AEI-2003-075
hep-th/0309064

Universal Features of Holographic Anomalies*

A. Schwimmer[a] and S. Theisen[b]

[a] *Department of Physics of Complex Systems, Weizmann Institute, Rehovot 76100, Israel*

[b] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, 14476 Golm, Germany*

Abstract

We study the mechanism by which gravitational actions reproduce the trace anomalies of the holographically related conformal field theories. Two universal features emerge: a) the ratios of type B trace anomalies in any even dimension are independent of the gravitational action being uniquely determined by the underlying algebraic structure b) the normalization of the type A and the overall normalization of the type B anomalies are given by action dependent expressions with the dimension dependence completely fixed.

September 2003

* Partially supported by GIF, the German-Israeli Foundation for Scientific Research, the Alexander-von-Humboldt Foundation, the European Commission RTN programmes HPRN-CT-2000-00131 and HPRN-CT-2000-00122, Minerva Foundation, the Center for Basic Interactions of the Israeli Academy of Sciences and the Einstein Center.

1. Introduction

The calculation of trace anomalies [1] provides a remarkable test [2][3] of the AdS/CFT correspondence [4]. Besides its very interesting result the calculation indicated a new, highly nontrivial mechanism by which an anomaly can appear in an essentially classical setup.

The algebraic structure underlying the anomaly calculation was studied in [5]. The Weyl transformation on the boundary CFT is embedded in a subgroup of the diffeomorphisms acting on the odd dimensional gravitational action ("PBH transformation" after Penrose [6] and Brown and Henneaux [7]). The PBH transformations act on a general solution $g_{ij}(x, \rho)$ of the equations of motion in a nonlinear fashion, constraining its form.

Though the functional dependence of g_{ij} is not completely determined one can isolate its part relevant for the anomaly calculation which is strongly constrained by the PBH transformations. These terms are singled out by a cohomological structure which was studied in [8]. In the present paper we study in detail the relation between the cohomologically nontrivial part of g_{ij} and the trace anomalies.

We show that the anomalies are related to the relevant part of g_{ij} linearly . The coefficients entering the relation depend on the gravitational action but have a universal dependence on the dimension. This relation provides a rationale for the existence of a nontrivial cohomology for g_{ij} and implicitly for the Fefferman-Graham("FG") ambiguity [9],[8]. Then the constraints imposed by the PBH transformation on g_{ij} get translated into relations between trace anomalies in the same dimension and different dimensions for a fixed gravitational action.

By studying the PBH transformation for the cohomologically non trivial part of g_{ij} we conclude that (i) the overall normalization of type A and B anomalies [10] are gravitational action dependent but the dimension dependence is universal; (ii) the ratios between the terms responsible for the various type B anomalies are completely fixed.

In Section 2 we discuss the general relation between trace anomalies and the cohomologically nontrivial part of g_{ij}. We use dimensional regularization which provides unique signatures for the two quantities allowing us to relate them linearly .

In Section 3 we calculate the exact expression implied by the PBH transformations for g_{ij} expanded to first order in curvature. We interpret this result as giving a unique relation between certain type B terms in all dimensions.

In Section 4 we calculate exactly using the PBH transformations all the type B terms relevant in $d = 6$ and we show that their coefficients are completely fixed. In conjunction with the results of Section 3 this indicates that all the coefficients of type B terms in all dimensions are fixed by the PBH transformation.

In Section 5 we check the universal results for the anomalies against the standard calculation for a gravitational action containing arbitrary terms quadratic in the curvatures.

In Section 6 we summarize our results and discuss the implications for the general structure of trace anomalies in conformal theories.

In an Appendix we review the relevant features of PBH transformations.

2. The relation between the cohomology of g_{ij} and trace anomalies

We review first the well understood signal for trace anomalies in dimensional regularization [10].

We start with the type A anomaly for which there is no true divergence in $d = 2n$. As a consequence the effective action which is Weyl invariant away from $2n$ dimensions can be decomposed into two pieces:

$$W_d(g) = W_d^{(\mathrm{nl})} + \frac{\mu^{d-2n}}{d-2n} \int d^d x \sqrt{g} E_{2n}(g) \tag{2.1}$$

where E_{2n} is the $2n$ dimensional Euler density and μ is a mass scale.

The first nonlocal term has a finite limit for $d = 2n$ and the second one has limit 0: in dimensional regularization the special relations valid in integer dimensions are implemented first.

The Weyl variation of the action in $d = 2n$ can be calculated as the variation (with negative sign) in d dimensions of the second, local term and it is proportional to E_{2n}.

For the type B anomaly the effective action in d dimensions has the generic form:

$$W_d(g) = \frac{1}{d-2n} \int d^d x \sqrt{g} C \ldots \Box^{-d/2+n} \ldots C - \frac{\mu^{d-2n}}{d-2n} \int d^d x \sqrt{g} C \ldots C \tag{2.2}$$

where we denoted symbolically by $C \ldots C$ a local expression which transforms under Weyl rescalings in a homogenous fashion with weight $2n$.

In this case the first term is Weyl invariant in d-dimensions and has a genuine ultraviolet divergence represented by the explicit pole term. In order to have a well defined limit in $d = 2n$ we need a local counter term which is the second term in (2.2) breaking explicitly the Weyl invariance. The Weyl variation which is finite comes now from the second term and gives an expression proportional to $C \dots C$.

We see that both the type A and type B anomalies are finally given (with negative signs) by variations in d dimensions of the local expressions represented by the second terms in the r.h.s. of (2.1), (2.2). We remark that exactly in $d = 2n$ the dependence on the scale μ disappears such that the anomaly does not violate global dilation invariance.

The above mechanism has an exact counterpart in the holographic context. The gravitational action evaluated on a solution of the equations of motions with boundary value $\overset{(0)}{g}_{ij}$ is invariant under diffeomorphisms which include as a subgroup the PBH transformations and therefore under Weyl transformations on the boundary. The potential anomalous violation arises due to the integration over ρ which is potentially divergent at $\rho = 0$. This infrared divergence replaces the ultraviolet divergence in the conformal field theory.

A calculation involving an exact integration over ρ between 0 and ∞ would produce in dimensional regularization the invariant terms in (2.1), (2.2). Alternatively one can produce directly the local terms from which the anomaly can be obtained following the procedure described above. For this one uses an expansion of the solution in integer powers of ρ multiplying local expressions of $\overset{(0)}{g}$ (the Fefferman-Graham expansion).

In addition the integration on ρ is limited between 0 and $\bar{\rho}$. Now the integration over ρ is explicit and in dimensional regularization $\rho = 0$ does not contribute. Therefore around $d = 2n$ one gets terms

$$\Delta W = \sum \frac{\bar{\rho}^{\frac{1}{2}(2n-d)}}{2n-d} \int d^d x \sqrt{\overset{(0)}{g}(x)}\, b_n(x) \tag{2.3}$$

where b_n are either E_{2n} for type A or one of the expressions transforming homogenously which we denoted by $C \dots C$ for type B. Obviously $\bar{\rho}$ plays the role of the mass scale μ^{-2} and the variations of the local terms (i.e. anomalies) become $\bar{\rho}$- independent for $d = 2n$.

The above calculation, being classical, allows, however an alternative path: derivatives of the action with respect to the initial conditions $\overset{(0)}{g}$ ("the energy momentum tensor") reduce to boundary terms in the usual Hamilton-Jacobi manner since the action is evaluated on a classical solution[1]. Since the derivatives of (2.3) with respect to $\overset{(0)}{g}$ have explicit poles they still carry the complete information about the anomalies. It follows that the boundary terms which are local expressions in terms of $g_{ij}(x, \bar{\rho})$ should have the same poles and should carry directly the information about anomalies.

[1] The Hamilton-Jacobi approach was used in the holographic context in [11][12][13].

In dimensional regularization the contribution of the boundary $\rho = 0$ being put to 0 the whole contribution will come from an expression involving $g_{ij}(x, \rho)$ evaluated at $\rho = \bar{\rho}$. Indeed in [8] poles were shown to appear in the Feffermann-Graham expansion of $g_{ij}(x, \rho)$ as a consequence of the existence of a nontrivial cohomology involving the PBH transformations. The non trivial classes are in one to one correspondence with the derivatives of the corresponding local anomaly terms:

there is a unique type A class for each even dimension $2n$:

$$A^E_{(n)ij} = \frac{1}{d-2n}\frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{ij}}\int d^dx\sqrt{g}E_{2n} \tag{2.4}$$

and several type B classes ("Bach tensors") corresponding to the type B anomalies:

$$A^B_{(n)ij} = \frac{1}{d-2n}\frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{ij}}\int d^dx\sqrt{g}C\ldots C \tag{2.5}$$

Their number depends on the total number of derivatives acting on the metric: there is one of order four, three of order six, etc. We note that $g^{ij}A^E_{(n)ij} = \frac{1}{2}E_{2n}$ and $g^{ij}A^B_{(n)ij} = \frac{1}{2}C\ldots C$.

Obviously, since the exact form of the boundary terms depends on the gravitational action the explicit relation between the trace anomalies and the non trivial cohomological classes (2.4),(2.5) will also depend on the action. We illustrate in detail this relation for the case of the simplest action which has an AdS solution:

$$S = \int d^dx d\rho\sqrt{G}(\hat{R}(G) - 2\Lambda) \tag{2.6}$$

where $\Lambda = \frac{1}{2}d(d-1)$.

With the FG ansatz (A.1) for the metric one finds

$$\begin{aligned}\sqrt{G} &= \frac{1}{2}\rho^{-1-d/2}\sqrt{g(\rho)}\\ \hat{R} &= d(d+1) + \rho R - 2(d-1)\rho g^{ij}g'_{ij} - 3\rho^2 g^{ij}g^{kl}g'_{ik}g'_{jl} + 4\rho^2 g^{ij}g''_{ij} + \rho^2(g^{ij}g'_{ij})^2\end{aligned} \tag{2.7}$$

In the second line and, until further notice, below, all quantities are computed with $g(\rho)$. Inserting this into the action gives

$$\begin{aligned}S = \frac{1}{2}\int d^dx d\rho\rho^{-1-d/2}\sqrt{g}\,\{&\rho R - 2(d-1)\rho g^{ij}\partial_\rho g_{ij} - 3\rho^2 g^{ij}g^{kl}\partial_\rho g_{ik}\partial_\rho g_{jl}\\ &+4\rho^2 g^{ij}\partial^2_\rho g_{ij} + \rho^2(g^{ij}\partial_\rho g_{ij})^2\}\end{aligned} \tag{2.8}$$

Varying the action w.r.t. to g_{ij} one obtains [2]

$$\delta S = \frac{1}{2}\int d^dx d\rho \sqrt{g}\rho^{-1-\frac{d}{2}} \left\{ \frac{1}{2}\rho R g^{ij} - \rho R^{ij} - 2\rho^2 (g'')^{ij} - \rho^2 (\mathrm{tr}g')(g')^{ij} - \frac{3}{2}\rho^2 \mathrm{tr}(g'^2) g^{ij} \right.$$
$$\left. -2\left(1-\frac{d}{2}\right)\rho (g')^{ij} + 2\rho^2 (g'^2)^{ij} + 2\left(1-\frac{d}{2}\right)\rho(\mathrm{tr}g')g^{ij} + 2\rho^2 \mathrm{tr}(g'')g^{ij} + \frac{1}{2}(\mathrm{tr}g')^2 g^{ij} \right\} \delta g_{ij} + \mathrm{b.t.} \tag{2.9}$$

where the boundary terms ($b.t.$) arise from the integrations by parts w.r.t. ρ. The expression which multiplies δg_{ij} are precisely the (ij) components of the Einstein equations which follow from the action (2.6). The boundary terms are

$$\mathrm{b.t.} = \int d^dx \sqrt{g} \left\{ 2\rho^{1-\frac{d}{2}} g^{ij}\delta g'_{ij} - \rho^{-\frac{d}{2}} g^{ij}\delta g_{ij} - \rho^{1-\frac{d}{2}}(g')^{ij}\delta g_{ij} \right\} \Big|_{\rho=0}^{\rho=\bar\rho} \tag{2.10}$$

Note that through the solution of the PBH equations all terms in the ρ-expansion of $g(x,\rho)$ are functions of $\overset{(0)}{g}_{ij}(x)$. The energy momentum tensor is then the functional derivative of (2.10) w.r.t. $\overset{(0)}{g}_{ij}$. In order to isolate the characteristic leading poles in $d-2n$ with the accompanying $\bar\rho^{d/2-n}$ powers in the functional derivative of (2.10) we should pick terms where one of the factors is $\overset{(n)}{g}_{ij}$, the coefficient of ρ^n in the expansion of $g_{ij}(x,\rho)$ around $\rho=0$, all the others being $\overset{(0)}{g}_{ij}$. After taking into account that the trace of $\overset{(n)}{g}_{ij}(x)$ does not have poles at $d=2n$ and therefore it is cohomologically trivial, we obtain for the term having the pole at $d=2n$:

$$\frac{1}{\sqrt{\overset{(0)}{g}}}\frac{\delta}{\delta \overset{(0)}{g}{}^{ij}}\ \mathrm{b.t.} = -n g^{(n)}_{ij}\bar\rho^{n-d/2} + \text{cohomologically trivial} \tag{2.11}$$

in agreement with the results of [14].

We remark that this way of doing the calculation is insensitive to the presence of explicit nonsingular boundary terms at $\rho=0$. It can also be straightforwardly applied to more general gravitational actions, as we will now demonstrate.

An example which we will need in the following and which appears as a gravity dual of $\mathcal{N}=2$ super-conformal field theories in four dimensions [15],[16] is the gravitational lagrangian containing general quadratic terms in the curvatures:

$$\mathcal{L} = \hat R - 2\Lambda + \alpha \hat R^2 + \beta \hat R_{\mu\nu}\hat R^{\mu\nu} + \gamma \hat R_{\mu\nu\rho\sigma}\hat R^{\mu\nu\rho\sigma} \tag{2.12}$$

[2] We use the following notation: $\mathrm{tr}g' = g^{ij}g'_{ij}$, $g'^{ij} = g^{ik}g^{jl}g'_{kl}$, etc.

Repeating the steps outlined above we obtain for the term having the pole at $d = 2n = 4$ and the dependence $\bar{\rho}^0$:

$$\frac{1}{\sqrt{\overset{(0)}{g}}}\frac{\delta}{\delta \overset{(0)}{g}{}^{ij}} \text{ b.t.} = -2(1 + 40\alpha + 8\beta - 4\gamma)\overset{(2)}{g}{}_{ij} + \text{coho trivial} \tag{2.13}$$

From equations (2.11),(2.13) the anomalies can be identified; once we isolate the pole terms the anomalies can be obtained also by taking the trace of these terms.

3. The solution of the PBH equations to leading order in the curvatures

In this section we start a systematic study of the PBH equations. The conclusion will be that all the type B cohomologically nontrivial contributions are uniquely determined to all orders in ρ.

We work to first order in the curvature. To start with we allow the most general covariant expression linear in the curvatures[3]

$$g_{ij}(\rho) = g_{ij} + \alpha(\rho\Box)\rho R_{ij} + \beta(\rho\Box)\rho g_{ij}R + \gamma(\rho\Box)\rho^2\nabla_i\nabla_j R + \mathcal{O}(R^2) \tag{3.1}$$

Here all quantities (R, ∇, etc.) are with respect to the metric on the boundary, $g_{ij} \equiv \overset{(0)}{g}{}_{ij}$, except where the ρ-dependence is explicitly given. We always suppress the x-dependence.

We calculate first the curvature independent piece in the Weyl transformation of (3.1):

$$\begin{aligned}\delta g_{ij}(\rho) = 2\sigma g_{ij} &+ \alpha(t)\rho[(d-2)\nabla_i\nabla_j\sigma + g_{ij}\Box\sigma] \\ &+ \beta(t)\rho g_{ij}[2(d-1)\Box\sigma] + \gamma(t)\rho^2[2(d-2)\nabla_i\nabla_j\Box\sigma]\end{aligned} \tag{3.2}$$

where we have defined $t = \rho\Box$. On the other hand, we can also expand the r.h.s. of eq.(A.2) to calculate the curvature independent piece and we find

$$\delta g_{ij}(\rho) = 2\sigma g_{ij} + \rho\nabla_i\nabla_j\sigma \tag{3.3}$$

Comparison of (3.2) and (3.3) gives

$$\begin{aligned} t\alpha(t) + 2(d-1)t\beta(t) = 0 \\ (d-2)\alpha(t) + 2(d-1)t\gamma(t) = 1\end{aligned} \tag{3.4}$$

[3] Note that each derivative ∇_i is accompanied by a factor $\sqrt{\rho}$.

If we define

$$t\bar{\beta} = 1 + 2(d-1)(d-2)\beta \tag{3.5}$$

we get

$$\begin{aligned}
\beta(t) &= -\frac{1}{2(d-1)(d-2)} + \frac{\bar{\beta}t}{2(d-1)(d-2)} \\
\alpha(t) &= \frac{1}{(d-2)} - \frac{\bar{\beta}t}{(d-2)} \\
\gamma(t) &= \frac{\bar{\beta}}{2(d-1)}
\end{aligned} \tag{3.6}$$

Inserting this into (3.1) we obtain

$$\begin{aligned}
g(\rho)_{ij} &= g_{ij} + \rho \overset{(1)}{g}_{ij} + \rho^2 \bar{\beta}(t) \left\{ \frac{1}{2(d-1)} \nabla_i \nabla_j R + \frac{1}{2(d-1)(d-2)} g_{ij} \Box R - \frac{1}{(d-2)} \Box R_{ij} \right\} + \mathcal{O}(R^2) \\
&\equiv g_{ij} + \rho \overset{(1)}{g}_{ij} + \rho^2 \bar{\beta}(t) X_{ij} + \mathcal{O}(R^2)
\end{aligned} \tag{3.7}$$

where, unlike the $\overset{(n)}{g}_{ij}$ for $n > 1$, $\overset{(1)}{g}_{ij}$ is uniquely fixed by (A.2) [5] and is

$$\overset{(1)}{g}_{ij} = \frac{1}{d-2} \left(R_{ij} - \frac{1}{2(d-1)} g_{ij} R \right) . \tag{3.8}$$

We remark that X_{ij} is simply

$$X_{ij} = -\frac{1}{d-3} \nabla^k \nabla^l C_{ikjl} \tag{3.9}$$

and therefore the terms appearing in (3.7) belong to the type B cohomologically nontrivial Bach tensors generated by $\int d^d x \sqrt{\overset{(0)}{g}} C^{ijkl} \Box^n C_{ijkl}$ in dimension $d = 2n + 4$.

Some basic properties of X_{ij}, which will be used below, are

$$\begin{aligned}
X_i{}^i &= 0 \\
\nabla^i X_{ij} &= 0 + \mathcal{O}(R^2)
\end{aligned} \tag{3.10}$$

What remains is to determine $\bar{\beta}(t)$. We will do this by comparing the Weyl variation of (3.7) with the expansion of eq.(A.2) to first order in the curvature. We start with the latter. It will be sufficient to work to first order in $\nabla_i \sigma$. This becomes clear once one realizes that the Weyl variations of the $\mathcal{O}(R^2)$ terms never generate any terms which are linear in R and with only one derivative acting on σ. In the computation one has to choose a basis for the possible terms. The basis we choose is that we always move all $\Box$'s to the left of explicit ∇_i's. One then has to use the explicit expression for $[\nabla_j \nabla_j, \Box]\sigma$ which produces terms $\mathcal{O}(R, \nabla_i \sigma)$. If we write

$$g_{ij}(\rho) = g_{ij} + Z_{ij}(\rho) + \mathcal{O}(R^2) \tag{3.11}$$

we obtain

$$g^{ij}(\rho) = g^{ij} - Z^{ij}(\rho) + \mathcal{O}(R^2) \qquad Z^{ij}(\rho) = g^{ik} g^{jl} Z_{kl}(\rho') \tag{3.12}$$

To this order (A.3) is

$$\begin{aligned} a^i(\rho) &= \frac{1}{2} \int_0^\rho d\rho' g^{ij}(x, \rho') \partial_j \sigma \\ &= \frac{1}{2} \int_0^\rho d\rho' \left\{ g^{ij} - Z^{ij}(\rho) \right\} \partial_j \sigma \end{aligned} \tag{3.13}$$

Eq.(A.2) also involves terms $g_{jk}(\rho) \nabla_i a^k(\rho)$. They must also be expanded to $\mathcal{O}(R)$. Doing all of this we find

$$\delta g_{ij}(\rho) = -\frac{1}{2} \int_0^\rho d\rho' \left\{ \nabla_i Z_j{}^l(\rho') + \nabla_j Z_i{}^l(\rho') \right\} \nabla_l \sigma + \frac{1}{2} \rho \nabla_l Z_{ij}(\rho) \nabla^l \sigma \tag{3.14}$$

Next, expand

$$\begin{aligned} Z_{ij}(\rho) &= \rho \overset{(1)}{g}_{ij} + \rho^2 \bar{\beta}(\rho \Box) X_{ij} \\ \bar{\beta}(\rho \Box) &= \sum_{n=0}^\infty \beta_n \rho^n \Box^n \end{aligned} \tag{3.15}$$

Inserting this into (3.14) we find

$$\delta g_{ij}(\rho) = \frac{1}{2} \sum_{n=0}^\infty \beta_{n-1} \rho^{n+2} \Box^{n-1} \left\{ \nabla_k X_{ij} - \frac{1}{(n+2)} \left(\nabla_i X_{jk} + \nabla_j X_{ik} \right) \right\} \nabla^k \sigma \tag{3.16}$$

This expression for $\delta g_{ij}(\rho)$, which is valid to $\mathcal{O}(R, \nabla \sigma)$, but to all orders in ρ, is our first result.

Next we compute $\delta(\bar{\beta}(\rho \Box) X_{ij})$. Here the main result is

$$\delta(\Box^n X_{ij}) = -(n+2)(2n+4-d) \Box^{n-1} \left\{ \nabla^k X_{ij} - \frac{1}{(n+2)} (\nabla_i X_j{}^k + \nabla_j X_i{}^k) \right\} \nabla_k \sigma \tag{3.17}$$

To derive it eq.(3.10) is used. Some intermediate results are

$$\begin{aligned} \delta(\Box^n X_{ij}) &= \Box^n \delta X_{ij} + \Box^{n-1} \left\{ n(d-4-2n) \nabla_k X_{ij} + 2n(\nabla_i X_{jk} + \nabla_j X_{ik}) \right\} \nabla^k \sigma \\ \Box^n \delta X_{ij} &= \Box^{n-1} \left\{ 2(d-4-2n) \nabla_k X_{ij} - (d-4)(\nabla_i X_{jk} + \nabla_j X_{ik}) \right\} \nabla^k \sigma \end{aligned} \tag{3.18}$$

Again, all calculations are to $\mathcal{O}(R, \nabla\sigma)$ in the basis where all $\Box$'s are moved all the way to the left.

Using (3.17) together with eq.(3.7) we obtain

$$\delta g(\rho)_{ij} = -\sum_{n=0}^{\infty} \rho^{n+2} \beta_n (n+2)(2n+4-d) \Box^{n-1} \left\{ \nabla^k X_{ij} - \frac{1}{(n+2)} (\nabla_i X_j{}^k + \nabla_j X_j{}^k) \right\} \nabla_k \sigma \tag{3.19}$$

Comparison of (3.16) and (3.19) gives

$$\beta_n = -\frac{1}{2(n+2)(2n+4-d)} \beta_{n-1} \tag{3.20}$$

with $\beta_0 = -\frac{1}{4(d-4)}$. Solving the recursion relation we finally get

$$\begin{aligned} g(\rho)_{ij} &= g_{ij} + \rho \overset{(1)}{g}_{ij} - \sum_{n=0}^{\infty} \rho^{n+2} \frac{2}{2^{(n+2)}} \frac{1}{(n+2)!} \frac{1}{(d-4)(d-6)\cdots(d-2(n+2))} \Box^n X_{ij} + \mathcal{O}(R^2) \\ &= g_{ij} + \rho \overset{(1)}{g}_{ij} - 2(d-2)\rho^2 \sum_{n=0}^{\infty} \frac{1}{2^{2(n+2)}} \frac{1}{(n+2)!} \frac{\Gamma(\frac{1}{2}d - (n+2))}{\Gamma(\frac{d}{2})} \rho^n \Box^n X_{ij} + \mathcal{O}(R^2) \end{aligned} \tag{3.21}$$

Therefore all the terms in the local FG expansion are uniquely determined to this order. This is a consequence of the fact that there are no terms linear in the curvature transforming homogenously whose coefficient would be undetermined. The terms have poles in even dimensions signaling their cohomologically nontrivial nature. The component $\overset{(n)}{g}$ has a leading pole at $d = 2n$ and poles at all the lower even dimensions. While these secondary poles should be present in $g_{ij}(x, \rho)$ in accordance with the FG ambiguity, they should not give rise to anomalies : ρ gives the correct scale dependence for $g_{ij}(x, \rho)$ at the pole but for an anomaly one would need a negative power dependence on the scale μ in contradiction with the analytical structure of CFT.

The knowledge of all the local terms in (3.21) allows the calculation of all the anomalies to this order in the curvture, the main purpose of the present paper. Having an exact solution enables us, however, to study as a byproduct the structure of the FG expansion and in particular the FG ambiguity on an all order in ρ expression.

The recursion relation (3.20) can be translated into a Bessel type differential equation for $\bar{\beta}$:

$$\left[4 \frac{d}{dt} \left(t^{1-d/2} \frac{d}{dt} t^2 \right) + t^{2-d/2} \right] \bar{\beta}(t) = 0 \,. \tag{3.22}$$

If we define the function $g(t)$ via $\bar{\beta}(t) = t^{-2} g(t)$ then $g(t)$ satisfies :

$$4t\, g'' + 4\left(1 - \frac{d}{2}\right)\, g' + g = 0\,. \tag{3.23}$$

The homogenous equation is supplemented with a matching condition to the first two terms in the expansion. The indicial equation for (3.23), $r(r - d/2) = 0$, is quadratic showing that there are two independent solutions. In particular for d an even dimension the indices for the two solutions differ by an integer signaling that one of the solutions contains a logarithm.

All these features can be seen explicitly by writing down the general solution for $g_{ij}(x, \rho)$ following from (3.22), (3.23):

$$\begin{aligned} g_{ij}(\rho) = & \overset{(0)}{g}_{ij} + \rho \overset{(1)}{g}_{ij} \\ & + \left\{ \frac{2(d-2)}{\Box^2} + \frac{\rho}{\Box} + \frac{c_1}{\Box^2}(\rho\Box)^{d/4} J_{-d/2}(\sqrt{\rho\Box}) + \frac{c_2}{\Box^2}(\rho\Box)^{d/4} J_{d/2}(\sqrt{\rho\Box}) \right\} X_{ij} \end{aligned} \tag{3.24}$$

where

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1+\nu+k)} \left(\frac{z}{2}\right)^{2k} \tag{3.25}$$

and $\overset{(1)}{g}_{ij}$ as given in (3.8); c_1, c_2 are arbitrary coefficients. We choose the constant c_1 such that when $c_2 = 0$ we recover $g_{ij}(\rho) = \overset{(0)}{g}_{ij} + \rho \overset{(1)}{g}_{ij} + \mathcal{O}(\rho^2)$. This gives

$$c_1 = -2(d-2)2^{-d/2}\Gamma(1 - \frac{d}{2}) = -\frac{2(d-2)2^{-d/2}\pi}{\sin(\frac{\pi d}{2})\Gamma(\frac{d}{2})} \tag{3.26}$$

The arbitrariness of c_2 is the FG ambiguity: in an even dimension $d = 2n$, one has a series starting with ρ^n whose normalization is not fixed and simultaneously the first series becomes singular. One can obtain a finite solution in an even dimension using the fact that the following limit exits

$$\lim_{d \to 2n} \frac{J_{d/2}(\sqrt{t}) \cos\left(\frac{\pi d}{2}\right) - J_{-d/2}(\sqrt{t})}{\sin\left(\frac{\pi d}{2}\right)} \equiv Y_n(\sqrt{t}) \tag{3.27}$$

Then choosing

$$c_2 = -\cos\left(\frac{\pi d}{2}\right) c_1 = \frac{2(d-2)}{2^{d/2}\Gamma(\frac{d}{2})} \pi \cot\left(\frac{\pi d}{2}\right) \tag{3.28}$$

the expression (3.24) will have a unique well-defined limit in even dimension which satisfies also the matching condition. The solution is nonlocal since Y_n has a logarithmic dependence of $\rho\Box$, c.f. e.g. [17].

We stress that this finite solution evaluated at $\rho = \bar{\rho}$ is not the renormalized expectation value of the energy momentum tensor: while it has the correct leading logarithmic $\bar{\rho}$ – independent term it contains also terms where the logarithm is multiplied by powers of $\bar{\rho}$, i.e. inverse powers of μ^2. The correct procedure is to use the regulated g_{ij} to evaluate the action. The integration over ρ should be extendable to infinity. The expression for the effective action in noninteger dimensions is therefore Weyl invariant the only possible poles reflecting the singularity at 0 of the ρ-integration. The counterterms needed will give now renormalized expressions which have correct analyticity as we reviewed at the beginning of Section 2. An explicit verification of the analytic structure in perturbation theory around flat space was done in [18]. For a discussion of "holographic renormalization" see [14][19][20].

4. The Solution of the PBH equations for $\overset{(3)}{g}$

The results of the previous section prove that the normalizations of the cohomologically nontrivial Bach tensors which contain terms with one curvature in all $\overset{(n)}{g}$ are uniquely determined by the PBH equations. In general there are more than one Bach tensors contributing to a given $\overset{(n)}{g}$ whose expressions start with $1, 2, \ldots, n$ curvature tensors. A complete proof that all the normalizations are related requires a determination of the relative contributions of these various tensors to $\overset{(n)}{g}$. We cannot offer a general proof. Instead we will check that for the first non-trivial case $\overset{(3)}{g}$ when three Bach tensors are present, their coefficients are completely fixed.

In order to solve the PBH equation for $\overset{(3)}{g}$ we expand its Weyl variation in terms of the lower $\overset{(n)}{g}$:

$$\begin{aligned}
\delta \overset{(3)}{g}_{ij} =& - 4\sigma \overset{(3)}{g}_{ij} - \frac{1}{4}(\nabla_k \overset{(1)}{g}_{ij}) \overset{(1)}{g}{}^{kl} \nabla_l \sigma - \frac{1}{4}\left[\overset{(1)}{g}_{ik} \nabla_j \overset{(1)}{g}{}^{kl} + \overset{(1)}{g}_{jk} \nabla_i \overset{(1)}{g}{}^{kl} \right] \nabla_l \sigma \\
&+ \frac{1}{2}\left[\nabla_l \overset{(2)}{g}_{ij} - \frac{1}{3}\left(\nabla_i \overset{(2)}{g}_{jl} + \nabla_j \overset{(2)}{g}_{il} \right) \right] \nabla^l \sigma + \frac{1}{6}\left(\nabla_i \overset{(1)}{g}{}^2{}_{jl} + \nabla_j \overset{(1)}{g}{}^2{}_{il} \right) \nabla^l \sigma \qquad (4.1) \\
&+ \frac{1}{3}\left(\overset{(2)}{g}_{jk} \nabla_i \nabla^k \sigma + \overset{(2)}{g}_{ik} \nabla_j \nabla^k \sigma \right) - \frac{1}{12}\left(\overset{(1)}{g}{}^2{}_{il} \nabla_j \nabla^l \sigma + \overset{(1)}{g}{}^2{}_{jl} \nabla_i \nabla^l \sigma \right)
\end{aligned}$$

where $\overset{(1)}{g}$ and $\overset{(2)}{g}$ are the solutions of the PBH equations presented in [5]. We remark that $\overset{(2)}{g}$ has two free parameters c_1, c_2 which reflects the fact that there are two symmetric tensors with four derivatives built from $\overset{(0)}{g}_{ij}$ which transform homogenously with weight two.

On the other hand $\overset{(3)}{g}_{ij}$ can be expanded in a general basis of tensors with six derivatives: $\Box\nabla_i\nabla_j R$, $\Box^2 R_{ij}$, $R\nabla_i\nabla_j R$, $R_{ij}\Box R$, etc. A complete basis, which contains 59 elements, can be found in [21]. All these terms transform with weight four under constant rescalings of the metric. Under local rescalings their transformations contain up to six derivatives of σ. In contrast to this (4.1) only contains at most two derivatives of σ. Matching the coefficients we obtain the solution for $\overset{(3)}{g}$, unique up to terms with six derivatives transforming homogenously. There are eight such symmetric tensors.

We isolate in the solution the contribution of the three Bach tensors $\mathcal{B}^\alpha$ defined as (c.f.(2.5))

$$\mathcal{B}^\alpha_{ij} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{ij}} \int d^d x \sqrt{g}\, \mathcal{C}_\alpha \tag{4.2}$$

for $\alpha = 1, 2, 3$. The expressions $\mathcal{C}_\alpha$ listed below transform homogenously with weight six under a Weyl transformation.

$$\mathcal{C}_1 = C^{klmn} C_{mnpq} C^{pq}{}_{kl} \tag{4.3}$$

$$\mathcal{C}_2 = C^{ijkl} C_i{}^m{}_k{}^n C_{jmln} \tag{4.4}$$

$$\mathcal{C}_3 = C^{ijkl} \Box C_{ijkl} + \ldots \tag{4.5}$$

The complete expression for $\mathcal{C}_3$ can e.g. be found in [22]. The solution for $\overset{(3)}{g}$ contains then the contributions:

$$\overset{(3)}{g}_{ij} = \frac{1}{576(d-6)} \left(7\mathcal{B}^{(1)}_{ij} + 4\mathcal{B}^{(2)}_{ij} + 6\mathcal{B}^{(3)}_{ij} \right) + \text{finite terms} \tag{4.6}$$

We remark that even though $\overset{(2)}{g}$ contains the arbitrary coefficients c_1, c_2 the Bach terms do not depend on them verifying our conjecture that all the type B coefficients in all the terms $\overset{(n)}{g}$ are uniquely determined.

The cohomologically nontrivial type A contribution, being given by a combination of homogenously transforming terms, is not determined by the PBH equations as discussed in [8]. If we solve the equations of motion for e.g. the simplest action (2.6) we obtain the additional pole term in (4.6) $\frac{\mathcal{J}_{ij}}{576(d-6)}$ where $\mathcal{J}_{ij}$ is the tensor corresponding to the Euler density in $d = 6$, i.e. $\frac{1}{d-6}\mathcal{J}_{ij} = A^E_{(3)ij}$, c.f. (2.4).

5. Anomalies for the General Quadratic Action

All the type B terms in $\overset{(n)}{g}_{ij}$ are determined as discussed in Sections 3 and 4. Combining this with the action dependent linear relation between $\overset{(n)}{g}_{ij}$ and the anomalies allows us to calculate directly the anomalies from the universal terms in $\overset{(n)}{g}_{ij}$.

For the type A anomaly we need to calculate the relevant action dependent contribution in the equation of motion and then use the same linear relation to the anomaly. Alternatively we can use the universal, action dependent relation proven in [5].

We exemplify this type of calculation for the action (2.12) containing general quadratic terms in the curvatures. The condition for this action to admit an AdS solution is:

$$\Lambda = \frac{1}{2}d(d-1) + (d-3)\left(\frac{\alpha}{2}d^2(d+1) + \frac{\beta}{2}d^2 + \gamma d\right) > 0 \tag{5.1}$$

The general relation between the anomalies and $g_{ij}^{(n)}$ is given for (2.12) by (2.13).

For the trace anomalies in $d = 4$ we need $\overset{(2)}{g}$ whose general expression is [8]:

$$\overset{(2)}{g}_{ij} = -\frac{B_{ij}}{16(d-4)} - \frac{a}{8(d-4)}\left(\frac{1}{4}C^2\overset{(0)}{g}_{ij} - C_{ij}^2\right) + \text{finite} \tag{5.2}$$

the type B being fixed while the type A has an action dependent parameter a. In order to find a one solves the equation of motion for $\overset{(2)}{g}$ isolating the type A combination

$$\Delta\overset{(2)}{g}_{ij} = b_1 C^2 \overset{(0)}{g}_{ij} + b_2 C_{ij}^2 \tag{5.3}$$

We obtain for b_1, b_2

$$\begin{aligned} b_1 &= -\frac{1}{32} - \frac{\gamma}{4(d-4)(1+40\alpha+8\beta-4\gamma)} + \text{finite as } d \to 4 \\ b_2 &= \frac{1}{8} + \frac{\gamma}{(d-4)(1+40\alpha+8\beta-4\gamma)} + \text{finite as } d \to 4 \end{aligned} \tag{5.4}$$

This gives

$$a = \frac{1+40\alpha+8\beta+4\gamma}{1+40\alpha+8\beta-4\gamma} \tag{5.5}$$

Combining with (2.13) this gives an expression for the anomalies

$$\langle T_i^i \rangle = -\frac{1}{8}\left\{(1+40\alpha+8\beta-4\gamma)C^2 - (1+40\alpha+8\beta+4\gamma)E_4\right\} \tag{5.6}$$

matching exactly the standard, “bulk” calculation [3],[15],[16],[23].

The coefficient of E_4 matches also the general formula of [5], namely it is essentially the value of the action evaluated on AdS space.

Another check can be made in $d = 6$ for the simplest action (2.6). From [5] we know that the normalization of the type A anomaly in $d = 2n$ dimensions is $\frac{4n}{2^{2n}(n!)^2}$ where the factor $4n = 2d$ is $R - 2\Lambda$ evaluated on AdS_{2d+1}. For $d = 6$ this gives $\frac{1}{192}E_6$. Using (4.6), the normalization of the Euler term $\overset{(3)}{g}_{ij} = \dfrac{\mathcal{J}_{ij}}{576(d-6)} + \ldots$ and the linear relation (2.11) which gives for $\overset{(3)}{g}$ a factor -3 we find perfect agreement for type A. The type B follows from the singular terms displayed in (4.6) and agrees with the results of [3].

6. Discussion

The use of PBH equations allows to uncover the universal features of trace anomalies in CFTs which have a holographic dual:

(i) The relative normalizations of the type B trace anomalies are completely fixed.

(ii) One is left with two action dependent overall normalizations for the type A and for the type B. Even though these normalizations are action dependent their dimensional dependence is fixed relating theories in different dimensions which are dual to the same gravitational lagrangian.

The features above appear in a classical context relating directly the nontrivial cohomology of the solutions of the PBH equations to the anomalies. A very much related manifestation of the same structure is the FG ambiguity [9].

Though the detailed structure of the equations is different, their spirit is very similar to the relation between chiral anomalies and the Chern-Simons lagrangians through the descent equations. This is satisfactory since in supersymmetric theories chiral and trace anomalies appear in the same supermultiplet.

Following this analogy the gravitational lagrangian corresponds to the elliptic genus, the coefficients of the curvatures being the analogues of the chiral matter representation dependent traces.

The question if the trace anomalies of every CFT can be represented by a holographic gravitational lagrangian is completely open. In particular the field theoretical meaning of the completely fixed ratios of type B anomalies is intriguing.

Acknowledgement: We thank D.Perini for his collaboration

Appendix: PBH transformations

Since the PBH transformations play an essential rôle, we will briefly review them; more details can be found in [5].

Following [9] and [3] we write the bulk metric in the form

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4}\left(\frac{d\rho}{\rho}\right) + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j \tag{A.1}$$

where the conformal boundary is at $\rho = 0$. We will set the length scale $l = 1$. The PBH transformations are those bulk-diffeomorphisms which leave the form of (A.1) invariant. They are parameterized by a scalar function $\sigma(x)$ and change $g_{ij}(x,\rho)$ as [5]

$$\delta g_{ij}(x,\rho) = 2\sigma(1-\rho\partial_\rho)g_{ij}(x,\rho) + \nabla_i a_j(x,\rho) + \nabla_j a_i(x,\rho) \tag{A.2}$$

where $a_i = g_{ij}a^j$ and

$$a^j = \frac{1}{2}\int_0^\rho d\rho' g^{jk}(x,\rho')\partial_k\sigma(x)\,. \tag{A.3}$$

The covariant derivatives in (A.2) are w.r.t. the metric $g_{ij}(x,\rho)$.

References

[1] M. J. Duff, “Twenty years of the Weyl anomaly,” Class. Quant. Grav. **11**, 1387 (1994), hep-th/9308075 and references therein.

[2] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.

[3] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP **9807**, 023 (1998), hep-th/9806087.

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. **323**, 183 (2000), hep-th/9905111.

[5] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, “Diffeomorphisms and holographic anomalies,” Class. Quant. Grav. **17**, 1129 (2000), hep-th/9910267.

[6] R.Penrose and W. Rindler, “Spinors and Spacetime,” CUP 1986, vol.2, chapter 9.

[7] J.D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: an Example from Three-Dimensional Gravity,” Commun. Math. Phys. **104** (1986) 207.

[8] A. Schwimmer and S. Theisen, “Diffeomorphisms, anomalies and the Fefferman-Graham ambiguity,” JHEP **0008**, 032 (2000), hep-th/0008082.

[9] C. Fefferman and R. Graham, “Conformal Invariants,” Astèrisque, hors série, 1995, p.95.

[10] S. Deser and A. Schwimmer, “Geometric Classification of Conformal Anomalies in Arbitrary Dimensions,” Phys. Lett. B **309**, 279 (1993), hep-th/9302047.

[11] J. de Boer, E. Verlinde and H. Verlinde, “On the holographic renormalization group,” JHEP **0008**, 003 (2000), hep-th/9912012.

[12] J. Kalkkinen, D. Martelli and W. Mück, “Holographic renormalisation and anomalies,” JHEP **0104**, 036 (2001), hep-th/0103111.

[13] D. Martelli and W. Mück, “Holographic renormalization and Ward identities with the Hamilton-Jacobi method,” Nucl. Phys. B **654**, 248 (2003), hep-th/0205061.

[14] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. **217**, 595 (2001), hep-th/0002230.

[15] S. Nojiri and S. D. Odintsov, “On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence,” Int. J. Mod. Phys. A **15**, 413 (2000) hep-th/9903033.

[16] M. Blau, K. S. Narain and E. Gava, "On subleading contributions to the AdS/CFT trace anomaly," JHEP **9909**, 018 (1999), hep-th/9904179.

[17] M Abramowitz and I. Stegun, "Handbook of Mathematical Functions," National Bureau of Standards Applied Mathematics Series 55, tenth printing 1972.

[18] G. Arutyunov and S. Frolov, "Three-point Green function of the stress-energy tensor in the AdS/CFT correspondence," Phys. Rev. D**60**, 026044 (1999), hep-th/9912210.

[19] M. Bianchi, D. Z. Freedman and K. Skenderis, "How to go with an RG flow," JHEP **0108**, 041 (2001), hep-th/0105276; "Holographic renormalization," Nucl. Phys. B **631**, 159 (2002), hep-th/0112119.

[20] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19**, 5849 (2002), hep-th/0209067.

[21] S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummings, "Normal forms for tensor polynomials: I. The Riemann tensor", Class. Quantum Grav. **9** (1992) 1151.

[22] J. Erdmenger, "Conformally covariant differential operators: Properties and applications," Class. Quant. Grav. **14**, 2061 (1997), hep-th/9704108.

[23] S. Theisen, unpublished.