

Fuchsian methods and spacetime singularities

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Abstract

Fuchsian methods and their applications to the study of the structure of spacetime singularities are surveyed. The existence question for spacetimes with compact Cauchy horizons is discussed. After some basic facts concerning Fuchsian equations have been recalled, various ways in which these equations have been applied in general relativity are described. Possible future applications are indicated.

1 Introduction

The singularity theorems of Penrose and Hawking showed that very often something goes wrong with the evolution of spacetimes according to the Einstein equations. They did not say what goes wrong and more than thirty years later we still have only limited information concerning the structure of general spacetime singularities. This situation represents an outstanding challenge to mathematical relativists. The most general rigorous results we possess on this subject have been obtained by Fuchsian methods. The purpose of this article is to give a portrait of the present state of development of the applications of these methods to general relativity. It also gives some indications as to what future improvements may be anticipated in this area.

How can we learn about singular solutions of differential equations? By looking at explicit solutions it is often possible to make a guess as to what the singularities might be like in general. In the case of equations arising in physics or other parts of science intuition coming from the applications may also make a contribution. Trial and error should not be neglected either. Once a conjectural picture of singularities of solutions of a particular equation has developed the next step is to confront it with the equation and to see if it is formally consistent. When formal consistency holds this is enough to make some consider the problem to be solved. Others see it as evidence for a certain solution of the problem but still want to have a definitive confirmation which can only be provided by rigorous mathematical theorems. Fuchsian methods allow the step from an intuitive to a rigorous result to be taken in many cases. When they apply they provide statements of the form that to each formal solution there corresponds a unique solution of the exact equation.

In the case of cosmological singularities in solutions of the Einstein equations the most influential heuristic work is that of Belinskii, Khalatnikov and Lifshitz (BKL). One important element of the picture they developed is the idea that near the singularity the dynamics at different spatial points should decouple. Thus the partial differential equations describing self-gravitating matter decouple asymptotically in the limit and reduce to ordinary differential equations. Furthermore, for many matter fields and assuming four spacetime dimensions, the limiting dynamics of generic solutions is that of a family of solutions of the Bianchi type IX vacuum equations (mixmaster model) with the spatial coordinates acting as parameters. The solutions of the mixmaster model have a complicated oscillatory behaviour and it is only recently that the principal features of their dynamics were understood rigorously [36], [37]. In view of this fact it is not surprising that the role of the mixmaster model as a template for the evolution of spatially inhomogeneous solutions of the Einstein equations has not been captured in mathematical theorems. It has been clearly exhibited in numerical calculations [6].

The BKL picture also predicts for which dimensions and combinations of matter fields oscillatory behaviour is not to be expected. In such cases the dynamics suggested by the BKL analysis is much simpler, with the most important geometric quantities varying in a monotone fashion near the singularity. It is in this situation that Fuchsian techniques can be expected to apply and indeed this has turned out to be the case in a variety of examples.

The paper is organized as follows. Section 2 describes the earliest application of Fuchsian techniques to general relativity known to the author, namely the construction of large classes of vacuum spacetimes with Cauchy horizons. In Section 3 a general introduction to Fuchsian techniques is given and an existence and uniqueness theorem of wide applicability is presented. A number of applications of this theorem to the construction of classes of spacetimes with certain kinds of singularities are described in Section 4. The last section lists existing generalizations of these results and suggests potential further developments.

2 Cauchy horizons

One of the most interesting issues concerning the nature of spacetime singularities is that of the strong cosmic censorship hypothesis. We recall the formulation of this hypothesis given by Eardley and Moncrief [14]. Corresponding to any initial data set for the Einstein-matter equations on a compact manifold there is a unique maximal Cauchy development which, intuitively, is the largest globally hyperbolic solution of the Einstein equations with the given initial data. The formulation of the strong cosmic censorship hypothesis which is of interest here is that for generic initial data on the given manifold the maximal Cauchy development should be inextendible. This means that for almost all initial data it is not possible to embed the maximal Cauchy development in a bigger solution of the Einstein-matter equations for which the hypersurface carrying the initial data remains a Cauchy surface. If this failed the original initial hypersurface

would not be a Cauchy surface for the extended spacetime in which the maximal Cauchy development was embedded. This means that the uniqueness of the time development in terms of initial data would no longer be guaranteed and that, from a physical point of view, predictability would break down.

In order to make the above formulation complete it is necessary to fix the notion of generic initial data precisely. Usually this is done by requiring this set of initial data to contain an open dense subset of the set of all initial data on the given manifold in a suitable topology. There are obvious candidates for the topology to be used. More problematic is the question of the choice of matter fields. As formulated the hypothesis depends implicitly on fixing a particular type of matter fields to describe the matter content of spacetime. In fact we cannot expect it to be true without limitations on the kind of matter chosen. One criterion is that the given matter model develops no singularities when considered as a test field in Minkowski space. Examples are a linear scalar field, the Maxwell field or collisionless matter described by the Vlasov equation. (See [33] for a general discussion of the Einstein-Vlasov system.) Examples of matter models which form singularities in flat space and which are therefore bad from the point of view of cosmic censorship include dust and perfect fluids with pressure. One way of avoiding the issue of the choice of matter fields is to concentrate on the vacuum case.

When it does happen that the maximal Cauchy development of certain initial data is extendible the boundary of the maximal Cauchy development in the extension is by definition a Cauchy horizon. It is null hypersurface with some (perhaps low) degree of regularity. A possible approach to learning more about cosmic censorship is to construct spacetimes which contain a Cauchy horizon and are otherwise as general as possible. A clean case to start with is that where the Cauchy horizon is compact and smooth.

In [21] and [22] Moncrief proved the existence of large classes of vacuum spacetimes with smooth compact Cauchy horizons. All these spacetimes have one spacelike Killing vector and are analytic (C^ω). He did so by solving a Cauchy problem for a class of singular partial differential equations. This is apparently the first occurrence in the literature on general relativity of Fuchsian methods, a technique which will be introduced in the next section. The presence of a Killing vector is not an accident and Moncrief and Isenberg [23] have shown that under suitable conditions the existence of a compact analytic Cauchy horizon automatically implies the existence of at least one Killing vector. Later it was found that analyticity could be replaced by the weaker condition of smoothness in the conditions of their theorem [15]. Moreover it has been shown that in this context a compact Cauchy horizon in a smooth spacetime is itself smooth [9]. Moncrief and Isenberg conjectured in [18] that any vacuum spacetime with a smooth compact Cauchy horizon has a Killing vector which is tangent to the generators of the horizon and whose integral curves are closed. This means that the Cauchy horizon, considered as an abstract manifold, admits an action of the circle S^1 . Not all three-dimensional manifolds admit such an action and thus a restriction on the topology of the manifold results. It must be a Seifert manifold. All these manifolds admit geometric structures in the sense of

Thurston and six of the Thurston geometries can be realized in this way. In [31] it was shown by a quite different method, using the Cheeger-Gromov theory of collapsing Riemannian manifolds with bounded curvature, that a smooth compact Cauchy horizon in a spacetime satisfying the Einstein equations coupled to certain matter models has a topology which is restricted to those which admit one of seven Thurston geometries. These are the six corresponding to Seifert manifolds together with the type Sol. In terminology more familiar to relativists this last type is related to Bianchi type VI₀. In [31] it was shown that all topologies of Seifert type allowed by this classification occur in locally spatially homogeneous solutions of the vacuum Einstein equations.

The results of Moncrief and Isenberg are limited to the case of the vacuum Einstein equations. It is interesting to ask to what extent similar results are true in the presence of matter. Some insight into this question can be obtained by looking at the spatially homogeneous case. There are (locally) homogeneous vacuum spacetimes which have a compact Cauchy surface and a compact Cauchy horizon (cf. [10].) In contrast to this, locally spatially homogeneous spacetimes with a compact Cauchy surface and phenomenological matter can be shown to have no Cauchy horizon in many cases [30]. This suggests that the generality of the spacetimes containing Cauchy horizons produced by Moncrief's construction is essentially dependent on the absence of matter sources in the field equations. If this is true then the Isenberg-Moncrief procedure is also limited to the source-free case. (Here the term source-free is meant to include the case of a source-free Maxwell field which acts as a source in the Einstein equations.)

3 A general existence and uniqueness theorem

This section describes the theory of Fuchsian equations. This class of equations has the form

$$t\partial u/\partial t + Nu = tf(t, x, u, u_x) \quad (1)$$

Here u is a function of a real number t , thought of as a time coordinate, and a point x in \mathbf{R}^n , thought of as a point of space. This function takes values in \mathbf{R}^k . The $k \times k$ matrix-valued function N depends only on x . It satisfies some positivity condition, as will be discussed in more detail later. The function f satisfies some regularity conditions which will also be specified later. The notation u_x is used as a shorthand for the collection of partial derivatives of u with respect to the variables x . For applications it is useful to note the following. Suppose that instead of (1) we had the equation

$$t\partial u/\partial t + Nu = t^\alpha f(t, x, u, u_x) \quad (2)$$

for some positive real number α . Defining $s = t^\alpha$ leads to the equation

$$s\partial u/\partial s + \alpha Nu = s[\alpha f(s^{1/\alpha}, x, u, u_x)] \quad (3)$$

which is of the form (1). Thus an equation of the form (2) can always be reduced to one of the form (1).

The assumption made on f is that it should be *regular* in a sense to be defined now. It is assumed that f is smooth for $t > 0$ and converges uniformly to zero on compact subsets of (x, u, u_x) -space as $t \rightarrow 0$. Moreover it is assumed that partial derivatives of f of any order with respect to x , u and u_x satisfy the same condition. This is the definition of regularity to be used in a smooth setting. In an analytic setting, which is where the strongest theorem is available, it is assumed in addition that f is continuous in t and analytic in the remaining arguments for $t > 0$ (cf. [20]). As to the positivity assumption on N , it would be very convenient to assume that N was positive definite. Unfortunately, this often does not hold in the applications and it is necessary to make do with a weaker condition. This can be expressed abstractly by requiring that the matrix exponential e^{tN} is bounded on compact sets of x -space for t close to zero. As has been proved in [1] a sufficient condition for this is that the eigenvalues of N are everywhere non-negative and that zero eigenvalues if they occur do not give rise to non-diagonal Jordan blocks.

With these preliminaries in hand we state the main general theorem.

Theorem If f is regular in the analytic sense and N , depending analytically on x , satisfies the condition that e^{tN} is bounded near $t = 0$ on compact subsets of x -space then the equation (1) has a unique solution u which is regular and tends to zero as $t \rightarrow 0$.

The definition of regularity of u used here is the analogue of that for f . This theorem was proved in [20]. Given that the aim here is to construct large classes of solutions it may seem strange that the theorem gives a unique solution. The explanation is that the equation which the theorem is applied to arises from the original equation by a reduction process where u represents a correction to the leading order behaviour. The leading order behaviour is given in terms of free functions and it is these functions which give rise to the generality of the construction. This will be seen in more detail later. There are analogues of the above theorem in the smooth setting but it is difficult to state a general result which is widely applicable. The difficulty is that in the smooth case the reduction process must give equations which are not only in Fuchsian form but are also hyperbolic in a suitable sense. Hyperbolicity is necessary to handle even the regular Cauchy problem.

4 Construction of generic classes of solutions of the Einstein equations

A key test case in the application of Fuchsian methods to general relativity is given by the Gowdy spacetimes with spatial topology T^3 . A heuristic analysis of the structure of singularities in this case was given by Grubišić and Moncrief [16]. The Gowdy spacetimes are solutions of the vacuum Einstein equations with two spatial Killing vectors and an additional discrete symmetry. The analysis of the structure of singularities in this class of spacetimes reduces essentially to

that of a pair of wave equations for two functions P and Q . The general solution depends on four functions of one space variable. Its asymptotic form is

$$P(t, \theta) = k(\theta) \log t + \phi(\theta) + t^\epsilon u(t, \theta) \quad (4)$$

$$Q(t, \theta) = Q_0(\theta) + t^{2k(\theta)}(\psi(\theta) + v(t, \theta)) \quad (5)$$

for some constant $\epsilon > 0$. A Fuchsian system is obtained by writing the Einstein equations for P and Q in terms of u and v . The pair (u, v) in this example plays the role of the function u in (1). The function k is known as the asymptotic velocity or simply velocity. The finding of Grubišić and Moncrief was that the above asymptotic form is formally consistent provided $0 < k(\theta) < 1$ for all θ . The inequalities imposed on k constitute the low velocity condition. In [20] it was proved that given analytic functions k , Q_0 , ϕ and ψ satisfying the low velocity condition there is a unique Gowdy solution for which P and Q have the above asymptotic form. In the case where the low velocity condition fails it is possible to obtain solutions depending on three free functions. The restriction which kills the fourth free function is that Q_0 is required to be constant.

The low velocity condition can be related to the notion of generalized Kasner exponents. Given a spacetime and a foliation by spacelike hypersurfaces, let λ_i be the eigenvalues of the second fundamental form of the leaves of the foliation. These define scalar functions on spacetime. If the mean curvature $\sum_i \lambda_i$ is everywhere non-zero then the generalized Kasner exponents (GKE) are defined to be $p_i = \lambda_i / \sum_j \lambda_j$. They satisfy $\sum_i p_i = 1$. In spacetimes with the asymptotics described above the GKE of the foliation by hypersurfaces of constant time converge to functions of the spatial coordinates as the singularity is approached. These functions satisfy the condition $\sum_i p_i^2 = 1$. In the Gowdy models on T^3 we can order the eigenvalues, and correspondingly the GKE, in such a way that the indices 2 and 3 correspond to the symmetry directions. Then the low velocity restriction is related to the condition that $p_1 < 0$.

There are other possible topologies for Gowdy spacetimes. They may be defined on $S^2 \times S^1$ or S^3 . In these cases the Killing vectors defining the symmetry have zeroes defining axes. On the axes the equations obtained by factoring out the symmetry directions have singularities. The possibility of generalizing the results of [20] to Gowdy spacetimes with these other topologies was studied in [39]. Fuchsian techniques could be applied but in that case they did not give the full number of free functions. It is not clear whether the restriction on the number of free functions is an essential feature of the problem or a failure of the technique.

If the discrete symmetry which is part of the definition of the class of Gowdy spacetimes is dropped the class of T^2 -symmetric vacuum spacetimes is obtained. There is evidence to suggest that general spacetimes of this class have oscillatory singularities [7] and so cannot be treated by Fuchsian methods. Partially restoring the discrete symmetry leads to the class of vacuum spacetimes with polarized T^2 symmetry. In the latter class the singularity is not expected to be oscillatory. It was shown in [17] that Fuchsian techniques can be applied to analyse the structure of the singularity in that case in a class of spacetimes

depending on the maximum number of free functions. This class corresponds to high velocity in the sense that $p_1 > 0$.

The results on Gowdy spacetimes on T^3 have been generalized to the case of the Einstein equations coupled to certain matter fields motivated by string theory [25]. This is the Einstein-Maxwell-axion-dilaton system. The behaviour found is closely analogous to that in the vacuum case.

All the results discussed in this section up to now concern spacetimes with two Killing vectors and in that case the equations obtained by dividing out the symmetry involve only one spatial variable. It should be emphasized that a strength of Fuchsian techniques is that they do not impose a restriction on the number of space dimensions of the system to be analysed. Thus there is no a priori reason why they should not apply to solutions of the Einstein equations with less than two symmetries. The restrictions which occur are due to the dynamics of the classes of spacetimes being considered near their singularities, with oscillatory behaviour obstructing the use of Fuchsian methods.

In [19] the theorem stated in the last section was applied to construct large classes of solutions of the vacuum Einstein equations with $U(1)$ symmetry. In that case the equations obtained by factoring out the symmetry are equations in two space dimensions. General solutions with $U(1)$ symmetry are expected to show oscillatory behaviour near their initial singularities. This has been confirmed numerically by Berger and Moncrief [8]. The subclass of polarized models has singularities with monotone behaviour. These depend on two free functions, compared to the four functions of the general case with $U(1)$ symmetry. There is also an intermediate case, the half-polarized case, where the solutions depend on three free functions and where Fuchsian methods can be applied as shown in [19].

In the BKL picture singularities in general vacuum solutions in 3+1 dimensions typically show behaviour of mixmaster type. In most cases this is not changed by the addition of matter fields. An exception is a linear minimally coupled scalar field. This suppresses the oscillations within the BKL picture. This is independent of symmetries and indeed Fuchsian methods can be used to construct solutions of the Einstein-scalar field system without symmetries and depending on the maximal number of free functions which have simple singularities whose asymptotics can be described in great detail [1].

A heuristic analysis in the spirit of BKL shows that the oscillations near the singularity in general solutions of the vacuum Einstein equations disappear when the spacetime dimension is at least eleven [13]. A corresponding proof of the existence of large classes of solutions with the expected properties was given in [12]. In that paper similar results were shown in all dimensions and in the presence of a rather general class of field-theoretic matter models. The suppression of oscillatory behaviour is due to the presence of a dilaton, a scalar field usually present in models inspired by string theory. It is, however, important to note that a scalar field is not in itself enough to guarantee the disappearance of oscillations. When form fields (such as the Maxwell field) are present oscillatory behaviour may persist.

5 Extending the method

The results of the last section were confined to the case where the functions prescribed and the solutions obtained are analytic. This is an artificial restriction and it would be preferable to have results which apply to prescribed functions and solutions which are smooth (C^∞) or only of finite differentiability. Only the smooth case will be discussed although the proofs are such that it is clear that they would also work for functions belonging to Sobolev spaces of sufficiently high order. The big step is that between C^ω and C^∞ , since it is at this level that it becomes important to use the hyperbolicity of the equations.

The results for Gowdy solutions on T^2 described in the last section were generalized to the case of C^∞ solutions in [32]. The methods of proof used are likely to be of much wider applicability but significant additional work is still required in any particular example. The only additional case in which this work has been done so far is that of Gowdy spacetimes with spherical topology [39].

There is older work in which equations of Fuchsian type were used to determine the structure of singularities in solutions of the Einstein equations without symmetry. These solutions are subject to another kind of restriction; they have isotropic singularities. The notion of an isotropic singularity arose in the context of Penrose's Weyl curvature hypothesis. Penrose proposed that initial singularities of solutions of the Einstein equations should have a special structure corresponding in some sense to a state of low entropy. This provides a selection criterion for initial data for cosmological models. With this motivation in mind it is interesting to know how many solutions of the Einstein equations there are with singularities satisfying this condition. The first general results on this question were developed in [26]. The required existence proofs were provided in [11]. They make heavy use of semigroup theory.

The results of [26] concerned solutions of the Einstein equations coupled to a perfect fluid with equation of state $p = \frac{1}{3}\rho$. They were generalized to a fluid with the more general equation of state $p = k\rho$ in [5]. In both cases the theorems obtained show that the solutions having an isotropic singularity depend on half the number of free functions allowed by general solutions of the Einstein equations. In [2] theorems were proved where the Euler equations are replaced by the Vlasov equation for collisionless matter. Note that the theorem of Section 3 does not apply directly to the Einstein-Vlasov system since this is a system of integrodifferential equations instead of a system of differential equations as in the hypotheses of that theorem. The picture of isotropic singularities obtained in the case of collisionless matter is quite different from that for a fluid. There is much more freedom to give initial data. The initial phase space density of particles can be prescribed almost without restriction. This discrepancy suggested looking at the case of matter described by the Boltzmann equation, which is in a sense intermediate between the Euler and Vlasov descriptions. It turns out that the behaviour of the collision cross section for large momenta plays a crucial role. If the cross section grows fast enough in the limit of large momenta the initial phase space density is forced to be in equilibrium which effectively reduces this case to the fluid case. If, on the other hand, the collision cross sec-

tion has slow growth the freedom in the initial phase space density is as great as in the Vlasov case. This has been shown on the level of formal expansions by Tod [41] although no existence proofs are yet available.

At several points in the above we have used function counting as a criterion for the generality of solutions. This is not very satisfactory and has been used up to now since it is the only thing which can presently be proved. What is desirable is to have theorems which say that the solutions having a particular asymptotic form near their singularities include all solutions arising from a non-empty open set of initial data on a regular Cauchy surface. It would be even better if this open set could also be described so as to see what kind of restriction on initial data it really represents. So far these goals have only been reached in one case, namely in the work of Ringström [38] concerning Gowdy solutions on T^3 .

One obstruction to the application of Fuchsian methods has already been described. This is the fact that the solutions may show a complicated oscillatory behaviour near the singularity. It is not the only obstruction. Another phenomenon which arises naturally within the BKL picture is that small-scale spatial structures can be formed as the singularity is approached. In situations where Fuchsian techniques can be applied it is typical that spatial derivatives of important geometrical quantities blow up at a rate which is comparable to the rate with which the undifferentiated quantities blow up as the singularity is approached. In the phenomenon of formation of small-scale spatial structure this property of the spatial derivatives no longer holds. The simplest instance is the formation of spikes in Gowdy solutions [35]. In that paper families of examples were constructed which depend on the maximum number of free functions. Thus in the sense of function counting Gowdy solutions with spikes are as common as those without. The development of spikes is associated with the presence of velocities greater than one. The low velocity condition in the Fuchsian existence theorems for Gowdy solutions prevents the occurrence of spikes. It is to be expected on the basis of the BKL picture that the formation of small-scale structure will occur more generally and there is numerical evidence supporting this [8].

It has already been mentioned that the possible applications of Fuchsian methods to field-theoretic matter models have been worked out in some generality in [12]. Symmetry assumptions can lead to further tractable cases and some of these were looked at in [25]. Much less has been done for phenomenological matter models such as perfect fluids and kinetic theory. Apart from the work on isotropic singularities the only available results are those of Anguige [3], [4] on certain solutions of the Einstein-Euler system with two spacelike Killing vectors. What happens for spacetimes without symmetry with a scalar field and a fluid or kinetic matter?

There is an application of Fuchsian methods to the asymptotics of expanding cosmological models which can be envisaged. Cosmological models with positive cosmological constant Λ often exhibit an inflationary phase of exponential expansion. At least formally the asymptotics obtained seems to be such as to be amenable to a Fuchsian treatment. As yet there is no analysis of this kind in

the literature. The idea of a mathematical similarity between certain spacetime singularities and inflationary expanding phases of cosmological models opens up the perspective of an interesting exchange of information between apparently unrelated areas of research. A formal analysis of the asymptotics of expanding cosmological models with $\Lambda > 0$ has been given in [40]. A corresponding treatment of the case of power-law inflation, where the cosmological constant is replaced by a scalar field with exponential potential, can be found in [24].

To finish, we mention a different kind of application of Fuchsian methods in relativity. In this case no spacetime singularities are involved. Instead the singularity in the equations arises from a coordinate singularity due to the use of polar coordinates. Consider a spherically symmetric solution of the Einstein equations coupled to some matter fields. If the solution is in addition static the Einstein equations reduce to a system of ordinary differential equations in a radial variable r . It has a singularity at the centre of symmetry, $r = 0$. In the case that the matter is a fluid existence theorems were obtained in [34]. The proofs were based on an existence theorem for singular ODE which, with hindsight, belongs to the class of Fuchsian equations. Corresponding results in the cases of collisionless matter and elastic solids were proved in [29] and [28] respectively. In the former case, integrodifferential equations occur, the theorem of [34] is not applicable, and existence has to be proved by hand. The same existence theorem for singular ODE has been applied in [27] to analyse the constraint equations of the Einstein-Vlasov-Maxwell system in the spherically symmetric case,

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