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Spinning membranes on $AdS_p \times S^q$

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Abstract

Minimal Surfaces in S^3 are shown to yield spinning membrane solutions in $AdS_4 \times S^7$.

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The AdS/CFT correspondence (see [1] for review) offers a powerful tool to study interesting aspects of supersymmetric large-N gauge theories beyond perturbation theory. The first stage of these developments relied mainly on the isomorphism between Kaluza-Klein states of classical type IIB supergravity compactified on $AdS_5 \times S^5$ and BPS observables of $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. Many variations on this theme involving theories with less supersymmetry, with and without conformal invariance, were also studied, leading to quantitative results about the spectrum and phase structure of QCD-like theories [2][3][4][5][6].

The problem to go beyond the SUGRA approximation is related to the difficulties to quantize string theory in Ramond-Ramond backgrounds. Even though a covariant quantization scheme has been developed [7], it has so far not been possible to use it to compute the string excitation spectrum on these backgrounds. An exception is the gravitational plane wave background which is obtained as the Penrose limit of the $AdS_5 \times S^5$ vacuum of type IIB string theory. In this background light-cone quantization leads to a free theory on the world-sheet whose spectrum is easily computed [8]. This opens the way to the duality between string theory and another sector of large-N SYM, which is characterized by large R-charge ($\sim \sqrt{N}$) and conformal weight ($\sim \sqrt{N}$). The extensive activity to which this has led was initiated in [9].

Studying time-dependent classical solutions of the string sigma-model in an $AdS_5 \times S^5$ target space-time and relating them to the dual conformal field theory, extends the testable features of the duality between string theory and $\mathcal{N} = 4$ SYM. This was proposed and demonstrated in [10]. Subsequent interesting developments are summarized and reviewed in [11].

A likely extension of these ideas is to go from strings to M-theory, where the fundamental objects are membranes rather than strings. In this case, maximally supersymmetric backgrounds, aside from eleven-dimensional Minkowski space, are $AdS_7 \times S^4$ and $AdS_4 \times S^7$. The former is the near-horizon limit of a stack of N coincident M5 branes with $\frac{1}{2}R_{AdS} = R_S = l_p(\pi N)^{1/3}$ and the latter is the near-horizon limit of a stack of N M2 branes with $2R_{AdS} = R_S = l_P(32\pi^2 N)^{1/6}$. The dualities between classical supergravity on these background and the conformal field theories on the world-volume of the branes which create them has been studied. In particular for the $AdS_4 \times S^7$ case, if the duality holds, nontrivial information about the $(0,2)$ conformal field theory of N interacting tensor multiplets in six dimensions has been obtained, e.g. its conformal anomaly has been computed [12][13]. Direct verifications have, however, so far been impossible, mainly due to the lack of knowledge of the interacting (0,2) theory.

The problems which one encounters in quantizing string theory on non-trivial backgrounds are of course much more severe in M-theory where quantization on any background is still elusive. The semiclassical analysis, which in the case of string theory provides valuable non-trivial information about the dual conformal field theory, can, however, be extended to M-theory. While the equations of motion of strings on $AdS_5 \times S^5$ reduce, for special symmetric configurations, to classical integrable systems [14][15], this is not as simple for membranes. Also, the integrable spin-chains which appear in the discussion of the dual gauge theory [16][17], have so far no known analogue in the (0,2) tensor theory.

In this letter we make a first step towards the semiclassical analysis of M-theory on $AdS_p \times S^q$ backgrounds. We will find that the equations of motion, upon imposing a suitable Ansatz (analogous to the corresponding string theory analysis, and similar to the Ansatz made in [18]), may be reduced to the equations describing minimal embeddings of 2-surfaces into higher spheres (as well as generalizations thereof).

Let us consider closed bosonic membranes in $AdS_p \times S^q$. Their dynamics is derived from the action

$$S = \int d^3\varphi \left(\sqrt{G} + \lambda(\vec{x}^2 - 1) + \tilde{\lambda}(y^2 - 1) \right) \tag{1}$$

where $y^\mu(\varphi^\alpha)$ $(\mu = 1, \ldots, p;\ \alpha = 0, 1, 2)$ and $x_k(\varphi^\alpha)$ $(k = 1, \ldots, q+1)$ are the embedding coordinates, $\vec{x}^2 = \sum_{k=1}^{q+1} x_k x_k$, $y^2 = y^\mu y^\nu \eta_{\mu\nu} = y_0^2 + y_p^2 - \sum_{\mu'=1}^{p-1} (y_{\mu'})^2$ and

$$G_{\alpha\beta} = \partial_\alpha y^\mu \partial_\beta y^\nu \eta_{\mu\nu} - \partial_\alpha \vec{x} \cdot \partial_\beta \vec{x}\,. \tag{2}$$

The constraints

$$y^2 = 1 = \vec{x}^2 \tag{3}$$

follow by varying (1) w.r.t. the Lagrange multipliers λ and $\tilde{\lambda}$ while variation w.r.t. y^μ and x_k yields the equations of motion

$$\partial_\alpha(\sqrt{G} G^{\alpha\beta} \partial_\beta y^\mu) = 2\tilde{\lambda} y^\mu\,, \tag{4}$$

$$\partial_\alpha(\sqrt{G} G^{\alpha\beta} \partial_\beta \vec{x}) = -2\lambda \vec{x}\,. \tag{5}$$

Note that we take the radii of the AdS spaces and the sphere to be equal. It is straightforward to generalize the discussion to the case of unequal radii, which is the situation in the M-theory context. Contracting (4) with y^μ and (5) with $\vec{x}$, respectively and using (3), one finds that

$$\begin{aligned} 2\tilde{\lambda} &= -\sqrt{G} G^{\alpha\beta} \partial_\alpha y^\mu \partial_\beta y^\nu \eta_{\mu\nu} \\ 2\lambda &= +\sqrt{G} G^{\alpha\beta} \partial_\alpha \vec{x} \cdot \partial_\beta \vec{x} \end{aligned} \tag{6}$$

implying

$$\begin{aligned}\lambda + \tilde{\lambda} &= -\frac{1}{2}\sqrt{G}G^{\alpha\beta}\left(\partial_\alpha y^\mu \partial_\beta y_\mu - \partial_\alpha \vec{x} \cdot \partial_\beta \vec{x}\right) \\ &= -\frac{3}{2}\sqrt{G}\,.\end{aligned} \tag{7}$$

Denoting φ^0 by t, let us make the Ansatz

$$\begin{aligned}&y_0 = \sin(\omega_0 t), \quad y_p = \cos(\omega_0 t), \quad y_{\mu'} = 0 \quad (\mu' = 1, \ldots, p-1) \\ &\vec{x}(t, \varphi^1, \varphi^2) = \mathcal{R}(t)\vec{m}(\varphi^1, \varphi^2)\end{aligned} \tag{8}$$

with

$$\mathcal{R}(t) = \begin{pmatrix} \cos(\omega_1 t) & -\sin(\omega_1 t) & & & \\ \sin(\omega_1 t) & \cos(\omega_1 t) & & & \\ & & \cos(\omega_2 t) & -\sin(\omega_2 t) & \\ & & \sin(\omega_2 t) & \cos(\omega_2 t) & \\ & & & & \ddots \end{pmatrix}. \tag{9}$$

Let us further demand $\dot{\vec{x}} \cdot \partial_1 \vec{x} = 0 = \dot{\vec{x}} \cdot \partial_2 \vec{x}$, which, writing $\vec{m}^{\mathrm{T}} = (r_1 \cos\theta_1, r_1 \sin\theta_1, r_2 \cos\theta_2, r_2 \sin\theta_2, \ldots)$ reads

$$\sum_{a=1}^{d \equiv [\frac{1}{2}(q+1)]} \omega_a r_a^2 \partial_1 \theta_a = 0 = \sum_{a=1}^{d} \omega_a r_a^2 \partial_2 \theta_a\,. \tag{10}$$

The world-volume metric is then block-diagonal

$$G_{\alpha\beta} = \begin{pmatrix} \omega_0^2 - \dot{\vec{x}}^2 & 0 & 0 \\ 0 & & \\ 0 & & -g_{rs} \end{pmatrix} \tag{11}$$

with $g_{rs} = \partial_r \vec{x} \cdot \partial_s \vec{x} = \partial_r \vec{m} \cdot \partial_s \vec{m}$ $(r, s = 1, 2)$ and $\dot{\vec{x}}^2 = \sum_{a=1}^d \omega_a^2 r_a^2$. As is not difficult to see, (4) implies that

$$\rho := \sqrt{G}G^{00} = \frac{\sqrt{g}}{\sqrt{\omega_0^2 - \sum_{a=1}^d \omega_a^2 r_a^2}} = \frac{g}{\sqrt{G}} \tag{12}$$

is (a) *time-independent* (density). In any case,

$$\sum_{a=1}^{d} \omega_a^2 r_a^2 + \frac{g}{\rho^2} = \omega_0^2 \tag{13}$$

has to hold and $\tilde{\lambda}$ is determined as $-\rho\omega_0^2/2$.

Let us now turn to the equation for $\vec{x}$ which determines $\vec{m}(\varphi^1, \varphi^2)$, i.e. the shape of the membrane that is being rotated inside S^q by the orthogonal matrix $\mathcal{R}(t)$ (cf. (9)), in order to yield an extremal three-manifold in $AdS_p \times S^q$. With (11), (5) becomes

$$\frac{1}{\rho}\partial_r\left(g\frac{g^{rs}}{\rho}\partial_s\vec{x}\right) = \ddot{\vec{x}} + \frac{2\lambda\vec{x}}{\rho}\,. \tag{14}$$

Due to eqs.(8),(9) and (6), implying $\ddot{\vec{x}} = \ddot{\mathcal{R}}(t)\vec{m}$,

$$\begin{aligned}\frac{2\lambda}{\rho} &= \dot{\vec{x}}^2 - \frac{\sqrt{G}}{\rho}g^{rs}\partial_r\vec{m}\cdot\partial_s\vec{m} \\ &= \sum_{a=1}^{d}\omega_a^2 r_a^2 - \frac{2g}{\rho^2}\end{aligned} \tag{15}$$

(14) reduces to

$$\{\{m_i, m_j\}, m_j\} = \left(-\omega_{(i)}^2 + \sum \omega_a^2 r_a^2 - \frac{2g}{\rho^2}\right) m_i \tag{16}$$

where $\omega_{(1)} = \omega_{(2)} := \omega_1$, $\omega_{(3)} = \omega_{(4)} := \omega_2$, etc.,

$$g = \det(\partial_r\vec{x}\cdot\partial_s\vec{x}) = \det(\partial_r\vec{m}\cdot\partial_s\vec{m}) = \rho^2\sum_{i<j}\{m_i, m_j\}^2$$

and the (Poisson) bracket is defined as ($\epsilon^{12} = -\epsilon^{21} = 1$)

$$\{f, g\} = \frac{1}{\rho}\epsilon^{rs}\partial_r f\partial_s g \tag{17}$$

for any two differentiable functions on the two-dimensional parameter manifold. The density ρ, though time-independent, was defined in (12) in a ‘dynamical’ way, i.e. depending on $\vec{x}(t, \varphi^1, \varphi^2)$. However, due to [19] we may assume it to be any given ‘non-dynamical’ density having the same ‘volume’ $\int \rho(\varphi^1, \varphi^2)d^2\varphi$. This frees (17) from its seeming $\vec{x}$-dependence while reducing the original (φ^1, φ^2)-diffeomorphism invariance to those preserving ρ.

Confining ourselves (for the time being) to solving (10) in a trivial way by letting the $\theta_a(\varphi^1, \varphi^2)$ be constants, i.e. independent of $\varphi^{1,2}$, the equations to be solved are

$$\{\{r_a, r_b\}, r_b\} = \left(-\omega_a^2 + \sum \omega_c^2 r_c^2 - \frac{2g}{\rho^2}\right) r_a\,, \qquad a = 1, \ldots, d \tag{18}$$

subject to (13) and to $\sum r_a^2 = 1$. In the case of the string, rather than the membrane, this equation becomes [14], for $d = 3$, the equation of motion of the Neumann system, namely the constrained motion of a three-dimensional harmonic oscillator on the surface of a two-sphere.

If the 'spatial' frequencies ω_a are chosen to be all equal, it follows that $\sum \omega_c^2 r_c^2 = \omega^2 =$ constant as well as (from (13)) $g/\rho^2 = \omega_0^2 - \omega^2 = \text{const}$. This simplifies (18) to

$$\{\{r_a, r_b\}, r_b\} = -2(\omega_0^2 - \omega^2) r_a \tag{19}$$

which can be explicitly solved by (known) minimal embeddings of two-surfaces into $d = [\frac{1}{2}(q+1)]$-dimensional unit spheres.

To see this, one could recall (12), which shows that (19), rewritten as

$$\frac{1}{\rho}\partial_s \left(g \frac{g^{su}}{\rho} \partial_u \vec{r} \right) = -2(\omega_0^2 - \omega^2)\vec{r}, \tag{20}$$

is identical to the standard 'minimal surface' equation

$$\frac{1}{\sqrt{g}} \partial_s (\sqrt{g} g^{su} \partial_u \vec{r}) = -2\vec{r}. \tag{21}$$

This is the Euler-Lagrange equation which one obtains if one varies

$$\int d^2\varphi \left(\sqrt{g} - \mu(\varphi)(\vec{r}^2 - 1) \right) \tag{22}$$

w.r.t. the embedding coordinates $r_a(\varphi^1, \varphi^2)$ and the local Lagrange multiplier $\mu(\varphi)$ (which guarantees $\vec{r}^2 = 1$).

Another way to show the equivalence of (20) (hence (19)) to (21) is as follows: the results of ref.[19] allow one to choose the coordinates φ^s in the diffeomorphism invariant equation (21) such that $\sqrt{g/(\omega_0^2 - \omega^2)}$ is equal to any given density with the same 'volume' (i.e. integral over $d^2\varphi$). Choosing it to be ρ shows that solutions of (21) give solutions of (20). To show the converse, one notes that (20) automatically implies that $\frac{g}{\rho^2} = \omega_0^2 - \omega^2$ (multiply (20) by $\vec{r}$, and use $\vec{r}^2 = 1$ three times: once on the r.h.s., once for $\vec{r} \cdot \partial_u \vec{r} = 0$ and, finally, to write $\vec{r} \cdot \partial_s \partial_u \vec{r}$ as $-g_{su}$).

Concerning explicit solutions of (19), resp. (21) (from now on we put $\omega_0^2 - \omega^2 = 1$ by rescaling ρ) let us only mention the two simplest ones:

$$r_1 = \sin\theta\cos\varphi \quad r_2 = \sin\theta\sin\varphi\,, \quad r_3 = \cos\theta\,, \quad r_{a>3} = 0 \tag{23}$$

(the equator 2-sphere in $S^{d-1\geq 2}$, $\varphi^1 = \theta \in [0,\pi]$, $\varphi^2 = \varphi \in [0, 2\pi]$, $\rho = \sin\theta$) and

$$\vec{r} = \frac{1}{\sqrt{2}} (\cos\varphi_1, \sin\varphi_1, \cos\varphi_2, \sin\varphi_2, 0, \ldots, 0) \tag{24}$$

(the Clifford-torus in $S^{d-1\geq 3}$). Lawson [20] proved that there exist minimal embeddings into S^3 of any topological type. Minimal tori in S^7 are given in [21].

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