the fluid if and only if the hydrodynamic force per particle acting on the fluid is purely conservative, i.e. if $\underline{\mathfrak{F}} = 0$. In its more common dual form, this equation can be written as

$$\partial_t\boldsymbol{W} - \nabla\times(\boldsymbol{v}\times\boldsymbol{W}) = \nabla\times\boldsymbol{\mathfrak{F}}\,. \tag{37}$$

The Helmholtz vorticity conservation expresses the conservation of angular momentum of fluid particles, and we can equivalently derive it in its integrated form, namely the conservation of circulation as first shown by Kelvin. We consider a 2-surface Σ and define the circulation $\mathcal{C}$ around its boundary $\partial\Sigma$ as

$$\mathcal{C} \equiv \oint_{\partial\Sigma}\underline{p} = \oint_{\partial\Sigma} p_i\,dx^i\,. \tag{38}$$

Using Stoke's theorem, we see that the circulation around $\partial\Sigma$ is equivalent to the vorticity flux through the surface Σ, i.e.

$$\mathcal{C} = \int_\Sigma \underline{w} = \frac{1}{2}\int_\Sigma w_{ij}\,dx^i\wedge dx^j\,, \tag{39}$$

and the more familiar dual expression is found by inserting (30):

$$\mathcal{C} = \int_\Sigma \boldsymbol{W}\cdot d\boldsymbol{S}\,, \tag{40}$$

where the surface normal element $d\boldsymbol{S}$ is $dS_i \equiv \frac{1}{2}\epsilon_{ijk}\,dx^j\wedge dx^k$. Using (35) the comoving time derivative of the circulation $\mathcal{C}$ yields

$$\frac{d\mathcal{C}}{dt} \;=\; \frac{d}{dt}\oint_{\partial\Sigma}\underline{p} = \oint(\partial_t + \pounds_{\boldsymbol{v}})\,\underline{p} = \oint\underline{\mathfrak{F}}\,, \tag{41}$$

which is known as Kelvin's theorem of conservation of circulation. As we have already seen before, strict conservation only applies if the non-conservative force per particle $\underline{\mathfrak{F}}$ vanishes.

B. Vorticity and superfluids

The hydrodynamics of superfluids is characterized by two fundamental properties: on one hand by the absence of dissipative mechanisms like friction or viscosity, and on the other hand by irrotational flow. As we will see now, the hydrodynamic description of superfluids is therefore a natural subclass within the more general framework of multi-constituent hydrodynamics presented here. Let us assume that a constituent $X = \mathrm{S}$ is superfluid, with particle density $n_{\rm S}$, velocity $\boldsymbol{v}_{\rm S}$ and mass $m^{\rm S}$. The absence of microscopic dissipative mechanisms implies that the superfluid is not bound to any other constituents , i.e. it is a perfect conductor in the sense that it can flow freely even in the presence of other constituents. Dissipation-free flow is characterized by the absence of non-conservative forces acting on the bulk[3] of superfluid, i.e.

$$\mathfrak{F}^{\mathrm{S}} = 0\,. \tag{42}$$

As a consequence of (36) and (41) we see that the vorticity (and therefore circulation) of a superfluid is strictly conserved. The second constraint, which distinguishes a superfluid from a perfect fluid, is that a superfluid is locally *irrotational*, i.e. its vorticity is zero, so

$$\underline{w}^{\mathrm{S}} = 0\,, \quad \Longleftrightarrow \quad \boldsymbol{W}^{\mathrm{S}} = 0\,. \tag{43}$$

Due to the vorticity conservation of superfluids, this constraint remains automatically satisfied if it is true at some instant t, i.e. it is consistent with the hydrodynamic evolution.

The formulation most commonly found in the literature on superfluids and superconductors is based on the concept of the so-called "superfluid velocity", which is constrained to be irrotational [23, 24]. If one interpreted this as the actual transport-velocity $\boldsymbol{v}_{\mathrm{S}}$, such a constraint would generally not be consistent with the equations of motion, contrary to the natural conservation of the *momentum vorticity* $\underline{w}^{\mathrm{S}}$. This "orthodox" formulation of superfluidity, which goes back to Landau's two-fluid model for ^{4}He, is therefore a rather unfortunate misinterpretation of physical quantities, as the so-called "superfluid velocity" is necessarily to be interpreted as the rescaled *superfluid momentum* in order to make this constraint consistent with hydrodynamics. The fact that in Newtonian single-fluid contexts the particle momentum only differs by a constant mass factor from the velocity has unfortunately lead to a less than careful distinction between these fundamentally different quantities. This simple identification no longer holds true in more general contexts, like in the case of multi-fluids (e.g. superfluids) or even in the case of a single relativistic perfect fluid. The velocity-circulation is generally *not* conserved, contrary to the conservation of momentum circulation (41). The orthodox framework of superfluid hydrodynamics will be discussed in more detail in Sect. VI D.

In addition to the superfluid constraints of being dissipation-free and irrotational, there is a further important restriction, namely the quantization of circulation. An irrotational flow can still carry non-zero circulation in the presence of *topological defects* (such as vortices). In order to see this, we note that (as a consequence of (43)) we can write the superfluid momentum $\underline{p}^{\mathrm{S}}$ as the gradient of a *phase* φ, namely

$$\underline{p}^{\mathrm{S}} = \hbar\, d\varphi\,, \quad \text{i.e.} \quad \boldsymbol{p}^{\mathrm{S}} = \hbar\, \nabla\varphi\,. \tag{44}$$

The circulation (38) can therefore be non-zero if $\partial\Sigma$ encloses a topological defect in φ, i.e. a region where φ (and $\boldsymbol{p}^{\mathrm{S}}$) is not defined, as for example in the case of flow inside a torus. While in the case of a perfect irrotational fluid the resulting circulation could have any value, the superfluid phase φ is restricted to change only by a multiple of 2π after a complete tour around the defect. The resulting circulation is therefore quantized as

$$\mathcal{C} = 2N\pi\hbar\,, \quad \text{with} \quad N \in \mathbb{Z}\,, \tag{45}$$

which gives rise to the well-known quantized vortex structure of superfluids.

C. Helicity conservation

Contrary to the conservation laws derived in the previous sections, which have been known for more than a century, there is a further conserved quantity namely the so-called helicity, whose existence in hydrodynamics has only been pointed out comparatively recently by Moffat [25]. This quantity is analogous to the magnetic helicity conservation found in magneto-hydrodynamics [26], and it is related to the topological structure of the vorticity, i.e. its "knottedness" [27]. The relativistic analogue of this conservation has been shown by Carter [13, 28, 29], and generalizations have been discussed by Bekenstein [30].

We define the helicity 3-form $\underline{H}$ (with components H_{ijk}) as the exterior product of the momentum 1-form $\underline{p}$ with the vorticity 2-form $\underline{w}$, i.e.

$$\underline{H} \equiv \underline{p} \wedge \underline{w}\,, \tag{46}$$

which in components reads as $H_{ijk} = 3 p_{[i} w_{jk]}$. A 3-form in a 3-dimensional manifold is dual to a *scalar*, so we can define the helicity density h as

$$H_{ijk} = h\, \epsilon_{ijk}\,. \tag{47}$$

From the duality relation together with the definition (46), we see that the helicity scalar has the following explicit expression

$$h = \frac{1}{3!}\epsilon^{ijk} H_{ijk} = p_i\, \frac{1}{2}\epsilon^{ijk} w_{jk} = \underline{p} \rfloor \boldsymbol{W} = \boldsymbol{p}\cdot(\nabla\times\boldsymbol{p})\,. \tag{48}$$

Using (35) and (36), the comoving time-derivative of $\underline{H}$ can be expressed as

$$\begin{aligned}(\partial_t + \pounds_{\boldsymbol{v}})\, \underline{H} &= \left[(\partial_t + \pounds_{\boldsymbol{v}})\, \underline{p}\right] \wedge \underline{w} + \underline{p} \wedge \left[(\partial_t + \pounds_{\boldsymbol{v}})\, \underline{w}\right] \\ &= (dQ + \underline{\mathfrak{F}}) \wedge \underline{w} + \underline{p} \wedge d\underline{\mathfrak{F}} \\ &= d(Q\underline{w}) + \left[d(\underline{p} \wedge \underline{\mathfrak{F}}) + 2 d\underline{\mathfrak{F}} \wedge \underline{p}\right]\,. \end{aligned} \tag{49}$$

We see that, not surprisingly, the vanishing of the non-conservative force $\underline{\mathfrak{F}}$ is a necessary (albeit not sufficient) condition for the conservation of helicity. We introduce the total helicity $\mathcal{H}$ of a volume V as

$$\mathcal{H} \equiv \int_V \underline{H} = \int_V h\, dV\,, \tag{50}$$

[3] However, there *can* be a non-conservative force acting on the superfluid at a vortex-core if the vortex is pushed by another fluid. This mechanism gives rise to the so-called effect of "mutual friction".

and, assuming $\underline{\mathfrak{F}} = 0$, we find for the comoving time derivative of $\mathcal{H}$:

$$\frac{d\mathcal{H}}{dt} = \int_V (\partial_t + \pounds_{\boldsymbol{v}})\,\underline{H} = \oint_{\partial V} Q\,\underline{w} = \oint_{\partial V} Q\boldsymbol{W}\cdot d\boldsymbol{S}\,. \quad (51)$$

The helicity $\mathcal{H}$ of a volume V is therefore conserved under transport by the fluid if, in addition to $\underline{\mathfrak{F}} = 0$, the vorticity $\boldsymbol{W}$ vanishes on the surface ∂V surrounding this volume.

V. HYDRODYNAMICS

A. The Lagrangian of hydrodynamics

In the previous sections we have derived the most general form of the Euler-Lagrange equations (18) associated with the convective variational principle, together with the force densities (11) and energy transfer rates (12). We are now interested in a particular class of Lagrangian densities $\Lambda_{\rm H}$, namely those which describe Newtonian hydrodynamics. One can postulate the general form of the hydrodynamic Lagrangian $\Lambda_{\rm H}$ in analogy to canonical particle mechanics as

$$\Lambda_{\rm H}(n_X, \boldsymbol{n}_X) \equiv \sum m^X \frac{\boldsymbol{n}_X^2}{2n_X} - \mathcal{E}\,, \quad (52)$$

where $\mathcal{E}$ is a thermodynamic potential related to the internal energy (or "equation of state") of the system. We therefore find the following general form for the conjugate momenta p_0^X and $\boldsymbol{p}^X$ as defined in Eq. (4):

$$-p_0^X = \frac{1}{2} m^X \boldsymbol{v}_X^2 + \frac{\partial \mathcal{E}}{\partial n_X}\,, \quad \boldsymbol{p}^X = m^X \boldsymbol{v}_X - \frac{\partial \mathcal{E}}{\partial \boldsymbol{n}_X}\,. \quad (53)$$

We want to identify these conjugate momenta with the actual physical energy and momentum per fluid particle, which implies that under a Galilean boost $-\boldsymbol{V}$ inducing the transformations

$$\boldsymbol{v}_X' = \boldsymbol{v}_X + \boldsymbol{V}\,, \quad n_X' = n_X\,, \quad \partial_t{}' = \partial_t - \boldsymbol{V}\cdot\nabla\,, \quad (54)$$

these momenta should transform (e.g. see [23, 31]) as

$$-p_0^{X\,\prime} = -p_0^X + \boldsymbol{V}\cdot\boldsymbol{p}^X + \frac{1}{2} m^X \boldsymbol{V}^2\,, \text{ and } \boldsymbol{p}^{X\,\prime} = \boldsymbol{p}^X + m^X \boldsymbol{V}\,. \quad (55)$$

One can verify that in this case the hydrodynamic force densities $\boldsymbol{f}_{\rm H}^X$ defined in (14) are invariant under Galilean boosts as one should expect. The particle creation rates $\mathit{\Gamma}_X$ defined in (13) are also Galilean invariant, so that the transformation of the total force densities f^X of (11) is seen to be

$$\boldsymbol{f}^{X\,\prime} = \boldsymbol{f}^X + \boldsymbol{V}\, m^X \mathit{\Gamma}_X\,. \quad (56)$$

The equations of motions of an isolated system, i.e. $\sum \boldsymbol{f}^X = 0$, are therefore Galilean invariant if and only if the total mass is conserved, i.e. if

$$\sum m^X \mathit{\Gamma}_X = 0\,. \quad (57)$$

By using (55) we can show that the energy transfer rates (16) transform as

$$g^{X\,\prime} = g^X + \boldsymbol{V}\cdot\boldsymbol{f}^X + m^X \mathit{\Gamma}_X \frac{V^2}{2}\,, \quad (58)$$

and due to mass conservation (57) the total energy change rate therefore satisfies

$$\sum g^{X\,\prime} = \sum g^X + \boldsymbol{V}\cdot\boldsymbol{f}_{\rm ext}\,, \quad (59)$$

so that the total energy conservation of an isolated system is Galilean invariant.

In general the transformation properties (55) are only consistent with the conjugate momenta (53) if $\mathcal{E}$ is itself Galilean invariant, which is shown in Appendix C. This implies that the velocity dependence of $\mathcal{E}$ can only be of the form

$$\mathcal{E}(n_X, \boldsymbol{n}_X) = \mathcal{E}(n_X, \boldsymbol{\Delta}_{X\,Y})\,, \quad (60)$$

where $\boldsymbol{\Delta}_{X\,Y}$ is the relative velocity between fluid X and fluid Y, i.e.

$$\boldsymbol{\Delta}_{X\,Y} \equiv \boldsymbol{v}_X - \boldsymbol{v}_Y = \frac{\boldsymbol{n}_X}{n_X} - \frac{\boldsymbol{n}_Y}{n_Y}\,. \quad (61)$$

We note that a function $\mathcal{E}$ of the form (60) satisfies the identity

$$\sum n_X \frac{\partial \mathcal{E}}{\partial \boldsymbol{n}_X} = 0\,, \quad (62)$$

which can be used together with (53) to show that the hydrodynamic momentum density (23) satisfies

$$\boldsymbol{J}_{\rm H} = \sum n_X \boldsymbol{p}^X = \sum m^X \boldsymbol{n}_X = \boldsymbol{\rho}\,, \quad (63)$$

i.e. the hydrodynamic momentum density $\boldsymbol{J}_{\rm H}$ is equal to the total mass current $\boldsymbol{\rho}$ as a consequence of Galilean invariance.

In addition to the requirement of Galilean invariance we will restrict our attention to systems of "perfect" multi-constituent fluids in the sense that their energy function $\mathcal{E}$ is isotropic. This means that we consider only equations of state of the form

$$\mathcal{E}(n_X, \boldsymbol{\Delta}_{X\,Y}) = \mathcal{E}(n_X, \Delta_{X\,Y}^2)\,. \quad (64)$$

Summarizing we can now write the hydrodynamic Lagrangian density (52) for this class of perfect multi-fluid systems as

$$\Lambda_{\rm H}(n_X, \boldsymbol{n}_X) = \sum m^X \frac{\boldsymbol{n}_X^2}{2n_X} - \mathcal{E}(n_X, \Delta_{X\,Y}^2)\,. \quad (65)$$

It is interesting to note that contrary to the relativistic case, which is governed by a fully covariant hydrodynamic Lagrangian density (e.g. see [13]), the Newtonian Lagrangian (65) is *not* strictly Galilean invariant because of the kinetic energy term. The violation is sufficiently weak, however, that is does not affect the Galilean invariance of the resulting equations of motion.

B. Conjugate momenta and entrainment effect

The total differential of the energy function $\mathcal{E}(n_X, \Delta^2_{XY})$ represents the first law of thermodynamics for the given system, namely

$$d\mathcal{E} = \sum \mu^X \, dn_X + \frac{1}{2}\sum_{X,Y} \alpha^{XY} \, d\Delta^2_{XY} \,, \qquad (66)$$

which defines the chemical potentials μ^X and the symmetric *entrainment* matrix α^{XY} as the thermodynamical conjugates to n_X and Δ^2_{XY}. The conjugate momenta (53) are therefore explicitly found as

$$\boldsymbol{p}^X \;=\; m^X \boldsymbol{v}_X - \sum_Y \frac{2\alpha^{XY}}{n_X} \boldsymbol{\Delta}_{XY} \,, \qquad (67)$$

$$-p_0^X \;=\; \mu^X - m^X \frac{v_X^2}{2} + \boldsymbol{v}_X \cdot \boldsymbol{p}^X \,. \qquad (68)$$

The expression (67) for the momenta in terms of the velocities is interesting, as it shows that in general the momenta are not aligned with the respective fluid velocities, which is the so-called entrainment effect[4]. The simple single-fluid case, in which the momentum is just $\boldsymbol{p} = m\boldsymbol{v}$, is only recovered if there is no entrainment between the fluids (i.e. $\alpha^{XY} = 0$) or if all constituents move together (i.e. $\boldsymbol{\Delta}_{XY} = 0$). This phenomenon is well-known (albeit not under the name entrainment) in solid-state physics, for example the electron momentum in a crystal lattice is connected to its velocity by an *effective mass-tensor* (e.g. see [33]). For a more detailed discussion of the explicit relation between effective masses and entrainment in a two-fluid model we refer the reader to [34]. In the context of superfluid mixtures the importance of the interaction and the entrainment effect has first been recognized by Andreev&Bashkin [35], although expressed in the conceptually more confused orthodox framework of superfluidity. Substituting (65) together with (68) and (67) into (20), we can now relate the "generalized pressure" Ψ directly to the energy function $\mathcal{E}$, namely

$$\mathcal{E} + \Psi = \sum n_X \mu^X \,, \qquad (69)$$

and with (66) the total differential of $\Psi(\mu^X, \Delta^2_{XY})$ is found as

$$d\Psi = \sum n_X \, d\mu^X - \frac{1}{2}\sum_{X,Y} \alpha^{XY} \, d\Delta^2_{XY} \,. \qquad (70)$$

We can further express the hydrodynamic force density (14) more explicitly as

$$\boldsymbol{f}^X_{\rm H} = n_X \left(\partial_t + \boldsymbol{v}_X \cdot \nabla\right) \boldsymbol{p}^X + n_X \nabla \mu^X - \sum_Y 2\alpha^{XY} \Delta^j_{XY} \nabla v_{X\,j} \,, \qquad (71)$$

[4] Sometimes also referred to as "drag" in the superfluid literature. But as pointed out in [32], this is rather misleading, as entrainment is a purely non-dissipative effect, whereas "drag" in physics usually refers to a resistive drag.

and for the conserved hydrodynamic energy density (26) we find

$$E_{\rm H} = \sum_X m^X n_X \frac{v_X^2}{2} + \mathcal{E} - \sum_{X,Y} \alpha^{XY} \, \Delta^2_{XY} \,. \qquad (72)$$

This relation can be used to clarify the physical meaning of the thermodynamic potential $\mathcal{E}$. One might have expected to find the total energy density simply as the sum of kinetic energies plus $\mathcal{E}$. It is to be noted though that $E_{\rm H}$, which represents the Hamiltonian $\mathcal{H}_{\rm H}(n_X, \boldsymbol{p}^X)$ of the system, is naturally a function of the fluid momenta $\boldsymbol{p}^X$ as opposed to the velocities. Similarly it turns out that in order to find the actual "internal energy", we have to construct the thermodynamic potential that depends on the relative momenta instead of Δ_{XY}. We therefore define the "entrained" relative momenta $\boldsymbol{J}^{XY}$ as

$$\boldsymbol{J}^{XY} \equiv 2\alpha^{XY} \boldsymbol{\Delta}_{XY} \,, \qquad (73)$$

representing the momentum exchange between constituents X and Y due to entrainment, namely by using (67) the momentum density of the constituent X can be written as

$$n_X \boldsymbol{p}^X = n_X m^X \boldsymbol{v}_X - \sum_Y \boldsymbol{J}^{XY} \,. \qquad (74)$$

Using this definition of $\boldsymbol{J}^{XY}$, the first law (66) now takes the form

$$d\mathcal{E} = \sum \mu^X \, dn_X + \frac{1}{2}\sum_{X,Y} \boldsymbol{J}^{XY} \, d\boldsymbol{\Delta}_{XY} \,, \qquad (75)$$

We can therefore introduce the internal energy density $\tilde{\mathcal{E}}$ as the Legendre transformed (with respect to the momenta $\boldsymbol{J}^{XY}$) of the energy function $\mathcal{E}$, namely

$$\tilde{\mathcal{E}}(n_X, \boldsymbol{J}^{XY}) \equiv \mathcal{E} - \frac{1}{2}\sum_{X,Y} \boldsymbol{J}^{XY} \cdot \boldsymbol{\Delta}_{XY} \,, \qquad (76)$$

with the associated total differential

$$d\tilde{\mathcal{E}} = \sum \mu^X \, dn_X - \frac{1}{2}\sum_{X,Y} \boldsymbol{\Delta}_{XY} \, d\boldsymbol{J}^{XY} \,. \qquad (77)$$

We note that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ only differ in systems where the entrainment effect is present. Traditionally the quantity $\tilde{\mathcal{E}}$ is what one might call the actual "internal energy" density, which is a function of the momenta, while the conjugate thermodynamic potential $\mathcal{E}$ does not seem to have a well established name in the literature. We see that in terms of the internal energy $\tilde{\mathcal{E}}$, the total energy density (72) does indeed have the expected form of "kinetic plus internal" energy, namely

$$E_{\rm H} = \sum_X m^X n_X \frac{v_X^2}{2} + \tilde{\mathcal{E}} \,. \qquad (78)$$

C. Entropy and temperature

As noted earlier, entropy can be included quite naturally in this framework as a constituent. The corresponding density and current are $n_{\rm s} = s$ and $\boldsymbol{n}_{\rm s} = s\boldsymbol{v}_{\rm s}$ in terms of the entropy density s and its transport velocity $\boldsymbol{v}_{\rm s}$. The entropy is naturally mass-less, i.e. $m^{\rm s} = 0$. The thermodynamically conjugate variable to the entropy (its "chemical potential") is the temperature, i.e. $\mu^{\rm s} = T$, so (66) can be written as

$$d\mathcal{E} = T\,ds + \sum_{X\neq {\rm s}} \mu^X\,dn_X + \frac{1}{2}\sum_{X,Y} \alpha^{X\,Y}\,d\Delta^2_{X\,Y}\,. \qquad (79)$$

The thermal momenta $p^{\rm s}_0 = \Theta_0$ and $\boldsymbol{p}^{\rm s} = \boldsymbol{\Theta}$ of the entropy constituent are found from (67) and (68), namely

$$\boldsymbol{\Theta} \;=\; -\sum_Y \frac{2\alpha^{{\rm s}\,Y}}{s}\boldsymbol{\Delta}_{{\rm s}\,Y}\,, \qquad (80)$$

$$-\Theta_0 \;=\; T + \boldsymbol{v}_{\rm s}\cdot\boldsymbol{\Theta}\,. \qquad (81)$$

We see that although the entropy has zero rest mass, it can acquire a non-zero dynamical momentum $\boldsymbol{\Theta}$ due to entrainment. This can also be interpreted as the entropy having a non-zero "effective mass". The hydrodynamic entropy force density $\boldsymbol{f}^{\rm s}_{\rm H}$ and energy change rate $g^{\rm s}$ defined in (14) and (16) yield

$$\boldsymbol{f}^{\rm s}_{\rm H} = s\nabla T + s\,(\partial_t + \boldsymbol{v}_{\rm s}\cdot\nabla)\,\boldsymbol{\Theta} - \sum 2\alpha^{{\rm s}\,Y}\,\Delta^j_{{\rm s}\,Y}\nabla v_{{\rm s}j}\,, \qquad (82)$$

$$g^{\rm s} = \boldsymbol{v}_{\rm s}\cdot\boldsymbol{f}^{\rm s}_{\rm H} + (T + \boldsymbol{v}_{\rm s}\cdot\boldsymbol{\Theta})\;\mathit{\Gamma}_{\rm s}\,. \qquad (83)$$

We see that the temperature gradient is a driving force of the entropy constituent, as would be expected. We also recognize the term $T\mathit{\Gamma}_{\rm s}$ in the expression of the energy transfer rate $g^{\rm s}$, which represents the heat creation "$T\,dS$".

VI. APPLICATIONS

A. Single perfect fluids

As the first application of the foregoing formalism, we consider a single perfect fluid consisting of several comoving constituents. This multi–constituent fluid is described by the densities n_X which move with a single velocity $\boldsymbol{v}_X = \boldsymbol{v}$, and so the currents are $\boldsymbol{n}_X = n_X\,\boldsymbol{v}$. Obviously all the relative velocities vanish in this case, i.e. $\boldsymbol{\Delta}_{X\,Y} = 0$, and therefore there is no entrainment. Here we will explicitly write the entropy with its density s, and we do not include it in the constituent index set labelled by X, i.e. $X \neq$ s. The Lagrangian (65) for this system is

$$\Lambda_{\rm H} = \sum m^X n_X \frac{\boldsymbol{v}^2}{2} - \mathcal{E}(s, n_X)\,, \qquad (84)$$

and the energy and pressure differentials (66) and (70) simply read as

$$d\mathcal{E} = T\,ds + \sum \mu^X\,dn_X\,, \;\text{ and }\; dP = s\,dT + \sum n_X\,d\mu^X\,, \qquad (85)$$

where in the case of a single fluid, the generalized pressure Ψ simply reduces to the usual fluid pressure P. The fluid momenta (67) and (68) are

$$\boldsymbol{p}^X = m^X\,\boldsymbol{v}\,, \;\text{ and }\; -p^X_0 = \mu^X + m^X\frac{v^2}{2}\,, \qquad (86)$$

while for the entropy constituent we have with (80) and (81):

$$\boldsymbol{\Theta} = 0\,, \;\text{ and }\; -\Theta_0 = T\,. \qquad (87)$$

The explicit expression for the force densities (11) and energy transfer rates (16) are found as

$$\boldsymbol{f}^X \;=\; n_X m^X\,(\partial_t + \boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} + n_X\nabla\mu^X + m^X\mathit{\Gamma}_X\,\boldsymbol{v}\,, \qquad (88)$$

$$g^X \;=\; \boldsymbol{v}\cdot\boldsymbol{f}^X + \mathit{\Gamma}_X\mu^X - m^X\frac{v^2}{2}\,\mathit{\Gamma}_X\,, \qquad (89)$$

$$\boldsymbol{f}^{\rm s} \;=\; s\nabla T\,, \qquad (90)$$

$$g^{\rm s} \;=\; \boldsymbol{v}\cdot\boldsymbol{f}^{\rm s} + T\mathit{\Gamma}_{\rm s}\,, \qquad (91)$$

If we allow for an external force $\boldsymbol{f}_{\rm ext}$ and energy exchange rate $g_{\rm ext}$, the equations of motion (18) of the system are

$$\boldsymbol{f}^{\rm s} + \sum \boldsymbol{f}^X = \boldsymbol{f}_{\rm ext}\,, \;\text{ and }\; g^{\rm s} + \sum g^X = g_{\rm ext}\,. \qquad (92)$$

Inserting (88)–(91) and using mass conservation (57), we find the explicit equations of motion

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} + \frac{1}{\rho}\nabla P \;=\; \frac{1}{\rho}\boldsymbol{f}_{\rm ext}\,, \qquad (93)$$

$$T\mathit{\Gamma}_{\rm s} + \sum \mu^X\mathit{\Gamma}_X \;=\; g_{\rm ext} - \boldsymbol{v}\cdot\boldsymbol{f}_{\rm ext}\,, \qquad (94)$$

where we have used the thermodynamic relation (85) in order to rewrite the momentum equation in the familiar Euler form. The energy equation expresses the heat creation $T\mathit{\Gamma}_{\rm s}$ by chemical reactions $\mathit{\Gamma}_X$. For an *isolated* system, where $\boldsymbol{f}_{\rm ext} = 0$ and $g_{\rm ext} = 0$, that entropy can only increase due to the second law of thermodynamics, so $\mathit{\Gamma}_{\rm s} \geq 0$. From (94) we therefore obtain a constraint on the direction of the chemical reactions, namely

$$\sum \mathit{\Gamma}_X\mu^X \leq 0\,. \qquad (95)$$

If we consider for example the case of two constituents of equal mass, so that the mass-conservation (57) implies $\mathit{\Gamma}_1 + \mathit{\Gamma}_2 = 0$, then this constraint now reads as

$$\mathit{\Gamma}_1(\mu^1 - \mu^2) \leq 0\,, \qquad (96)$$

which shows that chemical reactions only proceeds in the direction of the lower chemical potential as would be expected.

B. "Potential vorticity" conservation: Ertel's theorem

We now consider the case without chemical reactions, in which the general perfect fluid discussed in the foregoing section can be described effectively as a fluid consisting only of a single matter constituent and entropy.

In this case we can show that the vorticity is generally not conserved, but that a weaker form of the vorticity conservation still holds. The fluid is described by the particle number density n, the mass per particle m and a comoving entropy density s. Mass conservation (57) in this case reduces to $\Gamma = 0$. If we assume the system to be isolated, i.e. $\boldsymbol{f} + \boldsymbol{f}^{\rm s} = 0$, then the only force per particle (33) acting on the matter constituent is the "thermal force" (90), namely

$$\frac{1}{n}\boldsymbol{f}_{\rm H} = -\widetilde{s}\,\nabla T\,, \tag{97}$$

where $\widetilde{s} \equiv s/n$ is the specific entropy. If $\widetilde{s}$ is constant everywhere, then this "thermal force" is conservative, i.e. $\boldsymbol{\mathfrak{F}} = 0$ and by (41) the circulation is therefore conserved. In the non-uniform case, however, we find

$$\frac{d\mathcal{C}}{dt} = \oint_{\partial\Sigma} \underline{\mathfrak{F}} = -\oint_{\partial\Sigma} \widetilde{s}\, dT\,, \tag{98}$$

which vanishes only if we integrate along a path $\partial\Sigma$ that lies completely in a surface of constant $\widetilde{s}$. We can also see this in the Helmholtz formulation, namely by applying an exterior derivative to (97), one obtains

$$d\underline{\mathfrak{F}} = -d\widetilde{s}\wedge dT\,, \quad \text{i.e.} \quad \nabla\times\boldsymbol{\mathfrak{F}} = -\nabla\widetilde{s}\times\nabla T\,, \tag{99}$$

and it follows therefore from (36) that the vorticity is no longer generally conserved in this case. However, the quantity $d\widetilde{s}\wedge d\underline{\mathfrak{F}}$, or its equivalent dual expression $\nabla\widetilde{s}\cdot(\nabla\times\boldsymbol{\mathfrak{F}})$, still vanishes identically. Based on this observation we construct the "potential vorticity" 3-form $\underline{\mathcal{Z}}$ as

$$\underline{\mathcal{Z}} \equiv d\widetilde{s}\wedge\underline{w}\,, \tag{100}$$

and the dual scalar z is

$$\mathcal{Z}_{ijk} = z\,\epsilon_{ijk}\,, \text{ and } z = \frac{1}{3!}\epsilon^{ijk}\mathcal{Z}_{ijk} = \nabla\widetilde{s}\cdot(\nabla\times\boldsymbol{p})\,, \tag{101}$$

where the last expression was found using (30). The evolution of the potential vorticity 3-form $\underline{\mathcal{Z}}$ under transport by the fluid is

$$(\partial_t + \pounds_v)\,\underline{\mathcal{Z}} = d\left[(\partial_t + \pounds_v)\widetilde{s}\right]\wedge\underline{w}\,, \tag{102}$$

and therefore $\underline{\mathcal{Z}}$ is conserved for isentropic flow, i.e. if

$$\Gamma_{\rm s} = 0 \iff (\partial_t + \pounds_v)\,\widetilde{s} = 0\,. \tag{103}$$

The dual version of (102), namely the conservation of the scalar z is then found as

$$\partial_t\, z + \nabla\cdot(z\boldsymbol{v}) = 0\,. \tag{104}$$

Traditionally this conservation law is often expressed in terms of the scalar $\alpha \equiv z/\rho$, which then results in the following form of the conservation law:

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\;\alpha = 0\,, \tag{105}$$

which is generally known as "Ertel's theorem" [36, 37].

C. Thermally conducting fluids

We have so far only considered perfect fluids, which are perfect heat insulators as the entropy is strictly carried along by fluid elements and no heat is exchanged between fluid elements. It is quite straightforward to extend this to thermally conducting fluids simply by dropping the assumption that the entropy flux is bound to the matter fluid flow, i.e. we just have to allow $\boldsymbol{v}_{\rm s} \neq \boldsymbol{v}$, where $\boldsymbol{v}_{\rm s}$ and $\boldsymbol{v}$ are the velocities of the entropy fluid and the matter fluid respectively. For simplicity we consider only a single matter constituent, described by its particle number density n, which by (57) is automatically conserved, i.e. $\Gamma = 0$.

From the general expressions (81) and (80) we see that the "entropy fluid" acquires a non-zero momentum due to the interaction with the matter fluid, via entrainment. However, this aspect does not usually seem to be taken into account in the traditional description of heat-conducting fluids (e.g. see [23]). The aim of the present section is only to show how to recover the standard equations for a heat-conducting fluid, and we therefore simply assume the entrainment to be negligible, i.e. $\alpha = 0$. It is certainly an interesting question if this neglect of entrainment is physically justified in all cases. With this assumption, the force density (82) and energy rate (83) of the entropy reduce to

$$\boldsymbol{f}^{\rm s} = s\nabla T\,, \text{ and } g^{\rm s} = \boldsymbol{v}_{\rm s}\cdot\boldsymbol{f}^{\rm s} + T\,\Gamma_{\rm s}\,. \tag{106}$$

As in the (isolated) perfect fluid case discussed previously, the equations of motion are again $\boldsymbol{f}^{\rm s} + \boldsymbol{f} = 0$ and $g^{\rm s} + g = 0$. This time, however, one force density, $\boldsymbol{f}^{\rm s}$ say, can be specified by the model due to the increased number of degrees of freedom, so we set it to $\boldsymbol{f}^{\rm s} = \boldsymbol{f}_R$, where $\boldsymbol{f}_R$ is a resistive force acting against the entropy flow. We obtain the Euler equation in the same form as in (93), but now the energy equation takes the form

$$T\Gamma_{\rm s} = (\boldsymbol{v} - \boldsymbol{v}_{\rm s})\cdot\boldsymbol{f}_R\,. \tag{107}$$

By the second law of thermodynamics, namely $\Gamma_{\rm s} \geq 0$, we can constrain the form of the resistive force $\boldsymbol{f}_R$ to

$$\boldsymbol{f}_R = -\eta\,(\boldsymbol{v}_{\rm s} - \boldsymbol{v})\,, \quad \text{with} \quad \eta \geq 0\,, \tag{108}$$

i.e. the friction force acting on the entropy fluid is always opposed to its flow relative to the matter fluid. Obviously the value of the resistivity η is not restricted to be a constant but will generally depend on the state of the system. Following the traditional description (e.g. [23]) we introduce the heat flux density $\boldsymbol{q}$ *relative to the matter fluid* as

$$\boldsymbol{q} \equiv Ts(\boldsymbol{v}_{\rm s} - \boldsymbol{v})\,. \tag{109}$$

By combining this with (106) and (108), we see that the heat flux current is constrained by the second law to be of the form

$$\boldsymbol{q} = -\kappa\nabla T\,, \quad \text{with} \quad \kappa \equiv \frac{Ts^2}{\eta} \geq 0\,, \tag{110}$$

where κ is the *thermal conductivity.* With (109) we can express the velocity of the entropy fluid $\boldsymbol{v}_{\rm s}$ in terms of the heat flux $\boldsymbol{q}$ and the matter velocity $\boldsymbol{v}$, so the entropy creation rate $\Gamma_{\rm s}$ can be expressed as

$$\Gamma_{\rm s} = \partial_t s + \nabla\cdot\left(s\boldsymbol{v} + \frac{\boldsymbol{q}}{T}\right) . \tag{111}$$

We further find for the hydrodynamic energy flux vector $\boldsymbol{Q}_{\rm H}$ of (26):

$$\begin{aligned} \boldsymbol{Q}_{\rm H} &= \sum(-p_0^X)\boldsymbol{n}_X = (\mu + m\frac{v^2}{2})n\boldsymbol{v} + sT\boldsymbol{v}_{\rm s} \\ &= n\boldsymbol{v}\left(m\frac{v^2}{2} + \mu + \widetilde{s}T\right) + \boldsymbol{q} , \end{aligned} \tag{112}$$

where the last equality was found using (109). We introduce the specific enthalpy as $\mathrm{w} \equiv \mu + \widetilde{s}T$, and using the first law[5], namely $dP = n\,d\mu + s\,dT$, we find the total variation of the specific enthalpy as

$$d\mathrm{w} = Td\widetilde{s} + \frac{1}{n}dP , \tag{113}$$

and so we recover the standard expression (e.g. cf. [23]) for the energy flux:

$$\boldsymbol{Q}_{\rm H} = n\boldsymbol{v}\left(m\frac{v^2}{2} + \mathrm{w}\right) + \boldsymbol{q} . \tag{114}$$

D. The two-fluid model for superfluid ^{4}He

We now consider the example of superfluid ^{4}He at a non-zero temperature T. Let n be the number density of ^{4}He atoms and s be the entropy density. The ^{4}He atoms move with a velocity $\boldsymbol{v}$, while the entropy (carried by a thermal gas of excitations such as phonons and rotons) transports heat without friction (i.e. $\boldsymbol{f}_R = 0$) at the velocity $\boldsymbol{v}_{\rm N}$, so the relative velocity is $\boldsymbol{\Delta} = \boldsymbol{v}_{\rm N} - \boldsymbol{v}$. In this context the entropy fluid is often referred to as the "normal fluid" as opposed to the superfluid mass flow. The two transport currents, namely that of ^{4}He atoms and of entropy, are respectively

$$\boldsymbol{n} = n\,\boldsymbol{v} , \quad \text{and} \quad \boldsymbol{s} = s\,\boldsymbol{v}_{\rm N} . \tag{115}$$

The ^{4}He atoms have mass m, so the mass density is $\rho = nm$, and the hydrodynamic Lagrangian density (65) reads as

$$\Lambda_{\rm H} = \frac{1}{2}nmv^2 - \mathcal{E}(n, s, \Delta^2) , \tag{116}$$

[5] In the absence of entrainment the entropy fluid does not carry momentum, therefore the matter fluid defines a unique frame in which the stress tensor (23) is purely isotropic. In this case the generalized pressure Ψ is identical with the usual perfect fluid notion of the pressure P.

where the energy function $\mathcal{E}$ determines the first law (66) as

$$d\mathcal{E} = \mu\,dn + T\,ds + \alpha\,d\Delta^2 , \tag{117}$$

which defines the chemical potential μ of ^{4}He atoms, the temperature T and the entrainment α. The conjugate momenta (67), (68) of the ^{4}He atoms are

$$\boldsymbol{p} = m\boldsymbol{v} + \frac{2\alpha}{n}\boldsymbol{\Delta} , \tag{118}$$

$$-p_0 = \mu - \frac{1}{2}mv^2 + \boldsymbol{v}\cdot\boldsymbol{p} , \tag{119}$$

while for the entropy fluid Eqs. (80) and (81) yield

$$\boldsymbol{\Theta} = -\frac{2\alpha}{s}\boldsymbol{\Delta} , \tag{120}$$

$$-\Theta_0 = T + \boldsymbol{v}_{\rm N}\cdot\boldsymbol{\Theta} . \tag{121}$$

The conservation of mass (57) implies

$$\Gamma = \partial_t n + \nabla\cdot\boldsymbol{n} = 0 . \tag{122}$$

In the absence of vortices, there are no direct forces acting between the two fluids, so the equations of motion in the absence of external forces (i.e. $\boldsymbol{f}_{\rm ext} = 0$) are simply

$$\boldsymbol{f} = \boldsymbol{f}_{\rm H} = 0 \quad \text{and} \quad \boldsymbol{f}^{\rm N} = 0 . \tag{123}$$

The energy equations are $g = 0$ and $g^{\rm N} = g_{\rm ext}$, and with (83) this leads to

$$-g_{\rm ext} = \Gamma_{\rm s}(\Theta_0 + \boldsymbol{v}_{\rm N}\cdot\boldsymbol{\Theta}) = -T\Gamma_{\rm s} , \tag{124}$$

where we have inserted (121). We see that this equation describes the rate of entropy creation by an external heat source, namely

$$\partial_t s + \nabla\cdot(s\boldsymbol{v}_{\rm N}) = \frac{1}{T}g_{\rm ext} . \tag{125}$$

As discussed in Sect. IV B, the superfluid ^{4}He is (locally) irrotational, i.e.

$$w_{ij} = 2\nabla_{[i}p_{j]} = 0 , \Longleftrightarrow \boldsymbol{W} = \nabla\times\boldsymbol{p} = 0 . \tag{126}$$

Using (14), the equation of motion (123) for the superfluid therefore reduces to

$$\partial_t\boldsymbol{p} - \nabla p_0 = 0 , \tag{127}$$

and with the explicit momenta (119) and (118) this yields

$$\partial_t\left(\boldsymbol{v} + \varepsilon\boldsymbol{\Delta}\right) + \nabla\left(\widetilde{\mu} + \frac{1}{2}v^2 + \varepsilon\boldsymbol{v}\cdot\boldsymbol{\Delta}\right) = 0 , \tag{128}$$

where we introduced the entrainment number ε and the specific chemical potential $\widetilde{\mu}$ as

$$\varepsilon \equiv \frac{2\alpha}{\rho} , \quad \text{and} \quad \widetilde{\mu} \equiv \frac{\mu}{m} . \tag{129}$$

The entropy fluid is governed by the momentum equation $\boldsymbol{f}^{\mathrm{N}} = 0$, and with (82) and the entropy momenta (121) and (120), we find

$$(\partial_t + \boldsymbol{v}_{\mathrm{N}} \cdot \nabla) \left(\frac{2\alpha}{s} \boldsymbol{\Delta} \right) - \nabla T + \frac{2\alpha}{s} \Delta_j \nabla v_{\mathrm{N}}^j + \frac{2\alpha}{s^2} \Gamma_{\mathrm{s}} \boldsymbol{\Delta} = 0 \,. \tag{130}$$

The two equations (128) and (130) represent the "canonical" formulation of the two-fluid model for superfluid ^{4}He. These equations do not seem to bear any obvious relation to the "orthodox" formulation of Landau's two-fluid model found in all textbooks on the subject (e.g. see [23, 24, 31]). Nevertheless, these equations are equivalent to the orthodox framework, as we will show now, but it is important to note that the orthodox formulation is based on a rather unfortunate confusion between the velocity and momentum of the superfluid which is inherent in the historic definition of the "superfluid velocity" by Landau.

We demonstrate the equivalence of these formulations by explicitly translating the canonical formulation into the orthodox language. The starting point of Landau's model is the statement that the "superfluid velocity" is irrotational. We write $\boldsymbol{\nu}_{\mathrm{S}}$ for the "superfluid velocity", which is not to be confused with the actual velocity $\boldsymbol{v}$ of ^{4}He atoms, so the starting point is

$$\nabla \times \boldsymbol{\nu}_{\mathrm{S}} = 0 \,. \tag{131}$$

From the general discussion about vorticity conservation in Sect. IV A and its particular role in superfluids (Sect. IV B) we have already seen that contrary to the momentum vorticity $\boldsymbol{W} = \nabla \times \boldsymbol{p}$, the velocity-rotation $\nabla \times \boldsymbol{v}$ is generally *not* conserved by the fluid flow, and in particular not in the presence of more than one fluid as is the case in superfluid ^{4}He at $T > 0$. The only possible interpretation we can give $\boldsymbol{\nu}_{\mathrm{S}}$ in order for the constraint (131) to be consistent with hydrodynamics and to remain true for all times is that it is really the rescaled superfluid *momentum* $\boldsymbol{p}$, so the "key" to our translation is the ansatz

$$\boldsymbol{\nu}_{\mathrm{S}} \equiv \frac{\boldsymbol{p}}{m} \,. \tag{132}$$

While this would be equivalent to the fluid velocity in a single perfect fluid, as seen in (86), this has no interpretation as the velocity of either the mass or the entropy in the case of the present two-fluid model as we can see in (118). Therefore we call $\boldsymbol{\nu}_{\mathrm{S}}$ a *pseudo velocity*, as it is a *dynamic* combination of both fluid velocities governed by the entrainment α between the superfluid ^{4}He and its excitations. With the explicit entrainment relation (118) we can now express the velocity $\boldsymbol{v}$ of the ^{4}He fluid in terms of the pseudo-velocity $\boldsymbol{\nu}_{\mathrm{S}}$ and the normal-fluid velocity $\boldsymbol{v}_{\mathrm{N}}$ as

$$\boldsymbol{v} = (1-\varepsilon)^{-1} \left(\boldsymbol{\nu}_{\mathrm{S}} - \varepsilon \boldsymbol{v}_{\mathrm{N}} \right) , \tag{133}$$

where we used the definition (129) of the entrainment number ε. With this substitution, the total mass current $\boldsymbol{\rho}$, which is equal to the total momentum density $\boldsymbol{J}_{\mathrm{H}}$ as seen in (63), can be written in the form

$$\boldsymbol{J}_{\mathrm{H}} = \rho \boldsymbol{v} = \left[\frac{\rho}{1-\varepsilon} \right] \boldsymbol{\nu}_{\mathrm{S}} + \left[\frac{-\varepsilon \rho}{1-\varepsilon} \right] \boldsymbol{v}_{\mathrm{N}} \,, \tag{134}$$

which suggests to introduce a "superfluid density" ϱ_{S} and a "normal density" ϱ_{N} as

$$\varrho_{\mathrm{S}} \equiv \frac{\rho}{1-\varepsilon} \,, \quad \text{and} \quad \varrho_{\mathrm{N}} \equiv \frac{-\varepsilon \rho}{1-\varepsilon} \,, \tag{135}$$

such that total mass density ρ and mass current $\boldsymbol{\rho} = \boldsymbol{J}_{\mathrm{H}}$ can now be written as

$$\rho = \varrho_{\mathrm{S}} + \varrho_{\mathrm{N}} \,, \quad \text{and} \quad \boldsymbol{J}_{\mathrm{H}} = \varrho_{\mathrm{S}} \boldsymbol{\nu}_{\mathrm{S}} + \varrho_{\mathrm{N}} \boldsymbol{v}_{\mathrm{N}} \,. \tag{136}$$

It is now obvious that this split is completely artificial, and ϱ_{N} and ϱ_{S} are only *pseudo densities*, as they do not represent the density of any (conserved) physical quantity and are not even necessarily positive. In fact neither of the two pseudo-densities and currents are conserved individually, contrary to the physical currents (115). We note that even Landau warned against taking too literally the interpretation of superfluid ^{4}He as a "mixture" of these two (pseudo-) "fluids" [23]. Contrary to the artificial orthodox split, however, the separation into entropy fluid and ^{4}He mass flow is physically perfectly meaningful, and the superfluid *can* be regarded as a two-fluid system in the literal sense in the canonical framework. The pseudo "mass density" ϱ_{N}, which the normal fluid seems to carry in the orthodox description is due to the fact that entrainment provides the entropy fluid with a non-vanishing *momentum* (120) in the presence of relative motion, even though it does not transport any mass. This lack of careful distinction between mass current and momentum leads to the paradoxical picture of the "superfluid counterflow": for example, in the simple case of heat flow through a static superfluid, the normal fluid associated with the heat flow carries a pseudo mass-current $\varrho_{\mathrm{N}} \boldsymbol{v}_{\mathrm{N}}$. But because there is no net mass current there has to be some superfluid "counterflow" of pseudo mass current $\varrho_{\mathrm{S}} \boldsymbol{\nu}_{\mathrm{S}} = -\varrho_{\mathrm{N}} \boldsymbol{v}_{\mathrm{N}}$. This apparently strange behavior is solely due to an awkward choice of variables and a loss of direct contact between the quantities used in the orthodox description and the actual conserved physical quantities of ^{4}He.

Further following the traditional orthodox framework, we define the relative (pseudo-)velocity $\boldsymbol{w}$ as

$$\boldsymbol{w} \equiv \boldsymbol{v}_{\mathrm{N}} - \boldsymbol{\nu}_{\mathrm{S}} \,, \tag{137}$$

which, using (133), can be expressed in terms of $\boldsymbol{\Delta}$ as

$$\boldsymbol{w} = (1-\varepsilon) \, \boldsymbol{\Delta} \,. \tag{138}$$

In order to relate the canonical thermodynamic quantities to the orthodox language, we follow Khalatnikov [31] and Landau [23] and consider the energy density in the "superfluid frame" K_0, which is defined by $\boldsymbol{\nu}_{\mathrm{S}}^{(0)} = 0$.

In this frame, the momentum density $\boldsymbol{J}_{\mathrm{H}}^{(0)}$ expressed in (136) is

$$\boldsymbol{J}_{\mathrm{H}}^{(0)} = \varrho_{\mathrm{N}}\,\boldsymbol{v}_{\mathrm{N}}^{(0)} = \varrho_{\mathrm{N}}\,\boldsymbol{w} = -2\alpha\,\boldsymbol{\Delta}\,, \tag{139}$$

and the transport velocity $\boldsymbol{v}$ of the superfluid ^{4}He atoms in this frame can be expressed using (140) as

$$\boldsymbol{v}^{(0)} = \boldsymbol{v} - \boldsymbol{\nu}_{\mathrm{S}} = \frac{\varrho_{\mathrm{N}}}{\rho}\boldsymbol{v}_{\mathrm{N}}^{(0)} = \frac{1}{\rho}\boldsymbol{J}_{\mathrm{H}}^{(0)}\,. \tag{140}$$

The hydrodynamic energy density E_{H} of the fluid system is given by (72), which reads in this case

$$E_{\mathrm{H}} = \frac{1}{2}\rho\boldsymbol{v}^2 + \mathcal{E} - 2\alpha\,\boldsymbol{\Delta}^2\,, \tag{141}$$

and using the previous translations together with the first law (117), we can write the total variation $dE^{(0)}$ of the energy density in K_0 as

$$dE_H^{(0)} = T\,ds + \widetilde{\mu}_{\mathrm{S}}\,d\rho + \boldsymbol{w}\cdot d\boldsymbol{J}_{\mathrm{H}}^{(0)}\,, \tag{142}$$

which defines the "superfluid chemical potential" $\widetilde{\mu}_{\mathrm{S}}$ as

$$\widetilde{\mu}_{\mathrm{S}} = \widetilde{\mu} - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{\nu}_{\mathrm{S}})^2\,. \tag{143}$$

Using these quantities, the canonical equation of motion (128) can now be translated into the orthodox form as

$$\partial_t\boldsymbol{\nu}_{\mathrm{S}} + \nabla\left(\frac{\nu_{\mathrm{S}}^2}{2} + \widetilde{\mu}_{\mathrm{S}}\right) = 0\,. \tag{144}$$

One can equally verify that the generalized pressure, defined in (69), is expressible in terms of the orthodox quantities as

$$\Psi = -\mathcal{E} + \rho\,\widetilde{\mu} + s\,T = -E_{\mathrm{H}}^{(0)} + T\,s + \rho\,\widetilde{\mu}_{\mathrm{S}} + \boldsymbol{w}\cdot\boldsymbol{J}_{\mathrm{H}}^{(0)}\,, \tag{145}$$

in exact agreement with the expressions found in [23, 31]. For the remaining momentum equation, the total momentum conservation (22) is traditionally preferred over the equation of motion (130) of the entropy fluid. We therefore conclude this section by the appropriate translation of the stress tensor (23) into the orthodox language. The canonical expression for the stress tensor of superfluid ^{4}He is

$$T_{\mathrm{H}}^{ij} = n^i\,p^j + s^i\,\Theta^j + \Psi\,g^{ij}\,, \tag{146}$$

and inserting the previous expressions for the explicit momenta and the translations to orthodox variables, one can write this in the form

$$T_{\mathrm{H}}^{ij} = \varrho_{\mathrm{S}}\,\nu_{\mathrm{S}}^i\nu_{\mathrm{S}}^j + \varrho_{\mathrm{N}}\,v_{\mathrm{N}}^i v_{\mathrm{N}}^j + \Psi\,g^{ij}\,, \tag{147}$$

which concludes our proof of equivalence between canonical and orthodox description.

E. A two-fluid model for the neutron star core

Here we consider a (simplified) model for the matter inside a neutron star core, which mainly consists of a (charge neutral) plasma of neutrons, protons and electrons. We focus on superfluid models, in which the neutrons are assumed to be superfluid, which allows them to freely traverse the fluid of charged components due to the absence of viscosity. As discussed in Sect. IV B, this also implies some extra complications due to the quantization of vorticity into microscopic vortices. Here we are interested in a macroscopic description, i.e. we consider fluid elements that are small compared to the dimensions of the total system, but which contain a large number of vortices. On this scale we can work with a smooth averaged vorticity instead of having to worry about individual vortices. One effect of the presence of the vortices will be a slight anisotropy in the resulting smooth averaged fluid [22, 38, 39], which can be ascribed to the tension of vortices, and which we will neglect here for simplicity. The second effect of the vortex lattice is that it allows a direct force between the superfluid and the normal fluid, mediated by the respective vortex interactions, and which is naturally described in the context of the two-fluid model as a mutual force. The model assumptions used here are fairly common to most current studies of superfluid neutrons stars (e.g. see [34, 40–42]).

The model therefore consists of comoving constituents $X \in \{\mathrm{e}, \mathrm{p}, \mathrm{s}\}$, corresponding to the electrons, protons and entropy, and we will label this fluid with 'c'. The second fluid consists only of the superfluid neutrons, i.e. $X = \mathrm{n}$. Charge conservation implies

$$\mathit{\Gamma}_{\mathrm{e}} = \mathit{\Gamma}_{\mathrm{p}}\,, \tag{148}$$

and for simplicity we will assume local *charge neutrality*, i.e.

$$n_{\mathrm{e}} = n_{\mathrm{p}}\,. \tag{149}$$

We assume the electrons and protons to be strictly moving together in this model (i.e. we consider timescales longer than the plasma oscillation timescale), so we can neglect electromagnetic interactions altogether. Another physical constraint is *baryon conservation*, i.e. we must have

$$\mathit{\Gamma}_{\mathrm{n}} + \mathit{\Gamma}_{\mathrm{p}} = 0\,, \tag{150}$$

and together with mass conservation (57), this leads to the requirement[6]

$$m^{\mathrm{n}} = m^{\mathrm{p}} + m^{\mathrm{e}} \equiv m\,. \tag{151}$$

[6] This relation is of course not exactly satisfied in reality, which shows a well-known shortcoming of Newtonian physics: mass has to be conserved separately from energy.

We can therefore write the mass densities of the two fluids as

$$\rho_{\rm n} = m\, n_{\rm n}\,, \quad \text{and} \quad \rho_{\rm c} = m\, n_{\rm p}\,. \tag{152}$$

The first law (66) of this model reads as

$$d\mathcal{E} = T\,ds + \mu^{\rm n}\,dn_{\rm n} + \mu^{\rm e}\,dn_{\rm e} + \mu^{\rm p}\,dn_{\rm p} + \alpha^{\rm en}\,d\Delta^2_{\rm en} + \alpha^{\rm pn}\,d\Delta^2_{\rm pn} + \alpha^{\rm sn}\,d\Delta^2_{\rm sn}\,. \tag{153}$$

Obviously there is only one independent relative velocity $\boldsymbol{\Delta}$, namely

$$\boldsymbol{\Delta} \equiv \boldsymbol{v}_{\rm c} - \boldsymbol{v}_{\rm n} = \boldsymbol{\Delta}_{\rm en} = \boldsymbol{\Delta}_{\rm pn} = \boldsymbol{\Delta}_{\rm sn}\,, \tag{154}$$

and we define the total entrainment α as

$$\alpha \equiv \alpha^{\rm en} + \alpha^{\rm pn} + \alpha^{\rm sn}\,. \tag{155}$$

In the case of the neutron star model, we are obviously also interested to include the effects of gravitation. We can therefore not assume the system to be isolated and we include the effect of the gravitational potential Φ as an external force. The minimal equations of motion (18) therefore read as

$$\boldsymbol{f}^{\rm n} + \boldsymbol{f}^{\rm c} = -\rho\nabla\Phi\,, \quad \text{and} \quad g^{\rm n} + g^{\rm c} = -\boldsymbol{\rho}\cdot\nabla\Phi\,, \tag{156}$$

where the force and energy rate of the 'c'-fluid are naturally given by $\boldsymbol{f}^{\rm c} \equiv \boldsymbol{f}^{\rm p} + \boldsymbol{f}^{\rm e} + \boldsymbol{f}^{\rm s}$ and $g^{\rm c} \equiv g^{\rm p} + g^{\rm e} + g^{\rm s}$. With (148) and (150) we can write the respective force densities more explicitly as

$$\boldsymbol{f}^{\rm n} = \boldsymbol{f}^{\rm n}_{\rm H} + \Gamma_{\rm n}\boldsymbol{p}^{\rm n}\,, \tag{157}$$

$$\boldsymbol{f}^{\rm c} = \boldsymbol{f}^{\rm c}_{\rm H} - \Gamma_{\rm n}(\boldsymbol{p}^{\rm e} + \boldsymbol{p}^{\rm p}) + \Gamma_{\rm s}\boldsymbol{\Theta}\,, \tag{158}$$

where we naturally defined $\boldsymbol{f}^{\rm c}_{\rm H} \equiv \boldsymbol{f}^{\rm p}_{\rm H} + \boldsymbol{f}^{\rm e}_{\rm H} + \boldsymbol{f}^{\rm s}_{\rm H}$. Similarly we can write the energy rates (16) as

$$g^{\rm n} = \boldsymbol{v}_{\rm n}\cdot\boldsymbol{f}^{\rm n}_{\rm H} - \Gamma_{\rm n}p^{\rm n}_0\,, \tag{159}$$

$$g^{\rm c} = \boldsymbol{v}_{\rm c}\cdot\boldsymbol{f}^{\rm c}_{\rm H} + \Gamma_{\rm n}(p^{\rm e}_0 + p^{\rm p}_0) - \Gamma_{\rm s}\Theta_0\,. \tag{160}$$

Because the gravitational acceleration is the same for all bodies (i.e. fluids), we can now simply absorb the effect of the gravitational potential into the definition of "extended" forces $\widehat{\boldsymbol{f}}$ and energy rates $\widehat{g}$ which simply incorporate the respective gravitational force density and work rate, i.e. we define

$$\widehat{\boldsymbol{f}}^X \equiv \boldsymbol{f}^X + \rho_X\nabla\Phi\,, \tag{161}$$

$$\widehat{\boldsymbol{f}}^X_{\rm H} \equiv \boldsymbol{f}^X_{\rm H} + \rho_X\nabla\Phi\,, \tag{162}$$

$$\widehat{g}^X \equiv g^X + \rho_X\boldsymbol{v}_X\cdot\nabla\Phi\,. \tag{163}$$

With these redefinitions, the minimal equations of motion (156) again take the form of an isolated system, i.e.

$$\widehat{\boldsymbol{f}}^{\rm n} + \widehat{\boldsymbol{f}}^{\rm c} = 0\,, \quad \text{and} \quad \widehat{g}^{\rm n} + \widehat{g}^{\rm c} = 0\,, \tag{164}$$

while for (157)–(160) we obtain exactly the same form, just for all forces and energy rates replaced by their "extended" version. Using the foregoing equations, we obtain

$$\widehat{\boldsymbol{f}}^{\rm c}_{\rm H} = -\widehat{\boldsymbol{f}}^{\rm n} + \Gamma_{\rm n}\boldsymbol{p}^{\rm c} - \Gamma_{\rm s}\boldsymbol{\Theta}\,, \tag{165}$$

and therefore

$$\widehat{g}^{\rm c} = -\boldsymbol{v}_{\rm c}\cdot\widehat{\boldsymbol{f}}^{\rm n}_{\rm H} - \Gamma_{\rm n}\left[\boldsymbol{v}_{\rm c}\cdot(\boldsymbol{p}^{\rm n} - \boldsymbol{p}^{\rm c}) - p^{\rm c}_0\right] - \Gamma_{\rm s}\Theta_0\,. \tag{166}$$

Substituting this and the "extended" version of (159) into the energy-rate equation (164), we find

$$T\Gamma_{\rm s} = \boldsymbol{\Delta}\cdot\widehat{\boldsymbol{f}}^{\rm n}_{\rm H} + \Gamma_{\rm n}\left[p^{\rm n}_0 - p^{\rm e}_0 - p^{\rm p}_0 + \boldsymbol{v}_{\rm c}\cdot(\boldsymbol{p}^{\rm n} - \boldsymbol{p}^{\rm e} - \boldsymbol{p}^{\rm p})\right]\,, \tag{167}$$

where we have used the explicit form (81) of Θ_0. In addition to the external force, the two-fluid model allows one to prescribe one of the fluid force densities. In the present case it is most convenient to specify the "extended" hydrodynamic force $\widehat{\boldsymbol{f}}^{\rm n}_{\rm H}$ on the neutrons. As this force can only originate from the second fluid, we will refer to it as the *mutual force* $\boldsymbol{f}_{\rm mut}$, so we set

$$\widehat{\boldsymbol{f}}^{\rm n}_{\rm H} = \boldsymbol{f}_{\rm mut}\,. \tag{168}$$

Substituting the explicit conjugate momenta (67) and (68), we obtain the final expression for the entropy creation rate (167) as

$$T\Gamma_{\rm s} = \boldsymbol{\Delta}\cdot\boldsymbol{f}_{\rm mut} + \Gamma_{\rm n}\beta\,. \tag{169}$$

The first term on the right hand side is the work done by the mutual force, and the second term is the entropy created by beta reactions between the two fluids, for which the term "transfusion" has been coined [32]. The deviation from beta equilibrium characterized by β is found as

$$\beta \equiv \mu^{\rm p} + \mu^{\rm e} - \mu^{\rm n} - \frac{1}{2}m\left(1 - \frac{4\alpha}{\rho_{\rm n}}\right)\Delta^2\,, \tag{170}$$

where the last term gives the correction to the chemical equilibrium due to relative motion $\boldsymbol{\Delta}$ of the two fluids. The second law of thermodynamics for an isolated system states that entropy can only increase, i.e. $\Gamma_{\rm s} \geq 0$. In order for this to be identically true in (169), the mutual force $\boldsymbol{f}_{\rm mut}$ and the reaction rate $\Gamma_{\rm n}$ have to be of the form

$$\begin{aligned} \Gamma_{\rm n} &= \Xi\,\beta\,, &&\text{with} \quad \Xi \geq 0\,, \\ \boldsymbol{f}_{\rm mut} &= \eta\,\boldsymbol{\Delta} + \boldsymbol{\kappa}\times\boldsymbol{\Delta}\,, &&\text{with} \quad \eta \geq 0\,, \end{aligned} \tag{171}$$

where $\boldsymbol{\kappa}$ is an arbitrary vector characterizing a non-dissipative Magnus-type force orthogonal to the relative velocity. Further substituting the conjugate momenta in the expression for the hydrodynamic force densities (14), we find their explicit form

$$\boldsymbol{f}^{\rm n}_{\rm H} = n_{\rm n}(\partial_t + \boldsymbol{v}_{\rm n}\cdot\nabla)\left(m\boldsymbol{v}_{\rm n} + \frac{2\alpha}{n_{\rm n}}\boldsymbol{\Delta}\right) + n_{\rm n}\nabla\mu^{\rm n} + 2\alpha\Delta_j\nabla v^j_{\rm n}\,, \tag{172}$$

$$\boldsymbol{f}^{\rm c}_{\rm H} = n_{\rm p}(\partial_t+\boldsymbol{v}_{\rm c}\cdot\nabla)\left(m\boldsymbol{v}_{\rm c}-\frac{2(\alpha^{\rm en}+\alpha^{\rm pn})}{n_{\rm p}}\boldsymbol{\Delta}\right)+n_{\rm p}\nabla(\mu^{\rm p}+\mu^{\rm e})$$
$$-2\alpha\Delta_j\nabla v^j_{\rm c} - s(\partial_t+\boldsymbol{v}_{\rm c}\cdot\nabla)\left(\frac{2\alpha^{\rm sn}}{s}\boldsymbol{\Delta}\right)+s\nabla T\,. \quad (173)$$

We now make the simplifying assumption that we can neglect the entrainment of entropy, i.e. we assume that all the entrainment between the two fluids is due to the neutron-proton and neutron-electron contributions, so we set $\alpha^{\rm sn}=0$, which implies $\boldsymbol{\Theta}=0$. Using (67) we find

$$\boldsymbol{p}^{\rm e}+\boldsymbol{p}^{\rm p}-\boldsymbol{p}^{\rm n} = m\,(1-\varepsilon_{\rm n}-\varepsilon_{\rm c})\,\boldsymbol{\Delta}\,, \quad (174)$$

where we have defined the entrainment numbers

$$\varepsilon_{\rm n}\equiv\frac{2\alpha}{\rho_{\rm n}}\,, \quad\text{and}\quad \varepsilon_{\rm c}\equiv\frac{2\alpha}{\rho_{\rm c}}\,. \quad (175)$$

Putting all the pieces together, we obtain the momentum equations (168) and (165) in the form

$$(\partial_t+\boldsymbol{v}_{\rm n}\cdot\nabla)(\boldsymbol{v}_{\rm n}+\varepsilon_{\rm n}\boldsymbol{\Delta})+\nabla\left(\widetilde{\mu}^{\rm n}+\Phi\right)+\varepsilon_{\rm n}\Delta_j\nabla v^j_{\rm n}=\frac{1}{\rho_{\rm n}}\boldsymbol{f}_{\rm mut}\,, \quad (176)$$

$$(\partial_t+\ \boldsymbol{v}_{\rm c}\cdot\nabla)(\boldsymbol{v}_{\rm c}-\varepsilon_{\rm c}\boldsymbol{\Delta})+\nabla\left(\widetilde{\mu}^{\rm c}+\Phi\right)-\varepsilon_{\rm c}\Delta_j\nabla v^j_{\rm c}+\frac{s}{\rho_{\rm c}}\nabla T$$
$$=-\frac{1}{\rho_{\rm c}}\boldsymbol{f}_{\rm mut}+(1-\varepsilon_{\rm c}-\varepsilon_{\rm n})\,m\frac{\Gamma_{\rm n}}{\rho_{\rm c}}\boldsymbol{\Delta}\,. \quad (177)$$

with the specific chemical potentials $\widetilde{\mu}^{\rm n}\equiv\mu^{\rm n}/m$ and $\widetilde{\mu}^{\rm c}\equiv(\mu^{\rm p}+\mu^{\rm e})/m$.

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APPENDIX A: EVALUATION OF CONVECTIVE VARIATIONS

We write the particle flowlines as

$$x^i = x^i(\boldsymbol{a},t)\,, \quad (A1)$$

where the "particle coordinates" a^i are used to label individual particles and can be taken, for example, to be their initial position, i.e.

$$a^i = x^i(\boldsymbol{a},0)\,. \quad (A2)$$

This introduces a time-dependent map (or "pull-back") between the "material space" a^i and physical space x^i, and the associated Jacobian matrix $\mathcal{J}$ is

$$\mathcal{J}^i{}_j \equiv \left.\frac{\partial x^i}{\partial a^j}\right|_t\,. \quad (A3)$$

We consider the variations of fluid variables induced by *active* infinitesimal spatial displacements $\xi^i(\boldsymbol{x},t)$ and temporal shifts $\tau(\boldsymbol{x},t)$ of the fluid particle flowlines (A1), namely

$$x'^i(\boldsymbol{a},t') = x^i(\boldsymbol{a},t)+\xi^i(\boldsymbol{x},t)\,, \quad\text{and}\quad t'=t+\tau(\boldsymbol{x},t)\,. \quad (A4)$$

We note that the transformation (A4) not only shifts flowlines in space, but also in time. A physical quantity of the flow, $Q(\boldsymbol{x},t)$ say, is changed to $Q'(\boldsymbol{x}',t')$, and we define the corresponding *Eulerian* and *Lagrangian* variations as[7]

$$\delta Q \equiv Q'(\boldsymbol{x},t)-Q(\boldsymbol{x},t)\,, \quad (A5)$$
$$\Delta Q \equiv Q'(\boldsymbol{a},t')-Q(\boldsymbol{a},t)=Q'(\boldsymbol{x}',t')-Q(\boldsymbol{x},t)\,. \quad (A6)$$

By expanding ΔQ to first order using the definition (A4) of $x^{i'}$ and t', we find the relation

$$\Delta Q = \delta Q+\xi^j\,\nabla_j Q(\boldsymbol{x},t)+\tau\,\partial_t Q(\boldsymbol{x},t)\,. \quad (A7)$$

Let us consider the induced (first order) variation of the velocity $v^i\equiv\partial_t x^i(\boldsymbol{a},t)$, namely

$$\begin{aligned} v'^i(\boldsymbol{a},t') &= \partial_{t'}x'^i(\boldsymbol{a},t') = \partial_{t'}x^i(\boldsymbol{a},t)+\partial_t\xi^i(\boldsymbol{a},t) \\ &= \partial_t x^i(\boldsymbol{a},t)\,\left.\frac{\partial t}{\partial t'}\right|_{\boldsymbol{a}}+\partial_t\xi^i(\boldsymbol{a},t) \\ &= v^i(\boldsymbol{a},t)-v^i\,\partial_t\tau(\boldsymbol{a},t)+\partial_t\xi^i(\boldsymbol{a},t)\,, \quad (A8)\end{aligned}$$

which by (A6) corresponds to the following Lagrangian variation of the velocity:

$$\Delta v^i = \left[\partial_t\xi^i+v^l\nabla_l\xi^i\right]-\left[v^i\partial_t\tau+v^iv^l\nabla_l\tau\right]\,, \quad (A9)$$

and with (A7) the Eulerian variation is found as

$$\delta v^i = \left[\partial_t\xi^i+v^l\nabla_l\xi^i-\xi^l\nabla_l v^i\right]-\left[\partial_t\left(v^i\tau\right)+v^iv^l\nabla_l\tau\right]\,. \quad (A10)$$

From the conservation of mass one can derive an expression for the particle density n in terms of the Jacobian (A3), namely

$$n(\boldsymbol{x},t)=\frac{n_0(\boldsymbol{a})}{\det\mathcal{J}}\,, \quad (A11)$$

[7] Contrary to the Eulerian variation, the Lagrangian variation can be defined in different (non-equivalent) ways. The definition used here is based on comparing the quantity Q in different points by parallel-transport. Another common definition (e.g. see [11, 20]) consists in using the Lie-transported quantity instead. Both definitions are equivalent for scalars but differ for vectors and higher order tensors.

where $n_0(\boldsymbol{a}) = n(\boldsymbol{a}, 0)$ is the initial density at $t = 0$. Using (A3), the change of the Jacobian matrix $\mathcal{J}$ induced by the flowline variation (A4) can be found as

$$\begin{aligned} \mathcal{J}'^i{}_j(\boldsymbol{a}, t') &= \frac{\partial x'^i(\boldsymbol{a}, t')}{\partial a^j} = \left.\frac{\partial x^i(\boldsymbol{a}, t)}{\partial a^j}\right|_{t'} + \frac{\partial \xi^i}{\partial a^j} \\ &= \frac{\partial x^i(\boldsymbol{a}, t)}{\partial a^j} + \frac{\partial x^i(\boldsymbol{a}, t)}{\partial t} \left.\frac{\partial t}{\partial a^j}\right|_{t'} + \frac{\partial \xi^i}{\partial a^j} \\ &= \mathcal{J}^i{}_j(\boldsymbol{a}, t) - v^i \frac{\partial \tau}{\partial a^j} + \frac{\partial \xi^i}{\partial a^j}\,, \end{aligned} \tag{A12}$$

with the resulting Lagrangian variation (A6) expressible as

$$\Delta \mathcal{J}^i{}_j = \mathcal{J}^l{}_j \left(\nabla_l \xi^i - v^i \nabla_l \tau \right) . \tag{A13}$$

The derivative of a determinant $\det A$ with respect to a matrix element A_{ij} is given by

$$\frac{\partial \det A}{\partial A_{ij}} = \det(A)\, \left(A^{-1}\right)^{ij} , \tag{A14}$$

and therefore we can write the Lagrangian variation of the Jacobian determinant as

$$\Delta \left(\det \mathcal{J}\right) = \det(\mathcal{J}) \left(\mathcal{J}^{-1}\right)^j{}_i \, \Delta \mathcal{J}^i{}_j \,. \tag{A15}$$

The flowline variation (A4) therefore induces the Lagrangian change of the Jacobian

$$\frac{\Delta \left(\det J\right)}{\det J} = \nabla_l \xi^l - v^l \nabla_l \tau \,. \tag{A16}$$

Using (A11), the induced density variation is therefore found as

$$\Delta n = -n \nabla_l \xi^l + n v^l \nabla_l \tau \,, \tag{A17}$$

and with (A7) the corresponding Eulerian expression is found as

$$\delta n = -\nabla_l \left(n \xi^l\right) + \left[n v^l \nabla_l \tau - \tau \partial_t n \right] . \tag{A18}$$

By combining the results for velocity and density variations we find the variations of the current $n^i = n v^i$ as

$$\Delta n^i = \left[n\, \partial_t \xi^i(\boldsymbol{x}, t) + n^l \nabla_l \xi^i - n^i \nabla_l \xi^l \right] - n^i\, \partial_t \tau \,, \tag{A19}$$

$$\delta n^i = \left[n \partial_t \xi^i(\boldsymbol{x}, t) + n^l \nabla_l \xi^i - \nabla_l \left(n^i \xi^l\right) \right] - \partial_t \left(n^i \tau\right) . \tag{A20}$$

APPENDIX B: NOETHER IDENTITIES OF THE VARIATIONAL PRINCIPLE

In addition to the flowline variations considered so far, we will now also allow for *metric variations* δg_{ij}. Although we only consider Newtonian physics here, there is a-priori no reason to restrict ourselves to flat space. Most importantly, however, including metric variations allows us to obtain the form of the stress tensor T^{ij}_{H} and the associated momentum conservation (22) directly from the variational principle as a Noether identity, as opposed to constructing it from the equations of motion as we have done in Sec. III. Therefore we extend the variation (4) of the Lagrangian to

$$\delta \Lambda_{\mathrm{H}} = \sum p_0^X \,\delta n_X + \sum \boldsymbol{p}^X \cdot \delta \boldsymbol{n}^X + \frac{\partial \Lambda_{\mathrm{H}}}{\partial g_{ij}} \,\delta g_{ij} \,. \tag{B1}$$

Next consider the density change δn^X induced by a metric variation δg_{ij} at constant flowlines, i.e. constant $\mathcal{J}^i{}_j$. First we note that we can express the Jacobian as

$$\det \mathcal{J} = \epsilon_{ijk}\, \mathcal{J}^i{}_1\, \mathcal{J}^j{}_2\, \mathcal{J}^k{}_3 \,, \tag{B2}$$

and using (A14) the variation of the volume form $\epsilon_{ijk} = \sqrt{g}\,[ijk]$ induced by metric changes is expressible as

$$\delta \epsilon_{ijk} = \frac{1}{2} \epsilon_{ijk}\, g^{lm} \delta g_{lm} \,. \tag{B3}$$

Therefore we have

$$\left.\frac{\partial \det \mathcal{J}}{\partial g_{ij}}\right|_{\mathcal{J}} = \frac{1}{2} \det(\mathcal{J})\, g^{ij} \,, \tag{B4}$$

and using (A11) and (A18) we can write the variation of the density induced by spatial displacements $\boldsymbol{\xi}$ and metric variations δg_{ij} as

$$\delta n = -\nabla_l \left(n \xi^l\right) - \frac{1}{2} n g^{ij}\, \delta g_{ij} \,. \tag{B5}$$

$$\Delta n = -n \nabla_l \xi^l - \frac{1}{2} n g^{ij}\, \delta g_{ij} \,, \tag{B6}$$

where we have used the fact that with our definition of the Lagrangian variation (A7) we have

$$\Delta g_{ij} = \delta g_{ij} + \xi^l \nabla_l g_{ij} = \delta g_{ij} \,, \tag{B7}$$

as the metric is by definition constant under parallel transport. A metric change with fixed flowlines does not change the local velocity v^i, therefore the current variation can be written using (B5) and (A20) as

$$\delta n^i = \left[n\, \partial_t \xi^i(\boldsymbol{x}, t) + n^l \nabla_l \xi^i - \nabla_l \left(n^i \xi^l\right) \right] - \frac{1}{2} n^i g^{lj} \delta g_{lj} \,, \tag{B8}$$

$$\Delta n^i = \left[n\, \partial_t \xi^i(\boldsymbol{x}, t) + n^l \nabla_l \xi^i - n^i \nabla_l \xi^l \right] - \frac{1}{2} n^i\, g^{lj} \delta g_{lj} \,. \tag{B9}$$

When allowing for metric variations it is convenient (e.g. see [39]) to introduce the "diamond variation" $\lozenge \Lambda_{\mathrm{H}}$ as

$$\lozenge \Lambda_{\mathrm{H}} \equiv \frac{1}{\sqrt{g}} \delta \left(\sqrt{g}\, \Lambda_{\mathrm{H}} \right) = \delta \Lambda_{\mathrm{H}} + \frac{1}{2} \Lambda_{\mathrm{H}}\, g^{ij} \delta g_{ij} \,, \tag{B10}$$

such that the variation of the action (3) can now be written as (noting that $dV = \sqrt{g}\, d^3x$):

$$\delta \mathcal{I} = \int \lozenge \Lambda_{\mathrm{H}}\, dV\, dt \,. \tag{B11}$$

Substituting (B1), (B5) and (B8) and integrating by parts, $\Diamond\Lambda_{\rm H}$ can be cast in the form

$$\Diamond\Lambda_{\rm H} = -\sum f_i^X\,\xi_X^i + \frac{1}{2}T_{\rm H}^{ij}\,\delta g_{ij} + \nabla_l R^l + \partial_t R\,, \quad \text{(B12)}$$

where the canonical forces $\boldsymbol{f}_X$ have the explicit expression (11) and we defined the tensor $T_{\rm H}^{ij}$ as

$$T_{\rm H}^{ij} \equiv 2\frac{\partial\Lambda_{\rm H}}{\partial g_{ij}} + \Psi\, g^{ij}\,, \quad \text{(B13)}$$

using our earlier definition (20) of the generalized pressure Ψ.

Now consider a common displacement $\boldsymbol{\xi}$ of the *whole* system including the background metric, which induces a metric change

$$\delta g_{ij} = -2\nabla_{(i}\xi_{j)}\,, \quad \text{(B14)}$$

where (ij) indicates symmetric averaging, i.e. $2A_{(i}B_{j)} = A_iB_j + A_jB_i$. The corresponding Lagrangian variations (B9) and (B6) are found as

$$\Delta n_X \;=\; 0\,, \quad \text{(B15)}$$

$$\Delta n_X^i \;=\; n_X\left(\partial_t\xi^i + v_X^l\nabla_l\xi^i\right)\,. \quad \text{(B16)}$$

Substituting this into (B1), the induced $\Delta\Lambda_{\rm H}$ is

$$\Delta\Lambda_{\rm H} = \left(\sum n_X^i p^{X\,j} - 2\frac{\partial\Lambda_{\rm H}}{\partial g_{ij}}\right)\nabla_i\xi_j + J_{\rm H}^i\,\partial_t\xi_i\,, \quad \text{(B17)}$$

where we have used the definition (23) of the momentum density $\boldsymbol{J}_{\rm H}$. It is well known that contrary to the fully covariant Lagrangian for relativistic hydrodynamics (e.g. [13]), the Newtonian Lagrangian is not strictly Galilean invariant under boosts. This is due to the velocity dependence of the kinetic energy, as can be seen in the explicit form (52). We can therefore only demand strict invariance, i.e. $\Delta\Lambda_{\rm H} = 0$, for time-independent displacements, namely $\partial_t\boldsymbol{\xi} = 0$, which leads to the Noether identity

$$\frac{\partial\Lambda_{\rm H}}{\partial g_{ij}} = \frac{1}{2}\sum n_X^i p^{X\,j}\,. \quad \text{(B18)}$$

The left-hand side is manifestly symmetric in i and j, therefore we see that

$$\sum n_X^i p^{Xj} = \sum n_X^j p^{Xi}\,, \quad \text{(B19)}$$

and we can now write the (symmetric) stress tensor (B13) explicitly as

$$T^i_{{\rm H}j} = \sum n_X^i\, p_j^X + \Psi\, g^i{}_j\,. \quad \text{(B20)}$$

This tensor is identical to the expression (23) found earlier by construction from the equations of motion. It remains to be shown however, how the momentum conservation law (22) is directly obtainable as a Noether identity from the variational principle. Using (B17), (A7) and (B12) we can explicitly express the diamond variation as

$$\Diamond\Lambda_{\rm H} = -(\partial_t J^j)\,\xi_j - \nabla_l(\Lambda_{\rm H}\,\xi^l) + \partial_t(J_{\rm H}^l\,\xi_l)\,, \quad \text{(B21)}$$

which has to be identical to the expression (B12) for a common displacement $\boldsymbol{\xi}$ of the whole system, which after some partial integrations takes the form

$$\Diamond\Lambda_{\rm H} = \left(-\sum f^{X\,j} + \nabla_l T_{\rm H}^{lj}\right)\xi_j + \nabla_l(...)^l + \partial_t(...)\,. \quad \text{(B22)}$$

The requirement that the previous two expressions have to be identical (up to divergences and time derivatives) leads to the Noether identity

$$\partial_t J_{\rm H}^i + \nabla_j T_{\rm H}^{ij} = f^i_{\rm ext}\,, \quad \text{(B23)}$$

which is the momentum conservation law (22).

APPENDIX C: GALILEAN INVARIANCE OF $\mathcal{E}$

In this section we show that requiring the conjugate momenta p_0^X and $\boldsymbol{p}^X$ of (53) to transform as (55) under Galilean boosts (54) implies that the internal energy $\mathcal{E}$ has to be Galilean invariant. We assume that $\mathcal{E}(n_X, \boldsymbol{n}_X)$ transforms into $\mathcal{E}'(n_X, \boldsymbol{n}'_X)$ under a Galilean boost, where

$$\boldsymbol{n}'_X = \boldsymbol{n}_X + n_X\boldsymbol{V}\,. \quad \text{(C1)}$$

Therefore the conjugate momenta (53) in the frame moving with speed $-\boldsymbol{V}$ are of the form

$$-p_0^{X\prime} \;=\; \frac{1}{2}m^X\boldsymbol{v}_X^2 + m^X\boldsymbol{v}_X\cdot\boldsymbol{V} + \frac{1}{2}m^X\boldsymbol{V}^2 + \frac{\partial\mathcal{E}'}{\partial n_X}\,, \quad \text{(C2)}$$

$$\boldsymbol{p}^{X\prime} \;=\; m^X\boldsymbol{v}_X + m^X\boldsymbol{V} - \frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}'_X}\,, \quad \text{(C3)}$$

Using (53) to eliminate all terms containing $\boldsymbol{v}_X$, we arrive at

$$-p_0^{X\prime} = -p_0^X + \boldsymbol{V}\cdot\boldsymbol{p}^X + \frac{1}{2}m^X\boldsymbol{V}^2 + \left[\frac{\partial\mathcal{E}'}{\partial n_X} - \frac{\partial\mathcal{E}}{n_X} + \boldsymbol{V}\cdot\frac{\partial\mathcal{E}}{\boldsymbol{n}_X}\right]\,, \quad \text{(C4)}$$

$$\boldsymbol{p}^{X\prime} = \boldsymbol{p} + m^X\boldsymbol{V} + \left[\frac{\partial\mathcal{E}}{\partial\boldsymbol{n}_X} - \frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}'_X}\right]\,. \quad \text{(C5)}$$

By comparing with the required transformation properties (55) we see that a necessary and sufficient condition for this is the vanishing of the terms in brackets in (C4) and (C5). We can rewrite the partial derivatives of the energy function as follows

$$\frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}'_X} = \frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}_X}\cdot\left.\frac{\partial\boldsymbol{n}_X}{\partial\boldsymbol{n}'_X}\right|_{n_X} = \frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}_X}\,, \quad \text{(C6)}$$

and

$$\left.\frac{\partial\mathcal{E}'}{\partial n_X}\right|_{\boldsymbol{n}'_X} = \left.\frac{\partial\mathcal{E}'}{\partial n_X}\right|_{\boldsymbol{n}_X} + \frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}_X}\cdot\left.\frac{\partial\boldsymbol{n}_X}{\partial n_X}\right|_{\boldsymbol{n}'_X} = \left.\frac{\partial\mathcal{E}'}{\partial n_X}\right|_{\boldsymbol{n}_X} - \boldsymbol{V}\cdot\frac{\partial\mathcal{E}'}{\partial\boldsymbol{n}_X}\,. \quad \text{(C7)}$$

Inserting these identities into (C4) and (C5), the invariance requirement can be expressed as

$$\left.\frac{\partial \mathcal{E}}{\partial n_X}\right|_{\boldsymbol{n}_X} = \left.\frac{\partial \mathcal{E}'}{\partial n_X}\right|_{\boldsymbol{n}_X}, \quad \text{and} \quad \frac{\partial \mathcal{E}}{\partial \boldsymbol{n}_X} = \frac{\partial \mathcal{E}'}{\partial \boldsymbol{n}_X}, \quad \text{for all } X\,, \tag{C8}$$

therefore $\mathcal{E}'$ can only differ from $\mathcal{E}$ by a constant, which is unimportant because the absolute value of the energy scale is arbitrary. This shows that energy function $\mathcal{E}$ has to be Galilean invariant under the above assumptions.

APPENDIX D: NEWTONIAN LIMIT OF THE RELATIVISTIC LAGRANGIAN

As shown in the relativistically covariant framework by Carter [13], the equations of motion for conducting multi-constituent fluids can be derived from a covariant Lagrangian density of the form

$$\Lambda_{\rm cov} = -\rho c^2\,, \tag{D1}$$

where the scalar ρ is now the total mass-energy density of the system. For simplicity we consider here a two-fluid system, as generalizations to more fluids are straightforward while making the notation more cumbersome. The two fluids, A and B say, are described by the two 4-current densities $n_{\rm A}^\mu$, $n_{\rm B}^\mu$, and therefore the scalar $\Lambda_{\rm cov}(n_{\rm A}^\mu, n_{\rm B}^\mu)$ can only depend on the three independent scalar combinations of these two currents, for example

$$n_{\rm A} = \frac{1}{c}\sqrt{-g_{\mu\nu} n_{\rm A}^\mu n_{\rm A}^\nu}\,, \quad n_{\rm B} = \frac{1}{c}\sqrt{-g_{\mu\nu} n_{\rm B}^\mu n_{\rm B}^\nu}\,,$$

and

$$x = \frac{1}{c}\sqrt{-g_{\mu\nu} n_{\rm A}^\mu n_{\rm B}^\nu}\,, \tag{D2}$$

and so generally $\Lambda_{\rm cov} = \Lambda_{\rm cov}(n_{\rm A}, n_{\rm B}, x)$. Instead of x we can equivalently choose as a third independent quantity the combination

$$\frac{\Delta^2}{c^2} \equiv 1 - \left(\frac{n_{\rm A} n_{\rm B}}{x^2}\right)^2\,. \tag{D3}$$

We are interested here only in the purely hydrodynamic content of this framework, so we assume a flat space-time, i.e. a metric of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2\, dt^2 + d\boldsymbol{x}^2\,, \tag{D4}$$

with the time-coordinate $x^0 = t$ and so $g_{00} = -c^2$. When taking the Newtonian limit as $c \to \infty$, the metric becomes singular. The reason for this singular limit obviously lies in the fact that a locally Lorentzian theory reduces to a Galilean invariant theory, therefore the Lorentz invariance has to be broken in the limit. As the non-invertible metric no longer fully determines the space-time, we now have to *choose*[8] a preferred time coordinate, t say, in which to take the limit and which will reduce to the Newtonian absolute time.

The relation between the scalar rest-frame particle densities n_X and the densities n_X^0 in the preferred-time frame can be expressed from (D2) and (D4):

$$n_X = \frac{1}{c}\sqrt{c^2 (n_X^0)^2 - \boldsymbol{n}_X^2} = n_X^0\left[1 - \frac{1}{2}\left(\frac{\boldsymbol{v}_X}{c}\right)^2\right] + \mathcal{O}\left(c^{-4}\right)\,, \tag{D5}$$

where $(\boldsymbol{n}_X)^i = n_X^i$ is the spatial part of the 4-current n_X^μ in the preferred time frame, and the relation to the Newtonian 3-velocity $\boldsymbol{v}_X$ is simply $\boldsymbol{n}_X = n_X^0\, \boldsymbol{v}_X$. We see from this equation that if we choose the densities n_X^0 to represent the Newtonian particle number densities independent of c, then in the limit we find

$$\lim_{c\to\infty} n_X = n_X^0\,. \tag{D6}$$

We further note that the quantity Δ introduced in (D3) reduces to the relative velocity in the Newtonian limit, namely

$$\Delta^2 = (\boldsymbol{v}_{\rm A} - \boldsymbol{v}_{\rm B})^2 + \mathcal{O}\left((v/c)^2\right)\,. \tag{D7}$$

We now turn to the covariant Lagrangian $\Lambda_{\rm cov}$ of (D1) which we can quite generally be written as

$$\Lambda_{\rm cov} = -(n_{\rm A} m_{\rm A} + n_{\rm B} m_{\rm B})\, c^2 - \mathcal{E}(n_{\rm A}, n_{\rm B}, \Delta^2)\,, \tag{D8}$$

where the first term represents the rest-mass energy in the fluid frame, while $\mathcal{E}$ contains the "equation of state", i.e. the internal-energy function of the fluid. When we write this in the preferred time-frame using (D5), we obtain

$$\begin{aligned}\Lambda_{\rm cov} = & -(n_{\rm A}^0 m_{\rm A} + n_{\rm B}^0 m_{\rm B}) c^2 + \frac{1}{2} m_{\rm A}\, n_{\rm A}^0 \boldsymbol{v}_{\rm A}^2 + \frac{1}{2} m_{\rm B} n_{\rm B}^0 \boldsymbol{v}_{\rm B}^2 \\ & - \mathcal{E}(n_{\rm A}^0, n_{\rm B}^0, \Delta^2) + \mathcal{O}\left((v/c)^2\right)\,.\end{aligned} \tag{D9}$$

We see that this Lagrangian obviously diverges in the Newtonian limit $c \to \infty$ due to the rest-mass energies $n_X^0\, m_X\, c^2$. Before we can take this limit, we therefore have to renormalize the Lagrangian density by subtracting a finite counter-term that will make the limit finite. The most natural choice is obviously to subtract the mass-energy in the preferred-time frame that will determine the Newtonian absolute time. We therefore define the renormalized Lagrangian density $\Lambda_{\rm ren}$ as

$$\Lambda_{\rm ren} \equiv \Lambda_{\rm cov} + (n_{\rm A}^0 m_{\rm A} + n_{\rm B}^0 m_{\rm B})\, c^2\,. \tag{D10}$$

In $\Lambda_{\rm ren}$ we have explicitly broken Lorentz invariance by choosing a preferred time frame, and when taking the Newtonian limit we obtain the finite Lagrangian

$$\lim_{c\to\infty} \Lambda_{\rm ren} = m_{\rm A}\, \frac{\boldsymbol{n}_{\rm A}^2}{2\, n_{\rm A}} + m_{\rm B}\, \frac{\boldsymbol{n}_{\rm B}^2}{2\, n_{\rm B}} - \mathcal{E}(n_{\rm A}, n_{\rm B}, \Delta^2)\,, \tag{D11}$$

[8] See [15] for a more detailed discussion of this limit and how to construct a fully space-time covariant Newtonian framework.

which corresponds exactly to the Newtonian hydrodynamic Lagrangian Λ_{H} of (65).

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