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TWO-LOOP COMMUTING CHARGES AND THE STRING/GAUGE DUALITY

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We briefly review the status quo of the application of integrable systems techniques to the AdS/CFT correspondence in the large charge approximation, a rapidly evolving topic. Intricate string and gauge computations of, respectively, energies and scaling dimensions agree at the one and two-loop level, but disagree starting from three loops. To add to this pattern, we present further computations which demonstrate that for folded and circular spinning strings the full tower of infinitely many hidden commuting charges, responsible for the integrability, also agrees up to *two*, but not three, loops.

1. The status quo

A fresh approach to the AdS/CFT correspondence was initiated in [1]. The proposal involved e.g. the study of composite operators in the planar $\mathcal{N} = 4$ gauge theory containing as constituents complex scalar fields Z,Φ of the form

$$\mathrm{Tr}\, Z^{J_1} \Phi^{J_2} + \dots\,, \tag{1}$$

and to consider the situation where $J = J_1 + J_2$ becomes large, while J_2 remains small. The dots indicate arbitrary orderings of the scalars inside the trace. It was argued that in this limit the conjectured dual string becomes essentially free, and the spectrum therefore explicitly calculable. Furthermore, the string energies turned out to be *analytic* at small BMN coupling

$$\lambda' = \frac{\lambda}{J^2}\,, \tag{2}$$

where λ is the usual large color 't Hooft coupling. The string energy is interpreted on the gauge side as the anomalous dimension of the corresponding operator eq.(1). Expanding in λ' therefore *looks* like a perturbative expansion of the dimension in the dual gauge theory. And indeed the one- and two-loop predictions for the BMN operators eq.(1) were successfully reproduced in [1],[2].

The large J limit of [1] was given in [3] an alternative interpretation as a semi-classical approximation to the string sigma model. This way of thinking allows for a generalization of the BMN limit: One may, more generally, assume several charges to be large [4] (see also [5], and the closely related earlier work [6]). In the case of our operators in eq.(1) one thus assumes both J_1 and J_2 to be large. It was then shown [4] that the semi-classical computations of string energies could be performed exactly, leading to predictions for the anomalous dimensions of these operators which are rather intricate as they generically depend in mathematically complicated ways on both the coupling λ' and the ratio

$$\alpha = \frac{J_2}{J}\,. \tag{3}$$

In subsequent papers it was understood that the reason for the exact solvability of the semi-classical string motions may be traced back to *integrability* [7]. For a string propagating on a five-sphere the evolution equations are those of the O(6) sigma model. In light-cone coordinates (ξ, η), where $\tau = \xi + \eta$ and $\sigma = \xi - \eta$, they read

$$X_{\xi\eta} + (X_\xi \cdot X_\eta)X = 1\,, \qquad (X \cdot X) = 1. \tag{4}$$

Here X is a six-dimensional vector describing the embedding of the string world-sheet in S^5. The equations are supplemented by the Virasoro constraints. The simplest solutions correspond to rigid strings, i.e. string configurations with a time-independent shape. Rigid strings are naturally classified in terms of the Neumann integrable system [7], which is a special one-dimensional reduction of eqs.(4).

The string equations of motion inherit their integrability from the one of the string sigma model [8]. The underlying reason for the solvability is then an infinite family of mutually commuting Pohlmeyer charges $\{\mathcal{Q}_k\}$, where $\{\mathcal{Q}_2\}$ denotes the string energy.

How to check these predictions for the $\mathcal{N} = 4$ gauge theory? The problem is a priori difficult, even at one-loop, due to the complicated mixing between operators differing by the ordering of the scalars inside the trace. To deal with these problems, operatorial methods were developed in [9], not necessarily restricted to the planar case. However, for the planar case, a very important feature was noticed in [10]: The one-loop anomalous dimensions may be obtained by the diagonalization of an *integrable* quantum spin chain. In the example of the operators of eq.(1) the Z and Φ fields correspond to "up" and "down" spins. This integrable structure leads to the appearance of an infinite set of mutually commuting one-loop gauge charges $\{Q_k^{(1)}\}$, which generate (planar) symmetries which, mysteriously and yet-to-be-understood, are *not* part of the known symmetries of the superconformal $\mathcal{N} = 4$ theory. With one exception: $\{Q_2^{(1)}\}$, which is the one-loop approximation to the model's dilatation operator. Excitingly, investigating two-loop corrections, it was found that the charges continue to commute beyond the one-loop level [11]. This led to the conjecture [11] that the one-loop charges are actually only the leading order approximation of coupling constant dependent commuting charges $\{Q_k(\lambda)\}$, and that the exact dilatation operator $\{Q_2(\lambda)\}$ might thus be integrable.

The integrability of the one-loop planar gauge theory for scalar operators leads to the existence of a Bethe Ansatz for the anomalous dimensions [10]. Incidentally, one can show that this remains true for the set of all planar one-loop anomalous dimensions [12], as the one-loop dilatation operator, not necessarily restricted to the planar limit, is known for the most general $\mathcal{N} = 4$ composite operator [13]. The Bethe Ansatz allows one to quantitatively address the problem of the large J_1,J_2 limit of the operators in eq.(1), and to subsequently compare to the semi-classical string results. And indeed, this programme recently enjoyed considerable success. Investigating the two simplest types of string motions corresponding to the operators eq.(1) (the *folded* and the *circular* string), it was shown in [14],[16] that the one-loop string energies [4],[15],[7] and the gauge theory anomalous dimensions agree: $\mathcal{Q}_2^{(1)} \sim Q_2^{(1)}$. For agreement for other types of motion, see the string [17] and gauge computations [18]; for a review, see also [19].

In [20] it was shown, for folded and circular strings, that *all* one-loop charges $\mathcal{Q}_{2k}^{(1)} \sim Q_{2k}^{(1)}$ agree, establishing a direct connection between the respective integrable structures. This involved applying a Bäcklund transformation to the string motions, and extracting the charges from the Bethe Ansatz for the gauge theory. For a beautiful extension of this approach to the case involving three complex scalars, see [21]. In [22] it was shown that the two-loop energies $\mathcal{Q}_2^{(2)} \sim Q_2^{(2)}$ agree. This required a generalization of the spin chain picture to a long range chain, and an extension of the Bethe

Ansatz to higher loops[a].

Below we will extend these matchings by proving that also the two-loop charges $\mathcal{Q}^{(2)}_{2k} \sim Q^{(2)}_{2k}$ coincide.

While the matching is done for the two specific cases we are convinced that the relations between string and gauge charges are universal. As we will see, they are not only independent of the type of motion, but also of the parameter α in eq.(3). Of course it would be nice to present a general, solution independent proof. A first important step into this direction was undertaken, at one loop, in [24].

Very recently the most general one- and two-loop solution of the large J gauge theory scaling dimensions for the operators eq.(1) was obtained by Kazakov, Marshakov, Minahan and Zarembo [25]. Furthermore, a Bethe equation for the classical string sigma model was derived, and it was argued that the quantum spin chain Bethe equations may be obtained from it, mapping the integrable structures up to two loops for the most general solution. These results based on the monodromy approach are completely consistent with our extension of the Bäcklund approach to two loops described in this work below.

Not all is well, however. Serious problems appear at the *three* loop level. Indeed, the three-loop dilatation operator for the operators eq.(1) had been conjectured in [11], see also [23], and rigorously derived in [26]. Completely independent confirmation comes from a conceptually, but not technically related study of plane wave matrix theory at three loops [27]. Strictly speaking, the three loop dilatation generator is only known up to two unknown constants: The procedure of [26] allows, to the considered order, to add to $Q^{(3)}_2$, with arbitrary coefficients, the second ($Q^{(1)}_2$) and fourth ($Q^{(1)}_4$) one-loop conserved charge. However, both additions would be inconsistent with the perturbative BMN limit, and therefore certainly also with the large J_1,J_2 "spinning limit". However, three-loop integrability is *proven* in [26]. Therefore, the Bethe Ansatz of [22] applies, but the obtained three loop string and gauge energies *disagree*, $\mathcal{Q}^{(3)}_2 \nsim Q^{(3)}_2$, and the same is true for all other charges. Below we will show that the difference is nevertheless "small" in the sense that it may always be accounted for

[a] The procedure applied in [22], using a long range chain invented by Inozemtsev, accurately diagonalizes the dilatation operator for the states eq.(1) up to three loops. At four loops, a violation of BMN scaling was detected. This might either indicate that BMN scaling is indeed violated in the gauge theory, or alternatively, that a further integrable long range chain, different from Inozemtsev, exists. Some evidence for the second scenario comes from the results of [23]. However, this issue is beyond the present scope.

by a simple non-local combination of charges, generalizing the "curious observation" of [22]. An earlier three-loop string-gauge disagreement was reported for the easier (on the gauge side) case of $1/J$ corrections to the BMN limit in [28].

Let us assume that the AdS/CFT correspondence is not a near-symmetry, but actually correct. One possible explanation for the disagreement might be the fact that the parameter λ' in eq.(2) is not obtained in the same fashion in string and gauge theory. This possibility was first discussed, to our knowledge, in [22]. For an earlier discussion in a slightly different but related context, see [29]. In the "semiclassical" string theory calculations we take λ and J to be large in a coupled fashion, and form the finite coupling λ'. In *perturbative* gauge theory, we first assume λ to be small, and subsequently take J to infinity. The difference in procedure might be expressible as certain non-perturbative corrections on either side of the correspondence. Taking them into account will hopefully eventually show that indeed IIB string theory on the background $AdS_5 \times S^5$ and $\mathcal{N} = 4$ gauge theory are one and the same thing.

2. Two-loop string/gauge matching of infinitely many higher commuting charges

Let us now show explicitly that the integrable structures on both sides of the correspondence agree, up to two loops, in the two distinct cases of the folded and the circular spinning string. This will be done by combining the results of [20] and [22] in order to show that the full tower of commuting charges matches up to $\mathcal{O}(\lambda'^2)$. In the following we will concentrate on the new, two-loop aspects of this matching, and refer to [20],[22] for further background, technical details, and precise notations. On the string side the generating function of charges (resolvent) was found by applying the Bäcklund transformation to the solutions of the classical string equations of motion, *cf.* eqs.(4). The result for the "nearly improved" string resolvent reads, in both cases,

$$\tilde{\mathcal{E}}(\mu) = \frac{4\mu^3}{\pi} \frac{\sqrt{(1-z)(1-tz)}}{\sqrt{z}}\, \Pi(tz, t). \tag{5}$$

Here μ is the string spectral parameter, t is the string modulus, and the function $\Pi(tz, t)$ is the complete elliptic integral of the third kind. The auxiliary parameter z depends on μ, t, and $\mathcal{J} = J/\sqrt{\lambda}$, i.e. the charge in units of the string tension. In order to expand this expression, it is convenient to use the form

$$\tilde{\mathcal{Q}}(\varphi,\lambda') = \frac{1}{2} - \frac{1}{2}\ \mu^{-2}\ \frac{\tilde{\mathcal{E}}(\mu)}{\mathcal{J}} = \frac{1}{2} - 2\varphi\frac{\sqrt{\mp(1-z)(1-tz)}}{\sqrt{z}}\ \Pi(tz,t)\ , \quad (6)$$

The string resolvent then becomes a function of the rescaled spectral parameter φ and the BMN coupling constant λ',

$$\varphi^2 = \mp\frac{\mu^2}{\pi^2\mathcal{J}^2} \qquad \text{and} \qquad \lambda' = \frac{1}{\mathcal{J}^2}\ . \quad (7)$$

Here and in the following the sign $\mp$ depends on the type of string motion (folded: upper sign, circular: lower sign), which is entirely due to our conventions. The modulus $t = t(\lambda';\alpha)$ depends on the coupling constant λ', and also encodes information on the filling fraction $\alpha = J_2/J$, *cf.* eqs.(19),(20) and eqs.(31),(32). The auxiliary parameter $z = z(\varphi,\lambda';\alpha)$ also slightly differs in the two considered cases. It is determined by eqs. (21),(33) below.

This "nearly improved" string resolvent generates a set of string charges $\{\tilde{\mathcal{Q}}_{2k}(\lambda')\}$

$$\tilde{\mathcal{Q}}(\varphi,\lambda') = \sum_{k=0}^{\infty}\tilde{\mathcal{Q}}_{2k}(\lambda')\ \varphi^{2k}, \quad (8)$$

such that the lowest (zeroth) charge is linearly related to the string energy $\mathcal{Q}_2(\lambda') = \mathcal{E}/\mathcal{J}$ by $\tilde{\mathcal{Q}}_0(\lambda') = \frac{1}{2}(1-\mathcal{Q}_2(\lambda'))$. Of course we can alternatively, or in addition, expand in the coupling constant λ':

$$\tilde{\mathcal{Q}}(\varphi,\lambda') = \sum_{n=0}^{\infty}\tilde{\mathcal{Q}}^{(n+1)}(\varphi)\ \lambda'^n = \sum_{n,k=0}^{\infty}\tilde{\mathcal{Q}}_{2k}^{(n+1)}\ \lambda'^n\ \varphi^{2k}. \quad (9)$$

On the gauge side the resolvent for a set of gauge charges $\{\bar{Q}_{2k}(\lambda')\}$ was also found perturbatively, up to three loops, by using a Bethe Ansatz for a long range spin chain originally invented by Inozemtsev [30],[31]:

$$H(\varphi,\lambda') = \pm\sum_{k=1}^{\infty}\bar{Q}_{2k}(\lambda')\ \varphi^{2k}\ . \quad (10)$$

Here φ is a gauge spectral parameter which is obtained from the scaled rapidity of the Inozemtsev-Bethe Ansatz. Expanding in λ', we have

$$H(\varphi,\lambda') = \pm\sum_{n=0}^{2}\bar{Q}^{(n+1)}(\varphi)\ \lambda'^n = \pm\sum_{n=0}^{2}\sum_{k=1}^{\infty}\bar{Q}_{2k}^{(n+1)}\ \lambda'^n\ \varphi^{2k}. \quad (11)$$

It is important to note that the charges $\bar{Q}_{2k}^{(n)}$ are not the correct "observables" of the spin chain. E.g. the three-loop gauge anomalous dimension is obtained by the following specific linear combination:

$$Q_2(\lambda') = 1 \mp \frac{\lambda'}{4\pi^2}\,\bar{Q}_2(\lambda') - \frac{3\lambda'^2}{64\pi^4}\,\bar{Q}_4(\lambda') \mp \frac{5\lambda'^3}{512\pi^6}\,\bar{Q}_6(\lambda'). \qquad (12)$$

Likewise, while in the spin chain any linear combination of local charges leads again to a set of local charges, in the gauge theory these linear superpositions are essentially fixed: At one loop, the charges are required to obey BMN scaling [20], and loop corrections are then determined by quantum field theory. The proper three-loop charges will thus look like ($k > 0$)

$$Q_{2k}(\lambda') = \bar{Q}_{2k}(\lambda') \mp e_{k,1}\lambda'\,\bar{Q}_{2k+2}(\lambda') + e_{k,2}\lambda'^2\,\bar{Q}_{2k+4}(\lambda')\,. \qquad (13)$$

While it is certainly possible to find these (universal) numbers $e_{k,1}$, $e_{k,2}$, it is easily seen that we do not require them for the present purposes of two-loop matching: All we need to show[b] is that a linear map from the set of "nearly improved" string charges $\{\tilde{\mathcal{Q}}_{2k}(\lambda')\}$ to the set of gauge charges $\{\bar{Q}_{2k}(\lambda')\}$ exists to leading and next-to-leading order in λ'! By explicit inspection of the first few two-loop string and gauge charges, one finds that ($k \geq 0$, and $\bar{Q}_0^{(2)} \equiv 0$)

$$\tilde{\mathcal{Q}}_{2k}^{(2)} = \pm\bar{Q}_{2k}^{(2)} \pm \frac{1}{8\pi^2}(2k+1)\,\bar{Q}_{2k+2}^{(1)}\,. \qquad (14)$$

This leads to the following proposition

$$\tilde{\mathcal{Q}}(\varphi,\lambda') = \bar{Q}^{(1)}(\varphi) \pm \lambda'\left[\bar{Q}^{(2)}(\varphi) + \frac{1}{8\pi^2}\frac{\partial}{\partial\varphi}\frac{1}{\varphi}\,\bar{Q}^{(1)}(\varphi)\right] + \mathcal{O}(\lambda'^2) \qquad (15)$$

It can be written in an elegant fashion as follows:

$$\tilde{\mathcal{Q}}(\varphi,\lambda') = \Big(1 \mp \frac{\lambda'}{8\pi^2\varphi^2}\Big) H\Big(\varphi \pm \frac{\lambda'}{8\pi^2\varphi}, \lambda'\Big) + \mathcal{O}(\lambda'^2)\,. \qquad (16)$$

This is our main new result, and will be proven in the next sections for the case of the folded and circular string. It proves our assertion of the two-loop matching of string and gauge charges. Although derived starting from particular solutions, we believe that (16) is universal in the sense that

[b] This also relieves us from applying the procedure of "full improvement" to the string resolvent eq.(5), as discussed in [20], p. 24. At any rate, the latter would only assure us of the correct leading BMN scaling behavior of the charges; one can show that the obtained charges do *not* agree with the "proper" gauge charges at the two-loop level. Thus a further linear, upper triangle redefinition would be required.

it does not depend on specific solutions[c], and that it expresses the general matching of string/gauge integrable structures at two loops.

Now we discuss the three-loop case. As was already noticed in [22] the gauge/string energies disagree starting from three loops. It is however remarkable that the disagreement between the whole towers of gauge/string commuting charges admits a uniform description.

First, we expand eq.(6) up to ${\lambda'}^2$ and identify the term $\tilde{\mathcal{Q}}^{(3)}(\varphi)$. Then we perform the Gauss-Landen transformation (30) to bring this term to the gauge theory frame. Our two-loop matching formula (14) would seem to suggest that the matching of the three-loop gauge and string resolvents should again be given by a linear relation

$$\tilde{\mathcal{Q}}_{2k}^{(3)} \stackrel{?}{=} \bar{Q}_{2k}^{(3)} + \alpha_{k,1}\ \bar{Q}_{2k+2}^{(2)} + \alpha_{k,2}\ \bar{Q}_{2k+4}^{(1)} \,. \tag{17}$$

Working out explicitly the first few Inozemtsev charges one may verify that this proposal does not work, since the coefficients $\alpha_{k,1}$ and $\alpha_{k,2}$ then appear to be functions of the modular parameter q_0. Thus, linear combinations of Inozemtsev charges with constant coefficients cannot reproduce the string result. This motivates us to extend the set of charges by adding also their products. By trial and error we found the following remarkable formula ($k \geq 0$, and $\bar{Q}_0^{(2)} \equiv \bar{Q}_0^{(3)} \equiv 0$)

$$\begin{aligned}\tilde{\mathcal{Q}}_{2k}^{(3)} = \bar{Q}_{2k}^{(3)} + \frac{2k+1}{8\pi^2}\ \bar{Q}_{2k+2}^{(2)} + \frac{(2k+1)(2k+3)}{128\pi^4}\ \bar{Q}_{2k+4}^{(1)} \\ - \frac{1}{8\pi^2}\ \bar{Q}_2^{(1)}\ \bar{Q}_{2k}^{(2)}\end{aligned} \tag{18}$$

which describes the relation between the gauge and string towers of commuting charges up to and including three loops.

2.1. *Folded string*

The classical motion of the folded string leads to the following parametric result for the all-loop string energy $\mathcal{Q}_2$:

$$\lambda' = \frac{\pi^2}{4\mathrm{K}(t_0)^2}\left[\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 - \left(\frac{\mathrm{E}(t_0)}{\mathrm{E}(t)}\right)^2\right], \tag{19}$$

$$\mathcal{Q}_2 = \frac{\mathrm{K}(t)}{\mathrm{K}(t_0)}\sqrt{(1-t)\left(\frac{\mathrm{E}(t_0)}{\mathrm{E}(t)}\right)^2 + t\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2}\,.$$

[c] The sign flips are entirely due to a slight difference in the normalization of gauge theory charges for the folded and circular cases, *cf.* [20]. Defining the charges in an identical fashion in both cases removes this difference.

Here the modulus t contains the information about the coupling λ'. In addition, its constant piece t_0 parametrizes the "filling fraction"

$$\alpha = 1 - \frac{\mathrm{E}(t_0)}{\mathrm{K}(t_0)}\,. \tag{20}$$

The charges are found by the Bäcklund transformation; the final result for the string resolvent is given in eq.(6), where the "auxiliary" and true spectral parameters z, φ are related through

$$\varphi^2 + \left(\frac{1}{4\mathrm{K}(t_0)}\frac{\mathrm{E}(t_0)}{\mathrm{E}(t)}\right)^2 \frac{z(1-tz)}{1-z} + \frac{\lambda'}{4\pi^2} tz = 0\,. \tag{21}$$

Expanding this in λ' one finds

$$\tilde{\mathcal{Q}}(\varphi, \lambda') = \tilde{\mathcal{Q}}^{(1)}(\varphi) + \lambda' \tilde{\mathcal{Q}}^{(2)}(\varphi) + \mathcal{O}(\lambda'^2)\,. \tag{22}$$

Here the leading piece is

$$\tilde{\mathcal{Q}}^{(1)}(\varphi) = \frac{1}{2} - 2\varphi \frac{\sqrt{(1-z_0)(t_0 z_0 - 1)}}{\sqrt{z_0}}\, \Pi(t_0 z_0, t_0)\,. \tag{23}$$

where

$$\varphi^2 + \left(\frac{1}{4\mathrm{K}(t_0)}\right)^2 \frac{z_0(1-t_0 z_0)}{1-z_0} = 0\,. \tag{24}$$

The next term $\tilde{\mathcal{Q}}^{(2)}(\varphi)$ is rather complicated and we will not write it out.

The Bethe Ansatz for the spin chain describing two-loop gauge theory in the large J limit quite generally leads to a singular integral equation. In the present case this equation is of elliptic type, and closely related to the one appearing in the so-called O(N) matrix model, see e.g. [32]. Its solution leads to the following generating functions for the one-loop charges $\bar{Q}^{(1)}_{2k}$

$$\bar{Q}^{(1)}(\varphi) = \frac{1}{4} - \frac{a_0^2}{b_0}\sqrt{\frac{b_0^2 - \varphi^2}{a_0^2 - \varphi^2}}\; \Pi\left(-q_0 \frac{\varphi^2}{a_0^2 - \varphi^2}, q_0\right) \tag{25}$$

and two-loop charges $\bar{Q}^{(2)}_{2k}$

$$\bar{Q}^{(2)}(\varphi) = \frac{1}{32\pi^2\varphi^2}\left(1 - \frac{b_0}{2a_0}\sqrt{\frac{a_0^2-\varphi^2}{b_0^2-\varphi^2}} - \frac{a_0}{2b_0}\sqrt{\frac{b_0^2-\varphi^2}{a_0^2-\varphi^2}}\right)\,. \tag{26}$$

Here the natural modulus q_0 parametrizes the filling fraction through

$$\alpha = \frac{1}{2} - \frac{1}{2\sqrt{1-q_0}}\frac{\mathrm{E}(q_0)}{\mathrm{K}(q_0)}\,. \tag{27}$$

while the leading-order rapidity boundaries a_0, b_0 are given by

$$a_0 = \frac{1}{4\mathrm{K}(q_0)}, \qquad b_0 = \frac{1}{4\sqrt{1-q_0}\ \mathrm{K}(q_0)}\,. \tag{28}$$

It is now straightforward to calculate the expression

$$\bar{Q}^{(2)}(\varphi) + \frac{1}{8\pi^2}\frac{\partial}{\partial\varphi}\frac{1}{\varphi}\ \bar{Q}^{(1)}(\varphi) = \frac{a_0^2 + b_0^2 - 8a_0 b_0^2 \mathrm{E}(q_0)}{64\pi^2 a_0 b_0 \sqrt{(a_0^2-\varphi^2)(b_0^2-\varphi^2)}}\,. \tag{29}$$

As was shown in [20] after a Gauss-Landen transformation of the moduli

$$t_0 = -\frac{(1-\sqrt{1-q_0})^2}{4\sqrt{1-q_0}}\,, \tag{30}$$

eq.(23) becomes identical to eq.(25). Now we observe that the same transformation nicely turns the rather complicated expression $\tilde{\mathcal{Q}}^{(2)}(\varphi)$ in eq.(22) into the r.h.s. of eq.(29), proving our main assertion eq.(15).

2.2. *Circular string*

The analysis of the circular string is very similar to one for the folded case. The BMN coupling constant λ' and the string energy $\mathcal{Q}_2$ are given by

$$\lambda' = \frac{\pi^2 t}{4t_0^2 \mathrm{K}(t_0)^2}\left[\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 - \left(\frac{\mathrm{E}(t_0)-(1-t_0)\mathrm{K}(t_0)}{\mathrm{E}(t)-(1-t)\mathrm{K}(t)}\right)^2\right], \tag{31}$$

$$\mathcal{Q}_2 = \frac{t\mathrm{K}(t)}{t_0\mathrm{K}(t_0)}\sqrt{\frac{1}{t}\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 - \frac{1-t}{t}\left(\frac{\mathrm{E}(t_0)-(1-t_0)\mathrm{K}(t_0)}{\mathrm{E}(t)-(1-t)\mathrm{K}(t)}\right)^2},$$

where the parameter t_0 is determined via the filling fraction as follows

$$\alpha = 1 - \frac{1}{t_0} + \frac{1}{t_0}\ \frac{\mathrm{E}(t_0)}{\mathrm{K}(t_0)}. \tag{32}$$

The string spectral parameters z and φ are related through

$$\varphi^2 - \frac{t^2}{t_0^2}\left(\frac{1}{4\mathrm{K}(t_0)}\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 \frac{z(1-z)}{1-tz} - \frac{\lambda'}{4\pi^2}\frac{(1-t)z}{1-tz} = 0\,. \tag{33}$$

On the gauge theory side the configuration of Bethe roots corresponding to the circular string is described by two cuts on the imaginary axis: $ic < i\varphi < id$ and $-id < i\varphi < -ic$, with a constant condensate in between. For the one-loop problem we denote the endpoints of the cut as c_0 and d_0 and introduce the gauge theory modulus as $r_0 = \frac{c_0^2}{d_0^2}$, where $c_0 = \frac{1}{8\mathrm{K}(r_0)}$. The modulus

r_0 is related to the filling fraction α as follows $\alpha = \frac{1}{2} - \frac{1}{2\sqrt{r_0}} + \frac{1}{2\sqrt{r_0}}\frac{\mathrm{E}(r_0)}{\mathrm{K}(r_0)}$. Then the one-loop resolvent is

$$\bar{Q}^{(1)}(\varphi) = \frac{1}{4} - \frac{2}{d_0}\sqrt{(d_0^2-\varphi^2)(c_0^2-\varphi^2)}\;\Pi\left(\frac{\varphi^2}{d_0^2}, r_0\right). \qquad (34)$$

The two-loop correction found from the Inozemtsev-Bethe Ansatz reads

$$\bar{Q}^{(2)}(\varphi) = \frac{1}{32\pi^2\varphi^2}\left(1 - \frac{d_0}{2c_0}\sqrt{\frac{c_0^2-\varphi^2}{d_0^2-\varphi^2}} - \frac{c_0}{2d_0}\sqrt{\frac{d_0^2-\varphi^2}{c_0^2-\varphi^2}}\right). \qquad (35)$$

Using the two formulae above one finds

$$\bar{Q}^{(2)}(\varphi) + \frac{1}{8\pi^2}\frac{\partial}{\partial\varphi}\frac{1}{\varphi}\;\bar{Q}^{(1)}(\varphi) = \frac{c_0^2 - d_0^2 + 16c_0d_0^2\mathrm{E}(r_0)}{64\pi^2c_0d_0\sqrt{(c_0^2-\varphi^2)(d_0^2-\varphi^2)}} \qquad (36)$$

Expanding in λ' the resolvent (6) corresponding to the circular string one identifies the leading and the subleading terms, which are $\tilde{\mathcal{Q}}^{(1)}(\varphi)$ and $\tilde{\mathcal{Q}}^{(2)}(\varphi)$ respectively. Quite remarkably, the Gauss-Landen transformation

$$t_0 = -\frac{4\sqrt{r_0}}{(1-\sqrt{r_0})^2} \qquad (37)$$

transforms these expressions into eqs.(34) and (36).

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