Scalar fields on $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ and $H^2 \times \mathbf{R}$ geometric spacetimes and linear perturbations

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Abstract

Using appropriate harmonics, we study the future asymptotic behavior of massless scalar fields on a class of cosmological vacuum spacetimes. The spatial manifold is assumed to be a circle bundle over a higher genus surface with a locally homogeneous metric. Such a manifold corresponds to the $\widetilde{\mathrm{SL}}(2,\mathbf{R})$-geometry (Bianchi VIII type) or the $H^2 \times \mathbf{R}$-geometry (Bianchi III type). After a technical preparation including an introduction of suitable harmonics for the circle-fibered Bianchi VIII to separate variables, we derive systems of ordinary differential equations for the scalar field. We present future asymptotic solutions for these equations in a special case, and find that there is a close similarity with those on the circle-fibered Bianchi III spacetime. We discuss implications of this similarity, especially to (gravitational) linear perturbations. We also point out that this similarity can be explained by the *fiber term dominated behavior* of the two models.

1 Introduction

Linear perturbation analysis of Einstein's equation has fundamental importance in general relativity. It expands the significance of an exact solution that usually has a large symmetry by providing additional properties of the solution, especially those about stability or instability when the solution deviates from the symmetric configuration. In particular, detailed properties of the perturbations are expected to be served as useful pieces of information with which global nonlinear analyses, like those for the global existence problem [2, 16] and the cosmic censorship conjecture [18, 14] or conformal dynamics by Einstein's flow [1, 9], can develop.

Linear perturbation analysis can be however very difficult and often requires a formidable effort to carry out for many of the solutions we are interested in. Among them are the spatially homogeneous solutions [8, 20] which are not isotropic in any limit keeping the homogeneity. Linear stability properties are

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not known for most of those solutions. Recently, however, a complete perturbation analysis for a vacuum Bianchi III solution, which belongs to this class of anisotropic solutions, has been carried out [26, 22] after an enormous effort. In connection with this, we in this paper propose an indirect approach that extends the Bianchi III result to another model. More precisely, we make use of an analogy to derive perturbation properties for a "Bianchi VIII" solution.

The setting is as follows. The spacetime manifold we consider is the direct product of a spatial manifold M and time $\mathbf{R}$, for which M is a circle bundle over a higher genus surface Σ_g with genus $g > 1$. The background spatial metric is assumed to be locally homogeneous. In this situation, there are two kinds of possible locally homogeneous metrics on M depending upon whether M is a nontrivial bundle or a trivial bundle. If the bundle is nontrivial the metric becomes the Bianchi VIII type (Thurston's $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ [19, 28]), while if it is trivial it becomes the Bianchi III type (Thurston's $H^2 \times \mathbf{R}$).

The key fact is that the two kinds of vacuum spacetimes behave the "same" way in the future asymptotics in a special case; the evolutions of the fibers become asymptotically the same, and the evolutions of the base also become asymptotically the same. We will call this coincidence of the two backgrounds the *background asymptotic degeneracy*. (This "degeneracy" makes sense only when considering a class of fibered spatial manifolds with the same fiber and base manifold.) With this kind of coincidence, it is of great interest to compare test fields like scalar fields on these spacetimes and see if there is a substantial difference in the asymptotic behaviors of the fields. That is, if we find that there is no substantial difference in those behaviors — we will call this coincidence the *scalar field asymptotic degeneracy*, then this may be regarded as an evidence that other kinds of fields like the linear perturbations on the two kinds of spacetimes will also show the same asymptotic behavior or at least a very similar behavior to each other.

Based on this idea, in this paper we study the massless scalar field equation on the Bianchi VIII spacetime in detail. In particular, we separate the field equation using appropriate harmonics and find asymptotic solutions. We then make a comparison with the corresponding Bianchi III system, and will confirm the scalar field asymptotic degeneracy between the two systems. As mentioned, since the basic properties of the perturbations of the Bianchi III solution are already known, the comparison provides us with a conjecture about the Bianchi VIII perturbations.

One of the unfamiliar techniques needed to carry out our analysis may be the harmonics for the circle-fibered Bianchi VIII manifold. Since there does not seem to exist a prior work on this subject (see however [4, 11] for related work), we start with introducing appropriate ones for this manifold in §§.2-3. Then, we separate the field equation using those harmonics and find asymptotic solutions of the reduced equations in §§.4-5. We then compare the results with those of the Bianchi III system in §.6. The final section is devoted to conclusions.

2 $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ actions

In this section we introduce some basic objects associated with the Bianchi VIII manifolds as a preliminary for developing the harmonics.

Let $G = \widetilde{\mathrm{SL}}(2,\mathbf{R})$ be the universal covering group of $\mathrm{SL}(2,\mathbf{R})$, the real special linear group of rank 2. This group is also known as the Bianchi VIII group. Our spatial manifold M is the quotient $\Gamma\backslash G$ of G by a discrete subgroup $\Gamma \subset G$ acting freely from the left. M is assumed to be *closed* (= compact without boundary). In such a case, the manifold M is known to become a (nontrivial) circle bundle over a higher genus hyperbolic surface Σ_g.[1] ($g \geq 2$ is the genus.)

For given element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}), \tag{1}$$

it acts on the space of direct product $\Omega_+ \times \mathbf{T}$ of the upper half plane

$$\Omega_+ \equiv \{\zeta \in \mathbf{C} |\, \mathrm{Im}\zeta > 0\} \tag{2}$$

and the circle $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$,

$$g : \Omega_+ \times \mathbf{T} \to \Omega_+ \times \mathbf{T}, \tag{3}$$

as

$$g \cdot (\zeta, z) = (\varphi_g(\zeta), z - \vartheta_g(\zeta)), \tag{4}$$

where $(\zeta, z) = (x + iy, z) \in \Omega_+ \times \mathbf{T}$, and

$$\varphi_g(\zeta) \equiv \frac{a\zeta + b}{c\zeta + d}, \quad \vartheta_g(\zeta) \equiv 2 \arg(c\zeta + d). \tag{5}$$

In fact, it is a direct computation to confirm the homomorphism

$$(gg') \cdot (\zeta, z) = g \cdot (g' \cdot (\zeta, z)), \tag{6}$$

where gg' is understood as the usual matrix product. Since this natural action is simply transitive, i.e., for arbitrary $\mathbf{x}, \mathbf{x}' \in \Omega_+ \times \mathbf{T}$ there exists unique $g \in \mathrm{SL}(2, \mathbf{R})$ such that $g \cdot \mathbf{x} = \mathbf{x}'$, we can identify the space $\Omega_+ \times \mathbf{T}$ with the group $\mathrm{SL}(2, \mathbf{R})$ by associating $g \in \mathrm{SL}(2, \mathbf{R})$ with its action on the origin $\mathbf{x}_0 = (0+1i, 0)$; $g \simeq g \cdot \mathbf{x}_0$. If we take the universal cover of $\Omega_+ \times \mathbf{T}$, it apparently becomes $\Omega_+ \times \mathbf{R}$, which can be also identified with the space $\mathbf{R}^3_+ \equiv \{(x, y, z) \in \mathbf{R}^3 | y > 0\}$, therefore

$$G = \widetilde{\mathrm{SL}}(2, \mathbf{R}) \simeq \Omega_+ \times \mathbf{R} \simeq \mathbf{R}^3_+. \tag{7}$$

This identification is understood throughout this paper. The following explicit formula for the correspondence $G \simeq \mathbf{R}_+$ is sometimes useful;

$$\frac{1}{\sqrt{y}} \begin{pmatrix} y \cos\frac{z}{2} - x \sin\frac{z}{2} & y \sin\frac{z}{2} - x \cos\frac{z}{2} \\ -\sin\frac{z}{2} & \cos\frac{z}{2} \end{pmatrix} \simeq (x, y, z). \tag{8}$$

To find generators of G acting on $\mathbf{R}^3_+$, it is convenient to introduce the following one-parameter subgroups of $\mathrm{SL}(2, \mathbf{R})$;

$$n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \; a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \; u_\theta = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}, \tag{9}$$

[1] More precisely, M can be a *Seifeld fiber space* over a hyperbolic orbifold [19], but in this paper we assume that M is one of usual fiber bundles as a typical case.

where $\theta \in [0, 4\pi)$, $t \in \mathbf{R}$, and $s \in \mathbf{R}$. Then, any element $g \in \mathrm{SL}(2, \mathbf{R})$ is uniquely expressed in the form ([21], Proposition 1.3, Chap V)

$$g = u_\theta a_t n_s. \tag{10}$$

This decomposition is known as the Iwasawa decomposition. Let ξ_1, ξ_2, and ξ_3 be the generators of n_s, a_t, and u_θ, respectively; so, e.g., for $\mathbf{x} \in \mathbf{R}^3_+$,

$$\xi_1 f(\mathbf{x}) = \frac{d}{ds} f(n_s \cdot \mathbf{x})\Big|_{s=0} = \frac{\partial}{\partial x} f(\mathbf{x}). \tag{11}$$

Similarly, we can compute the others, and find

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial x}, \\ \xi_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ \xi_3 &= \frac{1}{2}\left(x^2 - y^2 + 1\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}. \end{aligned} \tag{12}$$

These generators satisfy the following algebra;

$$[\xi_1, \xi_2] = \xi_1, \quad [\xi_2, \xi_3] = -\xi_1 + \xi_3, \quad [\xi_3, \xi_1] = -\xi_2. \tag{13}$$

Although this is not of the canonical form that is usually assumed as the algebra of Bianchi VIII, it is easy to see that the linear combinations

$$\xi'_1 = \frac{1}{\sqrt{2}}(-\frac{3}{2}\xi_1 + \xi_3), \quad \xi'_2 = \xi_2, \quad \xi'_3 = \frac{1}{\sqrt{2}}(\frac{1}{2}\xi_1 + \xi_3), \tag{14}$$

satisfy

$$[\xi'_I, \xi'_J] = C^K{}_{IJ}\xi'_K \tag{15}$$

with the structure constants $C^K{}_{IJ} = -C^K{}_{JI}$ for which the nonzero independent components are given by

$$C^1{}_{23} = C^2{}_{31} = 1, \; C^3{}_{12} = -1. \tag{16}$$

This coincides with the canonical choice of structure constants of Bianchi VIII (e.g., [8]).

The generators ξ_I above are regarded as generating the *left action* of G on $G \simeq \mathbf{R}^3_+$. Another important kind of objects are generators of the *right action*, which we denote as χ_I. They are defined as differential operators such that they commute with all the left generators ξ_I

$$[\chi_I, \xi_J] = 0, \quad (I, J = 1 \sim 3), \tag{17}$$

and possess the same structure constants (16). They are given by

$$\begin{aligned} \chi_1 &= y\left(\sin z\frac{\partial}{\partial x} - \cos z\frac{\partial}{\partial y}\right) - \sin z\frac{\partial}{\partial z}, \\ \chi_2 &= y\left(\cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\right) - \cos z\frac{\partial}{\partial z}, \\ \chi_3 &= \frac{\partial}{\partial z}. \end{aligned} \tag{18}$$

The commutativity (17) means that χ_I are invariant under the left action of G as seen from the vanishing of the Lie derivatives, $\mathcal{L}_{\xi_I}\chi_J = 0$. We therefore call χ_I the *invariant vectors*. The dual one-forms of these vectors, the *invariant one-forms*, can be expressed as

$$\begin{aligned} \sigma^1 &= \frac{1}{y}(\sin z dx - \cos z dy), \\ \sigma^2 &= \frac{1}{y}(\cos z dx + \sin z dy), \\ \sigma^3 &= \frac{dx}{y} + dz. \end{aligned} \tag{19}$$

These one-forms are invariant in the sense $\mathcal{L}_{\xi_I}\sigma^J = 0$, and satisfy the following Maurer-Cartan relation

$$d\sigma^I = -\frac{1}{2}C^I{}_{JK}\sigma^J \wedge \sigma^K, \tag{20}$$

with respect to the same structure constants (16). Now, a homogeneous metric on G can be expressed as $q = q_{IJ}\sigma^I\sigma^J$ with q_{IJ} being constants, since we apparently have $\mathcal{L}_{\xi_I} q = 0$ for this metric. The generators ξ_I of the left actions are *Killing vectors* for these homogeneous metrics. When the metric is of the form $q = q_1((\sigma^1)^2 + (\sigma^2)^2) + q_3(\sigma^3)^2$, where q_1 and q_3 are constants, this metric possesses a fourth Killing vector

$$\xi_4 = \frac{\partial}{\partial z}, \tag{21}$$

and the metric is said to be *locally rotationally symmetric (LRS).* Note that this additional Killing vector for an LRS metric coincides with one of the invariant vectors, χ_3;

$$\xi_4 = \chi_3. \tag{22}$$

An importance of the invariant vectors χ_I is that they generate the *regular representation* $(T, L^2(\Gamma\backslash G))$ of G on the Hilbert space $L^2(\Gamma\backslash G)$ of all square-integrable functions on $\Gamma\backslash G$. By regular representation, we mean the "right regular representation."[2] It is apparent that χ_I generate the regular representation on $L^2(G)$, since χ_I are generators of the right action (of G) on G. Note that the invariant vectors χ_I are naturally well defined on the quotient $\Gamma\backslash G$, as well, since χ_I are by definition invariant under the action of $\Gamma \subset G$. (Therefore for the covering map $\pi : G \to \Gamma\backslash G$, the induced vectors $\pi^*\chi_I$ on $\Gamma\backslash G$ are well defined, which however we simply denote as χ_I.) Moreover, since the right and left actions commute each other the right action of G on the left quotient $\Gamma\backslash G$ is also well defined. Therefore the regular representation $(T, L^2(\Gamma\backslash G))$ is well defined and χ_I are its generators.

With the aid of Eq.(8), it is easy to explicitly compute the right action $\mathbf{x}\cdot g$ of $g \in \mathrm{SL}(2,\mathbf{R})$ on $\mathbf{x} \in \mathbf{R}^3_+$. (Compute the usual matrix product of the left hand side of Eq.(8) and g, and then read off the components in $\mathbf{R}_+$, using

[2]For given $g \in G$, we can define the map $T_g : f(\mathbf{x}) \to f(\mathbf{x}\cdot g)$, where $\mathbf{x} \in G \simeq \mathbf{R}^3_+$. The right action $\mathbf{x}\cdot g$ is naturally defined viewing $\mathbf{x}$ as the corresponding element in G. It is easy to confirm that the map $T : g \to T_g$ is a homomorphism; $T_g T_{g'} = T_{gg'}$. This homomorphism $(T, L^2(G))$ is called the (right) *regular representation* of G on $L^2(G)$.[21]

the correspondence (8).) The most important action is that of the compact subgroup u_θ, which is found to cause the translations $(x,y,z)\to(x,y,z+\theta)$. Its generator is therefore found to coincide with χ_3, which we call the *fiber generator*.

Eq.(22) is saying that if the metric is LRS, the fiber generator χ_3 is also a Killing vector. Therefore an LRS metric of Bianchi VIII type is U(1)-*symmetric* (as well as being locally homogeneous), since there exists isometries along the circle ($\simeq$ U(1)) fibers. Beware that non-LRS Bianchi VIII metrics are *not* U(1)-symmetric in any sense despite that they are circle-fibered and locally homogeneous. This is apparent from the fact that none of the generic Killing vectors ξ_I ($I=1\sim 3$) are well defined on the quotient manifold $\Gamma\backslash G$, since ξ_I are not commutative with each other. (Remember that the action of Γ is generated by ξ_I. This noncommutativity therefore implies the noncommutativity between Γ and ξ_I, which in turn implies ξ_I are not well defined on $\Gamma\backslash G$.)

The regular representation $(T, L^2(\Gamma\backslash G))$ is, as shown in the next section, unitary, but not irreducible. As well known (e.g., [21]), harmonics are basis vectors of *irreducible components* of a regular representation. We are hence interested in irreducible components in $(T, L^2(\Gamma\backslash G))$, which are the subject of the next section.

3 The harmonics

The most important entity to consider an irreducible unitary representation is the Casimir operator, which is an operator which commutes with all the generators of the representation, since from Schur's lemma, such an operator must be a constant when acting on an irreducible space. For the group G, it is given by

$$\Box \equiv (\chi_1)^2+(\chi_2)^2-(\chi_3)^2. \tag{23}$$

In fact, it is a direct computation to check the condition $[\Box,\chi_I]=0$ for all $I=1\sim 3$. In particular, it commutes with the fiber generator χ_3;

$$[\Box,\chi_3]=0. \tag{24}$$

It is therefore possible to simultaneously diagonalize these two operators;

$$\begin{aligned}\Box\phi_{m,\Lambda} &= -\Lambda\phi_{m,\Lambda},\\ \chi_3\phi_{m,\Lambda} &= im\phi_{m,\Lambda}.\end{aligned} \tag{25}$$

We call Λ the *Casimir eigenvalue*, m the *fiber eigenvalue*.

We assume that the closed manifold $\Gamma\backslash G$ coincides with a compactification of $\mathrm{SL}(2,\mathbf{R})$ (rather than $G=\widetilde{\mathrm{SL}}(2,\mathbf{R})$). This is equivalent to assuming that the discrete subgroup Γ has the subgroup $2\pi\mathbf{Z}$ generated by the action $(x,y,z)\to(x,y,z+2\pi)$. Any two points (x,y,z) and $(x,y,z+2\pi)$ should therefore be identified, which forces

$$m\in\mathbf{Z}. \tag{26}$$

If we considered a p-fold covering of the manifold, we would instead have $m\in\mathbf{Z}/p\equiv\{n/p|n\in\mathbf{Z}\}$. For simplicity, however, we do not consider this generalization in this paper.

We assume that each $\phi_{m,\Lambda}$ is normalized to unity;

$$||\phi_{m,\Lambda}||^2 \equiv (\phi_{m,\Lambda}, \phi_{m,\Lambda}) = 1, \tag{27}$$

with respect to the inner product (f_1, f_2) in $L^2(\Gamma\backslash G)$ defined by

$$(f_1, f_2) \equiv \int_{\Gamma\backslash G} f_1 f_2^* d\mu_0. \tag{28}$$

Here, f^* is the complex conjugate of f. The measure defined by

$$d\mu_0 \equiv \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \tag{29}$$

is the natural left-invariant measure of G, therefore the above integral on the quotient $\Gamma\backslash G$ is well defined. Since the measure is also right-invariant (i.e., it is "unimodular"), the above inner product is invariant under the right action of G, i.e., for $g \in G$,

$$\int_{\Gamma\backslash G} f_1(\mathbf{x}\cdot g) f_2^*(\mathbf{x}\cdot g) d\mu_0 = \int_{\Gamma\backslash G} f_1(\mathbf{x}) f_2^*(\mathbf{x}) d\mu_0. \tag{30}$$

This implies that the regular representation $(T, L^2(\Gamma\backslash G))$ is certainly *unitary*.

Our main purpose here is to find the harmonics belonging to given Λ, i.e., to find all the basis functions $\{\phi_{m',\Lambda}\}_{m'}$ of an irreducible space labeled by Λ. (The number of the irreducible spaces belonging to the same Λ, the "multiplicity," can be more than one. So, Λ is not a complete label to specify the irreducible space. More about the multiplicity will be mentioned at the end of this section.) This task of finding the harmonics can be done starting from one function $\phi_{m,\Lambda}$ satisfying Eqs.(25), as shown below.

First, let us define

$$\mathcal{A}_1 \equiv \frac{1}{\sqrt{2}}(\chi_1 - i\chi_2), \quad \mathcal{A}_2 \equiv \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2), \quad \mathcal{A}_3 \equiv -i\chi_3. \tag{31}$$

We have the following properties about adjointness:

Lemma 1 *In the Hilbert space $L^2(\Gamma\backslash G)$, it holds*

$$\mathcal{A}_1^\dagger = -\mathcal{A}_2, \quad \mathcal{A}_3^\dagger = \mathcal{A}_3. \tag{32}$$

Proof. We first show that the invariant operators χ_I $(I = 1 \sim 3)$ are anti-selfadjoint; $\chi_I^\dagger = -\chi_I$. Actually, if it is the case, $\mathcal{A}_1^\dagger = \frac{1}{\sqrt{2}}(\chi_1 - i\chi_2)^\dagger = \frac{1}{\sqrt{2}}(-\chi_1 - i\chi_2) = -\mathcal{A}_2$, and $\mathcal{A}_3^\dagger = -(i\chi_3)^\dagger = -i\chi_3 = \mathcal{A}_3$. To show the anti-selfadjointness of χ_I, note the identity

$$(\chi_I f_1, f_2) = \mathcal{I}_I - (f_1, \chi_I f_2), \tag{33}$$

where we want to show the integral

$$\mathcal{I}_I \equiv \int_{\Gamma\backslash G} \chi_I(f_1 f_2^*) d\mu_0 \tag{34}$$

vanishes for all $I = 1 \sim 3$. Actually, we have

$$\mathcal{I}_I = \int_{\Gamma\backslash G} \chi_I(f_1 f_2^*)\sigma^1 \wedge \sigma^2 \wedge \sigma^3 = \frac{1}{2}\int_{\Gamma\backslash G} d(f_1 f_2^* \epsilon_{IJK}\sigma^J \wedge \sigma^K), \tag{35}$$

which is confirmed using the identity

$$df = (\chi_1 f)\sigma^1 + (\chi_2 f)\sigma^2 + (\chi_3 f)\sigma^3, \tag{36}$$

and the relation

$$d(\epsilon_{IJK}\sigma^J \wedge \sigma^K) = 0, \tag{37}$$

where ϵ_{IJK} is the unit skew symmetric symbol; $\epsilon_{IJK} = \epsilon_{[IJK]}$, $\epsilon_{123} = 1$. Then, from Stokes theorem the quantity $\mathcal{I}_I$ does vanish, since the manifold $\Gamma\backslash G$ is assumed to be closed. ■

Next, note that the commutation relations among $\mathcal{A}_I$ become

$$[\mathcal{A}_3, \mathcal{A}_1] = \mathcal{A}_1, \quad [\mathcal{A}_3, \mathcal{A}_2] = -\mathcal{A}_2, \quad [\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3. \tag{38}$$

This in particular shows that operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are, respectively, the *raising and lowering operator*. In fact, since

$$\mathcal{A}_3\mathcal{A}_1\phi_m = ([\mathcal{A}_3, \mathcal{A}_1] + \mathcal{A}_1\mathcal{A}_3)\phi_m = (\mathcal{A}_1 + \mathcal{A}_1\mathcal{A}_3)\phi_m = (1+m)\mathcal{A}_1\phi_m, \tag{39}$$

$\mathcal{A}_1\phi_m$ is an eigenfunction for $m' = m+1$; $\mathcal{A}_1\phi_m \propto \phi_{m+1}$. Similarly, $\mathcal{A}_2\phi_m \propto \phi_{m-1}$. We can therefore assume that

$$\begin{aligned} \mathcal{A}_1\phi_m &= a_m\phi_{m+1}, \\ \mathcal{A}_2\phi_m &= b_m\phi_{m-1}, \\ \mathcal{A}_3\phi_m &= m\phi_m, \end{aligned} \tag{40}$$

for appropriate coefficients a_m and b_m.

On the other hand, one can easily check the identity

$$\mathcal{A}_1\mathcal{A}_2 = \frac{1}{2}(\Box - (\mathcal{A}_3)^2 + \mathcal{A}_3). \tag{41}$$

Together with the assumption (40) we find

$$a_{m-1}b_m = \frac{-1}{2}\left((m-\frac{1}{2})^2 + \Lambda - \frac{1}{4}\right). \tag{42}$$

From Eq.(32), we have

$$\begin{aligned} (\mathcal{A}_1\mathcal{A}_2\phi_{m,\Lambda}, \phi_{m,\Lambda}) &= (\mathcal{A}_2\phi_{m,\Lambda}, -\mathcal{A}_2\phi_{m,\Lambda}) \\ &= -||\mathcal{A}_2\phi_{m,\Lambda}||^2 \\ &= -|b_m|^2, \end{aligned} \tag{43}$$

where we have used the normalization (27). On the other hand,

$$\begin{aligned} (\mathcal{A}_1\mathcal{A}_2\phi_{m,\Lambda}, \phi_{m,\Lambda}) &= a_{m-1}b_m(\phi_{m,\Lambda}, \phi_{m,\Lambda}) \\ &= a_{m-1}b_m. \end{aligned} \tag{44}$$

Therefore, using Eq.(42),

$$|b_m|^2 = -a_{m-1}b_m = \frac{1}{2}\left((m-\frac{1}{2})^2 + \Lambda - \frac{1}{4}\right). \tag{45}$$

In particular, since $|b_m|^2 \geq 0$, for any possible m and Λ, it must hold that

$$(m - \frac{1}{2})^2 + \Lambda - \frac{1}{4} \geq 0. \tag{46}$$

Since an $m = 0$ mode is always contained in any irreducible components,[3] from the above condition Λ should be nonnegative

$$\Lambda \geq 0. \tag{47}$$

Also,

$$b_m^* = -a_{m-1}. \tag{48}$$

We can determine the coefficients a_m and b_m from Eqs.(45) and (48). For convenience, let us define

$$s \equiv \pm\sqrt{|\Lambda - \frac{1}{4}|}. \tag{49}$$

(i) Case $\Lambda \geq \frac{1}{4}$
In this case, it is convenient to choose

$$\begin{aligned} a_m &= \frac{1}{\sqrt{2}}(|m + \frac{1}{2}| + is), \\ b_m &= \frac{-1}{\sqrt{2}}(|m - \frac{1}{2}| - is). \end{aligned} \tag{50}$$

(ii) Case $0 \leq \Lambda < \frac{1}{4}$
In this case, we can choose

$$\begin{aligned} a_m &= \frac{1}{\sqrt{2}}\left((m + \frac{1}{2})^2 - s^2\right)^{1/2}, \\ b_m &= \frac{-1}{\sqrt{2}}\left((m - \frac{1}{2})^2 - s^2\right)^{1/2}. \end{aligned} \tag{51}$$

When $\Lambda = 0$ ($s^2 = 1/4$), from Eqs.(51) we have $a_0 = b_0 = 0$, reflecting the fact that this representation is trivial (see below) and the representation space is spanned by only one constant function $\phi_{0,0}$. The other representations are all infinite dimensional.

The differential representations (40) with the coefficients (50) or (51) above provide the heart of the relations used to separate the field equations.

In the classification of irreducible unitary representations of SL$(2, \mathbf{R})$, there are five kinds of series (e.g., [21, 27]). The case (i) above belongs to the class called the *first principal series*, while the case (ii) the *complementary series*. The other series may not generically occur from the representation we are considering, for which the fact (26) holds.

[3] This can be seen from Eq.(26) and the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ raises or lower the fiber eigenvalue by 1. That is, Spec$\mathcal{A}_3$, the spectra of $\mathcal{A}_3$ for given Λ, always coincides with the whole range of possible m; Spec$\mathcal{A}_3 = \mathbf{Z}$. ($\Lambda = 0$ is an exceptional case, but $m = 0$ exists in this case, too.) Remark however that this would not be the case if $m \in \mathbf{Z}/p$ ($p > 1$), which was as mentioned possible if considering a covering.

Note that the recursion relations (40) imply that $\phi_{m,\Lambda}$ can be constructed by successively applying appropriate operators on $\phi_{0,\Lambda}$. Explicitly, the relation is given by

$$\phi_{m,\Lambda} = \begin{cases} (a_{m-1}^{-1}\mathcal{A}_1)(a_{m-2}^{-1}\mathcal{A}_1)\cdots(a_0^{-1}\mathcal{A}_1)\phi_{0,\Lambda} & (m>0) \\ (b_{m+1}^{-1}\mathcal{A}_2)(b_{m+2}^{-1}\mathcal{A}_2)\cdots(b_0^{-1}\mathcal{A}_2)\phi_{0,\Lambda} & (m<0). \end{cases} \tag{52}$$

The function $\phi_{0,\Lambda}$ is a z-independent function, since from definition $\chi_3\phi_{0,\Lambda} = \partial\phi_{0,\Lambda}/\partial z = 0$. Moreover, let us observe the fact that when acting on a z-independent function $\hat{f}(x,y)$, the Casimir operator $\Box$ degenerates to a two-dimensional Laplacian $\hat{\triangle}_0$. (We use hat $\hat{}$ for quantities on a surface or z-independent functions.) In fact, since we can compute, from the expression (18) of χ_I,

$$(\chi_1)^2 + (\chi_2)^2 = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - 2y\frac{\partial^2}{\partial x\partial z} + \frac{\partial^2}{\partial z^2}, \tag{53}$$

we have

$$\begin{aligned} \Box\hat{f} &= ((\chi_1)^2 + (\chi_2)^2 - (\chi_3)^2)\hat{f} \\ &= ((\chi_1)^2 + (\chi_2)^2)\hat{f} \\ &= \hat{\triangle}_0\hat{f}, \end{aligned} \tag{54}$$

where

$$\hat{\triangle}_0 = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}). \tag{55}$$

The operator $\hat{\triangle}_0$ does coincide with the Laplacian, associated with the standard two-dimensional hyperbolic metric $h \equiv (dx^2+dy^2)/y^2 = (\sigma^1)^2 + (\sigma^2)^2$ on Σ_g.

Therefore we can summarize the construction of the harmonics on $\Gamma\backslash G$ as follows; we first consider a solution of the two-dimensional eigenvalue equation

$$\hat{\triangle}_0\hat{\phi}_\Lambda = -\Lambda\hat{\phi}_\Lambda \tag{56}$$

on the hyperbolic surface (Σ_g, h), and then apply the formula (52) assuming $\phi_{0,\Lambda} = \hat{\phi}_\Lambda$ to obtain an irreducible set of harmonics. Repeat this procedure for all independent solutions for given Λ. Repeat this for all possible values of Λ, then we obtain a complete set of harmonics.

This construction implies a simple but remarkable fact [27]; the *multiplicity* in an irreducible representation specified by Λ, occurring in $(T, L^2(\Gamma\backslash G))$, is directly determined by the multiplicity in the corresponding eigenstate of the two-dimensional Laplacian $\hat{\triangle}_0$.

The spectrum of (minus) the Laplacian $-\hat{\triangle}_0$ on a hyperbolic closed surface has been one of the major subjects in Riemannian geometry [5]. In particular, those in the range $(0, 1/4)$ are called *small eigenvalues*, and in what conditions they appear has been one of the central issues. We do not discuss this further here, but remark that small eigenvalues appear only when Σ_g takes a particular "shape" (specified by some "Teichmüler parameters"), in particular they do not necessary exist for an arbitrarily "shape" of Σ_g. The eigenvalue $\Lambda = 0$ corresponds to the trivial case, so the multiplicity (in the irreducible representation for $\Lambda = 0$) is always one.

4 The background solutions

We assume that the background spacetime metric is (locally) of the form

$$\mathrm{g} = -N^2(t)dt^2 + q_1(t)(\sigma^1)^2 + q_2(t)(\sigma^2)^2 + q_3(t)(\sigma^3)^2. \tag{57}$$

No general vacuum solution for this metric is known. However, see [17] for the future asymptotic behavior.

On the other hand, the general vacuum solution for the LRS metric, i.e., the case $q_1(t) = q_2(t)$, is known. It is a special case of the so-called NUT solutions [15]. The Bianchi VIII NUT solution is given by

$$N(t)^2 = U(t)^{-1}, \quad q_1(t) = q_2(t) = t^2 + l^2, \quad q_3(t) = 4l^2 U(t), \tag{58}$$

where

$$U(t) \equiv \frac{t^2 - l^2 + 2\mu t}{t^2 + l^2}, \tag{59}$$

with $l > 0$ and μ being constant parameters of solution. (This form of solution is obtained as a special case of the metric derived in [6].) Note that the conformal metric $h = (\sigma^1)^2 + (\sigma^2)^2$ is the hyperbolic metric in the upper half plane; $h = (dx^2 + dy^2)/y^2$, thus the conformal factor $q_1 (= q_2)$ is the scale factor for the base surface, while q_3 is the one for the circle fibers. To describe the future asymptotic behavior of these scale factors, let us introduce a proper time τ for this LRS solution by

$$\begin{aligned} \tau &= \int N dt = \int U^{-1/2} dt = \int \left(1 - \frac{\mu}{t} + O(\frac{1}{t^2})\right) dt \\ &= t - \mu \log t + O(\frac{1}{t}). \end{aligned} \tag{60}$$

This implies

$$\begin{aligned} t &= \tau + \mu \log t + O(\frac{1}{t}) \\ &= \tau + \mu \log(\tau + \mu \log t + O(\frac{1}{t})) + O(\frac{1}{t}) \\ &= \tau + \mu \log \tau + O(\frac{\log \tau}{\tau}). \end{aligned} \tag{61}$$

Then, it is easy to confirm the following expressions.

$$\begin{aligned} q_1(\tau) &= q_2(\tau) = (\tau + \mu \log \tau)^2 + O(\log \tau), \\ q_3(\tau) &= 4l^2 \left(1 + \frac{2\mu}{\tau}\right) + O(\frac{\log \tau}{\tau^2}). \end{aligned} \tag{62}$$

In particular, note that the fiber length

$$L \equiv \int_{\text{fiber}} \sqrt{q_3}\sigma^3 = \int_0^{2\pi} \sqrt{q_3} dz = 2\pi\sqrt{q_3} = 2\pi(2l + O(\frac{1}{\tau})) \tag{63}$$

approaches constant $L_\infty \equiv \lim_{\tau\to\infty} L = 4\pi l$. Therefore the parameter l is $(4\pi)^{-1}$ times the fiber length at the future infinity.

5 Future asymptotics of scalar fields

Let us consider the massless scalar field equation[4]

$$\mathrm{g}^{ab}\nabla_a\nabla_b\Psi = 0, \tag{64}$$

where ∇_a is the covariant derivative operator associated with a spacetime metric g_{ab}.

For our shift-free, Bianchi VIII metric (57),

$$\mathrm{g}^{ab}\nabla_a\nabla_b\Psi = \left(\frac{-1}{\sqrt{-\mathrm{g}}}\frac{\partial}{\partial t}\left(\sqrt{-\mathrm{g}}N^{-2}\frac{\partial}{\partial t}\right) + \triangle_q\right)\Psi, \tag{65}$$

where $\sqrt{-\mathrm{g}} \equiv \sqrt{-\det \mathrm{g}_{ab}} = N\sqrt{q_1q_2q_3}$ and $\triangle_q$ is the Laplacian with respect to the spatial metric q_{ab}, which can be expressed using the invariant operators as

$$\triangle_q = \sum_{I=1}^{3} q_I^{-1}(\chi_I)^2. \tag{66}$$

It is convenient to write the Laplacian $\triangle_q$ in terms of $\mathcal{A}_I$ and divide it into two parts, the *homogeneous part* $\triangle_q^{(0)}$ and the *inhomogeneous part* $\triangle_q^{(\mathrm{I})}$,

$$\triangle_q = \triangle_q^{(0)} + \triangle_q^{(\mathrm{I})}, \tag{67}$$

where

$$\begin{aligned}\triangle_q^{(0)} &= \frac{1}{2}(q_1^{-1}+q_2^{-1})(\mathcal{A}_1\mathcal{A}_2+\mathcal{A}_2\mathcal{A}_1) - q_3^{-1}(\mathcal{A}_3)^2,\\ \triangle_q^{(\mathrm{I})} &= \frac{1}{2}(q_1^{-1}-q_2^{-1})((\mathcal{A}_1)^2+(\mathcal{A}_2)^2).\end{aligned} \tag{68}$$

The "homogeneous" part does not change the index m when it acts on a single mode function $\phi_m = \phi_{m,\Lambda}$, while the "inhomogeneous" part does (and gives rise to inhomogeneous terms in field equations). In fact, we find

$$\begin{aligned}\triangle_q^{(0)}\phi_m &= \left(\frac{1}{2}(q_1^{-1}+q_2^{-1})(a_{m-1}b_m + a_m b_{m+1}) - q_3^{-1}m^2\right)\phi_m\\ &= -K_m(t)\phi_m,\\ \triangle_q^{(\mathrm{I})}\phi_m &= \frac{1}{2}(q_1^{-1}-q_2^{-1})(a_m a_{m+1}\phi_{m+2} + b_m b_{m-1}\phi_{m-2}),\end{aligned} \tag{69}$$

where

$$K_m(t) \equiv \frac{1}{2}(q_1^{-1}+q_2^{-1})(m^2+\Lambda) + q_3^{-1}m^2. \tag{70}$$

Here, we have used Eq.(45) to deform the first equation of (69).

We expand the field component $\Psi = \Psi_\Lambda$ belonging to an irreducible space specified by Λ as

$$\Psi(t,\mathbf{x}) = \sum_m \psi_m(t)\phi_m(\mathbf{x}). \tag{71}$$

[4] In this section we employ the abstract index notation [29] and use leading Latin letters $a, b, \cdots$ to denote abstract indices for vectors and tensors.

Then, the field equation (64) reduces to the following equations:

$$\frac{N}{(q_1q_2q_3)^{1/2}}\frac{d}{dt}\left(\frac{(q_1q_2q_3)^{1/2}}{N}\frac{d\psi_m}{dt}\right)+N^2K_m(t)\psi_m=I_m(t), \tag{72}$$

where

$$I_m(t)\equiv\frac{N^2}{2}(q_1^{-1}-q_2^{-1})(a_{m-2}a_{m-1}\psi_{m-2}+b_{m+2}b_{m+1}\psi_{m+2}). \tag{73}$$

These equations form two sets of infinitely many simultaneous ODEs — one for $\{\psi_m\}_{m=\mathrm{odd}}$ and one for $\{\psi_m\}_{m=\mathrm{even}}$. The term I_m, which contains $\psi_{m\pm2}$, works as an inhomogeneous term if we view the above single equation as a dynamical equation for ψ_m.

The LRS case. Let us consider the LRS background case (58), in which, due to the vanishing of the inhomogeneous term, each mode equation (72) becomes independent. The equation becomes of the form

$$\ddot{\psi}_m+\frac{\dot{f}}{f}\dot{\psi}_m+Z_m\psi_m=0, \tag{74}$$

where

$$Z_m(t)\equiv\frac{m^2}{4l^2}\left(\frac{t^2+l^2}{f}\right)^2+\frac{m^2+\Lambda}{f}, \tag{75}$$

and

$$f(t)\equiv t^2-l^2+2\mu t. \tag{76}$$

Here, dot ($\dot{}$) stands for d/dt. We are interested in the future ($t\to+\infty$) asymptotic solution. As emphasized in [22], to this it is necessary to transform the time variable to one that is suitable for the analysis. Let us define the new time variable T by

$$\frac{dT}{dt}=\sigma(t)>0, \tag{77}$$

using a positive function $\sigma(t)$, which is to be determined. We transform the unknown function ψ_m to another function X_m so that the new equation in terms of X_m, dX_m/dT, and d^2X_m/dT^2 has a vanishing dX_m/dT term. It is easy to do this for given $\sigma(t)$; The transformation is given by

$$\psi_m=\alpha(t)X_m, \tag{78}$$

where

$$\alpha(t)\equiv\sigma^{-1/2}f^{-1/2}. \tag{79}$$

The resulting equation is given by

$$\frac{d^2X_m}{dT^2}+WX_m=0, \tag{80}$$

where

$$W(T)\equiv\frac{1}{\sigma^2}\left(\frac{\ddot{\alpha}}{\alpha}+\frac{\dot{f}}{f}\frac{\dot{\alpha}}{\alpha}+Z_m\right). \tag{81}$$

(Again, dot ($\dot{}$) stands for d/dt, *not* d/dT.) If the function $W(T)$ in this equation approaches a constant $C\neq0$ as $T\to\infty$ ($t\to\infty$), then it may be natural to expect that the equation (80) has fundamental solutions approaching $e^{\pm i\sqrt{C}T}$ in case $C > 0$, or $e^{\pm\sqrt{|C|}T}$ in case $C < 0$. More precisely, using the standard symbol $o(1)$ to signify a function such that $\lim_{t\to\infty} o(1) = 0$, we expect that the equation has fundamental solutions of the form $e^{\pm i\sqrt{C}T}(1 + o(1))$ if $C > 0$, or $e^{\pm\sqrt{|C|}T}(1 + o(1))$ if $C < 0$. Actually, this is the case if and only if the function $W(t)$ satisfies the following finiteness condition (e.g.,[7]);

$$\int^{\infty} |W - C| dT = \int^{\infty} |W - C| \sigma dt < \infty. \tag{82}$$

In the present case (with $m \neq 0$), it is confirmed that this condition is satisfied with the choice $C = m^2/4l^2 > 0$ and

$$\sigma(t) = 1 - \frac{2\mu}{t}. \tag{83}$$

With this,

$$T = \int \sigma(t) dt = t - 2\mu \log t, \tag{84}$$

and

$$\alpha(t) = \frac{1}{t} + O(\frac{1}{t^3}). \tag{85}$$

Recalling Eq.(78), we thus have the following:

Proposition 2 *The generic, $m \neq 0$, mode equation for massless scalar field on the LRS vacuum Bianchi VIII solution, satisfying Eq.(74), has the following fundamental solutions:*

$$X_m^{(\pm)}(t) = t^{-1} e^{\pm i |\frac{m}{2l}| (t - 2\mu \log t)} (1 + o(1)). \tag{86}$$

We must consider the $m = 0$ case, separately. In this case, it is possible to choose

$$\sigma(t) = \frac{1}{t} \tag{87}$$

so that

$$W(t) = \Lambda - \frac{1}{4} + O(\frac{1}{t}). \tag{88}$$

Therefore, as long as $\Lambda \neq 1/4$, we can again apply the criterion mentioned above, since

$$|W - (\Lambda - \frac{1}{4})|\, \sigma = O(\frac{1}{t^2}), \tag{89}$$

implying the integral (82) is finite with $C = \Lambda - 1/4 \neq 0$. Note that there are both possibilities of C being positive and negative, depending on which we have two cases:

Proposition 3 *The* U(1)*-symmetric, $m = 0$, mode equation for massless scalar field on the LRS vacuum Bianchi VIII solution, satisfying Eq.(74), has the following fundamental solutions:*

$$X_0^{(\pm)}(t) = \begin{cases} t^{-\frac{1}{2}} e^{\pm i|s| \log t}(1 + o(1)) & (\Lambda > \frac{1}{4}) \\ t^{-\frac{1}{2} \pm |s|}(1 + o(1)) & (0 \leq \Lambda < \frac{1}{4}), \end{cases} \tag{90}$$

where $|s| \equiv \sqrt{|\Lambda - 1/4|}$.

We are interested in comparing with the Bianchi III model [26, 22]. To compare the asymptotic solutions, it is convenient to use a proper time τ as canonical time. Since our time coordinate t asymptotically approaches proper time, it might be expected that the formulas remain the same, but this is not exactly the case. As confirmed in the following, the phase velocity of a part suffers a slight modification.

Let τ be the proper time for the LRS solution, defined by Eq.(60). Using the formula (61), it is easy to confirm, e.g.,

$$t^{-1} = \tau^{-1} + O(\frac{\log \tau}{\tau^2}), \tag{91}$$

and

$$\begin{aligned} t - 2\mu \log t &= \tau + \mu \log \tau + O(\frac{\log \tau}{\tau}) - 2\mu \log \tau (1 + O(\frac{\log \tau}{\tau})) \\ &= \tau - \mu \log \tau + O(\frac{\log \tau}{\tau}). \end{aligned} \tag{92}$$

Beware that in the last equation the coefficient of $\log \tau$ is half the one of $\log t$ in the left hand side. Using these formulas, it is easy to rewrite the asymptotic solutions in Propositions 2 and 3 in terms of proper time τ. As a result, we have the following.

Theorem 4 *A massless scalar field on a spatially compactified LRS vacuum Bianchi VIII solution can be decomposed into its mode components that are independent from each other and each of which is specified by the fiber eigenvalue m and the Casimir eigenvalue Λ. Their fundamental solutions (at the future asymptotics) are given in terms of proper time τ as follows:*

$$X_m^{(\pm)}(\tau) = \begin{cases} \tau^{-1} e^{\pm i |\frac{m}{2l}|(\tau - \mu \log \tau)} (1 + o(1)) & (m \neq 0), \\ \tau^{-\frac{1}{2}} e^{\pm i |s| \log \tau} (1 + o(1)) & (m = 0, \Lambda > 1/4) \\ \tau^{-\frac{1}{2} \pm |s|} (1 + o(1)) & (m = 0, 0 \leq \Lambda < 1/4) \end{cases} \tag{93}$$

Here, l and μ are the parameters in the LRS solution (58), and $|s| \equiv \sqrt{|\Lambda - 1/4|}$.

6 Comparison with the Bianchi III model

We make a comparison between the Bianchi VIII and Bianchi III systems. We first compare the backgrounds, and then compare the asymptotic behaviors of scalar field.

The Bianchi III background solution. The LRS vacuum metric for Bianchi III is given by

$$\mathrm{g}^{(\mathrm{III})} = -N^2(t) dt^2 + q_1(t)((\sigma^1)^2 + (\sigma^2)^2) + q_3(t)(\sigma^3)^2 \tag{94}$$

where

$$N^2(t) = \frac{t - \mu}{t + \mu}, \quad q_1(t) = (t - \mu)^2, \quad q_3(t) = 4l^2 \frac{t + \mu}{t - \mu}, \tag{95}$$

and μ and l are real parameters of solution. The invariant 1-forms σ^I ($I = 1 \sim 3$) here are those for Bianchi III and must satisfy the following canonical relations;

$$d\sigma^1 = \sigma^1 \wedge \sigma^2, \quad d\sigma^2 = 0, \quad d\sigma^3 = 0. \tag{96}$$

Our convention in terms of coordinates is

$$\sigma^1 = dx/y, \quad \sigma^2 = dy/y, \quad \sigma^3 = dz. \tag{97}$$

The invariant (dual) vectors are given by

$$\chi_1 = y\,\partial/\partial x, \quad \chi_2 = y\,\partial/\partial y, \quad \chi_3 = \partial/\partial z. \tag{98}$$

In this section vectors ξ_I, χ_I, one-forms σ^I, and metric functions $N(t)$, $q_I(t)$ all refer to those for Bianchi III (not for Bianchi VIII), unless otherwise stated.

See Appendix A, [26], for details of the compactification of this solution. We assume an *orthogonal* [26] compactification, i.e., *each* hyperbolic $z =$ constant surface is compactified to a higher genus surface Σ_g. Although we do not repeat the details of the compactification, one of the most important facts for our discussion is that when compactified, each z-axis descends to circle fibers, and therefore the invariant vector χ_3 is called the *fiber generator*, like in the Bianchi VIII case.

As easily confirmed from the above metric, the conformal base metric $h^{(\mathrm{III})} = (\sigma^1)^2 + (\sigma^2)^2$ is the same as the one for the Bianchi VIII; they both coincide with the standard hyperbolic metric $h \equiv (dx^2 + dy^2)/y^2$. Since the 1-form σ^3 is dual to the fiber generator χ_3, $q_3(t)$ is the scale factor for the fibers, while $q_1(t)$ is the one for the base. This interpretation of the metric functions $q_I(t)$ is the same as that for the LRS Bianchi VIII solution (58).

The above metric is essentially the same one adopted in [26]. However, we have made two alterations. One is that we have replaced the parameter k with $-\mu$. As we will see, this makes the correspondence to the Bianchi VIII metric (58) better. The other alteration is the introduction of a new parameter l. At first, this parameter might seem redundant, since we could set $4l^2(\sigma^3)^2 \to (\sigma^3)^2$ by the induced map of the scaling diffeomorphism $z \to z/(2l)$. However, we should notice that this diffeomorphism is *not* well defined on the compactified manifold. To see this, let us focus on the fiber submanifold $(\mathbf{R}, q_3(\sigma^3)^2)$. To compactify we use the action by the translation $z \to z + 2\pi$. This map generates a group, $2\pi\mathbf{Z}$. The circle fiber $\mathcal{F}$ can therefore be expressed as $2\pi\mathbf{Z}\backslash(\mathbf{R}, q_3(\sigma^3)^2)$. The translation however does not commute with the scaling diffeomorphism, which means that the scaling is not well defined on the compactified manifold. The parameter l is therefore *not* redundant for the compactified manifold. The significance of this parameter is apparent if we consider the length of the fiber, which is given by

$$L^{(\mathrm{III})} = \int_{\mathrm{fiber}} \sqrt{q_3}\sigma^3 = \int_0^{2\pi} \sqrt{q_3}dz = 4\pi l\sqrt{\frac{t+\mu}{t-\mu}}. \tag{99}$$

In particular, $L_\infty^{(\mathrm{III})} \equiv \lim_{t\to\infty} L^{(\mathrm{III})} = 4\pi l$. Therefore l is the $(4\pi)^{-1}$ times the fiber length at the future infinity, like the LRS Bianchi VIII case. (See Eq.(63).)

We comment that we could also use the fiber metric $q_3'(\sigma^3)^2$ that is obtained by setting $4l^2 = 1$ as the universal cover metric, to express the compactified fiber. To this, we need to allow the covering group (rather than the metric) to have the parameter l, i.e., to compactify the fiber we consider the one-parameter translation $z \to z + 4\pi l$. Then, it is easy to see that the resulting fiber $\mathcal{F}' = 4\pi l\mathbf{Z}\backslash(\mathbf{R}, q_3'(\sigma^3)^2)$ is equivalent to the fiber $\mathcal{F}$. For example, the fiber length is computed as

$$\int_{\mathrm{fiber}} \sqrt{q_3'}\sigma^3 = \int_0^{4\pi l} \sqrt{q_3'}dz = 4\pi l\sqrt{\frac{t+\mu}{t-\mu}}, \tag{100}$$

which agrees with $L^{(\mathrm{III})}$. Remark that the parameter l is a relevant parameter in any case. In general, given a compactified locally homogeneous manifold, there are two ways to express it depending upon whether we fix the covering group or not. See [25] for a treatment of spatially compactified models with fixed covering groups (i.e., with no dynamical degrees of freedom in the covering group acting on spacetime), and [24] for one with varying covering groups with the smallest number of parameters in the universal cover metric. (See also [3, 13, 12] for related discussions.) Note that in [26] the background spacetime is expressed in the latter view point, while in this paper the former point of view has been exploited.

The reason we introduce the two-parameter metric (95) is that it has a straightforward correspondence to the (LRS) Bianchi VIII metric (58). In particular, as mentioned, the parameter l has the same meaning that it is $(4\pi)^{-1}$ times the fiber length at the future infinity.

Let τ be the proper time defined by

$$\begin{aligned} \tau &= \int N(t)dt = \int \sqrt{\frac{t-\mu}{t+\mu}}dt = \int \left(1 - \frac{\mu}{t} + O(\frac{1}{t^2})\right) dt \\ &= t - \mu \log t + O(\frac{1}{t}). \end{aligned} \tag{101}$$

This implies

$$t = \tau + \mu \log \tau + O(\frac{\log \tau}{\tau}). \tag{102}$$

Therefore, from Eqs.(95) we have

$$\begin{aligned} q_1(\tau) &= (\tau + \mu \log \tau)^2 + O(\log \tau), \\ q_3(\tau) &= 4l^2 \left(1 + \frac{2\mu}{\tau}\right) + O(\frac{\log \tau}{\tau^2}). \end{aligned} \tag{103}$$

Comparing these equations with Eqs.(62) it is confirmed that the asymptotic behavior of the LRS Bianchi III model is the same as that of the Bianchi VIII at least up to second leading terms. That is, the future asymptotic behaviors of base surface and circle fiber are the same for both LRS Bianchi III and VIII solutions. This establishes the *background degeneracy* of the two models.

Asymptotics of scalar fields on the Bianchi III. To compare the behaviors of scalar field we need to summarize the asymptotic properties of scalar field on the LRS Bianchi III background. As far as the compactified spatial manifold is of the orthogonal type, the harmonics can be constructed by simply making products of those $c_m(z)$ on the fiber and those $\hat{S}_\lambda(x,y)$ on the base hyperbolic surface. Those harmonics are defined by the following eigenvalue equations

$$\begin{aligned} \chi_3 c_m &= imc_m, \\ \hat{\triangle}_0 \hat{S}_\lambda &= -\lambda^2 \hat{S}_\lambda. \end{aligned} \tag{104}$$

Here, $\hat{\triangle}_0$ is the Laplacian with respect to the standard hyperbolic metric $h = (\sigma^1)^2 + (\sigma^2)^2$. The harmonics (mode functions) on an orthogonal closed Bianchi III manifold are the products

$$S_{\lambda,m} = c_m \hat{S}_\lambda. \tag{105}$$

We call m the *fiber eigenvalue*, and λ^2 the *base eigenvalue*. Remember that since we take the two-parameter metric (95), the identifications along the fibers are taken with the fixed step 2π, and as a result we have

$$m \in \mathbf{Z}, \tag{106}$$

since the solution of the first equation in Eqs.(104) is given by e^{imz}.

The mode-decomposed massless scalar field equation, computed with the one-parameter metric, is given in Eq.(243), [26]. To convert this equation to the one computed with the two-parameter metric (95), it is enough to perform the replacement

$$m \to \frac{m}{2l}. \tag{107}$$

Then, the equation reads

$$\ddot{\psi} + \frac{2t}{t^2-\mu^2}\dot{\psi} + \left(\frac{\lambda^2}{t^2-\mu^2} + \left(\frac{m}{2l}\right)^2 \frac{(t-\mu)^2}{(t+\mu)^2} \right)\psi = 0. \tag{108}$$

The unknown function $\psi = \psi_{\lambda,m}(t)$ is the field component for the mode $S_{\lambda,m}$. The exact solutions for the $\mu = 0$ background are given in Appendix B, [26]. No results are however presented there for general μ. Fortunately, however, it is not difficult to obtain asymptotic solutions for the equation, following the same procedure shown in the previous section. We just present the result here.

Proposition 5 *Consider the massless scalar field equation on the LRS Bianchi III vacuum metric (94), given by Eq.(108). The fundamental solutions of the equation are given by*

$$Y_m^{(\pm)}(t) = \begin{cases} t^{-1} e^{\pm i \left|\frac{m}{2l}\right|(t-2\mu\log t)}(1+o(1)) & (m \neq 0) \\ t^{-\frac{1}{2}} e^{\pm i|s|\log t}(1+o(1)) & (m=0, \lambda^2 > 1/4) \\ t^{-\frac{1}{2}\pm|s|}(1+o(1)) & (m=0, 0 \le \lambda^2 < 1/4), \end{cases} \tag{109}$$

where

$$|s| \equiv \sqrt{|\lambda^2 - \frac{1}{4}|}. \tag{110}$$

It is easy to convert t into the proper time τ in this solution. Using Eq.(102), we obtain the following.

Theorem 6 *A massless scalar field on a spatially compactified LRS vacuum Bianchi III solution can be decomposed into its mode components that are independent from each other and each of which is specified by the fiber eigenvalue m and the base Laplacian eigenvalue λ^2. Their fundamental solutions (at the future asymptotics) are given in terms of proper time τ as follows:*

$$Y_m^{(\pm)}(\tau) = \begin{cases} \tau^{-1} e^{\pm i \left|\frac{m}{2l}\right|(\tau-\mu\log \tau)}(1+o(1)) & (m \neq 0) \\ \tau^{-\frac{1}{2}} e^{\pm i|s|\log \tau}(1+o(1)) & (m=0, \lambda^2 > 1/4) \\ \tau^{-\frac{1}{2}\pm|s|}(1+o(1)) & (m=0, 0 \le \lambda^2 < 1/4), \end{cases} \tag{111}$$

Here, k and l are the parameters in the LRS solution (95), and $|s| \equiv \sqrt{|\lambda^2 - 1/4|}$.

Again, only the difference from the version expressed in terms of the coordinate time t is that the numerical factor of $-2\mu\log t$, in the phase part of the solution for $m \neq 0$, becomes half, $-\mu\log\tau$.

Comparison. Comparing the above theorem with Theorem 4, we see that the asymptotic solutions in the Bianchi VIII and Bianchi III models completely agree, provided the correspondence $\lambda^2 \leftrightarrow \Lambda$ is understood. We call this agreement the *scalar field asymptotic degeneracy* of the two models.

In the following we give a short account how this degeneracy can be understood. First, note that an arbitrary scalar field solution Ψ can be decomposed into two parts, the U(1)-symmetric part $\Upsilon(t,x,y)$ and the rest part $\Psi_{(\mathrm{gen})}(t,x,y,z) \equiv \Psi - \Upsilon$;

$$\Psi = \Psi_{(\mathrm{gen})} + \Upsilon. \tag{112}$$

(We do not have to mode-decompose each part here.) We first consider the U(1)-symmetric part Υ. Remember that the U(1)-symmetry means the translation symmetry along the fibers. As mentioned, as long as the background is LRS the background is also U(1)-symmetric along the fibers for both Bianchi types. Therefore we can consistently contract the fibers and obtain a reduced scalar field system on a $(2+1)$-dimensional spacetime. The 2-dimensional spatial manifold is the higher genus surface Σ_g, and the spacetime manifold becomes $\mathbf{R}\times\Sigma_g$ for both models. It is easy to write down the field equation on this contracted manifold for the Bianchi VIII system. From Eqs.(65), (66) and (53), we have

$$\mathrm{g}^{ab}\nabla_a\nabla_b\Upsilon = \left(\frac{-1}{\sqrt{-\mathrm{g}}}\frac{\partial}{\partial t}\left(\sqrt{-\mathrm{g}}N^{-2}\frac{\partial}{\partial t}\right) + q_1^{-1}\hat{\triangle}_0\right)\Upsilon, \tag{113}$$

where $\hat{\triangle}_0$ is the Laplacian (55) with respect to the standard hyperbolic metric. Thus, the field equation on the contracted manifold is given by

$$\left(-\frac{N}{q_1\sqrt{q_3}}\frac{\partial}{\partial t}\left(\frac{q_1\sqrt{q_3}}{N}\frac{\partial}{\partial t}\right) + N^2 q_1^{-1}\hat{\triangle}_0\right)\Upsilon = 0. \tag{114}$$

In this equation, the functions $q_I(t)$ and $N(t)$ should be thought of as the functions of time given in Eq.(58). The equation for the Bianchi III is also obtained in the same way, and we find that the above equation is exactly valid for this case, too, provided that $q_I(t)$ and $N(t)$ are those for the LRS Bianchi III solution (95). This equivalence explains the degeneracy for the U(1)-symmetric modes, since we have the background degeneracy, which means that the coefficient functions in Eq.(114) show asymptotically the same behavior for both cases.

Next, consider the rest part $\Psi_{(\mathrm{gen})}$, which we call the "generic part." Note that the spatial Laplacians for the two LRS backgrounds are expressed as

$$\triangle_q = \begin{cases} q_1^{-1}((\chi_1)^2 + (\chi_2)^2) + q_3^{-1}(\chi_3)^2 & \text{(VIII)} \\ q_1^{-1}\hat{\triangle}_0 + q_3^{-1}(\chi_3)^2. & \text{(III)} \end{cases} \tag{115}$$

The key fact is that in bath cases, q_3^{-1} dominates q_1^{-1}; [5]

$$\lim_{t\to\infty}\frac{q_1^{-1}(t)}{q_3^{-1}(t)} = 0. \tag{116}$$

[5] It is noteworthy that the dominance of q_3^{-1} also holds for the non-LRS Bianchi VIII background, i.e., as we can check using a result in [17], it holds $\lim q_1^{-1}/q_3^{-1} = \lim q_2^{-1}/q_3^{-1} = 0$.

Since from the assumption we have $\chi_3\Psi_{(\mathrm{gen})} \neq 0$, this may suggest that the term in q_3^{-1} will dominate the term in q_1^{-1} in both Laplacians. If this can be justified, we have the simplification

$$\triangle_q \to q_3^{-1}(\chi_3)^2, \quad \text{(VIII and III)} \tag{117}$$

resulting in the same asymptotic equation for both systems again;

$$\left(-\frac{N}{q_1\sqrt{q_3}}\frac{\partial}{\partial t}\left(\frac{q_1\sqrt{q_3}}{N}\frac{\partial}{\partial t}\right) + N^2 q_3^{-1}\chi_3\right)\Psi_{(\mathrm{gen})} = 0. \tag{118}$$

This explains the degeneracy for the generic modes.

We call the last equation the *fiber term dominated (FTD) equation* of the scalar field equation.[6] Beware that in contrast to the U(1)-symmetric equation (114), the above equation *cannot* be justified as a symmetry reduction. That is, although we dropped off χ_1 and χ_2-dependent terms, it is not to impose the additional conditions $\chi_1\Psi = \chi_2\Psi = 0$. Indeed, if this is the case for the Bianchi VIII we must have $\chi_3\Psi = -[\chi_1, \chi_2]\Psi = 0$, implying only spatially constant configurations are allowed. Therefore Ψ has no symmetry and in particular depends on all spatial coordinates $\Psi = \Psi(t, x, y, z)$.

Note that although Eq.(117) looks natural, we need a proof for its justification, which is what we have done with the mode decomposed equations.

7 Conclusions

We have separated the scalar field equation on a Bianchi VIII background using the harmonics for the circle-fibered closed Bianchi VIII manifold. The reduced wave equations form two sets of infinite number of simultaneous ordinary differential equations (ODEs). This is a result of the fact that irreducible representations of a noncompact Lie group like the Bianchi VIII group are in general infinite dimensional. When the background is LRS, however, each single wave equation becomes closed itself due to the additional symmetry. We have analyzed this closed equation and obtained future asymptotic solutions, as in Theorem 4. In particular, for the $m = 0$ (U(1)-symmetric) and $\Lambda < 1/4$ ("small eigenvalue") case the solution is non-oscillatory, while the other cases are oscillatory. They are all decaying (except one of the two fundamental solutions for the "zero-mode" with $m = \Lambda = 0$).

We have seen that this result completely agree with that of the Bianchi III model. We interpret this "scalar field asymptotic degeneracy" as an evidence that other linear fields, including electromagnetic fields and linear perturbations, on the Bianchi VIII background also have the same asymptotic behaviors as those on the Bianchi III. Since many results are already known for the Bianchi III [26, 22], we can, based on them, conjecture corresponding properties. Of our interest is that of the linear perturbations. The asymptotic solution of the stability measures (a kind of normalized gauge-invariant variables) for the Bianchi III, given in Theorem 2.5, [22], shows that the Bianchi III vacuum solution is asymptotically unstable, meaning that the perturbed spacetime asymptotically

[6] This equation can be considered as an analogy of the asymptotically velocity term dominated (AVTD) equations [10] of Einstein's equation valid for the opposite time direction.

becomes more and more inhomogeneous. It is therefore plausible that the (LRS) Bianchi VIII vacuum solution is also asymptotically unstable in the same sense.

One however needs to be careful about the difference between the properties of the vector or tensor harmonics applied to the two models. For example, the Bianchi III tensor harmonics are split into four kinds, the "even" ones, the "odd" ones, the "harmonic" ones, and the "transverse-traceless (TT)" ones [26]. Accordingly, there are four kinds of (independent) perturbations in the Bianchi III case. On the other hand, we will have no such splitting in the Bianchi VIII case. This difference comes from whether the invariant frame $\{\sigma^I, \chi_I\}$ is well defined on the compactified manifold considered, since if it is well defined we can construct all vector or tensor harmonics from the scalar harmonics with the help of the frame. See [23] for an explicit example, where a Bianchi II case is treated. Because of this property, the Bianchi VIII tensor harmonics will not have a splitting like the Bianchi III one.

Because of this difference, the reduced Bianchi VIII perturbation equations will have somewhat different properties. In particular, while, in the Bianchi III case, the reduced perturbation equations are given as independent second order ODEs (or independent systems of two simultaneous first order ODEs), in the Bianchi VIII case they will be given as independent systems of *two* simultaneous second order ODEs (or independent systems of *four* simultaneous first order ODEs). (Here, we are assuming an LRS background for both Bianchi types.) We can interpret this increase of variables in an independent system of equations as a consequence of the coupling between an "even" and "odd" mode, like the Bianchi II case [23].

The result we should refer to to obtain a corresponding property about the Bianchi VIII perturbations should be the one for the "even" perturbations of the Bianchi III solution. This is because they are the dominant perturbations among the even and odd ones. It is apparent that the Bianchi VIII perturbations will have nothing to do with the "harmonic" and "TT" ones of the Bianchi III solution, since those perturbations are connected to the modes that cannot be produced from the scalar harmonics. As a result, we can conclude that the conjectured growth rate of the stability measure for the generic Bianchi VIII perturbations is $O(\tau)$ (Cf. Theorem 2.5, [22]), meaning it is unstable.

Finally, we comment on non-LRS cases. Remember that the degeneracy between the Bianchi VIII and III systems concerns the LRS backgrounds. Although, since a Bianchi III manifold cannot be compactified unless it is LRS, we do not consider non-LRS Bianchi III system, the non-LRS Bianchi VIII system may be of great interest itself. However, the apparent difficulty is that we must solve infinite number of simultaneous ODEs in this case.

We point out that this difficulty might not be so crucial. We have seen that the scalar field asymptotic degeneracy (for the generic modes) was a consequence of the fiber term dominated (FTD) behavior. Since the dominance of the scale factor function q_3^{-1} over q_1^{-1} and q_2^{-1} continues to hold for the non-LRS background, we can expect that an FTD behavior holds for this case, too. If this is justifiable, it implies that each single mode becomes (virtually) independent like in the LRS case. (This is what can be easily confirmed by inspecting the non-LRS case equation.) Therefore the FTD behavior may be the key to find asymptotic solutions of the non-LRS case field equations.

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