

# Multi-spin strings on $AdS_5 \times T^{1,1}$ and operators of $\mathcal{N} = 1$ superconformal theory

Nakwoo Kim\*

*Max-Planck-Institut für Gravitationsphysik  
 Albert-Einstein-Institut  
 Am Mühlenberg 1, D-14476 Golm, Germany*

We study rotating strings with multiple spins in the background of  $AdS_5 \times T^{1,1}$ , which is dual to a  $\mathcal{N} = 1$  superconformal field theory with global symmetry  $SU(2) \times SU(2) \times U(1)$  via the AdS/CFT correspondence. We analyse the limiting behaviour of macroscopic strings and discuss the identification of the dual operators and how their anomalous dimensions should behave as the global charges vary. A class of string solutions we find are dual to operators in  $SU(2)$  subsector, and our result implies that the one-loop planar dilatation operator restricted to the  $SU(2)$  subsector should be equivalent to the hamiltonian of the integrable Heisenberg spin chain.

PACS numbers: 11.25.-w

## I. INTRODUCTION

According to the AdS/CFT correspondence [1], the quantum string spectrum in anti-de Sitter (AdS) space is identical to that of a certain conformal field theory (CFT) formulated on its boundary. The direct check of this conjecture is plagued by the difficulty of superstring quantization in curved backgrounds so it has been a challenge to perform a quantitative test beyond the supersymmetric subsector and their protected data.

A way to get around this problem was suggested recently and proved to be very successful. One makes use of classical string solitons, not necessarily supersymmetric, and performs semiclassical quantization of the string theory to compare with the computations in the dual gauge theory. The inverse of a large global charge plays the role of a new expansion parameter, enabling perturbative computations on both sides of the duality. The celebrated BMN limit [2] considers small deformations of half-BPS operators with large conformal dimensions, which are dual to pointlike strings orbiting in the  $S^5$ . It amounts to taking the Penrose limit of the given backgrounds and by exploiting the fact that the string theory in the plane-wave becomes free in the light-cone gauge [3], one obtains all-loop results for a large class of operators in the planar limit of the dual conformal field theory.

This program can be extended to other classical string solutions. The implication of the macroscopic spinning string solutions on the dual conformal field theory was first discussed in [4], and further generalized and refined in many subsequent publications [5], and see [6] for a review and more complete list of references. More precise comparison of the spectra has been made possible thanks to the crucial observation that the one-loop dilatation operator for pure scalar operators is isomorphic to the integrable  $SO(6)$  spin chains [7]. The anomalous dimensions for very long operators can be computed by solving

the Bethe ansatz equations in the thermodynamic limit, which are then compared to the energy of string solitons and impressive quantitative agreements have been observed [8]. There have been efforts to directly relate the quantum spin chain models with the string nonlinear sigma model, by comparing the higher conserved charges [9], or by considering the continuum limits of the spin chains [10, 11], to derive the nonlinear sigma model.

A natural question then is how much of the above developments, especially the agreement at the quantitative level, can be extended to other examples of the AdS/CFT correspondence, which typically have less supersymmetries. In this paper we choose to study the IIB string theory of  $AdS_5 \times T^{1,1}$ , which is dual to a  $\mathcal{N} = 1$  superconformal field theory with  $U(N) \times U(N)$  gauge group and bifundamental matter multiplets, as first described in [12]. Since the string dynamics in  $AdS_5$  should be identical to the maximally supersymmetric case, we restrict ourselves on the strings moving in the squashed sphere,  $T^{1,1}$ .

We stress that unlike the maximally supersymmetric case of  $AdS_5 \times S^5$ , the isometry of  $T^{1,1}$ ,  $SU(2) \times SU(2) \times U(1)$ , is *not* a consequence of the supersymmetry. It renders the nature of our analysis highly dynamical. As  $T^{1,1}$  is mapped to the moduli space of the gauge theory, the spinning strings in  $T^{1,1}$  are dual to pure scalar operators. We will find the on-shell relations between the conformal dimensions and the global charges expressed implicitly in terms of elliptic integrals. We should remark that some spinning string solutions of  $AdS_5 \times T^{1,1}$  have been studied already in [13]. In this paper more solutions are covered and we also discuss the dual operator to each rotating string solution. We hope our results can direct the gauge theory computation towards the dynamical confirmation of the generalized AdS/CFT correspondence.

This article is organised as follows. In Sec. 2 we will briefly review the duality of  $AdS_5 \times T^{1,1}$  and the dual  $\mathcal{N} = 1$  superconformal field theory. In Sec. 3 we present the multi-spin string solutions in  $T^{1,1}$ . In Sec. 4 we consider the limiting cases when the conserved quantities become large, and discuss the dual operators of  $\mathcal{N} = 1$

---

\*Electronic address: kim@aei.mpg.de

conformal field theory. In Sec. 5 we end with discussions and concluding remarks. The conventions and basic properties of elliptic integrals which were used in this paper can be found in the appendix.

## II. $T^{1,1}$ AND THE DUAL $\mathcal{N} = 1$ SUPERCONFORMAL FIELD THEORY

In this paper we will study strings moving in  $T^{1,1}$ , which is a homogeneous space  $(SU(2) \times SU(2))/U(1)$ , with  $U(1)$  chosen to be a diagonal subgroup of the maximal torus in  $SU(2) \times SU(2)$ . The explicit form of the metric is best written as a  $U(1)$  bundle over  $S^2 \times S^2$ ,

$$ds^2 = a(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + b(d\psi + p \cos \theta_1 d\phi_1 + q \cos \theta_2 d\phi_2)^2, \quad (1)$$

where  $\theta_i, \phi_i$  are the coordinates of two  $S^2$ , and the  $U(1)$  is denoted by  $\psi \in [0, 4\pi]$ . The space is an Einstein manifold if  $a = \frac{1}{6}, b = \frac{1}{9}$ , and if we further choose  $p = q = 1$  the space becomes supersymmetric:  $T^{1,1}$  provides the angular part of a singular Calabi-Yau manifold. One can easily see from Eq. (1) that the isometry is  $SU(2) \times SU(2) \times U(1)$ . The three mutually commuting Killing vectors can be chosen as  $\partial_{\phi_1}, \partial_{\phi_2}, \partial_{\psi}$ .

The dual conformal field theory with  $\mathcal{N} = 1$  supersymmetry, as identified in [12], has gauge group  $U(N) \times U(N)$ , and two chiral multiplets  $A_i$  in  $(N, \bar{N})$  and another two,  $B_i$ , in  $(\bar{N}, N)$ . This theory obviously has  $SU(2) \times SU(2)$  global symmetry which act separately on the doublets  $A_i, B_i$ , and also an anomaly-free  $U(1)$  R-symmetry. Altogether, these global symmetries are to be identified with the isometry group of  $T^{1,1}$ . The theory is also equipped with a quartic superpotential which is invariant under the global symmetries,

$$W = \frac{g}{2} \epsilon^{ij} \epsilon^{kl} \text{tr} A_i B_k A_j B_l. \quad (2)$$

Combined with the conformal invariance we see that the conformal dimension of  $A_i, B_i$  should be  $3/4$  at the conformal fixed point.

In this paper we are interested in the closed strings rotating in  $T^{1,1}$  with radius  $\lambda^{1/4}$ . As well known, the AdS/CFT correspondence relates  $\lambda$  to the 't-Hooft coupling constant of the dual gauge field theory. We choose to work with the Polyakov action in the conformal gauge. We choose the ansatz that the string is spinning along the three commuting Killing directions, and at rest along all other directions.

$$t = \kappa\tau, \quad \phi_1 = \omega_1\tau, \quad \phi_2 = \omega_2\tau, \quad \psi = \nu\tau,$$

and the remaining two angles,  $\theta_1, \theta_2$  depend only on  $\sigma$ . Then it is easy to show that the gauge fixing constraint  $\dot{X}X' = 0$  is trivially satisfied, while  $\dot{X}^2 + X'^2 = 0$  becomes

$$\kappa^2 = a(\theta_1'^2 + \theta_2'^2 + \omega_1^2 \sin^2 \theta_1 + \omega_2^2 \sin^2 \theta_2) + b(\nu + \omega_1 \cos \theta_1 + \omega_2 \cos \theta_2)^2, \quad (3)$$

which is just an integrated form of the equations of motion for  $\theta_i$ . Clearly we can treat the reduced string equations of motion as a classical mechanics system with *time*  $\sigma$ , and  $\kappa$  can be identified as the energy. We also have the periodicity condition  $\theta_i(\sigma + 2\pi) = \theta_i(\sigma)$ , up to the periodicity of the coordinates  $\theta_i$ .

The conserved quantities we are interested in are the energy  $E = \sqrt{\lambda}\kappa$ , in addition to the following three angular momenta,

$$J_A \equiv P_{\phi_1} = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \left[ \omega_1 (a \sin^2 \theta_1 + b \cos^2 \theta_1) + b(\nu + \omega_2 \cos \theta_2) \cos \theta_1 \right], \quad (4)$$

$$J_B \equiv P_{\phi_2} = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \left[ \omega_2 (a \sin^2 \theta_2 + b \cos^2 \theta_2) + b(\nu + \omega_1 \cos \theta_1) \cos \theta_2 \right], \quad (5)$$

$$J_R \equiv P_{\psi} = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} b(\nu + \omega_1 \cos \theta_1 + \omega_2 \cos \theta_2), \quad (6)$$

which are identified with the conformal dimension and other global charges of the dual operators.

From what we have discussed so far, one can easily infer the following *dictionary* which is crucial in the identification of dual operators to string solutions.

$$J_A \longleftrightarrow \frac{1}{2} \left[ \#(A_1) - \#(A_2) + \#(\bar{A}_2) - \#(\bar{A}_1) \right] \quad (7)$$

$$J_B \longleftrightarrow \frac{1}{2} \left[ \#(B_1) - \#(B_2) + \#(\bar{B}_2) - \#(\bar{B}_1) \right] \quad (8)$$

$$J_R \longleftrightarrow \frac{1}{4} \left[ \#(A_i) + \#(B_i) - \#(\bar{A}_i) - \#(\bar{B}_i) \right] \quad (9)$$

where for instance  $\#(A_1)$  counts how many times  $A_1$  appears in the dual composite operator.

In order to find general class of solutions with nontrivial  $\theta_1, \theta_2$  we need to know a constant of motion other than the energy. Whether it is possible or not is related to the question of integrability. Although this is certainly a very important issue, in this paper we consider solutions where only one coordinate, say  $\theta_1$ , is activated. The one-dimensional system is readily integrated, and it will be seen that we still have a rich class of nontrivial solutions whose dual operators are in general non-holomorphic combinations of the scalar fields  $A_i, B_i$ .

## III. SPINNING STRINGS IN $T^{1,1}$

### A. Single-Spin Solutions

Let us first consider the strings shrink to a point and orbiting with light velocity. It can be easily seen as the solution of the mechanical model when the particle is

at rest, i.e.  $\theta_i$  are fixed at 0 or  $\pi$ . If we evaluate the conserved quantities for instance at  $\theta_i = 0$ , we get the simple relation

$$bE^2 = J_A^2 = J_B^2 = J_R^2 = \lambda b^2(\nu + \omega_1 + \omega_2)^2 \quad (10)$$

i.e. all conserved quantities are equal up to sign, which is determined by the values of  $\theta_i$ . The linear relation between the charges implies that this string solution is in fact supersymmetric and dual to a chiral primary operator, which will be discussed in more detail in the next section. We remark that the semiclassical treatment of the string theory around the pointlike strings amounts to taking the Penrose limit of  $AdS_5 \times T^{1,1}$ , which was studied in [14].

Another class of simple solutions include rotating circular strings embedded in one of the two-spheres when  $\omega_1 = \omega_2 = 0$  and  $\nu \neq 0$ :

$$\theta_1 = n\sigma, \quad \theta_2 = 0 \text{ or } \pi, \quad (11)$$

in which case we have  $J_A = 0, J_B^2 = J_R^2 = \lambda b^2 \nu^2$ , and the energy is related to the angular momentum as follows:

$$E^2 = 9J_R^2 + \lambda \frac{n^2}{6}, \quad (12)$$

These solutions can be said to be single-spin solutions, as there is essentially only one nonvanishing component of spin.

Now we move to more general class of solutions with two spins, and in the following we will use  $\theta \equiv \theta_1, \omega \equiv \omega_1$  and set  $\theta_2 = 0, \omega_2 = 0$  to simplify the notations.

## B. Multi-Spin Solutions

The spinning string ansatz has been reduced to a one dimensional system with the following potential,

$$V(\theta) = a\omega^2 \sin^2 \theta + b(\nu + \omega \cos \theta)^2, \quad (13)$$

It is straightforward to integrate the equation for generic values of  $\omega, \nu$ , but before we present the full result below let us first consider the simpler case of  $\nu = 0$ .

For  $\nu = 0$ , the potential has the maximal value at  $\theta = \pi/2$  and without losing generality we can consider strings centered around the north pole. When  $y \equiv \kappa/\omega < \sqrt{a}$  we have a folded string, and the relevant integral is easily transformed into a complete elliptic integral, and from the periodicity condition  $\theta(\sigma + 2\pi) = \theta(\sigma)$  we get

$$\omega = \frac{2}{\pi} \sqrt{\frac{a}{a-b}} \mathbf{K} \left( \frac{y^2 - b}{a-b} \right). \quad (14)$$

It is also straightforward to express the nonvanishing components of angular momenta as functions of  $y$ ,

$$\frac{E}{\sqrt{\lambda}} = \frac{2y}{\pi} \sqrt{\frac{a}{a-b}} \mathbf{K} \left( \frac{y^2 - b}{a-b} \right), \quad (15)$$

$$\frac{J_A}{\sqrt{\lambda}} = \frac{2}{\pi} \sqrt{\frac{a}{a-b}} \left[ a \mathbf{K} \left( \frac{y^2 - b}{a-b} \right) - (a-b) \mathbf{E} \left( \frac{y^2 - b}{a-b} \right) \right], \quad (16)$$

$$\frac{J_B}{\sqrt{\lambda}} = \sqrt{\frac{a}{a-b}} b, \quad (17)$$

and  $J_R = J_B$ , which in fact holds for any solution considered in this section. The above expressions are valid for  $y^2 < a$ , and for larger values of  $y$  the range of  $\theta$  is not restricted and the string starts to wrap the circle parametrized by  $\theta$  completely. We call this class of solutions as the circular string. The result is again summarised in terms of the complete elliptic integrals.

$$\frac{E}{\sqrt{\lambda}} = \frac{2y}{\pi} \sqrt{\frac{a}{y^2 - b}} \mathbf{K} \left( \frac{a-b}{y^2 - b} \right), \quad (18)$$

$$\frac{J_A}{\sqrt{\lambda}} = \frac{2}{\pi} \sqrt{\frac{a}{y^2 - b}} \left[ y^2 \mathbf{K} \left( \frac{a-b}{y^2 - b} \right) - (y^2 - b) \mathbf{E} \left( \frac{a-b}{y^2 - b} \right) \right], \quad (19)$$

$$\frac{J_B}{\sqrt{\lambda}} = 0. \quad (20)$$

For generic values of  $\nu$ , the allowed range of  $\theta$  is determined from the value of  $\kappa$  compared to the maximum value of  $V(\theta)$ . For  $\kappa^2 < V_{\max}$  the string takes the shape of a folded arc, around the north-pole or south-pole, depending on the values of  $\omega, \nu$ . From the equation of motion and the periodicity condition  $\theta(\sigma + 2\pi) = \theta(\sigma)$  we get

$$2\pi\omega = \sqrt{\frac{a-b}{a}} \int \frac{d\theta}{\sqrt{(\cos \theta - \alpha)(\cos \theta - \beta)}}, \quad (21)$$

where  $\alpha, \beta$  are the two roots of the quadratic equation from the equation of motion, which we quote here for easy reference,

$$\frac{1}{a-b} \left\{ bx \pm \sqrt{abx^2 - (a-b)y^2 + a(a-b)} \right\}, \quad (22)$$

and we choose  $\alpha(\beta)$  to be the larger (smaller) one. We defined  $x \equiv \nu/\omega$ .

Folded string requires that this quadratic equation should have real roots, so we have the condition

$$y^2 \leq a + \frac{ab}{a-b} x^2. \quad (23)$$

The integral becomes simpler when  $\nu = 0$  and reduces to the solutions we have already discussed. Fortunately the integral can be facilitated in terms of elliptic functions also for  $\nu \neq 0$ , using the formulas presented in the appendix. It is thus possible to express the energy in terms of  $x, y$ , and the integrals leading to angular momenta can also be done. For definiteness we here consider folded strings centered around the north pole. Strings

spinning around the south pole can be covered by considering both signs for  $x \equiv \nu/\omega$ , as evident from the invariance of Eq. (13) under  $\theta \rightarrow \pi - \theta, \nu \rightarrow -\nu, \omega \rightarrow \omega$ .

$$\frac{E}{\sqrt{\lambda}} = \frac{4y}{\pi} \sqrt{\frac{a}{a-b}} \frac{1}{\sqrt{(1+\alpha)(1-\beta)}} \mathbf{K}(t), \quad (24)$$

$$\frac{J_A}{\sqrt{\lambda}} = \frac{2}{\pi} \sqrt{\frac{a}{a-b}} \frac{1}{\sqrt{(1+\alpha)(1-\beta)}} \left\{ \begin{aligned} &+2[a-bx+(a-b)\beta] \mathbf{K}(t) \\ &+(b-a)(1+\alpha)(1-\beta) \mathbf{E}(t) \\ &+2[(\alpha+\beta)(b-a)+2bx] \mathbf{\Pi}(k,t) \end{aligned} \right\}, \quad (25)$$

$$\frac{J_B}{\sqrt{\lambda}} = \frac{4}{\pi} \sqrt{\frac{a}{a-b}} \frac{b}{\sqrt{(1+\alpha)(1-\beta)}} \left\{ \begin{aligned} &+(x-1) \mathbf{K}(t) + 2 \mathbf{\Pi}(k,t) \end{aligned} \right\}, \quad (26)$$

where we have defined

$$t = \frac{(1-\alpha)(1+\beta)}{(1+\alpha)(1-\beta)}, \quad (27)$$

$$k = -\frac{1-\alpha}{1+\alpha}. \quad (28)$$

When  $y$  gets larger the quadratic equation does not have real roots, and the string is not of folded form any more and can completely wrap the two-sphere. In the evaluation of conserved charges the difference is that we now integrate over  $0 < \theta < 2\pi$ , not in the range dictated by the roots of the quadratic equation. For general circular strings the conserved quantities are expressed as real parts of the complete elliptic integrals evaluated at complex numbers since the roots  $\alpha, \beta$  become complex numbers.

#### IV. LARGE ENERGY LIMITS AND SCFT OPERATORS

We have presented several different classes of solutions in the last section. Now let us consider the limit of large conserved charges so that the classical relations can be a good approximation to the quantum strings we are eventually interested in.

We also want to identify the dual operators of  $\mathcal{N} = 1$  superconformal field theory, so that our results can be checked against gauge theory computation in the future. Strings moving in  $T^{1,1}$  are dual to pure scalar operators, which should not contain fermions, covariant derivatives or gauge field strengths. The scalar fields in the dual theory are in the bi-fundamental representations, so in order to construct gauge singlets the fields in  $(N, \bar{N})$  and in  $(\bar{N}, N)$  should appear alternatively. Most generally they take the form as

$$\text{tr}(AB \dots A\bar{A} \dots \bar{B}B \dots \bar{B}\bar{A} \dots). \quad (29)$$

We note that there exists an inequality between the bare dimension and the  $R$ -charge, when written in terms of the string variables,

$$E \geq 3|J_R|. \quad (30)$$

It is just the unitarity bound which results from the  $\mathcal{N} = 1$  superconformal algebra.

The simplest class of solutions are the pointlike strings, satisfying Eq. (10). In that case we find that the energy is exactly proportional to the angular momenta, implying protected state. Indeed, from the relevant quantum numbers we have, if we assume  $J_R > 0$ , the pointlike string at  $\theta_1 = \theta_2 = 0$  is identified as

$$\text{tr}(A_1 B_1)^{2J_R}, \quad (31)$$

and for instance when at  $\theta_1 = \theta_2 = \pi$ , as

$$\text{tr}(A_2 B_2)^{2J_R}. \quad (32)$$

For different signs of angular momenta we can also easily identify them as different chiral primaries. Their bare dimensions are indeed  $\frac{3}{4} \cdot 2J_R = 3J_R$ , consistent with the relation Eq. (10). Of course these long chiral primaries constitute the ground state of the plane-wave background as the Penrose limit of  $AdS_5 \times T^{1,1}$ , considered in [14].

The first non-supersymmetric example comes from the single-spin circular strings. When we expand Eq. (12) for large values of the spin,

$$E = 3J_R + \lambda \frac{n^2}{36J_R} - \lambda^2 \frac{n^4}{7776J_R^3} + \dots \quad (33)$$

And from the fact that  $J_A = 0, J_B = J_3$  we can identify the dual operator as

$$\text{tr}((A_1 B_1)^{J_R} (A_2 B_1)^{J_R} + \text{permutations}), \quad (34)$$

whose bare dimension  $4J_R \cdot \frac{3}{4} = 3J_R$  agrees with the leading part of Eq. (33), and since the correction terms are given as a regular series expansion in  $\lambda$ , we expect the subleading terms can be checked against perturbative gauge theory computations in the large  $N$  limit.

We remark here that the task of identifying the dual operators for single-spin strings has been greatly simplified by the fact that the unitarity bound Eq. (30) is saturated asymptotically.  $E = 3J_R$  first implies that among the four combinations  $AB, A\bar{A}, \bar{B}B, \bar{B}\bar{A}$  in  $(N, N)$  representations only  $AB$ 's should be included. In other words, the dual operators are holomorphic. Then  $J_B = J_R$  tells us that between  $B_1$  and  $B_2$ , only  $B_1$ 's should be used to construct the dual operator. Finally the filling fraction of  $A_i$ 's are governed by  $J_A$ . We also note that the conserved charges related to the remaining  $SU(2) \times SU(2)$  generators vanish, which means in Eq.(34) we are instructed to choose singlets of  $SU(2)$  concerning  $A_i$ 's. Different values of  $n$ , of course should denote different eigenvectors of the same quantum numbers.

Now let us consider the folded and circular strings with  $\nu = 0$ . From Eq. (15) and Eq. (18) we see that in the limit

$y^2 \rightarrow a$ , both  $E$  and  $J_B$  become large as the complete elliptic integral of the first kind,  $\mathbf{K}$ , develops a logarithmic divergence. Obviously our strategy is to expand  $E, J_A$  in terms  $y^2 - a$ , invert the expression for  $J_A$ , and substituting back into  $E$  to get a relation between  $E$  and  $J_A$ .

For that purpose it turns out convenient to employ so-called  $q$ -series expressions of elliptic integrals. One first defines  $K'(m) \equiv K(1-m)$  then for small  $m$  we have a regular series expansion for  $q$ ,

$$\begin{aligned} q &\equiv \exp[-\pi K'(m)/K(m)] \\ &= \frac{m}{16} + 8 \left(\frac{m}{16}\right)^2 + 84 \left(\frac{m}{16}\right)^3 + \dots \end{aligned} \quad (35)$$

and for  $q$ -series expansion of other elliptic integrals see the appendix.

For folded strings, we define

$$m = \frac{a - y^2}{a - b} \quad (36)$$

and for small  $m$  we have the following  $q$ -expansions, using the actual values of  $a, b$ ,

$$\begin{aligned} \frac{E}{\sqrt{\lambda}} &= -\frac{1}{\sqrt{2\pi}} \ln q \left(1 + \frac{4}{3}q + \frac{100}{9}q^2 + \dots\right) \\ \frac{J_A}{\sqrt{\lambda}} &= -\frac{2\sqrt{3}}{\pi} \left[ \ln q \left(\frac{1}{12} + \frac{1}{9}q + \frac{7}{9}q^2 + \dots\right) \right. \\ &\quad \left. + \left(\frac{1}{18} - \frac{2}{9}q + \frac{2}{3}q^2 + \dots\right) \right] \end{aligned} \quad (37)$$

For circular strings, we define instead

$$m = \frac{y^2 - a}{y^2 - b} \quad (39)$$

then the conserved quantities are

$$\begin{aligned} \frac{E}{\sqrt{\lambda}} &= -\frac{1}{\sqrt{2\pi}} \ln q \left(1 - \frac{4}{3}q + \frac{100}{9}q^2 + \dots\right) \\ \frac{J_A}{\sqrt{\lambda}} &= -\frac{2\sqrt{3}}{\pi} \left[ \ln q \left(\frac{1}{12} - \frac{1}{9}q + \frac{7}{9}q^2 + \dots\right) \right. \\ &\quad \left. + \left(\frac{1}{18} + \frac{2}{9}q + \frac{2}{3}q^2 + \dots\right) \right] \end{aligned} \quad (40)$$

In the limit of very massive strings the two different class of solutions exhibit very similar behaviour, both giving

$$E = \sqrt{6}J_A + \frac{\sqrt{2}}{3\pi}\sqrt{\lambda} + \frac{4\sqrt{6}}{27}J_A e^{-4\sqrt{3}\pi J_A/\sqrt{\lambda}} + \dots, \quad (42)$$

and the difference shows up in the sub-leading terms.

Now let us consider the dual operator, on the assumption that in the large  $E, J_A$  limit the leading behaviour  $E \sim \sqrt{6}J_A$  is given by the bare conformal dimension. But in this case we find it is not straightforward to identify the dual operators. The crucial difference is that the on-shell relation Eq. (42) does not approach the unitarity bound Eq. (30). As there are 4 scalar fields  $A_i, \bar{B}_i$  in  $(N, \bar{N})$

representation and their complex conjugates in  $(\bar{N}, N)$ , we have 16 different combinations in  $(N, N)$ , while we are given only 4 conserved quantities from string solutions. It is clear that the identification cannot be made without ambiguity, even up to mixing. For circular strings though, since  $J_B = J_R$  vanish identically one might conjecture that the dual operator does not contain any  $B$  or  $\bar{B}$ , giving the form  $\text{tr}(A\bar{A})^{2E/3}$ . Then  $E/J_A \rightarrow \sqrt{6}$  can be used to decide

$$\frac{\#(A_2) + \#(\bar{A}_1)}{\#(A_1) + \#(\bar{A}_2)} = \frac{11 - 4\sqrt{6}}{5}, \quad (43)$$

But one should bear in mind that it is also possible to consider more general forms of operators which are singlets of the  $SU(2)$  concerning  $B_i$ 's.

Now let us consider the general folded string solutions. We are again interested in the limits where the conserved quantities, especially the energy, become very large. One can see that here with general solutions there is another possibility for this, than exploiting the logarithmic divergence of elliptic integrals. The energy becomes large when  $y, -x \rightarrow \infty$ , while keeping  $t, k$  finite. Since the expansions do not involve logarithms, we expect a regular series expansion of  $E$  in terms of  $J_R$ . In order for folded strings to exist, at least one of the real roots in Eq. (22) should reside in  $[-1, 1]$ . It is possible when  $y, x$  are related asymptotically as

$$y^2 = b(x^2 + 2cx + \dots), \quad (44)$$

for  $-1 < c < 1$ . We can expand the charges in terms of  $1/x$ , invert it for  $J_B$ , and substitute back into the expressions for  $E, J_A$ . We will get series expansions of the following form,

$$\begin{aligned} E &= a_1 J_B + a_2 \frac{\lambda}{J_B} + a_3 \frac{\lambda^2}{J_B^3} + \dots, \\ J_A &= b_1 J_B + b_2 \frac{\lambda}{J_B} + b_3 \frac{\lambda^2}{J_B^3} + \dots. \end{aligned}$$

The coefficients of the expansion,  $a_n, b_n$  are determined by the parameters of  $y(x)$ . The first few coefficients turn out to depend only on the leading order correction term,  $c$ ,

$$a_1 = 3, \quad (45)$$

$$a_2 = \frac{4}{9\pi^2} \mathbf{K}(-z) (\mathbf{E}(-z) - \mathbf{K}(-z)), \quad (46)$$

$$b_1 = 1 - \frac{2}{z+1} \frac{\mathbf{E}(-z)}{\mathbf{K}(-z)}, \quad (47)$$

where we define

$$z = \frac{1+c}{1-c}. \quad (48)$$

We notice from  $a_1 = 3$  that the unitarity bound is asymptotically saturated, so the dual operators are holomorphic, and from  $J_B = J_R$  it is clear that  $B_2$ 's should

not be included. The dual operator thus takes the following form,

$$\text{tr} \left( (A_1 B_1)^{2r J_R} (A_2 B_1)^{2(1-r) J_R} + \text{permutations} \right). \quad (49)$$

And  $b_1$  gives the filling fraction of  $A_i$ 's, i.e.  $b_1 = 2r - 1$  and from Eq.(47)

$$r = 1 - \frac{1}{1+z} \frac{\mathbf{E}(-z)}{\mathbf{K}(-z)}. \quad (50)$$

So up to the problem of mixing, we can determine the dual operators without ambiguity, and the string soliton gives a prediction on their anomalous dimensions.

## V. DISCUSSIONS

In this paper we have studied spinning strings in  $T^{1,1}$  and obtained the on-shell relations between the conserved quantities which can be mapped to the conformal dimensions and other global charges of the dual operators.

Our main motivation was to find string solutions, the macroscopic limit of which can be compared with the perturbative gauge theory computations. We have provided two such solutions, and the dual operators are identified as Eq.(34) and Eq.(49). A general rule can be deduced empirically from our analysis. The solutions which satisfy asymptotically the unitarity bound of the  $\mathcal{N} = 1$  superconformal symmetry, are dual to holomorphic operators, and are amenable to perturbative treatment in the gauge theory. For the class of solutions which do not approach the unitarity bound, e.g. summarised as Eq. (42), it does not seem possible to confirm the string prediction using perturbative gauge theory computation. And it is not the only problem: it is highly ambiguous which gauge theory operators one should look at.

It is natural to expect that the two different types of asymptotic behaviour we have just observed has to do with the supersymmetry of each solution. We believe it would be illuminating to perform the  $\kappa$ -symmetry analysis with our solutions, as it was done for solutions on  $AdS_5 \times S^5$  in [15].

Finding general spinning solutions on  $AdS_5 \times S^5$  has been greatly facilitated by the observation that the spinning string ansatz leads to the integrable Neumann model, i.e. harmonic oscillators on the sphere [18]. The holomorphic operators we come across in this paper have excitations in only one of the  $SU(2)$  structure, i.e. they do not contain  $B_2$ . We reckon that solutions with both  $\theta_1, \theta_2$  nontrivial will lead to more general holomorphic operators not necessarily satisfying  $J_B = J_R$ . Integrability should be a key in such an extension.

An important question is then whether the string theory on  $AdS_5 \times T^{1,1}$  will turn out to be integrable or not. There are various indications that it is indeed the case for  $AdS_5 \times S^5$ , see e.g. [16]. Then the integrability of  $\mathcal{N} = 4$

super Yang-Mills should follow, via the AdS/CFT correspondence. Readers are referred to Ref. [17] on this issue.

To the best of our knowledge, the first hint of integrability of strings on  $AdS_5 \times S^5$  comes from the classic result of Pohlmeyer [19] that hamiltonian systems with quadratic constraints are classically integrable. It is well known that a conifold, i.e. a cone over  $T^{1,1}$ , can be embedded in  $\mathbf{C}^4$  with quadratic constraints, but in order to get the desired metric Eq. (1) one considers nontrivial Kähler potential. Without further analysis, we are thus not particularly optimistic on the integrability of the nonlinear sigma model on  $T^{1,1}$ . But it is presumably worth mentioning that the one-dimensional system we obtain using the spinning string ansatz, as summarised e.g. in Eq. (3), can indeed be rewritten as a motion on  $\mathbf{R}^4$  with flat metric and quadratic constraints. It is certainly very desirable to check the integrability of nonlinear sigma models on  $T^{1,1}$  and similar Einstein spaces relevant to Kaluza-Klein supergravity.

The integrable  $SO(6)$  spin chain is reduced to the Heisenberg  $XXX_{1/2}$  model when we consider a subset of  $SU(2) \subset SU(3) \subset SO(6)$ . When applied to the pure scalar operators of  $\mathcal{N} = 4$  super Yang-Mills, it corresponds to operators written in terms of two complex scalar fields. Recently this subsector has been extensively studied in the literature [8], mainly because they only mix among themselves to arbitrary orders of perturbation theory and it is relatively easier to extend the computations to higher orders.

In fact, the simplest nontrivial folded and circular strings on  $AdS_5 \times S^5$  turned out to be dual to operators in the  $SU(2)$  subsector just mentioned, and it is observed that the leading order behavior extracted from the string solutions agree with the solutions of the Bethe ansatz equations, for a review see Ref. [6].

The solutions we have found are also dual to a  $SU(2)$  subsector of the  $\mathcal{N} = 1$  superconformal field theory, as written in Eq. (49). We here point out that the leading order data, Eq. (46) and Eq. (47), are equivalent to the counterparts of the spinning strings on  $AdS_5 \times S^5$ . It can be seen by rewriting them using the identities Eq. (A4) and Eq. (A5),

$$a_2 = \frac{4}{9\pi^2} \mathbf{K}(z') (\mathbf{E}(z') - (1-z')\mathbf{K}(z')), \quad (51)$$

$$b_1 = 1 - 2 \frac{\mathbf{E}(z')}{\mathbf{K}(z')}, \quad (52)$$

with  $z' = \frac{z+1}{2}$ . They are identical to, for instance, Eq. (7.25) and Eq. (7.26) of Ref. [6], up to overall coefficients which can be absorbed into redefinition of the coupling constant. We thus conjecture that the planar one-loop dilatation operator of  $\mathcal{N} = 1$  superconformal field theory dual to  $AdS_5 \times T^{1,1}$ , when restricted to a  $SU(2)$  subsector, is isomorphic to the hamiltonian of Heisenberg  $XXX_{1/2}$  model. Since the full expressions governing the conserved quantities are certainly more involved and different from that of spinning strings on  $AdS_5 \times S^5$ , we

presume the universality we have observed here will be lifted in general at higher orders of perturbation theory.

The original motivation of this work was to see whether the integrability observed in perturbative  $\mathcal{N} = 4$  super Yang-Mills theory persists with other nonabelian gauge theories. The tests done with the BMN matrix model of M-theory plane-waves suggest that integrability might be a rather general feature of Yang-Mills theory in the planar limit [20], than naively expected. In order to check the integrability beyond the  $SU(2)$  subsector of  $\mathcal{N} = 1$  example we have considered in this paper, it will be nice to develop perturbative computations around nontrivial conformal fixed points, and find the effective vertex for the dilatation operator. We hope to be able to compare it to more general solutions of the nonlinear sigma model on  $T^{1,1}$ .

### Acknowledgments

It is a pleasure to thank G. Arutyunov for illuminating conversations and encouragement. This work is supported by the DFG - the German Science Foundation.

### APPENDIX A: PROPERTIES OF COMPLETE ELLIPTIC INTEGRALS

In this appendix we present some properties of the elliptic integrals which are used to derive the results of this note. The complete elliptic integrals are defined as following,

$$\mathbf{K}(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} \quad (\text{A1})$$

$$\mathbf{E}(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \varphi} d\varphi \quad (\text{A2})$$

$$\mathbf{\Pi}(k, m) = \int_0^{\pi/2} \frac{d\varphi}{(1 - k \sin^2 \varphi) \sqrt{1 - m \sin^2 \varphi}} \quad (\text{A3})$$

The following identities immediately follow from the definitions,

$$\mathbf{K}(-m) = \frac{1}{\sqrt{1+m}} \mathbf{K}\left(\frac{m}{m+1}\right), \quad (\text{A4})$$

$$\mathbf{E}(-m) = \sqrt{1+m} \mathbf{E}\left(\frac{m}{m+1}\right). \quad (\text{A5})$$

Integrals of the following form can be expressed in terms of the elliptic integrals,

$$I(n) = \int_{\alpha}^1 \frac{s^n ds}{\sqrt{(s-\alpha)(s-\beta)(1-s^2)}}, \quad (\text{A6})$$

which are needed to derive the results of this note.

$$I(0) = \frac{2}{\sqrt{(1+\alpha)(1-\beta)}} \mathbf{K}(t), \quad (\text{A7})$$

$$I(1) = \frac{2}{\sqrt{(1+\alpha)(1-\beta)}} \left[ 2\mathbf{\Pi}(k, t) - \mathbf{K}(t) \right], \quad (\text{A8})$$

$$I(2) = \frac{1}{\sqrt{(1+\alpha)(1-\beta)}} \left[ 2(\alpha + \beta)\mathbf{\Pi}(k, t), \right. \\ \left. - 2\alpha\mathbf{K}(t) + (1+\alpha)(1-\beta)\mathbf{E}(t) \right] \quad (\text{A9})$$

where we have defined

$$k = -\frac{1-\alpha}{1+\alpha}, \quad t = \frac{(1-\alpha)(1+\beta)}{(1+\alpha)(1-\beta)}. \quad (\text{A10})$$

In order to study the elliptic integrals near the logarithmic singularity, it is convenient to use the  $q$ -series, defined as

$$q \equiv \exp[-\pi K(1-m)/K(m)] \\ = \frac{m}{16} + 8 \left(\frac{m}{16}\right)^2 + 84 \left(\frac{m}{16}\right)^3 + 992 \left(\frac{m}{16}\right)^4 + \dots$$

Inverting, one obtains

$$m = 16(q - 8q^2 + 44q^3 - 192q^4 + \dots). \quad (\text{A11})$$

And the elliptic integrals are expressed in  $q$ -series

$$\mathbf{K}(m) = \frac{\pi}{2} (1 + 4q + 4q^2 + 4q^4 + \dots). \quad (\text{A12})$$

$$\mathbf{E}(m) = \frac{\pi}{2} (1 - 4q + 20q^2 + 96q^3 \dots). \quad (\text{A13})$$

- 
- [1] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323** (2000) 183 hep-th/9905111.  
[2] D. Berenstein, J. M. Maldacena and H. Nastase, JHEP **0204**, 013 (2002) hep-th/0202021.  
[3] R. R. Metsaev, Nucl. Phys. B **625** (2002) 70 [arXiv:hep-th/0112044].  
[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Nucl. Phys. B **636** (2002) 99 [arXiv:hep-th/0204051].  
[5] S. Frolov and A. A. Tseytlin, JHEP **0206** (2002) 007 [arXiv:hep-th/0204226].

- J. G. Russo, JHEP **0206** (2002) 038 [arXiv:hep-th/0205244].  
A. Armoni, J. L. F. Barbon and A. C. Petkou, JHEP **0206** (2002) 058 [arXiv:hep-th/0205280].  
J. A. Minahan, Nucl. Phys. B **648** (2003) 203 [arXiv:hep-th/0209047].  
A. A. Tseytlin, Int. J. Mod. Phys. A **18** (2003) 981 [arXiv:hep-th/0209116].  
A. Armoni, J. L. F. Barbon and A. C. Petkou, JHEP **0210** (2002) 069 [arXiv:hep-th/0209224].

- M. Alishahiha and A. E. Mosaffa, JHEP **0210** (2002) 060 [arXiv:hep-th/0210122].
- R. C. Rashkov and K. S. Viswanathan, arXiv:hep-th/0211197.
- J. M. Pons and P. Talavera, Nucl. Phys. B **665** (2003) 129 [arXiv:hep-th/0301178].
- S. Ryang, JHEP **0304** (2003) 045 [arXiv:hep-th/0303237].
- H. Dimov, V. Filev, R. C. Rashkov and K. S. Viswanathan, Phys. Rev. D **68** (2003) 066010 [arXiv:hep-th/0304035].
- A. A. Tseytlin, Nucl. Phys. B **664** (2003) 247 [arXiv:hep-th/0304139].
- S. Frolov and A. A. Tseytlin, Nucl. Phys. B **668** (2003) 77 [arXiv:hep-th/0304255].
- S. Frolov and A. A. Tseytlin, JHEP **0307** (2003) 016 [arXiv:hep-th/0306130].
- S. Frolov and A. A. Tseytlin, Phys. Lett. B **570** (2003) 96 [arXiv:hep-th/0306143].
- D. Aleksandrova and P. Bozhilov, JHEP **0308** (2003) 018 [arXiv:hep-th/0307113].
- D. Aleksandrova and P. Bozhilov, arXiv:hep-th/0308087.
- J. Engquist, J. A. Minahan and K. Zarembo, arXiv:hep-th/0310188.
- A. Mikhailov, arXiv:hep-th/0311019.
- L. A. P. Zayas, J. Sonnenschein and D. Vaman, arXiv:hep-th/0311190.
- B. Stefanski. Jr, arXiv:hep-th/0312091.
- [6] A. A. Tseytlin, arXiv:hep-th/0311139.
- [7] J. A. Minahan and K. Zarembo, JHEP **0303** (2003) 013 [arXiv:hep-th/0212208].
- [8] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B **664** (2003) 131 [arXiv:hep-th/0303060].
- N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, JHEP **0309** (2003) 010 [arXiv:hep-th/0306139].
- N. Beisert, arXiv:hep-th/0307015.
- N. Beisert and M. Staudacher, Nucl. Phys. B **670** (2003) 439 [arXiv:hep-th/0307042].
- N. Beisert, JHEP **0309** (2003) 062 [arXiv:hep-th/0308074].
- N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, JHEP **0310** (2003) 037 [arXiv:hep-th/0308117].
- N. Beisert, arXiv:hep-th/0310252.
- [9] G. Arutyunov and M. Staudacher, arXiv:hep-th/0310182.
- [10] A. Gorsky, arXiv:hep-th/0308182.
- [11] M. Kruczenski, arXiv:hep-th/0311203.
- [12] I. R. Klebanov and E. Witten, Nucl. Phys. B **536** (1998) 199 [arXiv:hep-th/9807080].
- [13] M. Schvellinger, arXiv:hep-th/0309161.
- [14] N. Itzhaki, I. R. Klebanov and S. Mukhi, JHEP **0203** (2002) 048 [arXiv:hep-th/0202153].
- J. Gomis and H. Ooguri, Nucl. Phys. B **635** (2002) 106 [arXiv:hep-th/0202157].
- [15] D. Mateos, T. Mateos and P. K. Townsend, arXiv:hep-th/0309114.
- [16] G. Mandal, N. V. Suryanarayana and S. R. Wadia, Phys. Lett. B **543** (2002) 81 [arXiv:hep-th/0206103].
- B. C. Vallilo, arXiv:hep-th/0307018.
- [17] L. Dolan, C. R. Nappi and E. Witten, JHEP **0310** (2003) 017 [arXiv:hep-th/0308089].
- [18] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, arXiv:hep-th/0307191.
- G. Arutyunov, J. Russo and A. A. Tseytlin, arXiv:hep-th/0311004.
- [19] K. Pohlmeyer, Commun. Math. Phys. **46** (1976) 207.
- [20] N. Kim, T. Klose and J. Plefka, Nucl. Phys. B **671** (2003) 359 [arXiv:hep-th/0306054].
- T. Klose and J. Plefka, arXiv:hep-th/0310232.